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1. The Haar integral (and measure) on locally compact groups

We begin with a number of definitions:

Definition 1.1. *Let f be a continuous function on a metric space X (or, more generally, on a topological space). The* **support** *of f; supp f is given as*

$$\text{supp } f = \overline{\{x \in X \mid f(x) \neq 0\}}.$$

Furthermore, $C_c(X)$ denotes the space of (complex) continuous functions whose support is compact.

Notice that the space of real-valued continuous functions with compact support is a real (but not complex) subspace of $C_c(X)$.

Definition 1.2. *A* linear functional *$\mathcal{L}$ on a vector space V is a linear map $\mathcal{L} : V \mapsto \mathbb{C}$. If $V = C_c(X)$, the linear functional $\mathcal{L}$ is* **positive** *if*

$$\forall f \geq 0 : \mathcal{L}(f) \geq 0.$$

The following theorem represents positive linear functionals on $C_c(X)$, the space of continuous complex valued functions of compact support. The Borel sets in the following statement refers to the σ-algebra generated by the open sets.

A non-negative countably additive Borel measure μ on a locally compact Hausdorff space X is regular iff

- $\mu(K) < \infty$ for every compact set K;
- For every Borel set E,
$$\mu(E) = \inf\{\mu(U) : E \subseteq U, U \text{ open}\}$$
- The relation
$$\mu(E) = \sup\{\mu(K) : K \subseteq E\}$$
holds whenever E is open or when E is Borel and $\mu(E) < \infty$.

Theorem 1.3. *Theorem. Let X be a locally compact Hausdorff space. For any positive linear functional $\mathcal{L}$ on $C_c(X)$, there is a unique Borel regular measure μ on X such that*

$$\mathcal{L}(f) = \int_X f(x)d\mu(x) \quad \textit{for all } f \in C_c(X).$$

In the case of a compact Hausdorff space, this is the so-called Riesz Representation Theorem.

Consider now a compact Lie group K. A positive linear functional I on $C(K)$ is called a left invariant Haar integral if

$$\forall f \in C(K); \forall g \in K : I(L(g)f) = I(f).$$

Similarly, it is a right invariant Haar integral if

$$\forall f \in C(K); \forall g \in K : I(R(g)f) = I(f).$$

Theorem 1.4. *On a compact Lie group K, there is a unique (up to multiplication by a positive constant) left Invariant Haar integral I. It is also right invariant. We write $I(f) = \int_K f d\mu = \int_K f(g)d\mu(g)$. The following equations hold; the first two are just the stated invariance; the third is an additional identity[1]. We set $\tilde{f}(g) = f(g^{-1})$:*

$$\forall f \in C(K); \forall h \in K :$$

$$\begin{aligned}
\int_K f(hg)d\mu(g) &= \int_K f(g)d\mu(g) \;\; \big(I(L(h^{-1})f) = I(f)\big) \quad \textit{left inv,} \\
\int_K f(gh)d\mu(g) &= \int_K f(g)d\mu(g) \;\; \big(I(R(h)f) = I(f)\big) \quad \textit{right inv.} \\
\int_K f(g^{-1})d\mu(g) &= \int_K f(g)d\mu(g) \;\; \Big(I(\tilde{f}) = I(f)\Big) \quad \textit{inversion inv.}
\end{aligned}$$

The measure, moreover, satisfies that the measure of K is finite. It is customary, and we shall also do this, to **normalize the Haar measure such that** $\mu(K) = 1$. The Haar measure satisfies furthermore that if f is a non-negative continuous function which is strictly positive in some point, then $I(f) > 0$.

We denote the space of square integrable functions on K by $L^2(K, d\mu)$ or just by $L^2(K)$. The inner product[2] is given by

$$\langle f_1, f_2 \rangle = \int_K f_1(g)\overline{f_2(g)}d\mu(g). \tag{1}$$

Since K is compact, and since the measure is finite, it is clear that $C(K) \subset L^2(K)$.

[1] The left regular and the right regular representation are given in (19) and (20), respectively

[2] Incidentally, our inner products are all linear in the left variable!

2. The Haar measure on some groups

2.1. **On $U(n)$.** The Cayley transform $\mathcal{C}$

$$\mathcal{C} : h \mapsto \frac{1 + \frac{1}{2}ih}{1 - \frac{1}{2}ih}$$

maps the space $H(n)$ of $n \times n$ hermitian matrices onto an open dense subset of $U(n)$. Using it, one may define an integral on $U(n)$ for each $p \in \mathbb{R}$ by

$$\int_{U(n)} f(u)du = \int_{H(n)} f\left(\frac{1 + \frac{1}{2}ih}{1 - \frac{1}{2}ih}\right) |\det(1 + \frac{1}{2}ih)|^p dh.$$

For a specific value of p ($p = -n$ as far as I remember!) this is the Haar integral. In the formula, dh is the Lebesgue measure on the real vector space $H(n)$ of dimension n^2.

2.2. **The Haar measure on $SU(2)$.** Recall that $SU(2) = S^3$ as a topological space. Specifically, write elements g of $SU(2)$ as

$$g = \begin{pmatrix} x_1 + ix_4 & x_2 + ix_3 \\ -x_2 + ix_3 & x_1 - ix_4 \end{pmatrix}.$$

Let us recall the definition of spherical coordinates in $\mathbb{R}^4$:

$$\begin{array}{rclrcl} x_1 & = & r\cos(\theta) & x_2 & = & r\sin(\theta)\cos(\psi) \\ x_3 & = & r\sin(\theta)\sin(\psi)\cos(\phi) & x_4 & = & r\sin(\theta)\sin(\psi)\sin(\phi) \end{array}$$

The Jacobian determinant ("change-of-coordinates" determinant) has absolute value

$$|\det(\frac{\partial x_1, x_2, x_3, x_4}{\partial r, \theta, \psi, \psi}| = r^3 \sin^2(\theta)\sin(\psi).$$

The condition on $g \in SU(2)$ is that $r^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. Thus, it follows that

$$\frac{1}{2\pi^2} \int_{\theta,\psi,\phi=0}^{\pi,\pi,2\pi} f(\theta, \psi, \phi)\sin^2(\theta)\sin(\psi)d\theta d\psi d\phi. \tag{2}$$

If f is constant in ψ, ϕ; i.e. if there exists a function F such that $\forall \theta, \psi, \phi : f(\theta, \psi, \phi) = F(\theta)$, we obtain

(3)

$$\frac{1}{2\pi^2} \int_{\theta,\psi,\phi=0}^{\pi,\pi,2\pi} f(\theta, \psi, \phi)\sin^2(\theta)\sin(\psi)d\theta d\psi d\phi = \frac{2}{\pi}\int_{\theta=0}^{\pi} F(\theta)\sin^2(\theta)d\theta.$$

This is relevant if for instance f is a class function[3].

2.3. **The Haar measure on $GL(n,\mathbb{R})$.** The Haar measure on the group $(\mathbb{R}\setminus\{0\},\cdot)$ is given by

$$\int_{x\neq 0} f(x)\frac{dx}{|x|}.$$

More generally, the Haar measure on $GL(n,\mathbb{R})$ is given as follows: Let $dx = dx_{11}\cdots dx_{n1}dx_{12}\cdots dx_{n2}\ldots dx_{nn}$ be the restriction of the Lebesgue measure on $\mathbb{R}^{n^2}$ the open subset $\{x=\{x_{ij}\}_{i,j=1}^{n} \mid \det(x)\neq 0\}$. Then the Haar measure is given as

$$\int f(x)\frac{dx}{|\det(x)|^n}.$$

3. Unitarity and complete reducibility

Definition 3.1. *A representation Π of a Matrix Lie Group is* **unitary** *if $\Pi : G \mapsto U(n)$ for some $n\in\mathbb{N}$. Equivalently, Π is a representation in a finite-dimensional Hilbert space $(V,\langle\cdot,\cdot\rangle)$ and each $\Pi(g)$ preserves the inner product $\langle\cdot,\cdot\rangle$:*

$$\forall v,w\in V;\forall g\in G : \langle\Pi(g)v,\Pi(g)w\rangle = \langle v,w\rangle.$$

