Topological Algebraic Geometry: A Workshop at The University of Copenhagen

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Lecture one: Schemes

This lecture establishes some basic language, including:

- Three ways of looking at schemes,
- Quasi-coherent sheaves,
- and the example of projective space.

Affine Schemes

Let A be a commutative ring. The affine scheme defined by A is the pair:

$$\mathrm{Spec}(A) = (\mathrm{Spec}(A), \mathcal{O}_A).$$

The underlying set of $\mathrm{Spec}(A)$ is the set of prime ideals $\mathfrak{p} \subseteq A$. If $I \subseteq A$ is an ideal, we define

$$V(I) = \{\, \mathfrak{p} \subseteq A \text{ prime} \mid I \not\subseteq \mathfrak{p} \,\} \subseteq \mathrm{Spec}(A).$$

These open sets form the *Zariski topology* with basis

$$V(f) = V((f)) = \{\, \mathfrak{p} \mid f \notin \mathfrak{p} \,\} = \mathrm{Spec}(A[1/f]).$$

The sheaf of rings $\mathcal{O}_A$ is determined by

$$\mathcal{O}_A(V(f)) = A[1/f].$$

Schemes as locally ringed spaces

$\mathrm{Spec}(A)$ is a *locally ringed space*: if $\mathfrak{p} \in \mathrm{Spec}(A)$, the stalk of $\mathcal{O}_A$ at $\mathfrak{p}$ is the local ring $A_{\mathfrak{p}}$.

Definition

A scheme $X = (X, \mathcal{O}_X)$ is a locally ringed space with an open cover (as locally ringed spaces) by affine schemes.
A morphism $f : X \to Y$ is a continuous map together with an induced map of sheaves

$$\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_Y$$

with the property that for all $x \in X$ the induced map of local rings

$$(\mathcal{O}_Y)_{f(x)} \longrightarrow (\mathcal{O}_X)_x$$

is **local***; that is, it carries the maximal ideal into the maximal ideal.*

Schemes as functors

If X is a scheme we can define a functor which we also call X from commutative rings to sets by by

$$X(R) = \mathbf{Sch}(\mathrm{Spec}(R), X).$$

$$\mathrm{Spec}(A)(R) = \mathbf{Rings}(A, R).$$

Theorem

A functor $X : Rings_c \longrightarrow \mathbf{Sets}$ is a scheme if and only if

1. *X is a sheaf in the Zariski topology;*
2. *X has an open cover by affine schemes.*

Example: projective space

Define a functor $\mathbb{P}^n$ from rings to sets: $\mathbb{P}^n(R)$ is the set of all split inclusions of R-modules

$$N \longrightarrow R^{n+1}$$

with N locally free of rank 1.

For $0 \le i \le n$ let $U_i \subseteq \mathbb{P}^n$ to be the subfunctor of inclusions j so that

$$N \xrightarrow{j} R^{n+1} \xrightarrow{p_i} R$$

is an isomorphism. Then the U_i form an open cover and $U_i \cong \mathbb{A}^n$.

Geometric points

If X is a (functor) scheme we get (locally ringed space) scheme $(|X|, \mathcal{O}_X)$ by:

1. $|X|$ is the set of (a *geometric points*) in X: equivalence classes of pairs $(\mathbb{F}, x)$ where $\mathbb{F}$ is a field and $x \in X(\mathbb{F})$.
2. An open subfunctor U determines an open subset of the set of geometric points.
3. Define $\mathcal{O}_X$ locally: if $U = \mathrm{Spec}(A) \to X$ is an open subfunctor, set $\mathcal{O}_X(U) = A$.

The geometric points of $\mathrm{Spec}(R)$ (the functor) are the prime ideals of R.
If X is a functor and R is a ring, then an R-point of X is an element in $X(R)$; these are in one-to-one correspondence with morphism $\mathrm{Spec}(R) \to X$.

Schemes as ringed topoi

This notion generalizes very well.

If X is a scheme let $\mathcal{X}$ denote the category of sheaves of sets on X. Then $\mathcal{X}$ is a *topos*:

1. $\mathcal{X}$ has all colimits and colimits commute with pull-backs (base-change);
2. $\mathcal{X}$ has a set of generators;
3. Coproducts in $\mathcal{X}$ are disjoint; and
4. Equivalence relations in C are effective.

If X is a scheme, $\mathcal{O}_X \in \mathcal{X}$ and the pair $(\mathcal{X}, \mathcal{O}_X)$ is a ringed topos.
[Slogan] A ringed topos is equivalent to that of a scheme if it is locally of the form $\mathrm{Spec}(A)$.

Quasi-coherent sheaves

Let $(X, \mathcal{O}_X)$ be a scheme and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. Then $\mathcal{F}$ is *quasi-coherent* if is locally presentable as an $\mathcal{O}_X$-module.

Definition

An $\mathcal{O}_X$ module sheaf is **quasi-coherent** *if for all $y \in X$ there is an open neighborhood U of y and an exact sequence of sheaves*

$$\mathcal{O}_U^{(J)} \longrightarrow \mathcal{O}_U^{(I)} \longrightarrow \mathcal{F}|_U \to 0.$$

If $X = \mathrm{Spec}(A)$, then the assignment $\mathcal{F} \mapsto \mathcal{F}(X)$ defines an equivalence of categories between quasi-coherent sheaves and A-modules.

Quasi-coherent sheaves (reformulated)

Let X be a scheme, regarded as a functor. Let $\mathbf{Aff}/X$ be the category of morphisms $a : \mathrm{Spec}(A) \to X$. Define

$$\mathcal{O}_X(\mathrm{Spec}(A) \to X) = \mathcal{O}_X(a) = A.$$

This is a sheaf in the Zariski topology.
A quasi-coherent sheaf $\mathcal{F}$ is sheaf of $\mathcal{O}_X$-modules so that for each diagram

$$\begin{tikzcd}
\mathrm{Spec}(B) \arrow[dd, "f"'] \arrow[dr, "b"] & \\
& X \\
\mathrm{Spec}(A) \arrow[ur, "a"'] &
\end{tikzcd}$$

the map

$$f^*\mathcal{F}(a) = B \otimes_A \mathcal{F}(a) \to \mathcal{F}(b)$$

is an isomorphism.

Example: $\mathcal{O}(k)$ on $\mathbb{P}^n$

Morphisms $a : \mathrm{Spec}(A) \to \mathbb{P}^n$ correspond to split inclusions

$$N \longrightarrow A^{n+1}$$

with N locally free of rank 1. Define $\mathcal{O}_{\mathbb{P}^n}$-module sheaves

$$\mathcal{O}(-1)(a) = N$$

and

$$\mathcal{O}(1)(a) = \mathrm{Hom}_A(N, A).$$

These are quasi-coherent, locally free of rank 1 and $\mathcal{O}(1)$ has canonical global sections x_i

$$\begin{tikzcd} N \arrow[r] & A^{n+1} \arrow[r, "p_i"] & A. \end{tikzcd}$$

Exercises

1. Show that the functor P^n as defined here indeed satisfies the two criteria to be a scheme.

2. Fill in the details of the final slide: define the global sections of sheaf and show that the elements x_i there defined are indeed global sections of the sheaf $\mathcal{O}(1)$ on P^n.

3. The definition of P^n given here can be extended to a more general statement: if X is a scheme, then the morphisms $X \to P^n$ are in one-to-one correspondence with locally free sheaves $\mathcal{F}$ of rank 1 over X generated by global sections s_i, $0 \leq i \leq n$.

4. Show that the functor which assigns to each ring R the set of finitely generated projective modules of rank 1 over R cannot be scheme.

Lecture 2: Derived schemes

In this lecture we touch briefly on the notion of derived schemes. Topics include:

- An axiomatic description of ring spectra;
- Jardine's definition of sheaves of spectra;
- Derived schemes (in the Zariski topology) and examples.

Spectra

We need a good model for the stable homotopy category. Let $\mathcal{S}$ be a category so that

1. $\mathcal{S}$ is a cofibrantly generated proper stable simplicial model model category Quillen equivalent to the Bousfield-Friedlander category of simplicial spectra;
2. $\mathcal{S}$ has a closed symmetric monoidal smash product which gives the smash product in the homotopy category;
3. the smash product and the simplicial structure behave well;
4. and so on.

Symmetric spectra (either simplicially or topologically) will do.

Commutative ring spectra

A commutative monoid A in $\mathcal{S}$ is a commutative ring spectra: there is a multiplication map

$$A \wedge A \longrightarrow A$$

and a unit map

$$S^0 \longrightarrow A$$

so that the requisite diagrams commutes.
There are A-modules with mulitplications $A \wedge M \to M$.
There are free commutative algebras:

$$\begin{aligned} \mathrm{Sym}(X) = \vee\ \mathrm{Sym}_n(X) &= \vee\ (X^{\wedge n})/\Sigma_n \\ &= \vee\ (E\Sigma_n)_+ \wedge_{\Sigma_n} X^{\wedge n}. \end{aligned}$$

These categories inherit model category structures

Sheaves of spectra

Let X be a scheme. A presheaf of spectra is a functor

$$\mathcal{F} : \{ \text{ Zariski opens in } X \ \}^{\mathrm{op}} \to \mathcal{S}.$$

Theorem (Jardine)

Presheaves of spectra form a simplicial model category where $\mathcal{E} \to \mathcal{F}$

- *is a weak equivalence if* $\mathcal{E}_{\mathfrak{p}} \to \mathcal{F}_{\mathfrak{p}}$ *is a weak equivalence for all* $\mathfrak{p} \in X$*;*
- $\mathcal{E} \to \mathcal{F}$ *is a cofibration in* $\mathcal{E}(U) \to \mathcal{F}(U)$ *is a cofibration for all* U*.*

A *sheaf of* (ring or module) spectra is a fibrant/cofibrant object. Jardine proves an analogous theorem for ring and module spectra for an arbitrary topos.

