TAG Lecture 1: Schemes

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Affine Schemes

Let A be a commutative ring. The affine scheme defined by A is the pair:

$$\mathrm{Spec}(A) = (\mathrm{Spec}(A), \mathcal{O}_A).$$

The underlying set of $\mathrm{Spec}(A)$ is the set of prime ideals $\mathfrak{p} \subseteq A$. If $I \subseteq A$ is an ideal, we define

$$V(I) = \{\ \mathfrak{p} \subseteq A \text{ prime} \mid I \not\subseteq \mathfrak{p}\ \} \subseteq \mathrm{Spec}(A).$$

These open sets form the *Zariski topology* with basis

$$V(f) = V((f)) = \{\ \mathfrak{p} \mid f \notin \mathfrak{p}\ \} = \mathrm{Spec}(A[1/f]).$$

The sheaf of rings $\mathcal{O}_A$ is determined by

$$\mathcal{O}_A(V(f)) = A[1/f].$$

Schemes as locally ringed spaces

$\mathrm{Spec}(A)$ is a *locally ringed space*: if $\mathfrak{p} \in \mathrm{Spec}(A)$, the stalk of $\mathcal{O}_A$ at $\mathfrak{p}$ is the local ring $A_{\mathfrak{p}}$.

Definition

A scheme $X = (X, \mathcal{O}_X)$ is a locally ringed space with an open cover (as locally ringed spaces) by affine schemes.
A morphism $f : X \to Y$ is a continuous map together with an induced map of sheaves

$$\mathcal{O}_Y \longrightarrow f_* \mathcal{O}_Y$$

with the property that for all $x \in X$ the induced map of local rings

$$(\mathcal{O}_Y)_{f(x)} \longrightarrow (\mathcal{O}_X)_x$$

is **local***; that is, it carries the maximal ideal into the maximal ideal.*

Schemes as functors

If X is a scheme we can define a functor which we also call X from commutative rings to sets by by

$$X(R) = \mathbf{Sch}(\mathrm{Spec}(R), X).$$

$$\mathrm{Spec}(A)(R) = \mathbf{Rings}(A, R).$$

Theorem

A functor $X : \mathbf{CRings} \longrightarrow \mathbf{Sets}$ is a scheme if and only if

1. *X is a sheaf in the Zariski topology;*
2. *X has an open cover by affine schemes.*

Example: projective space

Define a functor $\mathbb{P}^n$ from rings to sets: $\mathbb{P}^n(R)$ is the set of all split inclusions of R-modules

$$N \longrightarrow R^{n+1}$$

with N locally free of rank 1.

For $0 \leq i \leq n$ let $U_i \subseteq \mathbb{P}^n$ to be the subfunctor of inclusions j so that

$$N \xrightarrow{j} R^{n+1} \xrightarrow{p_i} R$$

is an isomorphism. Then the U_i form an open cover and $U_i \cong \mathbb{A}^n$.

Geometric points

If X is a (functor) scheme we get (locally ringed space) scheme $(|X|, \mathcal{O}_X)$ by:

1. $|X|$ is the set of (a *geometric points*) in X: equivalence classes of pairs $(\mathbb{F}, x)$ where $\mathbb{F}$ is a field and $x \in X(\mathbb{F})$.
2. An open subfunctor U determines an open subset of the set of geometric points.
3. Define $\mathcal{O}_X$ locally: if $U = \mathrm{Spec}(A) \to X$ is an open subfunctor, set $\mathcal{O}_X(U) = A$.

The geometric points of $\mathrm{Spec}(R)$ (the functor) are the prime ideals of R.

If X is a functor and R is a ring, then an R-point of X is an element in $X(R)$; these are in one-to-one correspondence with morphism $\mathrm{Spec}(R) \to X$.

Schemes as ringed topoi

This notion generalizes very well.

If X is a scheme let $\mathcal{X}$ denote the category of sheaves of sets on X. Then $\mathcal{X}$ is a *topos*:

1. $\mathcal{X}$ has all colimits and colimits commute with pull-backs (base-change);
2. $\mathcal{X}$ has a set of generators;
3. Coproducts in $\mathcal{X}$ are disjoint; and
4. Equivalence relations in C are effective.

If X is a scheme, $\mathcal{O}_X \in \mathcal{X}$ and the pair $(\mathcal{X}, \mathcal{O}_X)$ is a ringed topos.

[Slogan] A ringed topos is equivalent to that of a scheme if it is locally of the form $\mathrm{Spec}(A)$.

Quasi-coherent sheaves

Let $(X, \mathcal{O}_X)$ be a scheme and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. Then $\mathcal{F}$ is *quasi-coherent* if is locally presentable as an $\mathcal{O}_X$-module.

Definition

An $\mathcal{O}_X$ module sheaf is **quasi-coherent** *if for all $y \in X$ there is an open neighborhood U of y and an exact sequence of sheaves*

$$\mathcal{O}_U^{(J)} \longrightarrow \mathcal{O}_U^{(I)} \longrightarrow \mathcal{F}|_U \to 0.$$

If $X = \mathrm{Spec}(A)$, then the assignment $\mathcal{F} \mapsto \mathcal{F}(X)$ defines an equivalence of categories between quasi-coherent sheaves and A-modules.

Quasi-coherent sheaves (reformulated)

Let X be a scheme, regarded as a functor. Let $\mathbf{Aff}/X$ be the category of morphisms $a : \mathrm{Spec}(A) \to X$. Define

$$\mathcal{O}_X(\mathrm{Spec}(A) \to X) = \mathcal{O}_X(a) = A.$$

This is a sheaf in the Zariski topology.
A quasi-coherent sheaf $\mathcal{F}$ is sheaf of $\mathcal{O}_X$-modules so that for each diagram

$$\begin{array}{ccc} \mathrm{Spec}(B) & & \\ {\scriptstyle f}\downarrow & \searrow^{b} & \\ & & X \\ & \nearrow_{a} & \\ \mathrm{Spec}(A) & & \end{array}$$

the map

$$f^*\mathcal{F}(a) = B \otimes_A \mathcal{F}(a) \to \mathcal{F}(b)$$

is an isomorphism.

Example: $\mathcal{O}(k)$ on $\mathbb{P}^n$

Morphisms $a : \mathrm{Spec}(A) \to \mathbb{P}^n$ correspond to split inclusions

$$N \longrightarrow A^{n+1}$$

with N locally free of rank 1. Define $\mathcal{O}_{\mathbb{P}^n}$-module sheaves

$$\mathcal{O}(-1)(a) = N$$

and

$$\mathcal{O}(1)(a) = \mathrm{Hom}_A(N, A).$$

These are quasi-coherent, locally free of rank 1 and $\mathcal{O}(1)$ has canonical global sections x_i

$$N \longrightarrow A^{n+1} \xrightarrow{p_i} A.$$

Exercises

1. Show that the functor P^n as defined here indeed satisfies the two criteria to be a scheme.

2. Fill in the details of the final slide: define the global sections of sheaf and show that the elements x_i there defined are indeed global sections of the sheaf $\mathcal{O}(1)$ on P^n.

3. The definition of P^n given here can be extended to a more general statement: if X is a scheme, then the morphisms $X \to P^n$ are in one-to-one correspondence with locally free sheaves $\mathcal{F}$ of rank 1 over X generated by global sections s_i, $0 \leq i \leq n$.

4. Show that the functor which assigns to each ring R the set of finitely generated projective modules of rank 1 over R cannot be scheme.