TAG Lecture 6: The moduli stack of formal groups

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Periodic homology theories

Definition

Let E^* be a multiplicative cohomology theory and let

$$\omega_E = \tilde{E}^0 S^2 = E_2.$$

Then E is ***periodic*** *if*

1. $E_{2k+1} = 0$ *for all* k;
2. ω_E *is locally free of rank 1;*
3. $\omega_E \otimes_{E_0} E_{2n} \to E_{2n+2}$ *is an isomorphism for all* n.

A choice of generator $u \in \omega_E$ is an **orientation**; then

$$E_* = E_0[u^{\pm 1}].$$

The primordial example: complex K-theory.

Formal schemes

If X is a scheme and $\mathcal{I} \subseteq \mathcal{O}$ is a sheaf of ideals defining a closed scheme Z. The nth-**infinitesimal neighborhood** is

$$Z_n(R) = \{ f \in X(R) \mid f^* \mathcal{I}^n = 0 \}.$$

The associated formal scheme:

$$\widehat{Z} = \operatorname{colim} Z_n.$$

If $X = \mathrm{Spec}(A)$ and $\mathcal{I}$ defined by $I \subseteq A$, then

$$\widehat{Z} \overset{\mathrm{def}}{=} \mathrm{Spf}(\widehat{A}_I).$$

For example

$$\mathrm{Spf}(\mathbb{Z}[[x]])(R) = \text{the nilpotents of } R.$$

Formal groups

If E^* is periodic, then

$$G = \mathrm{Spf}(E^0 \mathbb{C}\mathrm{P}^\infty)$$

is a group object in the category of formal schemes – a commutative one-dimensional **formal group**.

If E^* is oriented, $E^0\mathbb{C}\mathrm{P}^\infty \cong E^0[[x]]$ and the group structure is determined by

$$\begin{gathered} E^0[[x]] \cong E^0\mathbb{C}\mathrm{P}^\infty \to E^0(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \cong E^0[[x, y]] \\ x \mapsto F(x, y) = x +_F y. \end{gathered}$$

The power series is a **formal group law**; the element x is a **coordinate**.

Example: elliptic spectra

Let $C : \mathrm{Spec}(R) \to \bar{\mathcal{M}}_{e\ell\ell}$ be étale and classify a generalized elliptic curve C. Hopkins-Miller implies that there is a periodic homology theory $E(R, C)$ so that

1. $E(R, C)_0 \cong R$;
2. $E(R, C)_2 \cong \omega_C$;
3. $G_{E(R,C)} \cong \widehat{C}_e$.

Hopkins-Miller says a lot more: the assignment

$$\{ \ \mathrm{Spec}(R) \xrightarrow[\text{étale}]{C} \bar{\mathcal{M}}_{e\ell\ell} \ \} \mapsto E(R, C)$$

is a sheaf of E_∞-ring spectra.

The moduli stack of formal groups

An Isomorphism of formal groups over a ring R

$$\phi : G_1 \to G_2$$

is an isomorphisms of group objects over R. Define $\mathcal{M}_{\mathbf{fg}}$ to be the moduli stack of formal groups.

If G_1 and G_2 have coordinates, then ϕ is determined by an invertible power series $\phi(x) = a_0 x + a_1 x^2 + \cdots$.

Theorem

There is an equivalence of stacks

$$\mathrm{Spec}(L) \times_\Lambda E\Lambda \simeq \mathcal{M}_{\mathbf{fg}}$$

where L is the Lazard ring and Λ is the group scheme of invertible power series.

Invariant differentials

Let $\begin{tikzcd} G \arrow[r, shift left, "q"] & S \arrow[l, shift left, "e"] \end{tikzcd}$ be a formal group. Then e identifies S with the 1st infinitesimal neighborhood defined the ideal of definition $\mathcal{I}$ of G. Define

$$\omega_G = q_*\mathcal{I}/\mathcal{I}^2 = q_*\Omega_{G/S}.$$

This gives an invertible quasi-coherent sheaf ω on $\mathcal{M}_{\mathbf{fg}}$:

- ω_G is locally free of rank 1, a generator is an invariant 1-form;
- if $S = \mathrm{Spec}(R)$ and G has a coordinate x, we can choose generator

$$\eta_G = \frac{dx}{F_x(0,x)} \in R[[x]]dx \cong \Omega_{G/S};$$

- if E is periodic, then $\omega_{G_E} \cong E_2 \cong \omega_E$.