Global sections

Let X be a scheme and $\mathcal{F}$ a sheaf on X. If $\{U_\alpha\}$ is an open cover, let $\mathcal{U}$ the associated category. Then

$$\begin{aligned} H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F}) = \mathcal{F}(X) &\cong \mathbf{Sh}(X,\mathcal{F}) \\ &\cong \lim_{\mathcal{U}} \mathcal{F}. \end{aligned}$$

If $\mathcal{F}$ is a sheaf of spectra these become:

$$\begin{aligned} R\Gamma(X,\mathcal{F}) &\simeq F_{\mathbf{Sh}_+}(X,\mathcal{F}) \\ &\simeq \operatorname{holim}_{\mathcal{U}} \mathcal{F}. \end{aligned}$$

And the derived nature of the subject begins to appear. There is a spectral sequence

$$H^s(X,\pi_t\mathcal{F}) \Longrightarrow \pi_{t-s}R\Gamma(X,\mathcal{F}).$$

Derived schemes

Theorem (Lurie)

Let X be a space and $\mathcal{O}$ a sheaf of ring spectra on X. Then $(X,\mathcal{O})$ is a ***derived scheme*** *if*

- $(X,\pi_0\mathcal{O})$ *is a scheme; and*
- $\pi_i\mathcal{O}$ *is a quasi-coherent* $\pi_0\mathcal{O}$ *module for all* i.

Remark

This looks like a definition, not a theorem. There is a better definition using topoi.

Derived affine schemes

Definition

Let A be a ring spectrum. Define $\mathrm{Spec}(A)$ by

- *Underlying space: $\mathrm{Spec}(\pi_0 A)$; and*
- *$\mathcal{O}$: sheaf associated to the presheaf*

$$V(f) = \mathrm{Spec}(\pi_0 A[1/f]) \mapsto A[1/f].$$

Remark

$A[1/f]$ is the localization of A characterized by requiring

$$\mathrm{Spec}(A[1/f], B) \subseteq \mathrm{Spec}(A, B)$$

to be subspace of components where f in invertible. Such localizations can be done functorially in the category of ring spectra.

Derived schemes: the category

Lurie's result above is actually part of an equivalence of categories:

Theorem (Lurie)

A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of derived schemes is a pair (f, ϕ) where

- *$f : X \to Y$ is a continuous map;*
- *$\phi : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is a morphism of sheaves of ring spectra*

so that

$$(f, \pi_0 \phi) : (X, \pi_0 X) \to (Y, \pi_0 Y)$$

is a morphism of schemes.

The collection of all morphisms $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a space.

Derived schemes as functors

If X is a derived scheme, we write

$$X : \text{Ring spectra} \to \textbf{Spaces}$$

for the functor

$$X(R) = \textbf{Dsch}(\mathrm{Spec}(R), X).$$

Example

The affine derived scheme $\mathbb{A}^1$ is characterized by

$$\mathbb{A}^1(R) = \Omega^\infty R.$$

The affine derived scheme Gl_1 is characterized

$$\mathrm{Gl}_1(R) \subseteq \Omega^\infty R.$$

to be the subsets of invertible components.

Example: Derived projective space

Define $\mathbb{P}^n(R)$ to be the *subspace* of the R-module morphisms

$$i : N \longrightarrow R^{n+1}$$

which split and so that $\pi_0 N$ is locally free of rank 1 as a $\pi_0 R$-module.

The underlying scheme of **derived** $\mathbb{P}^n$ is **ordinary** $\mathbb{P}^n$. The sub-derived schemes U_k, $0 \leq k \leq n$ of those q with

$$N \xrightarrow{i} R^{n+1} \xrightarrow{p_k} R$$

an equivalence *cover* $\mathbb{P}^n$. Note

$$U_k(R) \cong \mathbb{A}^n(R) \cong \Omega^\infty R^{\times n}.$$

Exercise

1. Let A be an E_∞-ring spectrum and M an A-module. Assume we can define the symmetric A-algebra $\mathrm{Sym}_A(M)$ and that it has the appropriate universal property. (What would that be?) Let $A = S$ be the sphere spectrum and let $M = \vee_n S$ ($\vee =$ coproduct or wedge). What is $\mathrm{Spec}(\mathrm{Sym}_S(M))$? That is, what functor does it represent?

2. Suppose $n = 1$ and $x \in \mathrm{Sym}_S(M)$ is represented by the inclusion $S = M \to \mathrm{Sym}_S(M)$. What is $\mathrm{Spec}(\mathrm{Sym}_S(M)[1/x])$?

Open-ended exercise: the tangent functor

3. If R is a ring, then $R[\epsilon] \overset{\mathrm{def}}{=} R[x]/(x^2)$. This definition makes sense for E_∞-ring spectra as well. If X is any functor on rings (or E_∞-ring spectra) the tangent functor T_X is given by

$$R \mapsto X(R[\epsilon]).$$

Explore this functor, for example:

1. Show that T_X is an abelian group functor over X;
2. If $x : \mathrm{Spec}(A) \to X$ is any A-point of X, describe the fiber

$$T_{X,x} = \mathrm{Spec}(A) \times_X T_X.$$

3. (More advanced) Show that this fiber is, in fact, an affine scheme.

Lecture 3: Flat, smooth, and étale maps

This lecture introduces some of the other standard topologies. We discuss:

- Descent and derived descent;
- smooth and étale maps;
- the new topologies and the sheaves in them; and
- briefly mention the cotangent complex.

Flat morphisms

A morphism of rings $A \to B$ is flat if $B \otimes_A (-)$ is exact. It is faithfully flat if it creates isomorphisms.

Definition

A morphism $f : X \to Y$ of schemes is flat if for all $x \in X$, $\mathcal{O}_x$ is a flat $\mathcal{O}_{f(x)}$-algebra. The morphism f is faithfully flat if is flat and surjective.

A morphism $A \to B$ of E_∞-ring spectra is flat if

1. $\pi_0 A \to \pi_0 B$ is a flat morphism of rings;
2. $\pi_0 B \otimes_{\pi_0 A} \pi_n A \cong \pi_n B$ for all n.

A bar construction

Let $X \to Y$ be a morphism of schemes and let

$$\epsilon : X_\bullet \longrightarrow Y$$

be the bar construction. Faithfully flat descent compares sheaves over Y with simplicial sheaves on

$$X_\bullet = \{\, X^{\bullet+1} \,\}.$$

If $X = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = Y$ are both affine; this is $\mathrm{Spec}(-)$ of the cobar construction

$$\eta : A \longrightarrow \{\, B^{\otimes \bullet+1} \,\}.$$

Descent

A module-sheaf $\mathcal{F}_\bullet$ on $X_\bullet$ is:

1. module sheaves $\mathcal{F}_n$ on X_n;
2. for each $\phi : [n] \to [m]$, a homomorphism $\theta(\phi) : \phi^*\mathcal{F}_n \to \mathcal{F}_m$;
3. subject to the evident coherency condition.

Definition

Such a module sheaf is **Cartesian** *if each $\mathcal{F}_n$ is quasi-coherent and $\theta(\phi)$ is an isomorphism for all ϕ.*

If $\mathcal{E}$ is a quasi-coherent sheaf on Y, $\epsilon^*\mathcal{E}$ is a Cartesian sheaf on $X_\bullet$.

Descent: If f is quasi-compact and faithfully flat, this is an equivalence of categories.

Derived Descent

A chain complex $\mathcal{F}_\bullet$ of simplicial module sheaves is the same as simplicial chain complex of module sheaves.

Definition

Let $\mathcal{F}_\bullet$ be a chain complex of simplicial module sheaves on $X_\bullet$. The $\mathcal{F}_\bullet$ is ***Cartesian*** *if*

1. *each $\theta(\phi) : \phi^* \mathcal{F}_n \to \mathcal{F}_m$ is an equivalence;*
2. *the homology sheaves $\mathcal{H}_i(\mathcal{F}_\bullet)$ are quasi-coherent.*

If $\mathcal{E}$ is a complex of quasi-coherent sheaves on Y, $\epsilon^* \mathcal{E}$ is a Cartesian sheaf on $X_\bullet$.

Derived descent: This if f is quasi-compact and faithfully flat, this is an equivalence of derived categories.

