TAG Lecture 3: Other Maps, Other Topologies

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17 June 2008

Flat morphisms

A morphism of rings $A \to B$ is flat if $B \otimes_A (-)$ is exact. It is faithfully flat if it creates isomorphisms.

Definition

A morphism $f : X \to Y$ of schemes is flat if for all $x \in X$, $\mathcal{O}_x$ is a flat $\mathcal{O}_{f(x)}$-algebra. The morphism f is faithfully flat if is flat and surjective.

A morphism $A \to B$ of E_∞-ring spectra is flat if

1. $\pi_0 A \to \pi_0 B$ is a flat morphism of rings;
2. $\pi_0 B \otimes_{\pi_0 A} \pi_n A \cong \pi_n B$ for all n.

A bar construction

Let $X \to Y$ be a morphism of schemes and let

$$\epsilon : X_\bullet \longrightarrow Y$$

be the bar construction. Faithfully flat descent compares sheaves over Y with simplicial sheaves on

$$X_\bullet = \{ X^{\bullet+1} \}.$$

If $X = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = Y$ are both affine; this is $\mathrm{Spec}(-)$ of the cobar construction

$$\eta : A \longrightarrow \{ B^{\otimes \bullet+1} \}.$$

Descent

A module-sheaf $\mathcal{F}_\bullet$ on $X_\bullet$ is:

1. module sheaves $\mathcal{F}_n$ on X_n;
2. for each $\phi : [n] \to [m]$, a homomorphism $\theta(\phi) : \phi^* \mathcal{F}_n \to \mathcal{F}_m$;
3. subject to the evident coherency condition.

Definition

Such a module sheaf is **Cartesian** *if each $\mathcal{F}_n$ is quasi-coherent and $\theta(\phi)$ is an isomorphism for all ϕ.*

If $\mathcal{E}$ is a quasi-coherent sheaf on Y, $\epsilon^* \mathcal{E}$ is a Cartesian sheaf on $X_\bullet$.

Descent: If f is quasi-compact and faithfully flat, this is an equivalence of categories.

Derived Descent

A chain complex $\mathcal{F}_\bullet$ of simplicial module sheaves is the same as simplicial chain complex of module sheaves.

Definition

Let $\mathcal{F}_\bullet$ be a chain complex of simplicial module sheaves on $X_\bullet$. The $\mathcal{F}_\bullet$ is **Cartesian** *if*

1. *each $\theta(\phi) : \phi^*\mathcal{F}_n \to \mathcal{F}_m$ is an equivalence;*
2. *the homology sheaves $\mathcal{H}_i(\mathcal{F}_\bullet)$ are quasi-coherent.*

If $\mathcal{E}$ is a complex of quasi-coherent sheaves on Y, $\epsilon^*\mathcal{E}$ is a Cartesian sheaf on $X_\bullet$.

Derived descent: This if f is quasi-compact and faithfully flat, this is an equivalence of derived categories.

Étale and smooth morphisms

There are not enough Zariski opens; there are too many flat morphisms, even finite type ones; therefore:

Suppose we are given any lifting problem in schemes

$$\begin{array}{ccc} \mathrm{Spec}(A/I) & \longrightarrow & X \\ {\scriptstyle\subseteq}\downarrow & \nearrow & \downarrow {\scriptstyle f} \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

with I **nilpotent** and f flat and locally finite. Then

Definition

1. *f is* **smooth** *if the problem always has a solution;*
2. *f is* **étale** *if the problem always has a unique solution.*

Examples: Étale and smooth morphisms

Theorem

$B = A[x, \dots, x_n]/(p_1, \dots, p_m)$ is

- *étale over A if $m = n$ and $\det(\partial p_i/\partial x_j)$ is a unit in B;*
- *smooth over A if $m \leq n$ and the $m \times m$ minors of the partial derivatives generate B.*

Any étale or smooth morphism is locally of this form.

1. $\mathbb{F}_p[x]/(x^{p^n} - x)$ is étale over $\mathbb{F}_p$.
2. Any finite separable field extension is étale.
3. $A[x]/(ax^2 + bx + c)$ is étale over A if $b^2 - 4ac$ is a unit in A.
4. $\mathbb{F}[x, y]/(y^2 - x^3)$ is not smooth over any field $\mathbb{F}$.