Definition 3.2. *A representation Π in V is completely decomposable or* **completely reducible** *if there exists a direct sum decomposition*

$$V = \oplus_{k=1}^{r} V_k$$

such that each V_k is an invariant subspace for Π and such that the resulting representation of G on V_k is irreducible for each k. Denote these representations by Π_k; $k=1,2,\ldots,r$, and set $V_k = V_{\Pi_k}$. We then may write

$$V_\Pi = \oplus_{k=1}^{r} V_{\Pi_k}, \text{ and } \Pi = \oplus_{k=1}^{r}\Pi_k. \tag{4}$$

Notice that the representations Π_k are not necessarily all inequivalent. The matrices representing the operators $\Pi(g)$ consists of k diagonal blocks when expressed according to the decomposition (4). We let $[\Pi:\Pi_k]$ denote the **multiplicity** *of Π_k in Π. This is the number of times Π_k (or representations equivalent to it) occurs in the sum (4).*

[3] A function f of a group G which satisfies: $\forall g,h\in G : f(hgh^{-1}) = f(g)$; i.e. which is constant on conjugacy classes.

Proposition 3.3. *If Π is a unitary representation on V and if W is an invariant subspace, then*

$$W^{\perp} = \{v \in V \mid \forall w \in W : \langle w, v\rangle = 0\}$$

is an invariant subspace.

Proof: We assume that it is known that $W^{\perp}$ indeed **is** a subspace and that $V = W \oplus W^{\perp}$ (otherwise: prove it!). We move directly to invariance: Let $v \in W^{\perp}, w \in W$. Then

$$\langle \Pi(g)v, w\rangle = \langle \Pi(g)v, \Pi(g)\Pi(g^{-1})w\rangle = \langle v, \Pi(g^{-1}) = 0.$$

Here, the first equality is just a re-writing, the second is unitarity, and the third follows from the invariance of W. The claim is thus verified. $\square$

The key observation now is the following:

Proposition 3.4. *If G is a compact matrix Lie Group, and Π is a finite-dimensional representation on V_{Π}, then there is an inner product $\langle \cdot, \cdot\rangle$ on V_{Π} which is invariant, i.e. such that Π is unitary with respect to this.*

Proof: This is where the Haar measure $d\mu(g)$ comes in! Let us namely start with an arbitrary inner product $\langle\langle \cdot, \cdot\rangle\rangle$ on V (such exist of course in abundance). Define

$$\langle v, w\rangle = \int_G \langle\langle \Pi(g)v, \Pi(g)w\rangle\rangle d\mu(g).$$

That this indeed is a positive-definite inner product is left as an exercise. The crucial computation is, for $h \in G$:

$$\begin{aligned}
\langle \Pi(h)v, \Pi(h)w\rangle &= \int_G \langle\langle \Pi(g)\Pi(h)v, \Pi(g)\Pi(h)w\rangle\rangle d\mu(g) \\
&= \int_G \langle\langle \Pi(gh)v, \Pi(gh)w\rangle\rangle d\mu(g) \\
&= \int_G \langle\langle \Pi(g)v, \Pi(g)w\rangle\rangle d\mu(g) \\
&= \langle v, w\rangle.
\end{aligned}$$

Here, the first equality is the definition, the second is saying that Π is a homomorphism, the third is the right invariance of the Haar measure, and the last is the definition. $\square$

Combining Proposition 3.3 with Proposition 3.4 we obtain the following result (which for finite groups is Maschke's Theorem)

Theorem 3.5. *Any finite-dimensional representation of a compact Matrix Lie Group is completely reducible.*

Notice that we only used that there was a right invariant Haar measure on G and that the total measure of G was finite.

For completeness, as well as additional information, we state

Theorem 3.6 (Schur's Lemma; special version).

- *If Π_V, Π_W are irreducible representations of a group G, and $\Phi : V \mapsto W$ is an intertwiner, i.e. if Φ is linear and*

$$\forall g \in G : \Phi \circ \Pi_V(g) = \Pi_W(g) \circ \Phi, \tag{5}$$

  then either Φ is a bijection (and then the two representations are equivalent) or $\Phi = 0$.
- *If Π_V is irreducible and if a linear map $\Psi : V \mapsto V$* commutes with *Π_V, i.e.*

$$\forall g \in G : \Psi \circ \Pi_V(g) = \Pi_V(g) \circ \Psi, \tag{6}$$

  then there is a $\lambda \in \mathbb{C}$ such that $\Psi = \lambda \cdot I_V$, where I_V is the identity operator on V.
- *Suppose G is a compact (Matrix Lie) Group. Then the converse to the previous point holds. That is, if Π_V is a finite-dimensional representation on V with the property that the only operators that commute with Π_V are the operators $\lambda \cdot I_V$, then Π_V is irreducible.*

Proof:(Sketched) The first two points are proved in Hall's book. As for the third, begin by assuming that the representation is unitary (Proposition 3.4). If W is an invariant subspace consider the orthogonal projection P_W onto W. Use Proposition 3.3 to prove that this operator commutes with the representation. Hence $P_W = 0$ or $P_W = I_V$. $\square$

4. Matrix coefficients and characters

Definition 4.1. *Let Π be a finite-dimensional complex representation of a group G on a vector space V. A function on G of the form*

$$F_{v,v^*} : G \ni g \mapsto (v^*, \Pi(g)v) \tag{7}$$

for any $v \in V$ and any $v^* \in V^*$ is a **matrix coefficient** *(of Π). We let*

$$\mathcal{M}_\Pi = Span\{(v^*, \Pi(\cdot)v, v) \mid v^* \in V^*, v \in V\}. \tag{8}$$

In the case of a Matrix Lie Group, these functions are clearly *continuous*.

Remark 4.2. *If the vector space $V = V_\Pi$ has an inner product $\langle \cdot, \cdot \rangle$ then, without additional assumptions about unitarity,*

$$\mathcal{M}_\Pi = Span\{\langle \Pi(\cdot)v_1, v_2\rangle \mid v_1, v_2 \in V\}. \tag{9}$$

The matrix coefficients are of fundamental importance, as we shall see below. However, we first introduce

Definition 4.3. *The* character χ^Π *of a representation* Π *of a group* G *on a finite-dimensional vector space* V *is the function from* G *to* $\mathbb{C}$ *given by*

$$\chi^\Pi(g) = \mathrm{Tr}_{V_\Pi}\Pi(g).$$

The following is obvious:

Proposition 4.4. *Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of V, and let $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ be the corresponding dual basis of V^*. Then*

$$\chi^\Pi(g) = \sum_{k=1}^{n}(\varepsilon_k, \Pi(g)e_k) = \sum_{k=1}^{n}(\Pi(g))_{kk}.$$

We will from now on set $\Pi_{kk}(g) = (\Pi(g))_{kk}$. Notice that χ^Π is a sum of matrix coefficients of Π.

In view of Theorem 3.5, the following is clear:

Lemma 4.5. *The character χ^Π of a finite-dimensional representation is a sum of the characters of the irreducible representations that occurs in its decomposition (c.f. Definition 3.2);*

$$\chi^\Pi = \sum_k \chi^{\Pi_k}.$$

4.1. **The character of the dual representation.** Recall from the chapter on Linear Algebra, specifically (40) on p.10 that the dual representation in matrix form is given as

$$\Pi'(g) = \Pi(g^{-1})^t. \tag{10}$$

Since we by Proposition 3.4 may assume that Π is unitary, it follows that we may assume that $\Pi(g^{-1}) = \Pi(g)^*$ (the complex conjugate matrix) and hence that $\Pi'(g) = \overline{\Pi(g)}$ (the matrix whose entries are the complex conjugates of the entries (in the same position) of $\Pi(g)$. In particular we get

Proposition 4.6.

$$\chi^{\Pi'}(g) = \sum_i \overline{\Pi_{ii}(g)} = \overline{\chi^{\Pi}(g)}.$$

4.2. **The character of a tensor product.** Let Π, Σ be two finite dimensional representations of G in V_Π, W_Σ, respectively.