Étale and smooth morphisms

There are not enough Zariski opens; there are too many flat morphisms, even finite type ones; therefore:

Suppose we are given any lifting problem in schemes

$$\begin{tikzcd}
\mathrm{Spec}(A/I) \arrow[r] \arrow[d, "\subseteq"'] & X \arrow[d, "f"] \\
\mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

with I **nilpotent** and f flat and locally finite. Then

Definition

1. *f is* ***smooth*** *if the problem always has a solution;*
2. *f is* ***étale*** *if the problem always has a unique solution.*

Examples: Étale and smooth morphisms

Theorem

$B = A[x, \dots, x_n]/(p_1, \dots, p_m)$ is

- *étale over A if $m = n$ and $\det(\partial p_i/\partial x_j)$ is a unit in B;*
- *smooth over A if $m \leq n$ and the $m \times m$ minors of the partial derivatives generate B.*

Any étale or smooth morphism is locally of this form.

1. $\mathbb{F}_p[x]/(x^{p^n} - x)$ is étale over $\mathbb{F}_p$.
2. Any finite separable field extension is étale.
3. $A[x]/(ax^2 + bx + c)$ is étale over A if $b^2 - 4ac$ is a unit in A.
4. $\mathbb{F}[x, y]/(y^2 - x^3)$ is not smooth over any field $\mathbb{F}$.

Étale maps as covering spaces

Theorem

Let $f : X \to Y$ be étale and separated and $U \subset Y$ be open. Any section s of

$$U \times_Y X \to U$$

is an isomorphism onto a connected component.

For the analog of normal covering spaces we have:

Definition

Let $f : X \to Y$ be étale and $G = \mathrm{Aut}_X(Y)$ (the Deck transformations). Then X is **Galois** *over$Y$ if we have an isomorphism*

$$\begin{aligned} G \times X &\longrightarrow X \times_Y X \\ (g, x) &\mapsto (g(x), x). \end{aligned}$$

Étale topology

Let X be a scheme. The étale topology has

- étale maps $U \to X$ as basic opens;
- a cover $\{ V_i \to U \}$ is a finite set of étale maps with $\coprod V_i \to U$ surjective.

Notes:

1. every open inclusion is étale; so an étale sheaf yields a Zariski sheaf;
2. Define $\mathcal{O}_X(U \to X) = \mathcal{O}_U(U)$; this is the étale structure sheaf.
3. There are module sheaves and quasi-coherent sheaves for the étale topology.

Zariski versus étale sheaves

The inclusion of a Zariski open $U \to X$ is rigid: $\mathrm{Aut}_X(U) = \{e\}$. An étale open $U \to X$ need not be rigid: $\mathrm{Aut}_X(U) \neq \{e\}$ in general.

Example

Let $\mathbb{F}$ be field and $X = \mathrm{Spec}(\mathbb{F})$.

- Module sheaves in the Zariski topology on X are $\mathbb{F}$- vector spaces.
- Module sheaves in the étale topology on X are twisted, discrete $\bar{\mathbb{F}} - \mathrm{Gal}(\bar{\mathbb{F}}/\mathbb{F})$-modules.

Étale morphisms for spectra

A morphism $f : A \to B$ of ring spectra is étale if

1. $\pi_0 A \to \pi_0 B$ is an étale morphism of rings; and
2. $\pi_0 B \otimes_{\pi_0 A} \pi_i A \to \pi_i B$ is an isomorphism.

Compare to:

Definition (Rognes)

Let $A \to B$ of ring spectra and let $G = \mathrm{Aut}_A(B)$. The morphism Galois if

- $B \wedge_A B \to F(G_+, B)$*; and*
- $A \to B^{hG} = F(G_+, B)^G$

are equivalences.

Hypotheses are needed: G finite or "stably dualizable".

The cotangent complex

Let $f : X \to Y$ be a morphism of schemes. Let $\mathrm{Der}_{X/Y}$ be the sheaf on X associated to the functor

$$\begin{tikzcd}
\mathrm{Spec}(R) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\mathrm{Spec}(R[\epsilon]) \arrow[r, "\epsilon=0"'] \arrow[ur, dashed] & Y
\end{tikzcd}$$

This is representable: $\mathrm{Der}_{X/Y} = \mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X)$. The **cotangent complex** $L_{X/Y}$ is the derived version.

Suppose f is locally finite and flat, then

1. f is étale if and only if $L_{X/Y} = 0$;
2. f is smooth if and only if $L_{X/Y} \simeq \Omega_{X/Y}$ and that sheaf is locally free.

Exercises

1. Let (A, Γ) be a Hopf algebroid. Assume Γ is flat over A. Then we get a simplicial scheme by taking $\mathrm{Spec}(-)$ of the cobar construction on the Hopf algebroid. Show that the category of Cartesian (quasi-coherent) sheaves on this simplicial scheme is equivalent to the category of (A, Γ)-comodules.

2. Let $A \to B$ be a morphism of algebras and M an A-module. Show that the functor on B-modules

$$M \to \mathbf{Def}_A(B, M)$$

is representable by a B-module $\Omega_{B/M}$. Indeed, if I is the kernel of the multiplication map $B \otimes_A B \to B$, then $\Omega_{B/A} \cong I/I^2$.

3. Calculate Let $B = \mathbb{F}[x, y]/(y^2 - x^3)$ where $\mathbb{F}$ is a field. Show that $\Omega_{B/\mathbb{F}}$ is locally free of rank 1 except at $(0, 0)$.

Lecture 4: Algebraic stacks

We introduce the notion of algebraic stacks and quasi-coherent sheaves thereon. Topic include:

- Stacks of G-torsors and quotient stacks;
- Projective space as a stacks;
- Quasi-coherent sheaves versus comodules;
- Deligne-Mumford stacks and their derived counterparts.

Sheaves of groupoids

Let S be a scheme (usually $\mathrm{Spec}(R)$). Stacks are built from sheaves of groupoids $\mathcal{G}$ on S.

Example

Let (A, Γ) be Hopf algebroid over R. Then

$$\begin{tikzcd} \mathcal{G} = \{ \; \mathrm{Spec}(\Gamma) \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathrm{Spec}(A) \arrow[l] \; \} \end{tikzcd}$$

is a sheaf of groupoids in all our topologies.

Given $U \to S$ and $x \in \mathcal{G}(U)$, get a presheaf Aut_x

$$\mathrm{Aut}_x(V \to U) = \mathrm{Iso}_{\mathcal{G}(V)}(x|_V, x|_V).$$

$\mathcal{G}$ is a prestack if this is sheaf. Hopf algebroids give prestacks.

Effective descent and stacks

Let $\mathcal{G}$ be a prestack on S and let $\mathcal{N}\mathcal{G}$ be its nerve; this is a presheaf of simplicial sets.

Definition

$\mathcal{G}$ is a **stack** *if $\mathcal{N}\mathcal{G}$ is a fibrant presheaf of simplicial sets.*

This is equivalent to $\mathcal{G}$ satisfying the following:

Effective Descent Condition: Given

1. a cover $V_i \to U$ and $x_i \in \mathcal{G}(U_i)$;
2. isomorphisms $\phi_{ij} : x_i|_{V_i \times_U V_j} \to x_j|_{V_i \times_U V_j}$;
3. subject to the evident cocyle condition;

Then there exists $x \in \mathcal{G}(U)$ and isomorphisms $\psi_i : x_i \to x|_{V_i}$.

Example: Principal G-bundles

Hopf algebroids hardly ever give stacks. Let's fix this.
Let Λ be a Hopf algebra over a ring k and

$$G = \mathrm{Spec}(\Lambda) \to \mathrm{Spec}(k) = S$$

the associated group scheme.

Definition

*A G-scheme $P \to U$ over U is a a G-**torsor** if it locally of the form $U \times_S G$.*

The functor from schemes to groupoids

$$U \mapsto \{\ G\text{-torsors over } U \text{ and their isos}\ \}$$

is a stack. This is the **classifying** stack BG.

Example: Algebraic homotopy orbits

Let X be a G-scheme. Form a functor to groupoids

$$U \mapsto \left\{ \begin{tikzcd} P \arrow[r, "G\text{-map}"] \arrow[d, "\text{torsor}"'] & X \\ U \end{tikzcd} \right\}$$

This is the **quotient stack** $X \times_G EG = [X/G/S]$.
If Λ is our Hopf algebra, A a comodule algebra, then $(A, \Gamma = A \otimes \Lambda)$ is a **split** Hopf algebroid and

$$\mathrm{Spec}(A) \times_G EG$$

is the associated stack to the sheaf of groupoids we get from (A, Λ).

Example: Projective space

Consider the action

$$\mathbb{A}^{n+1} \times \mathbb{G}_m \longrightarrow \mathbb{A}^{n+1}$$
$$(a_0, \ldots, a_n) \times \lambda \mapsto (a_0\lambda, \ldots, a_n\lambda)$$

Define $\mathbb{P}^n \to \mathbb{A}^{n+1} \times_{\mathbb{G}_m} E\mathbb{G}_m$ by

$$\{\, N \to R^{n+1} \,\} \mapsto \left\{ \begin{array}{ccc} \mathrm{Iso}(R, N) & \longrightarrow & \mathbb{A}^{n+1} \\ \downarrow & & \\ \mathrm{Spec}(R) & & \end{array} \right\}$$

We get an isomorphism

$$\mathbb{P}^n \cong (\mathbb{A}^{n+1} - \{0\}) \times_{\mathbb{G}_m} E\mathbb{G}_m.$$

Morphisms and pullbacks

A morphism of stack $\mathcal{M} \to \mathcal{N}$ is a morphism of sheaves of groupoids. A 2-commuting diagram

$$\begin{tikzcd}
 & \mathcal{N}_1 \arrow[dd, "p"'] \\
\mathcal{M} \arrow[ur, "f"] \arrow[dr, "g"'] & \\
 & \mathcal{N}_2
\end{tikzcd}$$

is specified natural isomorphism $\phi : pf \to g$.