Height of a formal group

Let G be a formal group over a scheme S over $\mathbb{F}_p$. There are recursively defined global sections

$$v_k \in H^0(S, \omega_G^{p^k-1})$$

so that we have a factoring

$$\begin{tikzcd} G \arrow[r, "F"'] \arrow[rr, bend left, "p"] & G^{(p^n)} \arrow[r, "V"'] & G \end{tikzcd}$$

if and only if $v_1 = v_2 = \cdots = v_{n-1} = 0$. Here F is the relative Frobenius.

Then G has **height** greater than or equal to n.

The height filtration

We get a descending chain of closed substacks over $\mathbb{Z}_{(p)}$

$$\begin{tikzcd}
\mathcal{M}_{\mathbf{fg}} & \mathcal{M}(1) \arrow[l, "p=0"'] & \mathcal{M}(2) \arrow[l, "v_1=0"'] & \mathcal{M}(3) \arrow[l, "v_2=0"'] & \cdots \arrow[l] & \mathcal{M}(\infty) \arrow[l]
\end{tikzcd}$$

and the complementary ascending chain of open substacks

$$\mathcal{U}(0) \subseteq \mathcal{U}(1) \subseteq \mathcal{U}(2) \subseteq \cdots \subseteq \mathcal{M}_{\mathbf{fg}}.$$

Over $\mathbb{Z}_{(p)}$ there is a homotopy Cartesian diagram

$$\begin{tikzcd}
\bar{\mathcal{M}}_{e\ell\ell} \arrow[r] \arrow[d] & \mathcal{M}_{\mathrm{Weier}} \arrow[d] \\
\mathcal{U}(2) \arrow[r] & \mathcal{M}_{\mathbf{fg}}
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Over $\mathbb{Z}_{(p)}$ there is a homotopy Cartesian diagram

$$\begin{tikzcd}
\bar{\mathcal{M}}_{e\ell\ell} \arrow[r] \arrow[d, "\text{flat}"'] & \mathcal{M}_{\mathrm{Weier}} \arrow[d, "\text{not flat}"] \\
\mathcal{U}(2) \arrow[r] & \mathcal{M}_{\mathbf{fg}}
\end{tikzcd}$$

Flat morphisms (LEFT)

Suppose $G : \mathrm{Spec}(R) \to \mathcal{M}_{\mathbf{fg}}$ is flat. Then there is an associated homology theory $E(R, G)$.

More generally: take a “flat” morphism $\mathcal{N} \to \mathcal{M}_{\mathbf{fg}}$ and get a family of homology theories.

Theorem (Landweber)

A representable and quasi-compact morphism $\mathcal{N} \to \mathcal{M}_{\mathbf{fg}}$ of stacks is flat if and only if v_n acts as a regular sequence; that is, for all n, the map

$$v_n : f_*\mathcal{O}/\mathcal{I}_n \to f_*\mathcal{O}/\mathcal{I}_n \otimes \omega^{p^n-1}$$

is an injection.

The realization problem

Suppose $\mathcal{N}$ is a Deligne-Mumford stack and

$$f : \mathcal{N} \to \mathcal{M}_{\mathbf{fg}}$$

is a flat morphism. Then the graded structure sheaf on

$$(\mathcal{O}_{\mathcal{N}})_* = \{\omega_{\mathcal{N}}^{\otimes *}\}$$

can be realized as a diagram of spectra in the homotopy category.

Problem

Can the graded structure sheaf be lifted to a sheaf of E_∞-ring spectra? That is, can $\mathcal{N}$ be realized as a derived Deligne-Mumford stack? If so, what is the homotopy type of the space of all such realizations?

Exercises

These exercises are intended to make the notion of height more concrete.

1. Let $f : F \to G$ be a homomorphism of formal group laws over a ring R of characteristic p. Show that if $f'(0) = 0$, then $f(x) = g(x^p)$ for some power series g. To do this, consider the effect of f in the invariant differential.

2. Let F be a formal group law of F and $p(x) = x +_F \cdots +_F x$ (the sum taken p times) by the p-series. Show that either $p(x) = 0$ or there in an $n > 0$ so that

$$p(x) = u_n x^{p^n} + \cdots .$$

3. Discuss the invariance of u_n under isomorphism and use your calculation to define the section v_n of $\omega^{\otimes p^n - 1}$.

An exercise about LEFT

4. One direction of LEFT is fairly formal: show that $G : \mathrm{Spec}(R) \to \mathcal{M}_{\mathbf{fg}}$ is flat that then the v_i form a regular sequence.

The other direction is a theorem and it depends, ultimately, on Lazard's calculation that there is an unique isomorphism classe of formal groups of height n over algebraically closed fields.