Proposition 4.7.

$$\chi^{\Pi\otimes\Sigma} = \chi^{\Pi} \cdot \chi^{\Sigma}.$$

Proof: Let $\{e_1, e_2, \ldots, e_n\}$ and $\{f_1, f_2, \ldots, f_m\}$ be bases of V_Π and W_Σ, respectively. Since $(\Pi \otimes \Sigma)(g)(e_i \otimes f_j) = \Pi(g)e_i \otimes \Sigma(g)f_j$, it follows that the coefficient of $(\Pi \otimes \Sigma)(g)(e_i \otimes f_j)$ in $e_i \otimes f_j$ is given as $\Pi_{ii}(g) \cdot \Sigma_{jj}(g)$. But then

$$\mathrm{Tr}_{V_\Pi \otimes W_\Sigma}(\Pi\otimes\Sigma)(g) = \sum_{i,j} \Pi_{ii}(g)\cdot\Sigma_{jj}(g) = \left(\sum_i \Pi_{ii}(g)\right)\cdot\left(\sum_j \Sigma_{jj}(g)\right).$$

The claim follows immediately from this. $\square$

4.3. **Schur orthogonality.**

Lemma 4.8. *Let Π_V and Π_W be complex representations of the compact Matrix Lie Group G. Let $\langle\cdot,\cdot\rangle$ be an inner product on Π_V (not assumed invariant). If $v_1 \in V, w_1 \in W$ then the map $T : V \mapsto W$ given by*

$$T(v) = \int_G \langle \Pi_V(g)v, v_1\rangle \Pi_W(g^{-1})w_1 \, d\mu(g)$$

is G-equivariant (an intertwining map between Π_V and Π_W).

Proof: The right invariance of the Haar measure gives that

$$T(\Pi_V(h)v) = \int_G \langle \Pi_V(gh)v, v_1\rangle \Pi_W(g^{-1})w_1 \, d\mu(g) = \Pi_W(h)T(v).$$

$\square$

Theorem 4.9 (Schur orthogonality 1)**.** [4]*Suppose that Π_V, Π_W are irreducible representations of the compact group G. Either every matrix coefficient of Π_V is orthogonal in $L^2(G)$ to every matrix coefficient of Π_W, or the two representations are equivalent.*

[4] See e.g. D. Bump: "Lie Groups" (Graduate Texts in Mathematics, Springer Verlag 2004)

Proof: Let us assume that there are *invariant* inner products $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$ on V and W, respectively. We wish to prove that if some matrix coefficients are not orthogonal, then the representations are equivalent. It suffices to take matrix coefficients of the form (9): $\langle \Pi_V(g)v_1, v_2\rangle_V$ and $\langle \Pi_W(g)w_1, w_2\rangle_W$. Our assumption is that for some v_1, v_2, w_1, w_2

$$
\begin{aligned}
(11)\qquad 0 &\neq \int_G \langle \Pi_V(g)v_1, v_2\rangle_V \overline{\langle \Pi_W(g)w_1, w_2\rangle_W} d\mu(g) \\
(12)\qquad &= \int_G \langle \Pi_V(g)v_1, v_2\rangle_V \langle \Pi_W(g^{-1})w_2, w_1\rangle_W d\mu(g) \\
(13)\qquad &= \langle T(v_1), w_1\rangle,
\end{aligned}
$$

where T is defined as in Lemma 4.8 in terms of the vectors v_2, w_2. But this means that T is not zero, hence, by Schur's Lemma, T is an isomorphism, i.e. the representations are equivalent. $\square$

Theorem 4.10 (Schur orthogonality 2)**.** [5]*Let Π_V be an irreducible representation of a compact group G with invariant inner product $\langle \cdot, \cdot \rangle_V$. Then, for all $v_1, v_2, v_3, v_4 \in V$:*

(14)
$$\int_G \langle \Pi_V(g)v_1, v_2\rangle_V \overline{\langle \Pi_V(g)v_3, v_4\rangle_V} d\mu(g) = \frac{1}{\dim V}\langle v_1, v_3\rangle \langle v_4, v_2\rangle.$$

Theorem 4.11 (Schur orthogonality 3)**.** *Let Π_V, Π_W be irreducible representations of the Matrix Lie Group G. Then*

$$(15)\quad \langle \chi^V, \chi^W\rangle_{L^2(G)} = \int_G \chi^V(g)\overline{\chi^W(g)} d\mu(g) = \begin{cases} 1 & \textit{if } \Pi_V \equiv \Pi_W \\ 0 & \textit{otherwise} \end{cases}$$

Proof: As remarked after Proposition 4.4, the characters are sums of matrix coefficients, hence if Π_V and Π_W are inequivalent it follows from Theorem 4.9 that $\langle \chi^V, \chi^W\rangle_{L^2(G)} = 0$. In case $\Pi_V = \Pi_W$ it follows from Proposition 4.6 and Proposition 4.7 that

$$\langle \chi^V, \chi^V\rangle_{L^2(G)} = \int_G \chi^{\Pi\otimes\Pi'} d\mu(g).$$

Now notice that if Π is any finite-dimensional representation, then χ^Π is a sum of the characters of the irreducible summands:

$$\chi^\Pi = \sum_i \chi^{\Pi_i},$$

[5]For a proof see e.g. Bump's book p. 12 and p. 15

where the Π_i are the irreducible summands (the same representation may occur several times!) Since the character of the trivial representation is the constant function 1 and since this has integral 1 w.r.t. the normalized Haar measure, it follows that

$$\int_G \chi^\Pi d\mu(g) = [\Pi : 1],$$

where $[\Pi : 1]$ denotes the number of times the trivial representation 1 occurs in the decomposition of Π into irreducible representations. We now show that $[\Pi_V \otimes \Pi'_V : 1] = 1$, and then we are done:

We first have to investigate, what it means for the trivial representation to occur in $\Pi_V \otimes \Pi'_V$: This means precisely that there is a non-zero vector $w \in V \otimes V^*$ which is fixed by all $\Pi(g) \otimes \Pi'(g)$. And the multiplicity $m = [\Pi_V \otimes \Pi'_V : 1] = 1$ is the number of mutually orthogonal such vectors $w_1, \cdots, w_m$. We want to show that $m = 1$: Let w be any one of the $w_1, \cdots, w_m$ (we assume that there is at least one! - see later). We define a map $T_w : V \mapsto V$ by

$$\forall v \in V, v^* \in V^* : (v^*, T(v)) = \langle v \otimes v^*, w\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes an invariant inner product in $V \otimes V^*$. We claim that T commutes with the representation Π:

$$\begin{aligned} &\forall g \in G; \forall v \in V, v^* \in V^* : \\ &\qquad (v^*, T(\Pi(g)v)) &=& \ \langle \Pi(g)v \otimes \Pi'(g)\Pi'(g^{-1})v^*, w\rangle \\ = \langle v \otimes \Pi'(g^{-1})v^*, (\Pi(g) \otimes \Pi'(g))\, w\rangle &&=& \ (v \otimes \Pi'(g^{-1})v^*, w\rangle \\ = (\Pi'(g^{-1})v^*, T(v)) &&=& \ (v^*, \Pi(g)T(v)). \end{aligned}$$

So, by Schur's Lemma, T is a scalar multiple λ of the identity. This constant is clearly non-zero, and we may absorb it into w. But then we may assume:

$$\forall v \in V, v^* \in V^* : (v^*, v) = \langle v \otimes v^*, w\rangle. \tag{16}$$