Given $\mathcal{M}_1 \xrightarrow{f} \mathcal{N} \xleftarrow{g} \mathcal{M}_2$ the pull-back $\mathcal{M}_1 \times_{\mathcal{N}} \mathcal{M}_2$ has objects

$$(x \in \mathcal{M}_1, y \in \mathcal{M}_2, \phi : f(x) \to g(y) \in \mathcal{N}).$$

Representable morphisms

Definition

A morphism $\mathcal{M} \to \mathcal{N}$ is **representable** *if for all morphisms $U \to \mathcal{N}$ of schemes, the pull-back*

$$U \times_{\mathcal{N}} \mathcal{M}$$

is equivalent to a scheme.

A representable morphism of stacks $\mathcal{N} \to \mathcal{M}$ is smooth or étale or quasi-compact or $\cdots$ if

$$U \times_{\mathcal{N}} \mathcal{M} \longrightarrow U$$

has this property for all $U \to \mathcal{N}$.

Algebraic Stacks

Definition

A stack $\mathcal{M}$ is **algebraic** *if*

1. *all morphisms from schemes $U \to \mathcal{M}$ are algebraic; and,*
2. *there is a smooth surjective map $q : X \to \mathcal{M}$.*

$\mathcal{M}$ is *Deligne-Mumford* if P can be chosen to be étale.

$X \times_G EG$ is algebraic with presentation

$$X \longrightarrow X \times_G EG$$

if G is smooth. Deligne-Mumford if G is étale.

Quasi-coherent sheaves

Definition

A **quasi-coherent sheaf** $\mathcal{F}$ *on an algebraic stack* $\mathcal{M}$*:*

1. *for each smooth* $x : U \to \mathcal{M}$*, a quasi-coherent sheaf* $\mathcal{F}(x)$*;*
2. *for 2-commuting diagrams*

$$\begin{tikzcd}
V \arrow[dr, "y"] \arrow[dd, "f"'] & \\
& \mathcal{M} \\
U \arrow[ur, "x"'] &
\end{tikzcd}$$

coherent isomorphisms $\mathcal{F}(\phi) : \mathcal{F}(y) \to f^*\mathcal{F}(x)$*.*

Descent: If $X \to \mathcal{M}$ is a presentation then

$$\{ \text{ QC-sheaves on } \mathcal{M} \} \simeq \{ \text{ Cartesian sheaves on } X_\bullet \}$$

Example: Quasi-coherent sheaves and comodules

Suppose $\mathcal{M} = X \times_G EG$ where

- $G = \mathrm{Spec}(\Lambda)$ with Λ smooth over the base ring;
- $X = \mathrm{Spec}(A)$ where A is comodule algebra.

Then $X = \mathrm{Spec}(A) \to \mathcal{M}$ is a presentation and

$$\mathrm{Spec}(A) \times_{\mathcal{M}} \mathrm{Spec}(A) \cong \mathrm{Spec}(A \otimes \Lambda) = \mathrm{Spec}(\Gamma).$$

We have

$$\{ \text{ Cartesian sheaves on } X_\bullet \} \simeq \{ \text{ } (A,\Gamma)\text{-comodules } \}.$$

Derived Deligne-Mumford stacks

Theorem (Lurie)

Let $\mathcal{M}$ be a stack and $\mathcal{O}$ a sheaf of ring spectra on $\mathcal{M}$. Then $(\mathcal{M}, \mathcal{O})$ is a derived Deligne-Mumford stack if

1. *$(\mathcal{M}, \pi_0\mathcal{O})$ is a Deligne-Mumford stack; and*
2. *$\pi_i\mathcal{O}$ is a quasi-coherent sheaf on $(\mathcal{M}, \pi_0\mathcal{O})$ for all i.*

Exercise

Let $\mathbb{G}_m$ be the multiplicative group and $B\mathbb{G}_m$ its classifying stack: this assigns to each commutative ring the groupoid of $\mathbb{G}_m$-torsors over A. Show that $B\mathbb{G}_m$ classifies locally free modules of rank 1; that is, the groupoid of $\mathbb{G}_m$-torsors is equivalent to the groupoid of locally free modules of rank 1.

The proof is essentially the same as that of equivalence between line bundles over a space X and the principle $\mathrm{Gl}_1(\mathbb{R})$-bundles over X. Here are two points to consider:

1. If N is locally free of rank 1, then $\mathrm{Iso}_A(A, N)$ is a $\mathbb{G}_m$-torsor;

2. If P is a $\mathbb{G}_m$ torsor, choose a faithfully flat map $f : A \to B$ so that we can choose an isomorphism $\phi : f^*P \cong \mathbb{G}_m$. If $d_i : B \to B \otimes B$ are the two inclusions then ϕ determines an isomorphism $d_1^*\mathbb{G}_m \to d_0^*\mathbb{G}_m$ – which must be given by a $\mu \in (B \otimes_A)^\times$. Then (B, μ) is the descent data determining a locally free module of rank 1 over A.

Lecture 5: Elliptic curves

We discuss the compactified moduli stack of elliptic curves and its derived analog, thus introducing the Hopkins-Miller theorem and topological modular forms. Included are

- Weierstrass versus elliptic curves;
- an affine étale cover of $\bar{\mathcal{M}}_{\ell\ell}$;
- a brief discussion of modular forms.

Weierstrass curves

Definition

A **Weierstrass curve** $C = C_{\mathbf{a}}$ *over a ring* R *is a closed subscheme of* $\mathbb{P}^2$ *defined by the equation*

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

The curve C has a unique point $e = [0, 1, 0]$ when $z = 0$.

1. C is has at most one singular point;
2. C is always smooth at e;
3. the smooth locus $C_{\mathbf{sm}}$ is an abelian group scheme.

Elliptic curves

Definition

*An **elliptic curve** over a scheme S is a proper smooth curve of genus 1 over S* $C \underset{e}{\overset{q}{\rightleftarrows}} S$ *with a given section e.*

Any elliptic curve is an abelian group scheme:

if $T \to S$ is a morphism of schemes, the morphism

$$\begin{aligned} \{T\text{-points of } C\} &\longrightarrow \mathrm{Pic}^{(1)}(C) \\ P &\longmapsto \mathcal{I}^{-1}(P) \end{aligned}$$

is a bijection.

Comparing definitions

Let $C = C_{\mathbf{a}}$ be a Weierstrass curve over R. Define elements of R by

$$\begin{aligned} b_2 &= a_1^2 + 4a_2 \\ b_4 &= 2a_4 + a_1a_3 \\ b_6 &= a_3^2 + 4a_6 \end{aligned}$$

$$\begin{aligned} c_4 &= b_2^2 - 24b_4 \\ c_6 &= b_2^3 + 36b_2b_4 - 216b_6 \\ (12)^3\Delta &= c_4^3 - c_6^2 \end{aligned}$$

Then C is elliptic if and only if Δ is invertible. All elliptic curves are locally Weierstrass (more below).

Examples

1.) Legendre curves: over $\mathbb{Z}[1/2][\lambda, 1/\lambda(\lambda-1)]$:

$$y^2 = x(x-1)(x-\lambda)$$

2.) Deuring curves: over $\mathbb{Z}[1/3][\nu, 1/(\nu^3+1)]$:

$$y^2 + 3\nu xy - y = x^3$$

3.) Tate curves: over $\mathbb{Z}[\tau]$:

$$y^2 + xy = x^3 + \tau$$

∞.) The cusp: $y^2 = x^3$.

The stacks

Isomorphisms of elliptic curves are isomorphisms of pointed schemes. This yields a stack $\mathcal{M}_{ell}$.

Isomorphisms of Weierstrass curves are given by projective transformations

$$\begin{aligned} x &\mapsto \mu^{-2}x + r \\ y &\mapsto \mu^{-3}y + \mu^{-2}sx + t \end{aligned}$$

This yields an algebraic stack

$$\mathcal{M}_{\mathrm{Weier}} = \mathbb{A}^5 \times_G EG$$

where $G = \mathrm{Spec}(\mathbb{Z}[r, s, t, \mu^{\pm 1}])$.

Invariant differentials

Consider $C \underset{e}{\overset{q}{\rightleftarrows}} S$. Then e is a closed embedding defined by an ideal $\mathcal{I}$. Define

$$\omega_C = q_* \mathcal{I}/\mathcal{I}^2 = q_* \Omega_{C/S}.$$

- ω_C is locally free of rank 1; a generator is an invariant 1-form;
- if $C = C_{\mathbf{a}}$ is Weierstrass, we can choose the generator
$$\eta_{\mathbf{a}} = \frac{dx}{2y + a_1 x + a_3};$$
- if C is elliptic, a choice of generator defines an isomorphism $C = C_{\mathbf{a}}$; thus, all elliptic curves are locally Weierstrass.

Modular forms

The assignment $C/S \mapsto \omega_C$ defines a quasi-coherent sheaf on $\mathcal{M}_{e\ell\ell}$ or $\mathcal{M}_{\mathrm{Weier}}$.

