# TAG Lecture 7: Derived global sections

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## Coherent cohomology

**Definition**

*Let $X$ be an algebraic stack and $\mathcal{F}$ a quasi-coherent sheaf on $X$. The* **coherent cohomology** *of $\mathcal{F}$ is the right derived functors of global sections:*

$$H^s(X, \mathcal{F}) = H^s R\Gamma(\mathcal{F}).$$

**Warning:** I may need hypotheses on $X$, but I will be vague about this.

If $X$ is derived Deligne-Mumford stack, we have a **descent spectral sequence**

$$H^s(X, \pi_t \mathcal{O}_X) \Longrightarrow \pi_{t-s} R\Gamma(\mathcal{O}_X).$$

# Čech complexes

Suppose $X \to Y$ is faithfully flat. We have the simplicial bar construction

$$\epsilon : X_\bullet \longrightarrow Y.$$

We get a spectral sequence

$$\pi^s H^t(X_\bullet, \epsilon^* \mathcal{F}) \Longrightarrow H^{s+t}(Y, \mathcal{F}).$$

If $U$ is affine, $H^s(U, \mathcal{F}) = 0$ for $s > 0$.

If $X = \sqcup U_i$ is where $\mathcal{U} = \{U_i\}$ is a finite affine cover of $Y$ separated, we get an isomorphism with coherent cohomology and Čech cohomology

$$H^s(Y, \mathcal{F}) \cong \check{H}(\mathcal{U}, \mathcal{F}).$$



# Comodules and comodule Ext

Let $\mathcal{M}$ be a stack and suppose $\mathrm{Spec}(A) \to \mathcal{M}$ is a flat presentation with the property that

$$\mathrm{Spec}(A) \times_{\mathcal{M}} \mathrm{Spec}(A) \cong \mathrm{Spec}(\Lambda).$$

Then $(A, \Gamma)$ is a Hopf algebroid and we have

$$H^s(X, \mathcal{F}) \cong \mathrm{Ext}^s_\Lambda(A, M)$$

where $M = \epsilon^* \mathcal{F}$ is the comodule obtained from $\mathcal{F}$.

**Example:** $\mathcal{M} = X \times_G EG$ with $X = \mathrm{Spec}(A)$ and $G = \mathrm{Spec}(\Lambda)$. Here $\Gamma = A \otimes \Lambda$.

## Example: the Adams-Novikov $E_2$-term

Let $G : \mathrm{Spec}(L) \to \mathcal{M}_{\mathbf{fg}}$ classify the formal group of the universal formal group law. Then $E(L, G) = MUP$ is periodic complex cobordism. We have

$$\mathrm{Spec}(L) \times_{\mathcal{M}_{\mathbf{fg}}} \mathrm{Spec}(L) = \mathrm{Spec}(W) = \mathrm{Spec}(MUP_0 MUP)$$

where

$$W = L[a_0^{\pm 1}, a_1, a_2, \ldots].$$

Then

$$\begin{aligned} H^s(\mathcal{M}_{\mathbf{fg}}, \omega^{\otimes t}) &\cong \mathrm{Ext}^s_W(L, MUP_{2t}) \\ &\cong \mathrm{Ext}^s_{MUP_* MUP}(\Sigma^{2t} MUP_*, MUP_*). \end{aligned}$$

This is not really the $E_2$-term of the ANSS so we must talk about:



## Gradings: the basics

- A graded $R$-module is a $R[\mu^{\pm 1}]$-comodule;
- A graded ring gives an affine $\mathbb{G}_m$-scheme.

**Example**

1. The Lazard ring $L$ is graded:

$$x +_{F_\mu} y = \mu^{-1}((\mu x) +_F (\mu y)).$$

2. $W = L[a_0^{\pm 1}, a_1, \ldots]$ is graded:

$$(\phi\mu)(x) = \mu^{-1}\phi(\mu x).$$

3. Weierstrass curves: $x \mapsto \mu^{-2}x$, $y \mapsto \mu^{-3}y$.

# Gradings and cohomology

Let $H = \mathrm{Spec}(\Lambda)$ be an affine group scheme with an action of $\mathbb{G}_m$ and let

$$G = H \rtimes \mathbb{G}_m = \mathrm{Spec}(\Lambda[\mu^{\pm 1}])$$

be the semi-direct product. Let $X = \mathrm{Spec}(A)$ be an affine right $G$-scheme. Then $(A_*, \Gamma_* = A_* \otimes \Lambda_*)$ is a graded Hopf algebroid.

$$H^s(X \times_G EG, \mathcal{F}) = \mathrm{Ext}^s_{\Gamma_*}(A_*, M_*).$$

where $M_* = \mathcal{F}(\mathrm{Spec}(A) \to X \times_G EG)$.