This, on the other hand, completely determines w: If two vectors have the same inner products with all vectors $v \otimes v^*$ (which, after all, span $V \otimes V^*$), then they must be equal. Actually, equation (16) determines uniquely a vector w as can be seen by inserting $e_i \otimes \varepsilon_j$ for $v \otimes v^*$ for all choices of i, j. The result is that

$$w = \sum_i e_i \otimes \varepsilon_i.$$

This proves that the trivial representation occurs exactly once. $\square$

4.4. **Representations from actions; the left (and the right) regular representation.** Let G be a group which acts on a set M i.e. such that there is a map $G \times M \ni (g_0, m) \mapsto g_o \star m$ satisfying

$$(g_1 g_2) \star m = g_1 \star (g_2 \star m). \tag{17}$$

We can turn any such action into a representation U_a of G in the vector space $\mathcal{F}(M)$ of complex valued functions on M by the formula

$$(U_a(g)f)(m) = f(g^{-1} \star m). \tag{18}$$

Let G be any group. The **left regular representation** L is defined in the vector space $\mathcal{F}(G)$ of complex valued functions on G from the left action $g_0 \star_L g = g_0 g$ of the group on itself. Thus,

$$(L(g_0)f)(g) = f(g_0^{-1} g). \tag{19}$$

The **right regular representation** R is defined in the vector space $\mathcal{F}(G)$ of complex valued functions on G from the ("right") action $g_0 \star_R g = g(g_0)^{-1}$ of the group on itself. Thus,

$$(R(g_0)f)(g) = f(g g_0). \tag{20}$$

In the case of a compact group one usually considers L, R in $L^2(G, d\mu)$ and call these representations, which by means of the invariance of the Haar measure are unitary, the left, resp. right, regular representations.

The crucial observations are the following:

Proposition 4.12. *The matrix coefficients F_{v,v^*} in (7) belong to $L^2(G, d\mu)$ and satisfy*

$$L(g_0)F_{v,v^*} = F_{v,\Pi'(g_0)v^*} \ ; \ R(g_0)F_{v,v^*} = F_{\Pi(g_0)v,v^*}. \tag{21}$$

Proof: The functions F_{v,v^*} are continuous, hence bounded and measurable, hence belong to $L^2(G)$. The two identities are proved in much the same way, so we only do one of them: $(L(g_0)F_{v,v^*})(g) = (v^*, \Pi(g_0^{-1} g)v) = (\Pi'(g_0)v^*, v) = F_{v,\Pi'(g_0)v^*}$.

Lemma 4.13. *Let Π_V be an irreducible finite-dimensional representation of the compact (Matrix Lie) Group G. Then $\Pi_V \otimes \Pi'_V$ is an irreducible representation of $G \otimes G$.*

Proof: (Sketched) We use Theorem 3.6. Suppose an operator T on $V \otimes V^*$ commutes with $\Pi_V \otimes \Pi'_V$. Write T in block form corresponding to (60) in the Notes on Linear Algebra. (We are **not** assuming that T is of the form $A \otimes B$ for some A, B!) Using the irreducibility of the two representations, it follows from the fact that T commutes in particular with all operators of the form $\Pi(g) \otimes I_{V^*}$ that the blocks of T must be scalar multiples λ_{ij} of the identity in V^*. Then use the operators $I_V \otimes \Pi(g)$ to conclude that $\lambda_{ij} = \lambda \cdot \delta_{ij}$ for some $\lambda \in \mathbb{C}$. □

Combining Proposition 4.12 and Lemma 4.13, the following is obtained:

Proposition 4.14. *Let $\Pi = \Pi_V$ be an irreducible finite-dimensional representation of the compact (Matrix Lie) Group G. The linear map $\Psi_\Pi : V \otimes V^* \mapsto L^2(G, d\mu)$ given by its action on the tensors $v \otimes v^*$ as*

$$(\Psi_\pi(v \otimes v^*))(g) = F_{v,v^*}(g) = (v^*, \Pi(g)v), \tag{22}$$

is an intertwiner between the representations of $G \times G$ given by, respectively,

$$(g_1, g_2) \mapsto \Pi(g_1) \otimes \Pi'(g_2)$$

on $V \otimes V^*$ and

$$(g_1, g_2) \mapsto R(g_1) \otimes L(g_2)$$

on $L^2(G, d\mu)$, that is

$$\forall (g_1, g_2) \in G \times G : \Psi \circ (\Pi \otimes \Pi')(g_1, g_2) = (R \otimes L)(g_1, g_2) \circ \Psi. \tag{23}$$

5. Further notes on Representations of compact groups

5.1. The Peter-Weyl Theorem, proof in the case of compact matrix groups.

We begin by introducing the space $\widehat{G}$ of equivalence classes of irreducible finite-dimensional representations of the compact (Matrix Lie) Group G.

Theorem 5.1. *Let G be a compact group. Then*

$$\begin{aligned} L^2(G) &= \bigoplus_{\Pi \in \widehat{G}} \mathcal{M}_\Pi \\ L &= \bigoplus_{\Pi \in \widehat{G}} \underbrace{(\Pi \oplus \cdots \oplus \Pi)}_{\dim(V_\Pi)} \quad (\textit{orthogonal direct sum}) \\ L \otimes R^6 &= \bigoplus_{\Pi \in \widehat{G}} \Pi \otimes \Pi' \quad (\textit{as representation of } G \times G \textit{ on } L_2(G)). \end{aligned}$$

To prove this fundamental theorem in the matrix case, we use another fundamental theorem, namely the Stone-Weierstrass Theorem. For this

[6] This notation is a little unfortunate, since it is not a usual tensor product representation, but just refers to the representation given in (23). The tensor products on the right hand side of the equation are "correct" tensor product representations!

we need the concept of a *-algebra $\mathcal{A}$ of continuous functions on a topological space X. By this we mean a *vector space $\mathcal{A}$ of continuous functions which has the additional property that $\forall f, g \in \mathcal{A} : f \cdot g \in \mathcal{A}$ and $f^* = \overline{f} \in \mathcal{A}$.* We say that $\mathcal{A}$ *separates points* if $\forall x, y \in X \exists f \in \mathcal{A} : f(x) \neq f(y)$. The theorem can now be formulated:

Theorem 5.2 (Stone-Weierstrass)**.** *Let $\mathcal{A}$ be a *-algebra of continuous functions on a compact topological space K which separates points and does not vanish identically at any point.*[7] *Then $\mathcal{A}$ is uniformly dense in the space $\mathcal{C}(K)$ of continuous functions on K;*

$$\overline{\mathcal{A}}^{\|\cdot\|_\infty} = \mathcal{C}(K).$$

Since uniform convergence implies L_2 convergence we of course have

Corollary 5.3. *Under the same assumptions,*

$$\overline{\mathcal{A}}^{\|\cdot\|_2} = L_2(K).$$

The application we are interested in is the following:

Corollary 5.4. *Suppose that K is a compact subset of $\mathbb{R}^n$. Equip K with the induced (subset) topology. Let $\mathcal{A} = \mathcal{P}^K_{\mathbb{C}}(\mathbb{R}^n)$ be the set of restrictions to K of real polynomials with complex coefficients. Then*

$$\begin{aligned} \overline{\mathcal{P}^K_{\mathbb{C}}(\mathbb{R}^n)}^{\|\cdot\|_\infty} &= \mathcal{C}(K), \text{ (and hence)} \\ \overline{\mathcal{P}^K_{\mathbb{C}}(\mathbb{R}^n)}^{\|\cdot\|_2} &= L_2(K). \end{aligned}$$

Proof: This follows from Theorem 5.2 since it is clear that polynomials are continuous on $\mathbb{R}^n$ and hence that their restrictions are continuous on K. Moreover, it is obviously a *-algebra, and it separates points $x \neq y \in K$ since if $x \neq y$ then for some coordinate number, say, i, $x_i \neq y_i$. But the polynomial $P_i(r_1, \cdots, r_n) = r_i$ then separates x and y. □

The Gram-Schmidt ortho-normalization process applied to a space of polynomial functions clearly results in a basis of functions which still are polynomials. It then follows from the preceding corollary that

Corollary 5.5. *There is an orthonormal basis of $L_2(K)$ consisting of restrictions of polynomials.*

We now consider a compact matrix group $K \subset GL(n, \mathbb{R})$ or $K \subset GL(n, \mathbb{C})$. In the first case we consider $K \subset \mathbb{R}^{n^2} = \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n}$