Definition

A **modular form** *of weight n is a global section of $\omega^{\otimes n}$.*

The classes c_4, c_6 and Δ give modular forms of weight 4, 6, and 12.

Theorem (Deligne)

There are isomorphisms

$$\mathbb{Z}[c_4, c_6, \Delta^{\pm 1}]/(c_4^3 - c_6^2 = (12)^3 \Delta) \to H^0(\mathcal{M}_{e\ell\ell}, \omega^{\otimes *})$$

and

$$\mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 = (12)^3 \Delta) \to H^0(\mathcal{M}_{\mathrm{Weier}}, \omega^{\otimes *})$$

The compactified Deligne-Mumford stack

We have inclusions

$$\mathcal{M}_{ell} \subseteq \bar{\mathcal{M}}_{ell} \subseteq \mathcal{M}_{\mathrm{Weier}}$$

where

1. $\mathcal{M}_{ell}$ classifies elliptic curves: those Weierstrass curves with Δ invertible;
2. $\bar{\mathcal{M}}_{ell}$ classifies those Weierstrass curves with a unit in (c_4^3, c_6^2, Δ).

Theorem

The algebraic stacks $\mathcal{M}_{ell}$ and $\bar{\mathcal{M}}_{ell}$ are Deligne-Mumford stacks.

Topological modular forms

Theorem (Hopkins-Miller-Lurie)

There is a derived Deligne-Mumford stack $(\bar{\mathcal{M}}_{ell}, \mathcal{O}^{\mathrm{top}})$ whose underlying ordinary stack is $\bar{\mathcal{M}}_{ell}$.

Define the spectrum of topological modular forms **tmf** to be the global sections of $\mathcal{O}^{\mathrm{top}}$.

There is a spectral sequence

$$H^s(\bar{\mathcal{M}}_{ell}, \omega^{\otimes t}) \Longrightarrow \pi_{2t-s}\mathbf{tmf}.$$

Exercises

1. Calculate the values of c_4 and Δ for the Legendre, Deuring, and Tate curves. Decide when the Tate curve is singular.

2. Show that the invariant differential $\eta_{\mathbf{a}}$ of a Weierstrass curve is indeed invariant; that is, if $\phi : C_{\mathbf{a}} \to C_{\mathbf{a}'}$ is a projective transformation from one curve to another, then $\phi^*\eta_{\mathbf{a}'} = \mu\eta_{\mathbf{a}}$.

3. The j-invariant $\bar{\mathcal{M}}_{e\ell\ell} \to \mathbb{P}^1$ sends an elliptic curve C to the class of the pair (c_4^3, Δ). Show this classifies the line bundle $\omega^{\otimes 12}$.

Remark: The j-invariant classifies isomorphisms; that, the induced map of sheaves $\pi_0\bar{\mathcal{M}}_{e\ell\ell} \to \mathbb{P}^1$ is an isomorphism.

Lecture 6: The moduli stack of formal groups

We introduce the moduli stack of smooth one-dimensional formal groups, whose geometry governs the chromatic viewpoint of stable homotopy theory. We include

- periodic homology theories;
- a brief discussion of formal schemes;
- the height filtrations; and
- the Landweber exact functor theorem.

Periodic homology theories

Definition

Let E^* be a multiplicative cohomology theory and let

$$\omega_E = \tilde{E}^0 S^2 = E_2.$$

Then E is **periodic** *if*

1. $E_{2k+1} = 0$ *for all k;*
2. ω_E *is locally free of rank 1;*
3. $\omega_E \otimes_{E_0} E_{2n} \to E_{2n+2}$ *is an isomorphism for all n.*

A choice of generator $u \in \omega_E$ is an **orientation**; then

$$E_* = E_0[u^{\pm 1}].$$

The primordial example: complex K-theory.

Formal schemes

If X is a scheme and $\mathcal{I} \subseteq \mathcal{O}$ is a sheaf of ideals defining a closed scheme Z. The nth-**infinitesimal neighborhood** is

$$Z_n(R) = \{ f \in X(R) \mid f^*\mathcal{I}^n = 0 \}.$$

The associated formal scheme:

$$\widehat{Z} = \operatorname{colim} Z_n.$$

If $X = \mathrm{Spec}(A)$ and $\mathcal{I}$ defined by $I \subseteq A$, then

$$\widehat{Z} \overset{\mathrm{def}}{=} \mathrm{Spf}(\widehat{A}_I).$$

For example

$$\mathrm{Spf}(\mathbb{Z}[[x]])(R) = \text{the nilpotents of } R.$$

Formal groups

If E^* is periodic, then

$$G = \mathrm{Spf}(E^0\mathbb{C}\mathrm{P}^\infty)$$

is a group object in the category of formal schemes – a commutative one-dimensional **formal group**.

If E^* is oriented, $E^0\mathbb{C}\mathrm{P}^\infty \cong E^0[[x]]$ and the group structure is determined by

$$\begin{aligned} E^0[[x]] \cong E^0\mathbb{C}\mathrm{P}^\infty &\to E^0(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \cong E^0[[x,y]] \\ x &\mapsto F(x,y) = x +_F y. \end{aligned}$$

The power series is a **formal group law**; the element x is a **coordinate**.

Example: elliptic spectra

Let $C : \mathrm{Spec}(R) \to \bar{\mathcal{M}}_{e\ell\ell}$ be étale and classify a generalized elliptic curve C. Hopkins-Miller implies that there is a periodic homology theory $E(R,C)$ so that

1. $E(R,C)_0 \cong R$;
2. $E(R,C)_2 \cong \omega_C$;
3. $G_{E(R,C)} \cong \widehat{C}_e$.

Hopkins-Miller says a lot more: the assignment

$$\{ \ \mathrm{Spec}(R) \xrightarrow[\text{étale}]{C} \bar{\mathcal{M}}_{e\ell\ell} \ \} \mapsto E(R,C)$$

is a sheaf of E_∞-ring spectra.

The moduli stack of formal groups

An Isomorphism of formal groups over a ring R

$$\phi : G_1 \to G_2$$

is an isomorphisms of group objects over R. Define $\mathcal{M}_{\mathbf{fg}}$ to be the moduli stack of formal groups.

If G_1 and G_2 have coordinates, then ϕ is determined by an invertible power series $\phi(x) = a_0 x + a_1 x^2 + \cdots$.

Theorem

There is an equivalence of stacks

$$\mathrm{Spec}(L) \times_{\Lambda} E\Lambda \simeq \mathcal{M}_{\mathbf{fg}}$$

where L is the Lazard ring and Λ is the group scheme of invertible power series.

Invariant differentials

Let $G \underset{e}{\overset{q}{\rightleftarrows}} S$ be a formal group. Then e identifies S with the 1st infinitesimal neighborhood defined the ideal of definition $\mathcal{I}$ of G. Define

$$\omega_G = q_*\mathcal{I}/\mathcal{I}^2 = q_*\Omega_{G/S}.$$

This gives an invertible quasi-coherent sheaf ω on $\mathcal{M}_{\mathbf{fg}}$:

- ω_G is locally free of rank 1, a generator is an invariant 1-form;
- if $S = \mathrm{Spec}(R)$ and G has a coordinate x, we can choose generator

$$\eta_G = \frac{dx}{F_x(0,x)} \in R[[x]]dx \cong \Omega_{G/S};$$

- if E is periodic, then $\omega_{G_E} \cong E_2 \cong \omega_E$.

Height of a formal group

Let G be a formal group over a scheme S over $\mathbb{F}_p$. There are recursively defined global sections

$$v_k \in H^0(S, \omega_G^{p^k-1})$$

so that we have a factoring

$$\begin{tikzcd}
G \arrow[r, "F"'] \arrow[rr, bend left, "p"] & G^{(p^n)} \arrow[r, "V"'] & G
\end{tikzcd}$$

if and only if $v_1 = v_2 = \cdots = v_{n-1} = 0$. Here F is the relative Frobenius.

Then G has **height** greater than or equal to n.

The height filtration

We get a descending chain of closed substacks over $\mathbb{Z}_{(p)}$

$$\begin{tikzcd}
\mathcal{M}_{\mathbf{fg}} & \mathcal{M}(1) \arrow[l, "p=0"'] & \mathcal{M}(2) \arrow[l, "v_1=0"'] & \mathcal{M}(3) \arrow[l, "v_2=0"'] & \cdots \arrow[l] & \mathcal{M}(\infty) \arrow[l]
\end{tikzcd}$$

and the complementary ascending chain of open substacks

$$\mathcal{U}(0) \subseteq \mathcal{U}(1) \subseteq \mathcal{U}(2) \subseteq \cdots \subseteq \mathcal{M}_{\mathbf{fg}}.$$

Over $\mathbb{Z}_{(p)}$ there is a homotopy Cartesian diagram

$$\begin{tikzcd}
\bar{\mathcal{M}}_{e\ell\ell} \arrow[r] \arrow[d] & \mathcal{M}_{\mathrm{Weier}} \arrow[d] \\
\mathcal{U}(2) \arrow[r] & \mathcal{M}_{\mathbf{fg}}
\end{tikzcd}$$

The height filtration

We get a descending chain of closed substacks over $\mathbb{Z}_{(p)}$

$$\begin{tikzcd}
\mathcal{M}_{\mathbf{fg}} & \mathcal{M}(1) \arrow[l, "p=0"'] & \mathcal{M}(2) \arrow[l, "v_1=0"'] & \mathcal{M}(3) \arrow[l, "v_2=0"'] & \cdots \arrow[l] & \mathcal{M}(\infty) \arrow[l]
\end{tikzcd}$$

and the complementary ascending chain of open substacks

$$\mathcal{U}(0) \subseteq \mathcal{U}(1) \subseteq \mathcal{U}(2) \subseteq \cdots \subseteq \mathcal{M}_{\mathbf{fg}}.$$

Over $\mathbb{Z}_{(p)}$ there is a homotopy Cartesian diagram

$$\begin{tikzcd}
\bar{\mathcal{M}}_{e\ell\ell} \arrow[r] \arrow[d, "\text{flat}"'] & \mathcal{M}_{\text{Weier}} \arrow[d, "\text{not flat}"] \\
\mathcal{U}(2) \arrow[r] & \mathcal{M}_{\mathbf{fg}}
\end{tikzcd}$$

Flat morphisms (LEFT)

Suppose $G : \mathrm{Spec}(R) \to \mathcal{M}_{\mathbf{fg}}$ is flat. Then there is an associated homology theory $E(R, G)$.