1. $H^s(\mathcal{M}_{\mathbf{fg}}, \omega^{\otimes t}) \cong \mathrm{Ext}^s_{MU_*MU}(\Sigma^{2t} MU_*, MU_*)$;
2. $H^s(\mathcal{M}_{\mathrm{Weier}}, \omega^{\otimes t}) \cong \mathrm{Ext}^s_{\Lambda_*}(\Sigma^{2t} A_*, A_*)$.
3. $H^*(\mathbb{A}^{n+1} \times_{\mathbb{G}_m} E\mathbb{G}_m, \mathcal{F}) = M_0$.



# Derived push-forward

Given $f : X \to Y$ and a sheaf $\mathcal{F}$ on $X$, then $f_*\mathcal{F}$ is the sheaf on $Y$ associated to

$$U \mapsto H^0(U \times_Y X, \mathcal{F}).$$

If $\mathcal{F}$ is quasi-coherent and $f$ is quasi-compact, $f_*\mathcal{F}$ is quasi-coherent.

There is a composite functor spectral sequence

$$H^s(Y, R^t f_* \mathcal{F}) \Longrightarrow H^{s+t}(X, \mathcal{F}).$$

If higher cohomology on $Y$ is zero:

$$H^0(Y, R^t f_* \mathcal{F}) \cong H^t(X, \mathcal{F}).$$

## Example: projective space

Let $S_* = \mathbb{Z}[x_0, \ldots, x_n]$ with $|x_i| = 1$.

**Theorem**

$$H^t(\mathbb{P}^n, \mathcal{O}(*)) \cong \begin{cases} S_* & t = 0; \\ S_*/(x_0^\infty, \ldots, x_n^\infty) & t = n. \end{cases}$$

We examine the diagram

$$\begin{array}{ccc} \mathbb{A}^{n+1} - \{0\} & \xrightarrow{j} & \mathbb{A}^{n+1} \\ \downarrow & & \downarrow \\ \mathbb{P}^n & \xrightarrow{i} & \mathbb{A}^{n+1} \times_{\mathbb{G}_m} E\mathbb{G}_m. \end{array}$$

We must calculate (the global sections) of

$$Rj_*j^*\mathcal{O}_{\mathbb{A}^{n+1}} \text{ “=” } Rj_*j^*S_*.$$



## Example: the affine case

Let $X = \mathrm{Spec}(R)$ and $j : U \to X$ the open defined by an ideal $I = (a_1, \ldots, a_n)$. If $\mathcal{F}$ is defined by the module $M$, then $j_*j^*\mathcal{F}$ is defined by $K$ where there is an exact sequence

$$0 \to K \to \prod_s M[\frac{1}{a_s}] \to \prod_{s<t} M[\frac{1}{a_s a_t}].$$

There is also an exact sequence

$$0 \to \Gamma_I M \to M \to K \to 0$$

where

$$\Gamma_I M = \{x \in M \mid I^n x = 0 \text{ for some } n\}.$$

# Local cohomology

If $i : U \to Y$ is the open complement of a closed sub-stack $Z \subseteq Y$ define local cohomology by the fiber sequence

$$R\Gamma_Z \mathcal{F} \to \mathcal{F} \to Ri_* i^* \mathcal{F}.$$

If $X = \mathrm{Spec}(A)$ and $Z$ is define by $I = (a_1, \ldots, a_k)$ local cohomology can be computed by the **Koszul complex**

$$M \to \prod_s M[\frac{1}{a_s}] \to \prod_{s<t} M[\frac{1}{a_s a_t}] \to \cdots \to M[\frac{1}{a_1 \ldots a_k}] \to 0.$$

Since $x_0, \ldots, x_n \in S_*$ is a regular sequence:

$$R^{n+1}\Gamma_{\{0\}} S_* = S_*/(x_0^\infty, \ldots, x_n^\infty)$$

and $R^t\Gamma_{\{0\}} S_* = 0$, $t \neq n+1$.