[7] This means that there is no point $k \in K$ such that $f(k) = 0$ for all $f \in \mathcal{A}$.

and in the second case $K \subset \mathbb{R}^{2n^2} = \underbrace{\mathbb{C}^n \times \cdots \times \mathbb{C}^n}_{n}$ - where we view (when it is convenient) $\mathbb{C} = \mathbb{R}^2$. Let us refer to these cases as the real and, respectively, complex case. In both cases we work with polynomials with complex coefficients. In the real case we work with polynomials in variables $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ where each $\mathbf{x}_i \in \mathbb{R}^n; i = 1, 2, \ldots, n$, and in the complex case we work with polynomials in variables $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$, $\overline{\mathbf{z}_1}, \overline{\mathbf{z}_2}, \cdots, \overline{\mathbf{z}_n}$ where each $\mathbf{z}_i \in \mathbb{C}^n; i = 1, 2, \ldots, n$ and the variables $z, \overline{z}$ are viewed as independent real variables on $\mathbb{C}$. Since in any real situation we have $K \subset GL(n, \mathbb{R}) \subset GL(n, \mathbb{C})$ we will from now on only work with the complex situation. Observe that for $g \in K$, left multiplication by g in K; $M_g : k \mapsto g \cdot k$ becomes

$$(\mathbf{z}_1, \cdots, \mathbf{z}_n, \overline{\mathbf{z}_1}, \cdots, \overline{\mathbf{z}_n}) \overset{M_g}{\mapsto} (g\mathbf{z}_1, \cdots, g\mathbf{z}_n, \overline{g}\,\overline{\mathbf{z}_1}, \cdots, \overline{g}\,\overline{\mathbf{z}_n}),$$

where by definition, $\overline{g\mathbf{z}} = \overline{g}\,\overline{\mathbf{z}}$.

Now observe that the set $\mathcal{P}_{all} = \mathcal{P}(\mathbf{z}_1, \cdots, \mathbf{z}_n, \overline{\mathbf{z}_1}, \cdots, \overline{\mathbf{z}_n})$ of all polynomials in the variables $\mathbf{z}_1, \ldots, \mathbf{z}_n, \overline{\mathbf{z}_1}, \cdots, \overline{\mathbf{z}_n}$ has the form

$$\mathcal{P}_{all} = \mathcal{P}(\mathbf{z}_1, \ldots, \mathbf{z}_n, \overline{\mathbf{z}_1}, \cdots, \overline{\mathbf{z}_n}) = \mathcal{P}(\mathbf{z}_1) \cdot \mathcal{P}(\mathbf{z}_2) \cdot \cdots \cdot \overline{\mathcal{P}(\mathbf{z}_n)}.$$

We obtain a representation U_K of K on $\mathcal{P}_{all}$ as in (18). Specifically, in suggestive notation,

$$\forall g \in K : (U_K(g)p)\,(z, \overline{z}) = p(g^{-1} \star z, \overline{g}^{-1} \star \overline{z}), \tag{24}$$

where $g \star z = gz$ and $\overline{g} \star \overline{z} = \overline{gz} = \overline{gz}$.

Next observe that any polynomial in $\mathcal{P}(\mathbf{z}_1)$ can be written as a sum of products of the form $\prod_{i=1}^r \langle \mathbf{z}_1, v_i \rangle$ for vectors $v_1, \ldots, v_r \in \mathbb{C}$. Let ρ_K denote the defining representation of K. The preceding analysis shows that any polynomial representation transforms according to a sub-representation of (possibly a sum of) representations of the form $(\rho_K)^{\otimes^r} \otimes (\rho'_K)^{\otimes^s}$ for certain non-negative integers r, s. (If e.g. $r = 0$, $(\rho_K)^{\otimes^r}$ becomes the trivial 1-dimensional representation. To be completely precise, $(\rho_K)^{\otimes^r}$ is the representation of K occurring in the space $\mathcal{P}_r(\overline{\mathbf{z}})$.

Now, to connect these spaces, consider more explicitly the restriction map $\mathcal{R}$ from $\mathcal{P}_{all}$ to $C(K)$; the space of continuous functions on K:

$$\mathcal{P}_{all} \ni p \mapsto \mathcal{R}p \in C(K) : \ \forall k \in K : (\mathcal{R}p)(k) = p(k). \tag{25}$$

Proposition 5.6. *$\mathcal{R}$ is an intertwiner between the representation U_K on $\mathcal{P}_{all}$ and the left regular representation L on $C(K)$.*

The proof of this is left to the reader. Please observe that one must also explain why $\mathcal{R}p$ is a continuous function.

EXAMPLES:

In case $K = S^1 = \{z \in \mathbb{C} \mid |z|^2 = z\overline{z} = 1\}$, a polymomial p in z has the form $p(z) = \sum_{k=0}^{n} a_k z^k$ and a polynomial q in $\overline{z}$ has the form $q(\overline{z}) = \sum_{\ell=0}^{r} b_\ell \overline{z}^\ell$. Here the coefficients a_k and b_ℓ are all complex numbers. The restrictions to K are in this case

$$
\begin{aligned}
p(e^{i\theta}) &= \sum_{k=0}^{n} a_k e^{ik\theta} \\
q(e^{-i\theta}) &= \sum_{\ell=0}^{r} b_\ell e^{-i\ell\theta}
\end{aligned} \tag{26}
$$

It is known from the theory of Fourier Series that both types of restrictions are needed to expand a continuous function.

In case $K = SU(2)$, we have the following perhaps unusual way of describing the group:

$$
\begin{aligned}
K &= SU(2) = \left\{ \begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix} \in Gl(2,\mathbb{C}) \mid z_1\overline{z}_1 + z_2\overline{z}_2 = 1, \right. \\
&\qquad \left. z_3\overline{z}_3 + z_4\overline{z}_4 = 1, z_1\overline{z}_3 + z_2\overline{z}_4 = 0 \textit{ and } z_1 z_4 - z_2 z_3 = 1 \right\}.
\end{aligned} \tag{27}
$$

Upon closer scrutiny it is revealed that in this case, because we also have that $SU(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \alpha \end{pmatrix} \in Gl(2,\mathbb{C}) \mid |\alpha|^2 + |\beta|^2 = 1 \right\}$, it suffices to consider polynomials in the variables z_1, z_2, z_3, z_4. That is, no conjugations are needed (because the right hand column involves the conjugates of the left hand column).

In case $K = O(3)$, we have the following way of describing the group:

$$
\begin{aligned}
&K = O(3) = \{O(Z_1, Z_2, Z_3) = (Z_1 \;\; Z_2 \;\; Z_3) \in Gl(3,\mathbb{C}) \mid \\
&\quad \forall i = 1,2,3: \; Z_i \textit{ is a column vector in } \mathbb{C}^3 \textit{ and} \\
&O(Z_1,Z_2,Z_3)^t O(Z_1,Z_2,Z_3) = 1. \textit{ Moreover, } \forall i = 1,2,3: Z_i = \overline{Z}_i.\}.
\end{aligned} \tag{28}
$$

Clearly, also in this case one only needs the polynomials, not the conjugates of such.

Because of Corollary 5.5 we have (essentially) proved the Peter-Weyl Theorem in the case of compact matrix groups. What remains in a little "cleaning up" regarding the multiplicities. This can be done in analogy with the case of a finite group and is left to the reader.

Instead, let us notice that we have in fact proved more:

Proposition 5.7. *For a compact matrix group G, any $\Pi \in \widehat{G}$ occurs in*

$$(\rho_K)^{\otimes^r} \otimes (\rho'_K)^{\otimes^s}$$

for some non-negative integers r, s.

