TAG Lecture 8: Topological Modular Forms

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Topological modular forms

Compute

$$H^s(\bar{\mathcal{M}}_{e\ell\ell}, \omega^{\otimes t}) \Longrightarrow \pi_{t-s}\mathbf{tmf}$$

when 2 is inverted. Can do $p = 2$ as well, but harder.
We have a Cartesian diagram

$$\begin{array}{ccc} U & \xrightarrow{j} & \mathbb{A}^5 \\ \downarrow & & \downarrow \\ \bar{\mathcal{M}}_{e\ell\ell} & \xrightarrow[i]{} & \mathcal{M}_{\mathrm{Weier}} \end{array}$$

where U is the open defined by the comodule ideal

$$I = (c_4^3, \Delta).$$

Cohomology of $\mathcal{M}_{\mathrm{Weier}}$, $p > 3$.

Any Weierstrass curve is isomorphic to a curve of the form

$$y^2 = x^3 - (1/48)c_4 x - (1/216)c_6$$

and the only remaining projective transformations are

$$(x, y) \mapsto (\mu^{-2}x, \mu^{-3}y).$$

Then

$$\mathrm{Spec}(\mathbb{Z}_{(p)}[c_4, c_6]) \to \mathcal{M}_{\mathrm{Weier}}$$

is a presentation. There is no higher cohomology and

$$H^0(\mathcal{M}_{\mathrm{Weier}}, \omega^{\otimes *}) \cong \mathbb{Z}_{(p)}[c_4, c_6]$$

with $|c_4| = 8$ and $|c_6| = 12$. Note $\Delta = (1/(12)^3)(c_4^3 - c_6^2)$.

Cohomology of $\mathcal{M}_{\mathrm{Weier}}$, $p = 3$.

Any Weierstrass curve is isomorphic to a curve of the form

$$y^2 = x^3 + (1/4)b_2 + (1/2)b_2 + (1/4)b_6$$

and the remaining projective transformations are

$$(x, y) \mapsto (\mu^{-2}x + r, \mu^{-3}y).$$

Then

$$\mathrm{Spec}(\mathbb{Z}_{(3)}[b_2, b_4, b_6]) \to \mathcal{M}_{\mathrm{Weier}}$$

is a presentation and

$$H^s(\mathcal{M}_{\mathrm{Weier}}, \omega^t) = \mathrm{Ext}^s_\Gamma(\Sigma^{2t}A_*, A_*)$$

with

$$A_* = \mathbb{Z}_{(p)}[b_2, b_4, b_6] \qquad \text{and} \qquad \Gamma_* = A_*[r]$$

with appropriate degrees.

Some familiar elements (Rezk)

In the Hopf algebroid (A_*, Γ_*), we have

$$\eta_R(b_2) = b_2 + 12r$$

so there is higher cohomology:

$$H^*(\mathcal{M}_{\mathrm{Weier}}, \omega^{\otimes *}) = \mathbb{Z}_{(3)}[c_4, c_6, \Delta][\alpha, \beta]/I$$

where $|\alpha| = (1, 4)$ and $|\beta| = (2, 12)$. Here I is the relations:

$$c_4^3 - c_6^2 = (12)^3 \Delta$$
$$3\alpha = 3\beta = 0$$
$$c_i\alpha = c_i\beta = 0.$$

Note: Δ acts "periodically".

A property of local cohomology

To compute

$$H^s(\mathcal{M}_{\mathrm{Weier}}, R^t i_* \omega^*) \Longrightarrow H^{s+t}(\bar{\mathcal{M}}_{e\ell\ell}, \omega^*)$$

we compute $R\Gamma_I A_*$ where $I = (c_4^3, \Delta)$.
Note: if $\sqrt{I} = \sqrt{J}$ then $R\Gamma_I = R\Gamma_J$.
For $p > 3$ we take $I = (c_4^3, \Delta)$ and $J = (c_4, c_6)$.

$$R^2\Gamma_I A_* = \mathbb{Z}_{(p)}[c_4, c_6]/(c_4^\infty, c_6^\infty).$$

and

$$H^s(\bar{\mathcal{M}}_{e\ell\ell}, \omega^*) \cong \begin{cases} \mathbb{Z}_{(p)}[c_4, c_6], & s = 0; \\ \mathbb{Z}_{(p)}[c_4, c_6]/(c_4^\infty, c_6^\infty), & s = 1. \end{cases}$$

Note the duality. The homotopy spectral sequence collapses.

Local cohomology for **tmf**, $p = 3$

At $p = 3$ there are inclusions of ideals

$$(c_4^3, \Delta) \subseteq (c_4, \Delta) \subseteq (c_4, e_6, \Delta) = J = \sqrt{I}$$

where

$$e_6^2 = 12\Delta.$$

Since J is **not** generated by a regular sequence we must use: if $J = (I, x)$ there is a fiber sequence

$$R\Gamma_J M \to R\Gamma_I M \to R\Gamma_I M[1/x].$$

We take $I = (c_4, e_6)$ and $J = (c_4, e_6, \Delta)$.

Duality at 3

Let $A_* = \mathbb{Z}_{(3)}[b_2, b_4, b_6]$.

Proposition

$R^s_J\, A_* = 0$ if $s \neq 2$ and $R^2_J\, A_*$ is the Δ-torsion in $A_*/(c_4^\infty, e_6^\infty)$.

Corollary (Duality)

$R^2\Gamma_J \omega^{-10} \cong \mathbb{Z}_{(3)}$ with generator corresponding to $12/c_4 e_6$ and there is non-degenerate pairing

$$R^2\Gamma_J\, \omega^{-t-10} \otimes \omega^t \to R^1\Gamma_J\, \omega^{-10} \cong \mathbb{Z}_{(3)}$$

We now can calculate π_***tmf**, at least it $p = 2$.

The differentials and extensions

The crucial differentials are classical:

$$d_5\Delta = \alpha\beta^2 \qquad \text{(Toda)}$$
$$d_9\Delta\alpha = \beta^4 \qquad \text{(Nishida)}$$

There is also an exotic extension in the multiplication: if z is the homotopy class detected by $\Delta\alpha$, then:

$$\alpha z = \beta^3.$$

In fact, $z = \langle \alpha, \alpha, \beta^2 \rangle$ so

$$\alpha z = \alpha\langle \alpha, \alpha, \beta^2 \rangle = \langle \alpha, \alpha, \alpha \rangle \beta^2 = \beta^3.$$