More generally: take a “flat” morphism $\mathcal{N} \to \mathcal{M}_{\mathbf{fg}}$ and get a family of homology theories.

Theorem (Landweber)

A representable and quasi-compact morphism $\mathcal{N} \to \mathcal{M}_{\mathbf{fg}}$ of stacks is flat if and only if v_n acts as a regular sequence; that is, for all n, the map

$$v_n : f_*\mathcal{O}/\mathcal{I}_n \to f_*\mathcal{O}/\mathcal{I}_n \otimes \omega^{p^n-1}$$

is an injection.

The realization problem

Suppose $\mathcal{N}$ is a Deligne-Mumford stack and

$$f : \mathcal{N} \to \mathcal{M}_{\mathbf{fg}}$$

is a flat morphism. Then the graded structure sheaf on

$$(\mathcal{O}_{\mathcal{N}})_* = \{\omega_{\mathcal{N}}^{\otimes *}\}$$

can be realized as a diagram of spectra in the homotopy category.

Problem

Can the graded structure sheaf be lifted to a sheaf of E_∞-ring spectra? That is, can $\mathcal{N}$ be realized as a derived Deligne-Mumford stack? If so, what is the homotopy type of the space of all such realizations?

Exercises

These exercises are intended to make the notion of height more concrete.

1. Let $f : F \to G$ be a homomorphism of formal group laws over a ring R of characteristic p. Show that if $f'(0) = 0$, then $f(x) = g(x^p)$ for some power series g. To do this, consider the effect of f in the invariant differential.

2. Let F be a formal group law of F and $p(x) = x +_F \cdots +_F x$ (the sum taken p times) by the p-series. Show that either $p(x) = 0$ or there in an $n > 0$ so that

$$p(x) = u_n x^{p^n} + \cdots.$$

3. Discuss the invariance of u_n under isomorphism and use your calculation to define the section v_n of $\omega^{\otimes p^n - 1}$.

An exercise about LEFT

4. One direction of LEFT is fairly formal: show that $G : \mathrm{Spec}(R) \to \mathcal{M}_{\mathbf{fg}}$ is flat that then the v_i form a regular sequence.

The other direction is a theorem and it depends, ultimately, on Lazard's calculation that there is an unique isomorphism class of formal groups of height n over algebraically closed fields.

Lecture 7: Derived global sections

In this lecture and the next we outline an argument for calculating the homotopy groups of **tmf**. Here we introduce:

- coherent cohomology and derived pushforward;
- cohomology versus comodule Ext;
- how to calculate the cohomology of projective space.

Coherent cohomology

Definition

Let X be an algebraic stack and $\mathcal{F}$ a quasi-coherent sheaf on X. The ***coherent cohomology*** *of $\mathcal{F}$ is the right derived functors of global sections:*

$$H^s(X, \mathcal{F}) = H^s R\Gamma(\mathcal{F}).$$

Warning: I may need hypotheses on X, but I will be vague about this.

If X is derived Deligne-Mumford stack, we have a **descent spectral sequence**

$$H^s(X, \pi_t \mathcal{O}_X) \Longrightarrow \pi_{t-s} R\Gamma(\mathcal{O}_X).$$

Čech complexes

Suppose $X \to Y$ is faithfully flat. We have the simplicial bar construction

$$\epsilon : X_\bullet \longrightarrow Y.$$

We get a spectral sequence

$$\pi^s H^t(X_\bullet, \epsilon^* \mathcal{F}) \Longrightarrow H^{s+t}(Y, \mathcal{F}).$$

If U is affine, $H^s(U, \mathcal{F}) = 0$ for $s > 0$.

If $X = \sqcup U_i$ is where $\mathcal{U} = \{U_i\}$ is a finite affine cover of Y separated, we get an isomorphism with coherent cohomology and Čech cohomology

$$H^s(Y, \mathcal{F}) \cong \check{H}(\mathcal{U}, \mathcal{F}).$$

Comodules and comodule Ext

Let $\mathcal{M}$ be a stack and suppose $\mathrm{Spec}(A) \to \mathcal{M}$ is a flat presentation with the property that

$$\mathrm{Spec}(A) \times_{\mathcal{M}} \mathrm{Spec}(A) \cong \mathrm{Spec}(\Lambda).$$

Then (A, Γ) is a Hopf algebroid and we have

$$H^s(X, \mathcal{F}) \cong \mathrm{Ext}^s_\Lambda(A, M)$$

where $M = \epsilon^*\mathcal{F}$ is the comodule obtained from $\mathcal{F}$.

Example: $\mathcal{M} = X \times_G EG$ with $X = \mathrm{Spec}(A)$ and $G = \mathrm{Spec}(\Lambda)$. Here $\Gamma = A \otimes \Lambda$.

Example: the Adams-Novikov E_2-term

Let $G : \mathrm{Spec}(L) \to \mathcal{M}_{\mathbf{fg}}$ classify the formal group of the universal formal group law. Then $E(L, G) = MUP$ is periodic complex cobordism. We have

$$\mathrm{Spec}(L) \times_{\mathcal{M}_{\mathbf{fg}}} \mathrm{Spec}(L) = \mathrm{Spec}(W) = \mathrm{Spec}(MUP_0 MUP)$$

where

$$W = L[a_0^{\pm 1}, a_1, a_2, \ldots].$$

Then

$$\begin{aligned} H^s(\mathcal{M}_{\mathbf{fg}}, \omega^{\otimes t}) &\cong \mathrm{Ext}^s_W(L, MUP_{2t}) \\ &\cong \mathrm{Ext}^s_{MUP_*MUP}(\Sigma^{2t} MUP_*, MUP_*). \end{aligned}$$

This is not really the E_2-term of the ANSS so we must talk about:

Gradings: the basics

- A graded R-module is a $R[\mu^{\pm 1}]$-comodule;
- A graded ring gives an affine $\mathbb{G}_m$-scheme.

Example

1. The Lazard ring L is graded:
$$x +_{F_\mu} y = \mu^{-1}((\mu x) +_F (\mu y)).$$
2. $W = L[a_0^{\pm 1}, a_1, \ldots]$ is graded:
$$(\phi\mu)(x) = \mu^{-1}\phi(\mu x).$$
3. Weierstrass curves: $x \mapsto \mu^{-2}x$, $y \mapsto \mu^{-3}y$.

Gradings and cohomology

Let $H = \mathrm{Spec}(\Lambda)$ be an affine group scheme with an action of $\mathbb{G}_m$ and let
$$G = H \rtimes \mathbb{G}_m = \mathrm{Spec}(\Lambda[\mu^{\pm 1}])$$
be the semi-direct product. Let $X = \mathrm{Spec}(A)$ be an affine right G-scheme. Then $(A_*, \Gamma_* = A_* \otimes \Lambda_*)$ is a graded Hopf algebroid.

$$H^s(X \times_G EG, \mathcal{F}) = \mathrm{Ext}^s_{\Gamma_*}(A_*, M_*).$$

where $M_* = \mathcal{F}(\mathrm{Spec}(A) \to X \times_G EG)$.

1. $H^s(\mathcal{M}_{\mathbf{fg}}, \omega^{\otimes t}) \cong \mathrm{Ext}^s_{MU_*MU}(\Sigma^{2t} MU_*, MU_*)$;
2. $H^s(\mathcal{M}_{\mathrm{Weier}}, \omega^{\otimes t}) \cong \mathrm{Ext}^s_{\Lambda_*}(\Sigma^{2t} A_*, A_*)$.
3. $H^*(\mathbb{A}^{n+1} \times_{\mathbb{G}_m} E\mathbb{G}_m, \mathcal{F}) = M_0$.

Derived push-forward

Given $f : X \to Y$ and a sheaf $\mathcal{F}$ on X, then $f_*\mathcal{F}$ is the sheaf on Y associated to

$$U \mapsto H^0(U \times_Y X, \mathcal{F}).$$

If $\mathcal{F}$ is quasi-coherent and f is quasi-compact, $f_*\mathcal{F}$ is quasi-coherent.