EXAMPLE:

Recall that the inverse of a matrix may be computed from the so-called Cramer's Rule. Specifically, If a matrix A is invertible,

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det A}$$

where C_{rs} is the determinant of the matrix obtained from A by placing zeros in row r and column s except that the position rs should have a 1. This is the rs "co-factor". Since a matrix U in $SU(n)$ satisfies $\det U = 1$ and $U = (U^*)^{-1}$ it follows by considering the inverse of U^* (!) that the last column in U satisfies:

$$U_{jn} = \det D_{jn},$$

where D_{jn} is the matrix obtained from $\overline{U}$ by replacing the jth row and nth column by zeros except for a 1 in the position jn. This implies that

$$U^n_{col} = F(\overline{U^1_{col}}, \overline{U^2_{col}}, \ldots, \overline{U^{n-1}_{col}}),$$

where F is a multilinear function. It follows from this in the same way as it did for $SU(2)$ that one only needs consider polynomials, not conjugations of polynomials (or: vice versa!). This leads immediately to the Corollary 5.8

It is also proved in a more advanced exercise that for $SU(n)$, the dual representation ρ'_K occurs in $(\rho_K)^{\otimes^{n-1}}$[8].

Corollary 5.8. *Any $\Pi \in \widehat{SU(n)}$ occurs in $(\rho_K)^{\otimes^r}$ for some non-negative integer r.*

Notice that the representations of K that occur in $C(K)$ (or $L^2(K)$) are those that are in the image of the intertwiner $\mathcal{R}$. If we have the full description of the representations in $\mathcal{P}_{all}$ this information is equivalent to knowing the answer to the following question - which is unknown in complete generality to me:

QUESTIONS: What is the kernel $\mathcal{K}$ of $\mathcal{R}$? We know from abstract algebra that it is a finitely generated ideal. But when is K equal to the zero set of this ideal? And, if not, is the zero set a group? (See the first two examples below.) What can be said about the quotient $\mathcal{P}_{all}/\mathcal{K}$?

[8]Notice that this implies that the trivial representation occurs in $(\rho_K)^{\otimes^n}$.

EXAMPLE: *In the $K = S^1$ example from before, one can see that if a polynomial*

$$\sum_{k,\ell=0}^{N,R} a_{k,\ell} z^k \overline{z}^\ell \mapsto 0 \ on\ S^1,$$

then there exists a polynomial q such that

$$\sum_{k,\ell=0}^{N,R} a_{k,\ell} z^k \overline{z}^\ell = q(z,\overline{z})(z\overline{z}-1).$$

In other words, the kernel is generated, as an ideal, by $(z\overline{z}-1)$.

If we now consider the representation theory of S^1 we can see (as expected) that $\mathcal{P}_{all}/\mathcal{K}$ has a basis consisting of the equívalence classes $[z^n]; n = 0, 1, 2, \ldots,$ toghether with the classes $[\overline{z}^k]; k = 1, 2, \ldots]$. Let $p_n(z) = z^n$ for $n = 0, 1, 2, \ldots$ and $q_k(\overline{z}) = \overline{z}^k$ for $k = 1, 2, \ldots$. Then the representation $U_K = U_{S^1}$ is given as follows:

$$(U_{S^1}(e^{i\theta})p_n)(z) = p_n(e^{-i\theta}z) \ = \ e^{-in\theta}z^n = e^{-in\theta}p_n(z) \tag{29}$$

$$(U_{S^1}(e^{i\theta})q_k)(\overline{z}) = q_k(\overline{e^{-i\theta}z}) \ = \ e^{ik\theta}\overline{z}^k = e^{ik\theta}q_k(\overline{z}). \tag{30}$$

In other words, we get the 1-dimensional representations of S^1:

$$e^{i\theta} \mapsto e^{ir\theta} \ ; \ r = 0, \pm 1, \pm 2, \ldots, \pm n, \ldots.$$

These are precisely the irreducible finite-dimensional reepresentations of S^1.

EXAMPLE/EXERCISE: *Let r, s be positive relatively prime integers. Carry out an analysis of the group*

$$K_1 = \left\{ \begin{pmatrix} e^{ir\theta} & 0 \\ 0 & e^{is\theta} \end{pmatrix} \mid \theta \in \mathbb{R} \right\} \tag{31}$$

along the lines of the previous example.

EXAMPLE: *Let Π_N and Π_R be two irreducible represnetations of $SU(2)$ in spaces $\mathcal{P}_N^{(2)}$ and $\mathcal{P}_R^{(2)}$ of homogeneous polynomials of degree N and R, respectively in 2 complex variables. Then, as representation spaces*

$$\mathcal{P}_N^{(2)} \otimes \mathcal{P}_R^{(2)} = \oplus_{i=0}^{\min\{N,R\}} \mathcal{P}_{|N-R|+2i}^{(2)} = \mathcal{P}_{|N-R|}^{(2)} \oplus \cdots \oplus \mathcal{P}_{N+R}^{(2)}. \tag{32}$$

In particular,

$$\Pi_N \otimes \Pi_= \oplus_{i=0}^{\min\{N,R\}} \Pi_{|N-R|+2i}^{(2)} = \Pi_{|N-R|} \oplus \cdots \oplus \Pi_{N+R}. \tag{33}$$

Proof: *The restriction map $\mathcal{S}$ (restriction to the "diagonal"): $\mathcal{P}_N^{(2)} \otimes \mathcal{P}_R^{(2)} \mapsto \mathcal{P}_{N+R}^{(2)}$ given by*

$$(\mathcal{S}(p_N \otimes q_R))(z_1, z_2) = p_N(z_1, z_2) q_R(z_1, z_2) \tag{34}$$

for $p_N \in \mathcal{P}_N^{(2)}$ and $q_R \in \mathcal{P}_R^{(2)}$ is clearly a non-trivial intertwiner into $\mathcal{P}_{N+R}^{(2)}$. (Contemplate this.) We extend the map $\mathcal{S}$ to all of $\mathcal{P}_{N+R}^{(4)}$ in the obvious way. Let us for a moment denote the variables of p_N by w_1, w_2 and those of q_R by w_3, w_4. The subspace $\mathcal{K}_2$ (ideal) of $\mathcal{P}_{N+R}^{(4)}$ generated by the second order homogeneous polynomial $w_1 w_4 - w_2 w_3$ (the similarity with the formula for a determinant is intended!) is clearly mapped to zero by the map $\mathcal{S}$. The map $\mathcal{E} : \mathcal{P}_{N+R-2}^{(4)} \mapsto \mathcal{P}_{N+R}^{(4)}$ given by

$$\mathcal{P}_{N+R-2}^{(4)} \ni p \mapsto (w_1 w_4 - w_2 w_3) p \in \mathcal{P}_{N+R}^{(4)} \tag{35}$$

is **injective** *because there is Unique Factorization in $\mathcal{P}_{N+R}^{(4)}$. The subspace $\mathcal{E}\left(\mathcal{P}_{N-1}^{(2)} \otimes \mathcal{P}_{R-1}^{(2)}\right) \subset \mathcal{P}_N^{(2)} \otimes \mathcal{P}_R^{(2)}$ has dimension NR, $\mathcal{P}_N^{(2)} \otimes \mathcal{P}_R^{(2)}$ has dimension $(N+1)(R+1) = NR + N + R + 1$ and $\mathcal{P}_{N+R}^{(2)}$ has dimension $N + R + 1$.*

It follows that there is an equivalence of representation spaces

$$\mathcal{P}_N^{(2)} \otimes \mathcal{P}_R^{(2)} = \mathcal{P}_{N-1}^{(2)} \otimes \mathcal{P}_{R-1}^{(2)} \oplus \mathcal{P}_{N+R}^{(2)}, \tag{36}$$

where, since we write $\oplus$, the summand $\mathcal{P}_{N+R}^{(2)}$ actually should be interpreted as an irreducible subspace in $\mathcal{P}_N^{(2)} \otimes \mathcal{P}_R^{(2)}$ isomorphic to this summand. We use here the complete reducibility for finite-dimensional representations of compact Matrix Lie Groups. The proof is now almost complete; an induction argument (downward) finishes the job. □

6. The representation theory of finite (or compact) groups

The representation theory of finite groups is covered by the above, since a finite group has a Haar measure:

Proposition 6.1. *The counting measure in a finite group G is left as well as right invariant.*

Proof: Multiplication by a fixed element $h \in G$, be it from the left or from the right, simply introduces a permutation of the group elements; it clearly preserves the magnitude of any subset. □

We will later list some special cases of the previous results.