Étale maps as covering spaces

Theorem

Let $f : X \to Y$ be étale and separated and $U \subset Y$ be open. Any section s of

$$U \times_Y X \to U$$

is an isomorphism onto a connected component.

For the analog of normal covering spaces we have:

Definition

Let $f : X \to Y$ be étale and $G = \mathrm{Aut}_X(Y)$ (the Deck transformations). Then X is **Galois** *overY if we have an isomorphism*

$$G \times X \longrightarrow X \times_Y X$$
$$(g, x) \mapsto (g(x), x).$$

Étale topology

Let X be a scheme. The étale topology has

- étale maps $U \to X$ as basic opens;
- a cover $\{ V_i \to U \}$ is a finite set of étale maps with $\coprod V_i \to U$ surjective.

Notes:

1. every open inclusion is étale; so an étale sheaf yields a Zariski sheaf;
2. Define $\mathcal{O}_X(U \to X) = \mathcal{O}_U(U)$; this is the étale structure sheaf.
3. There are module sheaves and quasi-coherent sheaves for the étale topology.

Zariski versus étale sheaves

The inclusion of a Zariski open $U \to X$ is rigid: $\mathrm{Aut}_X(U) = \{e\}$. An étale open $U \to X$ need not be rigid: $\mathrm{Aut}_X(U) \neq \{e\}$ in general.

Example

Let $\mathbb{F}$ be field.

- Quasi-coherent sheaves in the Zariski topology are $\mathbb{F}$-vector spaces.
- Quasi-coherent sheaves in the étale topology are twisted, discrete $\bar{\mathbb{F}} - \mathrm{Gal}(\bar{\mathbb{F}}/\mathbb{F})$-modules.

Étale morphisms for spectra

A morphism $f : A \to B$ of ring spectra is étale if

1. $\pi_0 A \to \pi_0 B$ is an étale morphism of rings; and
2. $\pi_0 B \otimes_{\pi_0 A} \pi_i A \to \pi_i B$ is an isomorphism.

Compare to:

Definition (Rognes)

Let $A \to B$ of ring spectra and let $G = \mathrm{Aut}_A(B)$. The morphism Galois if

- $B \wedge_A B \to F(G_+, B)$; *and*
- $A \to B^{hG} = F(G_+, B)^G$

are equivalences.

Hypotheses are needed: G finite or "stably dualizable".

The cotangent complex

Let $f : X \to Y$ be a morphism of schemes. Let $\mathrm{Der}_{X/Y}$ be the sheaf on X associated to the functor

$$\begin{array}{ccc} \mathrm{Spec}(R) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spec}(R[\epsilon]) & \xrightarrow[\epsilon=0]{} & Y \end{array}$$

This is representable: $\mathrm{Der}_{X/Y} = \mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X)$. The **cotangent complex** $L_{X/Y}$ is the derived version.

Suppose f is locally finite and flat, then

1. f is étale if and only if $L_{X/Y} = 0$;
2. f is smooth if and only if $L_{X/Y} \simeq \Omega_{X/Y}$ and that sheaf is locally free.

Exercises

1. Let (A, Γ) be a Hopf algebroid. Assume Γ is flat over A. Then we get a simplicial scheme by taking $\mathrm{Spec}(-)$ of the cobar construction on the Hopf algebroid. Show that the category of Cartesian (quasi-coherent) sheaves on this simplicial scheme is equivalent to the category of (A, Γ)-comodules.

2. Let $A \to B$ be a morphism of algebras and M an A-module. Show that the functor on B-modules

$$M \to \mathbf{Def}_A(B, M)$$

is representable by a B-module $\Omega_{B/M}$. Indeed, if I is the kernel of the multiplication map $B \otimes_A B \to B$, then $\Omega_{B/A} \cong I/I^2$.

3. Calculate Let $B = \mathbb{F}[x, y]/(y^2 - x^3)$ where $\mathbb{F}$ is a field. Show that $\Omega_{B/\mathbb{F}}$ is locally free of rank 1 except at $(0, 0)$.