There is a composite functor spectral sequence

$$H^s(Y, R^tf_*\mathcal{F}) \Longrightarrow H^{s+t}(X, \mathcal{F}).$$

If higher cohomology on Y is zero:

$$H^0(Y, R^tf_*\mathcal{F}) \cong H^t(X, \mathcal{F}).$$

Example: projective space

Let $S_* = \mathbb{Z}[x_0, \ldots, x_n]$ with $|x_i| = 1$.

Theorem

$$H^t(\mathbb{P}^n, \mathcal{O}(*)) \cong \begin{cases} S_* & t = 0; \\ S_*/(x_0^\infty, \ldots, x_n^\infty) & t = n. \end{cases}$$

We examine the diagram

$$\begin{tikzcd}
\mathbb{A}^{n+1} - \{0\} \arrow[r, "j"] \arrow[d] & \mathbb{A}^{n+1} \arrow[d] \\
\mathbb{P}^n \arrow[r, "i"] & \mathbb{A}^{n+1} \times_{\mathbb{G}_m} E\mathbb{G}_m .
\end{tikzcd}$$

We must calculate (the global sections) of

$$Rj_*j^*\mathcal{O}_{\mathbb{A}^{n+1}} \text{ “=” } Rj_*j^*S_*.$$

Example: the affine case

Let $X = \mathrm{Spec}(R)$ and $j : U \to X$ the open defined by an ideal $I = (a_1, \ldots, a_n)$. If $\mathcal{F}$ is defined by the module M, then $j_*j^*\mathcal{F}$ is defined by K where there is an exact sequence

$$0 \to K \to \prod_s M[\frac{1}{a_s}] \to \prod_{s<t} M[\frac{1}{a_s a_t}].$$

There is also an exact sequence

$$0 \to \Gamma_I M \to M \to K \to 0$$

where

$$\Gamma_I M = \{x \in M \mid I^n x = 0 \text{ for some } n\}.$$

Local cohomology

If $i : U \to Y$ is the open complement of a closed sub-stack $Z \subseteq Y$ define local cohomology by the fiber sequence

$$R\Gamma_Z\mathcal{F} \to \mathcal{F} \to Ri_*i^*\mathcal{F}.$$

If $X = \mathrm{Spec}(A)$ and Z is define by $I = (a_1, \ldots, a_k)$ local cohomology can be computed by the **Koszul complex**

$$M \to \prod_s M[\frac{1}{a_s}] \to \prod_{s<t} M[\frac{1}{a_s a_t}] \to \cdots \to M[\frac{1}{a_1 \ldots a_k}] \to 0.$$

Since $x_0, \ldots, x_n \in S_*$ is a regular sequence:

$$R^{n+1}\Gamma_{\{0\}}S_* = S_*/(x_0^\infty, \ldots, x_n^\infty)$$

and $R^t\Gamma_{\{0\}}S_* = 0$, $t \neq n+1$.

Lecture 8: Topological modular forms

This lecture computes the homotopy groups of **tmf** via the descent spectral sequence, emphasizing the role of Weierstrass curves and Serre duality.

The spectral sequence

Compute

$$H^s(\bar{\mathcal{M}}_{ell}, \omega^{\otimes t}) \Longrightarrow \pi_{t-s}\mathbf{tmf}$$

when 2 is inverted. Can do $p = 2$ as well, but harder. We have a Cartesian diagram

$$\begin{tikzcd}
U \arrow[r, "j"] \arrow[d] & \mathbb{A}^5 \arrow[d] \\
\bar{\mathcal{M}}_{ell} \arrow[r, "i"'] & \mathcal{M}_{\mathrm{Weier}}
\end{tikzcd}$$

where U is the open defined by the comodule ideal

$$I = (c_4^3, \Delta).$$

Cohomology of $\mathcal{M}_{\text{Weier}}$, $p > 3$.

Any Weierstrass curve is isomorphic to a curve of the form

$$y^2 = x^3 - (1/48)c_4 x - (1/216)c_6$$

and the only remaining projective transformations are

$$(x, y) \mapsto (\mu^{-2}x, \mu^{-3}y).$$

Then

$$\mathrm{Spec}(\mathbb{Z}_{(p)}[c_4, c_6]) \to \mathcal{M}_{\text{Weier}}$$

is a presentation. There is no higher cohomology and

$$H^0(\mathcal{M}_{\text{Weier}}, \omega^{\otimes *}) \cong \mathbb{Z}_{(p)}[c_4, c_6]$$

with $|c_4| = 8$ and $|c_6| = 12$. Note $\Delta = (1/(12)^3)(c_4^3 - c_6^2)$.

Cohomology of $\mathcal{M}_{\text{Weier}}$, $p = 3$.

Any Weierstrass curve is isomorphic to a curve of the form

$$y^2 = x^3 + (1/4)b_2 + (1/2)b_2 + (1/4)b_6$$

and the remaining projective transformations are

$$(x, y) \mapsto (\mu^{-2}x + r, \mu^{-3}y).$$

Then

$$\mathrm{Spec}(\mathbb{Z}_{(3)}[b_2, b_4, b_6]) \to \mathcal{M}_{\text{Weier}}$$

is a presentation and

$$H^s(\mathcal{M}_{\text{Weier}}, \omega^t) = \mathrm{Ext}^s_{\Gamma}(\Sigma^{2t}A_*, A_*)$$

with

$$A_* = \mathbb{Z}_{(p)}[b_2, b_4, b_6] \qquad \text{and} \qquad \Gamma_* = A_*[r]$$

with appropriate degrees.

Some familiar elements (Rezk)

In the Hopf algebroid (A_*, Γ_*), we have

$$\eta_R(b_2) = b_2 + 12r$$

so there is higher cohomology:

$$H^*(\mathcal{M}_{\mathrm{Weier}}, \omega^{\otimes *}) = \mathbb{Z}_{(3)}[c_4, c_6, \Delta][\alpha, \beta]/I$$

where $|\alpha| = (1, 4)$ and $|\beta| = (2, 12)$. Here I is the relations:

$$\begin{aligned} c_4^3 - c_6^2 &= (12)^3\Delta \\ 3\alpha = 3\beta &= 0 \\ c_i\alpha = c_i\beta &= 0. \end{aligned}$$

Note: Δ acts "periodically".

A property of local cohomology

To compute

$$H^s(\mathcal{M}_{\mathrm{Weier}}, R^t i_* \omega^*) \Longrightarrow H^{s+t}(\bar{\mathcal{M}}_{e\ell\ell}, \omega^*)$$

we compute $R\Gamma_I A_*$ where $I = (c_4^3, \Delta)$.
Note: if $\sqrt{I} = \sqrt{J}$ then $R\Gamma_I = R\Gamma_J$.
For $p > 3$ we take $I = (c_4^3, \Delta)$ and $J = (c_4, c_6)$.

$$R^2\Gamma_I A_* = \mathbb{Z}_{(p)}[c_4, c_6]/(c_4^\infty, c_6^\infty).$$

and

$$H^s(\bar{\mathcal{M}}_{e\ell\ell}, \omega^*) \cong \begin{cases} \mathbb{Z}_{(p)}[c_4, c_6], & s = 0; \\ \mathbb{Z}_{(p)}[c_4, c_6]/(c_4^\infty, c_6^\infty), & s = 1. \end{cases}$$

Note the duality. The homotopy spectral sequence collapses.

Local cohomology for **tmf**, $p = 3$

At $p = 3$ there are inclusions of ideals

$$(c_4^3, \Delta) \subseteq (c_4, \Delta) \subseteq (c_4, e_6, \Delta) = J = \sqrt{I}$$

where

$$e_6^2 = 12\Delta.$$

Since J is **not** generated by a regular sequence we must use: if $J = (I, x)$ there is a fiber sequence

$$R\Gamma_J M \to R\Gamma_I M \to R\Gamma_I M[1/x].$$

We take $I = (c_4, e_6)$ and $J = (c_4, e_6, \Delta)$.

Duality at 3

Let $A_* = \mathbb{Z}_{(3)}[b_2, b_4, b_6]$.

Proposition

$R^s_J\, A_* = 0$ if $s \neq 2$ and $R^2_J\, A_*$ is the Δ-torsion in $A_*/(c_4^\infty, e_6^\infty)$.

Corollary (Duality)

$R^2\Gamma_J\omega^{-10} \cong \mathbb{Z}_{(3)}$ with generator corresponding to $12/c_4e_6$ and there is non-degenerate pairing

$$R^2\Gamma_J\, \omega^{-t-10} \otimes \omega^t \to R^1\Gamma_J\, \omega^{-10} \cong \mathbb{Z}_{(3)}$$

We now can calculate π_***tmf**, at least it $p = 2$.

The differentials and extensions

The crucial differentials are classical:

$$\begin{aligned} d_5\Delta &= \alpha\beta^2 && \text{(Toda)} \\ d_9\Delta\alpha &= \beta^4 && \text{(Nishida)} \end{aligned}$$

There is also an exotic extension in the multiplication: if z is the homotopy class detected by $\Delta\alpha$, then:

$$\alpha z = \beta^3.$$

In fact, $z = \langle \alpha, \alpha, \beta^2 \rangle$ so

$$\alpha z = \alpha\langle \alpha, \alpha, \beta^2 \rangle = \langle \alpha, \alpha, \alpha \rangle \beta^2 = \beta^3.$$

Lectures 9 and 10: Lurie's realization result

This final (longer) lecture discusses

- *p*-divisible groups;
- how they arise in homotopy theory;
- Lurie's realization result;
- the impact of the Serre-Tate theorem; and
- gives a brief glimpse of the Behrens-Lawson generalizations of **tmf**.

p-divisible groups

Pick a prime p and work over $\mathrm{Spf}(\mathbb{Z}_p)$; that is, p is implicitly nilpotent in all rings. This has the implication that we will be working with p-complete spectra.