7. The representation theory of $SU(2)$ (once again)

7.1. **Special representations of $SU(2)$ - the group point of view.** We consider the representation Π_n on the space $\mathcal{P}_n$ of homogeneous polynomials of degree n in the complex variables z, w.[9] Then

$$(\Pi_n(\begin{pmatrix} a & b \\ c & d \end{pmatrix})p)(z,w) = p(dz - bw, -cz + aw). \tag{37}$$

If we let $p_{j,n}(z,w) = z^j w^{n-j}$ for $j = 0, 1, \dots, n$, we obtain in particular

$$(\Pi_n(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix})p_{j,n}) = e^{(n-2j)i\theta} p_{j,n}.$$

It follows that the character is given by

$$\begin{aligned}
\chi^{\Pi_n}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} &= \qquad (38)\\
e^{ni\theta} + e^{(n-2)i\theta} + \cdots + e^{(-n+2)i\theta} + e^{-ni\theta} &= \qquad (39)\\
\frac{e^{ni\theta} - e^{-ni\theta}}{e^{i\theta} - e^{-i\theta}} &= \frac{\sin(n+1)\theta}{\sin\theta}. \qquad (40)
\end{aligned}$$

We know that any element $g = \begin{pmatrix} x_1 + ix_4 & x_2 + ix_3 \\ -x_2 + ix_3 & x_1 - ix_4 \end{pmatrix} \in SU(2)$ is conjugate to a diagonal element $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ where, by considering traces, $x_1 = \cos\theta$, which means that the θ we use here is the same as in Section 2.2. Then, by the formula for the Haar measure,

$$\begin{aligned}
\langle \chi^{\Pi_n}, \chi^{\Pi_r} \rangle = \int_{SU(2)} \chi^{\Pi_n}\overline{\chi^{\Pi_r}} dg &= \qquad (41)\\
\frac{1}{2\pi^2}\int_{\theta,\psi,\phi} \frac{\sin(n+1)\theta}{\sin\theta}\overline{\frac{\sin(r+1)\theta}{\sin\theta}} \sin^2\theta \sin\psi d\psi d\phi d\theta &= \\
\frac{2}{\pi}\int_0^\pi \sin(n+1)\theta \sin(r+1)\theta d\theta &= \delta_{n,r}.
\end{aligned}$$

7.2. **Decomposition of tensor products.**

Lemma 7.1. *Assume that $n \geq m$. Then*

$$\chi^{\Pi_n} \cdot \chi^{\Pi_m} = \chi^{\Pi_{n+m}} + \chi^{\Pi_{n+m-2}} + \cdots + \chi^{\Pi_{n-m}}.$$

[9] More precisely, we consider *holomorphic* polynomials; i.e. expressions of the form $\sum_i a_i z^i w^{n-i}$.

Proof: We have that

$$\chi^{\Pi_n} \cdot \chi^{\Pi_m} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} =$$

$$(e^{ni\theta} + e^{(n-2)i\theta} + \cdots + e^{-ni\theta}) \quad \cdot \quad (e^{mi\theta} + e^{(m-2)i\theta} + \cdots + e^{-mi\theta})$$

It is easy to see that $e^{(n+m)i\theta}$ appears exactly once in this product, $e^{(n+m-2)i\theta}$ appears twice, $e^{(n+m-4)i\theta}$ appears three times,..., $e^{(n-m)i\theta}$, appears $m+1$ times, $e^{(n-m-2)i\theta}$ appears $m+1$ times, $e^{(n-m-4)i\theta}$ appears $m+1$ times, $e^{(-n+m)i\theta}$ appears $m+1$ times,..., $e^{-(n+m-4)i\theta}$ appears three times, $e^{-(n+m-2)i\theta}$ appears twice, and $e^{-(n+m)i\theta}$ appears exactly once. The claim follows from this, once one realizes that the decomposition in Lemma 4.5, by Theorem 4.11, is unique.. $\square$

Corollary 7.2.

$$\Pi_n \otimes \Pi_m = \Pi_{n+m} \oplus \Pi_{n+m-2} \oplus \cdots \oplus \Pi_{n-m}.$$

8. Unitarity; the algebra point of view.

We will in this section consider a number of situations, where one has a representation π of a 3-dimensional Lie algebra $\mathfrak{g} = \mathrm{Span}_{\mathbb{R}}\{L_1, L_2, L_3\}$ (but it easily generalizes to higher dimensional Lie algebras). Let us denote the representation space by V. We are not assuming that V necessarily is finite-dimensional. We will assume that there are 3 operators S, A, C that are complex linear combinations of the generators of the Lie algebra in the given representation, and which satisfy

$$[S, A] = 2 \cdot s \cdot A; \ [S, C] = -2 \cdot s \cdot C; \text{ and } [A, C] = S \tag{42}$$

for some real constant s. We are not directly assuming irreducibility, but we assume furthermore, that there is a non-zero vector v_0 that satisfies

$$A(v_0) = 0 \qquad \text{and } S(v_0) = \lambda \cdot v_0, \tag{43}$$

for some $\lambda \in \mathbb{R}$. We are interested in the space

$$V_C = \mathrm{Span}_{\mathbb{C}}\{C^n(v_0) \mid n = 0, 1, 2, \dots\}.$$

The space is "created" by C and its non-negative powers, and one often calls C a *creation operator*. Likewise, A is called an *annihilation operator*. There is no universal name for S; it depends on the specific situation. The vector v_0 is sometimes called the *highest weight vector*, while in physics, again depending on the situation, it might be called the *vacuum vector*.

We are interested in those situations, where there exists an inner product $\langle \cdot, \cdot \rangle$ (positive definite, of course, but no assumptions at the moment about completeness) on V_C.

Definition 8.1. *If T is a linear operator $V_C \mapsto V_C$, the* formal adjoint*, $T^\dagger$, is defined by*

$$\forall n, r \in \mathbb{N} \cup \{0\} : \langle T^\dagger C^n(v_0), C^r(v_0)\rangle = \langle C^n(v_0), TC^r(v_0)\rangle.$$

Definition 8.2. *The representation π of $\mathfrak{g}$ is* unitarizable *if the operators $\pi(L_k); k = 1, 2, 3$ are* formally skew adjoint*, i.e.*

$$\forall k = 1, 2, 3 : \ \pi(L_k)^\dagger = -\pi(L_k).$$

In the situations we are going to investigate, the assumption of unitarizability implies

$$\text{Either } C^\dagger = -A, \text{ or } C^\dagger = A. \tag{44}$$

Let us cover both cases by writing

$$C^\dagger = \varepsilon \cdot A \tag{45}$$

for some fixed $\varepsilon = \pm 1$. Now observe

Proposition 8.3. *With the above assumptions there is at most one inner product on V_C in which the representation is formally skew adjoint and such that v_0 is a unit vector in the corresponding norm.*

Proof: We need to compute $\langle C^n(v_0), C^r(v_0)\rangle$ for all n, r in $\mathbb{N} \cup \{0\}$. Assume that $n \geq r$ and observe (verify #1):

$$\langle C^n(v_0), C^r(v_0)\rangle = \varepsilon^n \langle v_0, [A^n, C^r](v_0)\rangle. \tag{46}$$

Then observe that (42) implies that

$$[A^n, C^r]v_0 = A^{n-r} L_{n,r} \cdot v_0, \text{ (verify \#2)} \tag{47}$$

where $L_{n,r} = L_{n,r}(\lambda)$ is a real constant, and where, as usual, $A^0 := 1$. But this means that we at most get something non-zero when $n = r$. The case $n < r$ can be handled similarly (verify #3). Finally, the constant $L_{n,r} =$ can be computed explicitly. $\square$

Remark 8.4. *Let $v_n = C^n(v_0)$ for all $n = 0, 1, 2, \ldots$. It follows from the proof above that*

$$r \neq n \Rightarrow \langle v_r, v_n\rangle = 0. \tag{48}$$

If $s \neq 0$[10]*, this could also have been proved by observing that these vectors are eigenvectors for S (verify #4).*

To proceed, we need to compute $[A^n, C^n]$.

