TAG Lecture 5: Elliptic Curves

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Weierstrass curves

Definition

A **Weierstrass curve** $C = C_{\mathbf{a}}$ *over a ring* R *is a closed subscheme of* $\mathbb{P}^2$ *defined by the equation*

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

The curve C has a unique point $e = [0, 1, 0]$ when $z = 0$.

1. C is has at most one singular point;
2. C is always smooth at e;
3. the smooth locus $C_{\mathbf{sm}}$ is an abelian group scheme.

Elliptic curves

Definition

An **elliptic curve** *over a scheme S is a proper smooth curve of genus 1 over S* $C \underset{e}{\overset{q}{\rightleftarrows}} S$ *with a given section e.*

Any elliptic curve is an abelian group scheme:

if $T \to S$ is a morphism of schemes, the morphism

$$\begin{aligned} \{T\text{-points of } C\} &\longrightarrow \mathrm{Pic}^{(1)}(C) \\ P &\longmapsto \mathcal{I}^{-1}(P) \end{aligned}$$

is a bijection.

Comparing definitions

Let $C = C_{\mathbf{a}}$ be a Weierstrass curve over R. Define elements of R by

$$\begin{aligned} b_2 &= a_1^2 + 4a_2 \\ b_4 &= 2a_4 + a_1 a_3 \\ b_6 &= a_3^2 + 4a_6 \end{aligned}$$

$$\begin{aligned} c_4 &= b_2^2 - 24b_4 \\ c_6 &= b_2^3 + 36b_2b_4 - 216b_6 \\ (12)^3\Delta &= c_4^3 - c_6^2 \end{aligned}$$

Then C is elliptic if and only if Δ is invertible. All elliptic curves are locally Weierstrass (more below).

Examples

1.) Legendre curves: over $\mathbb{Z}[1/2][\lambda, 1/\lambda(\lambda-1)]$:

$$y^2 = x(x-1)(x-\lambda)$$

2.) Deuring curves: over $\mathbb{Z}[1/3][\nu, 1/(\nu^3+1)]$:

$$y^2 + 3\nu xy - y = x^3$$

3.) Tate curves: over $\mathbb{Z}[\tau]$:

$$y^2 + xy = x^3 + \tau$$

∞.) The cusp: $y^2 = x^3$.

The stacks

Isomorphisms of elliptic curves are isomorphisms of pointed schemes. This yields a stack $\mathcal{M}_{e\ell\ell}$.

Isomorphisms of Weirstrass curves are given by projective transformations

$$x \mapsto \mu^{-2}x + r$$
$$y \mapsto \mu^{-3}y + \mu^{-2}sx + t$$

This yields an algebraic stack

$$\mathcal{M}_{\mathrm{Weier}} = \mathbb{A}^5 \times_G EG$$

where $G = \mathrm{Spec}(\mathbb{Z}[r, s, t, \mu^{\pm 1}])$.

Invariant differentials

Consider $C \underset{e}{\overset{q}{\rightleftarrows}} S$. Then e is a closed embedding defined by an ideal $\mathcal{I}$. Define

$$\omega_C = q_*\mathcal{I}/\mathcal{I}^2 = q_*\Omega_{C/S}.$$

- ω_C is locally free of rank 1; a generator is an invariant 1-form;
- if $C = C_{\mathbf{a}}$ is Weierstrass, we can choose the generator

$$\eta_{\mathbf{a}} = \frac{dx}{2y + a_1x + a_3};$$

- if C is elliptic, a choice of generator defines an isomorphism $C = C_{\mathbf{a}}$; thus, all elliptic curves are locally Weierstrass.

Modular forms

The assignment $C/S \mapsto \omega_C$ defines a quasi-coherent sheaf on $\mathcal{M}_{e\ell\ell}$ or $\mathcal{M}_{\text{Weier}}$.

Definition

A **modular form** *of weight n is a global section of $\omega^{\otimes n}$.*

The classes c_4, c_6 and Δ give modular forms of weight 4, 6, and 12.

Theorem (Deligne)

There are isomorphisms

$$\mathbb{Z}[c_4, c_6, \Delta^{\pm 1}]/(c_4^3 - c_6^2 = (12)^3\Delta) \to H^0(\mathcal{M}_{e\ell\ell}, \omega^{\otimes *})$$

and

$$\mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 = (12)^3\Delta) \to H^0(\mathcal{M}_{\text{Weier}}, \omega^{\otimes *})$$

The compactified Deligne-Mumford stack

We have inclusions

$$\mathcal{M}_{e\ell\ell} \subseteq \bar{\mathcal{M}}_{e\ell\ell} \subseteq \mathcal{M}_{\text{Weier}}$$

where

1. $\mathcal{M}_{e\ell\ell}$ classifies elliptic curves: those Weierstrass curves with Δ invertible;
2. $\bar{\mathcal{M}}_{e\ell\ell}$ classifies those Weierstrass curves with a unit in (c_4, c_6, Δ).

Theorem

The algebraic stacks $\mathcal{M}_{e\ell\ell}$ and $\bar{\mathcal{M}}_{e\ell\ell}$ are Deligne-Mumford stacks.

Topological modular forms

Theorem (Hopkins-Miller-Lurie)

There is a derived Deligne-Mumford stack $(\bar{\mathcal{M}}_{e\ell\ell}, \mathcal{O}^{\text{top}})$ whose underlying ordinary stack is $\bar{\mathcal{M}}_{e\ell\ell}$.

Define the spectrum of topological modular forms **tmf** to be the global sections of $\mathcal{O}^{\text{top}}$.

There is a spectral sequence

$$H^s(\bar{\mathcal{M}}_{e\ell\ell}, \omega^{\otimes t}) \Longrightarrow \pi_{2t-s}\mathbf{tmf}.$$

Exercises

1. Calculate the values of c_4 and Δ for the Legendre, Duering, and Tate curves. Decide when the Tate curve is singular.

2. Show that the invariant differential $\eta_{\mathbf{a}}$ of a Weierstrass curve is indeed invariant; that is, if $\phi : C_{\mathbf{a}} \to C_{\mathbf{a}'}$ is a projective transformation from one curve to another, then $\phi^* \eta_{\mathbf{a}'} = \mu \eta_{\mathbf{a}}$.

3. The j-invariant $\bar{\mathcal{M}}_{e\ell\ell} \to \mathbb{P}^1$ sends an elliptic curve C to the class of the pair (c_4^3, Δ). Show this classifies the line bundle $\omega^{\otimes 12}$.

Remark: The j-invariant classifies isomorphisms; that, the induced map of sheaves $\pi_0 \bar{\mathcal{M}}_{e\ell\ell} \to \mathbb{P}^1$ is an isomorphism.