Definition

*Let R be a ring and G a sheaf of abelian groups on R-algebras. Then G is a p-**divisible group** of **height** n if*

1. *$p^k : G \to G$ is surjective for all k;*
2. *$G(p^k) = \mathrm{Ker}(p^k : G \to G)$ is a finite and flat group scheme over R of rank p^{kn};*
3. *$\operatorname{colim} G(p^k) \cong G$.*

This definition is valid when R is an E_∞-ring spectrum.

Examples of p-divisible groups

Formal Example: A formal group over a field or complete local ring is p-divisible.

Warning: A formal group over an arbitrary ring may not be p-divisible as the height may vary “fiber-by-fiber”.

Étale Example: $\mathbb{Z}/p^\infty = \operatorname{colim} \mathbb{Z}/p^k$ with

$$\mathbb{Z}/p^k = \mathrm{Spec}(\mathrm{map}(\mathbb{Z}/p^n, R)).$$

Fundamental Example: if C is a (smooth) elliptic curve then

$$C(p^\infty) \overset{\mathrm{def}}{=} C(p^n)$$

is p-divisible of height 2.

A short exact sequence

Let G be p-divisible and G_{for} be the completion at e. Then G/G_{for} is étale ; we get a natural short exact sequence

$$0 \to G_{\mathrm{for}} \to G \to G_{\mathrm{et}} \to 0$$

split over fields, but not in general.

Assumption: We will always have G_{for} of dimension 1.

Classification: Over a field $\mathbb{F} = \bar{\mathbb{F}}$ a p-divisible group of height n is isomorphic to one of

$$\Gamma_k \times (\mathbb{Z}/p^\infty)^{n-k}$$

where Γ_k is the unique formal group of height k. Also

$$\mathrm{Aut}(G) \cong \mathrm{Aut}(\Gamma_k) \times \mathrm{Gl}_{n-k}(\mathbb{Z}_p).$$

Ordinary vs supersingular elliptic curves

Over $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) = p$, an elliptic curve C is **ordinary** if $C_{\mathrm{for}}(p^\infty)$ has height 1. If it has height 2, C is **supersingular**.

Theorem

Over an algebraically closed field, there are only finitely many isomorphism classes of supersingular curves and they are all smooth.

If $p > 3$, there is a modular form of A of weight $p - 1$ so that C is supersingular if and only if $A(C) = 0$.

p-divisible groups in stable homotopy theory

Let E be a $K(n)$-local periodic homology theory with associated formal group

$$\mathrm{Spf}(E^0\mathbb{C}\mathrm{P}^\infty)) = \mathrm{Spf}(\pi_0 F(\mathbb{C}\mathrm{P}^\infty, E)).$$

We have

$$F(\mathbb{C}\mathrm{P}^\infty, C) \cong \lim F(BC_{p^n}, E).$$

Then

$$G = \operatorname{colim} \mathrm{Spec}(\pi_0 L_{K(n-1)} F(BC_{p^n}, E))$$

is a p-divisible group with formal part

$$G_{\mathrm{for}} = \mathrm{Spf}(\pi_0 F(\mathbb{C}\mathrm{P}^\infty, L_{K(n-1)}E)).$$

Moduli stacks

Define $\mathcal{M}_p(n)$ to be the moduli stack of p-divisible groups

1. of height n and
2. with $\dim G_{\mathrm{for}} = 1$.

There is a morphism

$$\begin{aligned} \mathcal{M}_p(n) &\longrightarrow \mathcal{M}_{\mathbf{fg}} \\ G &\mapsto G_{\mathrm{for}} \end{aligned}$$

Remark

1. The stack $\mathcal{M}_p(n)$ is not algebraic, just as $\mathcal{M}_{\mathbf{fg}}$ is not. Both are “pro-algebraic”.
2. Indeed, since we are working over $\mathbb{Z}_p$ we have to take some care about what we mean by an algebraic stack at all.

Some geometry

Let $\mathcal{V}(k) \subseteq \mathcal{M}_p(n)$ be the open substack of p-divisible groups with formal part of height k. We have a diagram

$$\begin{tikzcd}
\mathcal{V}(k-1) \arrow[r] \arrow[d] & \mathcal{V}(k) \arrow[r] \arrow[d] & \mathcal{M}_p(n) \arrow[d] & \\
\mathcal{U}(k-1) \arrow[r] & \mathcal{U}(k) \arrow[r] & \mathcal{U}(n) \arrow[r] & \mathcal{M}_{\mathbf{fg}}
\end{tikzcd}$$

1. the squares are pull backs;
2. $\mathcal{V}(k) - \mathcal{V}(k-1)$ and $\mathcal{U}(k) - \mathcal{U}(k-1)$ each have one geometric point;
3. in fact, these differences are respectively

$$B\,\mathrm{Aut}(\Gamma_k) \times B\mathrm{Gl}_{n-k}(\mathbb{Z}_p) \text{ and } B\,\mathrm{Aut}(\Gamma_k).$$

Lurie's Theorem

Theorem (Lurie)

Let $\mathcal{M}$ be a Deligne-Mumford stack of abelian group schemes. Suppose $G \mapsto G(p^\infty)$ gives a representable and formally étale morphism

$$\mathcal{M} \longrightarrow \mathcal{M}_p(n).$$

Then the realization problem for the composition

$$\mathcal{M} \longrightarrow \mathcal{M}_p(n) \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

has a canonical solution. In particular, $\mathcal{M}$ is the underlying algebraic stack of derived stack.

Remark: This is an application of a more general representability result, also due to Lurie.

Serre-Tate and elliptic curves

Let $\mathcal{M}_{e\ell\ell}$ be the moduli stack of elliptic curves. Then

$$\mathcal{M}_{e\ell\ell} \longrightarrow \mathcal{M}_p(2) \qquad C \mapsto C(p^\infty)$$

is formally étale by the Serre-Tate theorem.

Let C_0 be an $\mathcal{M}$-object over a field $\mathbb{F}$, with $\mathrm{char}(\mathbb{F}) = p$. Let $q : A \to \mathbb{F}$ be a ring homomorphism with nilpotent kernel. A **deformation** of C_0 to R is an $\mathcal{M}$-object over A and an isomorphism $C_0 \to q^*C$. Deformations form a category $\mathbf{Def}_{\mathcal{M}}(\mathbb{F}, C_0)$.

Theorem (Serre-Tate)

We have an equivalence:

$$\mathbf{Def}_{e\ell\ell}(\mathbb{F}, C_0) \to \mathbf{Def}_{\mathcal{M}_p(2)}(\mathbb{F}, C_0(p^\infty))$$

Topological modular forms

If C is a singular elliptic curve, then $C_{\mathrm{sm}} \cong \mathbb{G}_m$ or

$$C_{\mathrm{sm}}(p^\infty) = \text{multiplicative formal group}$$

which has height 1, not 2. Thus

$$\mathcal{M}_{e\ell\ell} \longrightarrow \mathcal{M}_p(2)$$

doesn't extend over $\bar{\mathcal{M}}_{e\ell\ell}$; that is, the approach just outlined constructs $\mathbf{tmf}[\Delta^{-1}]$ rather than $\mathbf{tmf}$.

To complete the construction we could

1. handle the singular locus separately: "Tate K-theory is E_∞"; and
2. glue the two pieces together.

Higher heights

There are very few families of group schemes smooth of dimension 1. Thus we look for stackifiable families of abelian group schemes A of higher dimension so that

- There is a natural splitting $A(p^\infty) \cong A_0 \times A_1$ where A_0 is a p-divisible group with formal part of dimension 1; and
- Serre-Tate holds for such A: $\mathbf{Def}_{A/\mathbb{F}} \simeq \mathbf{Def}_{A_0/\mathbb{F}}$.

This requires that A support a great deal of structure; very roughly:

(E) $\mathrm{End}(A)$ should have idempotents; there is a ring homomorphism $B \to \mathrm{End}(A)$ from a certain central simple algebra;

(P) Deformations of $A(p^\infty)$ must depend only on deformations of A_0; there is a duality on A – a **polarization.**

Shimura varieties

Such abelian schemes have played a very important role in number theory.

Theorem (Behrens-Lawson)

For each $n > 0$ there is a moduli stack Sh_n (a ***Shimura variety****) classifying appropriate abelian schemes equipped with a formally étale morphism*

$$\mathrm{Sh}_n \longrightarrow \mathcal{M}_p(n).$$

In particular, the realization problem for the surjective morphism

$$\mathrm{Sh}_n \to \mathcal{U}(n) \subseteq \mathcal{M}_{\mathbf{fg}}$$

has a canonical solution.

The homotopy global sections of the resulting sheaf of E_∞-ring spectra is called **taf**: topological automorphic forms.