[10] This extra assumption was added on June 14 (why is it necessary?)

Proposition 8.5.

$$[A^n, C^n] = \sum_{k=0}^{n} C^k P_{n,k}(S) A^k, \tag{49}$$

where $\forall k = 0, 1, \ldots, k$, $P_{k,n}(S)$ *is a polynomial in* S*, and where (verify #5)*

$$P_{0,n}(S) = n! \cdot S \cdot (S - s) \cdots (S - (n-1)s)). \tag{50}$$

Corollary 8.6. *(verify #6)*

$$\langle v_n, v_n \rangle > 0 \Leftrightarrow \varepsilon^n \cdot \lambda \cdot (\lambda - s) \cdot \cdots \cdot (\lambda - (n-1)s) > 0. \tag{51}$$

Example 8.7. *To get started, one may notice that*

$$\begin{aligned} [A, C] = S,\ [A^2, C] = AS + SA &= 2A(S + s), \\ [A, C^2] = 2C(S - s),\ [A^2, C^2] &= 2S(S - s) + 4C(S - 2s)A. \end{aligned}$$

Also notice that (verify #7)

$$\forall n : [A, C^n] = n \cdot C^{n-1}(S - (n-1)s). \tag{52}$$

We will now use these general observations to examine some specific Lie algebras;

8.1. **The Heisenberg algebra.** The Heisenberg Lie algebra $\mathfrak{h}$ is generated by elements R, U, W satisfying

$$[R, U] = W,\ [W, R] = 0 = [W, U]. \tag{53}$$

Usually one considers the elements P, Q, Z in the set $i \cdot \mathfrak{h}$ given by $P = -iR$, $Q = -iU$, and $Z = -iW$. In terms of these, the commutators are

$$[Q, P] = iZ,\ [Z, Q] = 0 = [Z, P]. \tag{54}$$

One defines

$$A = \frac{1}{\sqrt{2}}(Q + iP),\ S = Z,\ \text{and}\ C = \frac{1}{\sqrt{2}}(Q - iP). \tag{55}$$

Then the triple A, S, C satisfies (42) with $s = 0$ (verify #8)[11] . Moreover, it follows that in a unitarizable representation, one must have

$$C^\dagger = A,\ \text{which implies that here}, \varepsilon = 1. \tag{56}$$

So (verify #9)

$$\text{There is unitarity if and only if } \lambda \geq 0. \tag{57}$$

[11] In this example, we denote by X both a Lie algebra element and its value in a representation

8.2. **The Lie algebra** $su(2)$**.** This Lie algebra is generated by three elements

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{58}$$

In the complexification one finds the elements (verify #10)

$$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{59}$$

In a representation[12] π set $C = \pi(Y), A = \pi(X), S = \pi(H)$.

This triple then satisfies (42). Now one has (verify #11) $s = 1$ and $\varepsilon = 1$, and any finite-dimensional representation of $su(2)$ is unitarizable according to the procedure given in this section (verify #12).

Before proceeding, observe that one has (verify #13)

Proposition 8.8. *The Lie algebras* $su(1,1)$*,* $sl(2,\mathbb{R})$*, and* $sp(1,\mathbb{R})$ *are isomorphic.*

8.3. **The Lie algebra** $su(1,1)$**.** This Lie algebra is generated by three elements

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{60}$$

In the complexification one finds the elements (verify #14)

$$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{61}$$

In a representation[13] π set $C = \pi(Y), A = \pi(X), S = \pi(H)$. This triple then satisfies (42) (verify #15). In contrast to the $su(2)$ case, however, one now has $s = 1$, and $\varepsilon = -1$ (verify #16), and there are infinite-dimensional unitarizable representations provided the constant λ satisfies a certain condition, which is left for the reader to determine (verify #17).

[12] June 19: The word unitary has been removed. What we mean is that we are looking *for* unitarizable representations π.

[13] June 19: The word unitary has been removed. What we mean is that we are looking *for* unitarizable representations π.

8.4. **The Heisenberg algebra AND the Lie algebra $su(1,1)$.** Let C, S, A be the triple from the discussion of the Heisenberg algebra. Set

$$\tilde{C} = \frac{i}{2}C^2,\ \tilde{A} = \frac{i}{2}A^2, \tilde{S} = -(CA + \frac{1}{2}) \tag{62}$$

Then this triple satisfies all the properties needed to define a unitarizable module for $su(1,1)$ provided that C, A, S come from a unitarizable representation of the Heisenberg algebra (verify $\#18$)[14]. There are two independent highest weight vectors (not necessarily both of norm 1), namely v_0 and $C(v_0)$, where v_0 is the highest weight vector for the representation of the Heisenberg algebra (verify $\#19$). More precisely, set

$$\begin{aligned} V_C^{\text{even}} &= \text{Span}\{C^n(v_0) \mid n = 0, 2, 4, \cdots\} \text{ and} \\ V_C^{\text{odd}} &= \text{Span}\{C^n(v_0) \mid n = 1, 3, 5, \cdots\}. \end{aligned}$$

Then these spaces are invariant under the $su(1,1)$ representation defined by the elements $\tilde{C}, \tilde{A}, \tilde{S}$ (verify $\#20$). The vectors $\tilde{v}_0 = v_0, \hat{v}_0 = C(v_0)$ fit into (43) for this triple. In particular, they are eigenvectors for $\tilde{S}$ of eigenvalues, $\tilde{\lambda}$ and $\hat{\lambda}$, respectively. This task is left for the reader (verify $\#21$).

[14] This is what is sometimes called "The Harmonic Representation" or "The Metaplectic Representation". It is connected to the "The Stone von Neumann Uniqueness Theorem" and to the fact that $Sp(1, \mathbb{R})$ acts by automorphisms on the Heisenberg algebra.

9. Higher dimensional examples, e.g. $SU(3)$

One is here led by the same sort of reasoning to complexification and to operators

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{63}$$

$$f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{64}$$

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{65}$$

(There are two more generators of $su(3)$, but they are not as important.) A finite dimensional irreducible unitary representation $(\pi, \mathcal{H}_\pi)$ of $SU(3)$ is given by a pair (n_1, n_2) of non-negative integers (and any such pair leads to a such representation) and is given by a **highest weight vector** v_0 satisfying

$$d\pi(e_1)v_0 = d\pi(e_2)v_0 = 0$$
$$d\pi(h_1)v_0 = n_1 v_0 \; ; \; d\pi(h_2)v_0 = n_2 v_0$$
$$\mathcal{H}_\pi = \mathrm{Span}\{d\pi(f_1)^{r_1} d\pi(f_2)^{s_1} \cdots d\pi(f_1)^{r_k} d\pi(f_2)^{s_k}(v_0)\}.$$

An irreducible representation of $U(3)$ is also an irreducible representation of $SU(3)$ and is simply given by such a representation tensored together with a 1-dimensional representation of the form $U(3) \ni u \mapsto \det u^p$ for $p \in \mathbb{Z}$. We here assume that $p \geq 0$. One usually represents such a representation by a triple $(p + n_1 + n_2, p + n_2, p)$ where (n_1, n_2) gives the $SU(3)$ representation. Furthermore, it is convenient to give it in the form of a "Young Diagram" with three rows of boxes with resp. $p + n_1 + n_2$, $p + n_2$, and p) boxes.

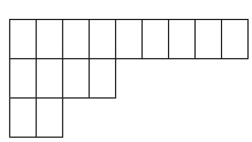

Figure 1. Young Diagram (9,4,2) corresponding to the irreducible representation of $SU(3)$ given by $(n_1, n_2) = (5, 2)$. As an irreducible representation of $U(3)$ it has an extra factor of $\det u^3$ (because of the 2 boxes in the third row)