TAG Lecture 2: Schemes

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Spectra

We need a good model for the stable homotopy category. Let $\mathcal{S}$ be a category so that

1. $\mathcal{S}$ is a cofibrantly generated proper stable simplicial model model category Quillen equivalent to the Bousfield-Friedlander category of simplicial spectra;
2. $\mathcal{S}$ has a closed symmetric monoidal smash product which gives the smash product in the homotopy category;
3. the smash product and the simplicial structure behave well;
4. and so on.

Symmetric spectra (either simplicially or topologically) will do.

Commutative ring spectra

A commutative monoid A in $\mathcal{S}$ is a commutative ring spectra: there is a multiplication map

$$A \wedge A \longrightarrow A$$

and a unit map

$$S^0 \longrightarrow A$$

so that the requisite diagrams commutes.
There are A-modules with mulitplications $A \wedge M \to M$.
There are free commutative algebras:

$$\begin{aligned} \mathrm{Sym}(X) = \vee\, \mathrm{Sym}_n(X) &= \vee\, (X^{\wedge n})/\Sigma_n \\ &= \vee\, (E\Sigma_n)_+ \wedge_{\Sigma_n} X^{\wedge n}. \end{aligned}$$

These categories inherit model category structures

Sheaves of spectra

Let X be a scheme. A presheaf of spectra is a functor

$$\mathcal{F} : \{\text{ Zariski opens in } X\ \}^{\mathrm{op}} \to \mathcal{S}.$$

Theorem (Jardine)

Presheaves of spectra form a simplicial model category where $\mathcal{E} \to \mathcal{F}$

- *is a weak equivalence if* $\mathcal{E}_\mathfrak{p} \to \mathcal{F}_\mathfrak{p}$ *is a weak equivalence for all* $\mathfrak{p} \in X$*;*
- $\mathcal{E} \to \mathcal{F}$ *is a cofibration in* $\mathcal{E}(U) \to \mathcal{F}(U)$ *is a cofibration for all* U*.*

A *sheaf of* (ring or module) spectra is a fibrant/cofibrant object. Jardine proves an analogous theorem for ring and module spectra for an arbitrary topos.

Global sections

Let X be a scheme and $\mathcal{F}$ a sheaf on X. If $\{U_\alpha\}$ is an open cover, let $\mathcal{U}$ the associated category. Then

$$H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F}) = \mathcal{F}(X) \cong \mathbf{Sh}(X,\mathcal{F}) \\ \cong \lim_{\mathcal{U}} \mathcal{F}.$$

If $\mathcal{F}$ is a sheaf of spectra these become:

$$R\Gamma(X,\mathcal{F}) \simeq F_{\mathbf{Sh}_{\mathcal{S}}}(X,\mathcal{F}) \\ \simeq \operatorname{holim}_{\mathcal{U}} \mathcal{F}.$$

And the derived nature of the subject begins to appear. There is a spectral sequence

$$H^s(X,\pi_t\mathcal{F}) \Longrightarrow \pi_{t-s}R\Gamma(X,\mathcal{F}).$$

Derived schemes

Theorem (Lurie)

Let X be a space and $\mathcal{O}$ a sheaf of ring spectra on X. Then $(X,\mathcal{O})$ is a **derived scheme** *if*

- *$(X,\pi_0\mathcal{O})$ is a scheme; and*
- *$\pi_i\mathcal{O}$ is a quasi-coherent $\pi_0\mathcal{O}$ module for all i.*

Remark

This looks like a definition, not a theorem. There is a technical condition on $\mathcal{O}$ which I am suppressing. There is a better definition using topoi which makes this condition arise naturally.

Derived affine schemes

Definition

Let A be a ring spectrum. Define $\mathrm{Spec}(A)$ by

- *Underlying space: $\mathrm{Spec}(\pi_0 A)$; and*
- *$\mathcal{O}$: sheaf associated to the presheaf*

$$V(f) = \mathrm{Spec}(\pi_0 A[1/f]) \mapsto A[1/f].$$

Remark

$A[1/f]$ is the localization of A characterized by requiring

$$\mathbf{Sp}(A[1/f], B) \subseteq \mathbf{Sp}(A, B)$$

to be subspace of components where f in invertible. Such localizations can be done functorially in the category of ring spectra.

Derived schemes: the category

Lurie's result above is actually part of an equivalence of categories:

Theorem (Lurie)

A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of derived schemes is a pair (f, ϕ) where

- *$f : X \to Y$ is a continuous map;*
- *$\phi : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is a morphism of sheaves of ring spectra*

so that

$$(f, \pi_0 \phi) : (X, \pi_0 X) \to (Y, \pi_0 Y)$$

is a morphism of schemes.

The collection of all morphisms $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a space.

Derived schemes as functors

If X is a derived scheme, we write

$$X : \text{Ring spectra} \to \textbf{Spaces}$$

for the functor

$$X(R) = \textbf{Dsch}(\mathrm{Spec}(R), X).$$

Example

The affine derived scheme $\mathbb{A}^1$ is characterized by

$$\mathbb{A}^1(R) = \Omega^\infty R.$$

The affine derived scheme $\mathbb{G}_m$ is characterized

$$\mathbb{G}_m(R) = \mathrm{Gl}_1(R) \subseteq \Omega^\infty R.$$

to be the subsets of invertible components.

Example: Derived projective space

Define $\mathbb{P}^n(R)$ to be the *subspace* of the R-module morphisms

$$i : N \longrightarrow R^{n+1}$$

which split and so that $\pi_0 N$ is locally free of rank 1 as a $\pi_0 R$-module.

The underlying scheme of **derived** $\mathbb{P}^n$ is **ordinary** $\mathbb{P}^n$. The sub-derived schemes U_k, $0 \leq k \leq n$ of those q with

$$N \xrightarrow{i} R^{n+1} \xrightarrow{p_k} R$$

an equivalence *cover* $\mathbb{P}^n$. Note

$$U_k(R) \cong \mathbb{A}^n(R) \cong \Omega^\infty R^{\times n}.$$

Exercise

1. Let A be an E_∞-ring spectrum and M an A-module. Assume we can define the symmetric A-algebra $\mathrm{Sym}_A(M)$ and that it has the appropriate universal property. (What would that be?) Let $A = S$ be the sphere spectrum and let $M = \vee_n S$ ($\vee =$ coproduct or wedge). What is $\mathrm{Spec}(\mathrm{Sym}_S(M))$? That is, what functor does it represent?

2. Suppose $n = 1$ and $x \in \mathrm{Sym}_S(M)$ is represented by the inclusion $S = M \to \mathrm{Sym}_S(M)$. What is $\mathrm{Spec}(\mathrm{Sym}_S(M)[1/x])$?

Open-ended exercise: the tangent functor

3. If R is a ring, then $R[\epsilon] \overset{\text{def}}{=} R[x]/(x^2)$. This definition makes sense for E_∞-ring spectra as well. If X is any functor on rings (or E_∞-ring spectra) the tangent functor T_X is given by

$$R \mapsto T_X(R[\epsilon]).$$

Explore this functor, for example:

1. Show that T_X is an abelian group functor over X;
2. If $x : \mathrm{Spec}(A) \to X$ is any A-point of X, describe the fiber

$$T_{X,x} = \mathrm{Spec}(A) \times_X T_X.$$

3. (More advanced) Show that this fiber is, in fact, an affine scheme.