ENDOTRIVIAL MODULES FOR FINITE GROUPS VIA HOMOTOPY THEORY

JESPER GRODAL

ABSTRACT. Classifying endotrivial \( kG \)-modules, i.e., elements of the Picard group of the stable module category for an arbitrary finite group \( G \), has been a long-running quest, which by deep work of Dade, Alperin, Carlson, Thévenaz, and others, has been reduced to understanding the subgroup consisting of modular representations that split as the trivial module direct sum a projective module when restricted to a Sylow \( p \)-subgroup. In this paper we identify this subgroup as the first cohomology group of the orbit category on non-trivial \( p \)-subgroups with values in the units \( k^\times \), viewed as a constant coefficient system. We then use homotopical techniques to give a number of formulas for this group in terms of the abelianization of normalizers and centralizers in \( G \), in particular verifying the Carlson–Thévenaz conjecture—this reduces the calculation of this group to algorithmic calculations in local group theory rather than representation theory. We also provide strong restrictions on when such representations of dimension greater than one can occur, in terms of the \( p \)-subgroup complex and \( p \)-fusion systems. We immediately recover and extend a large number of computational results in the literature, and further illustrate the computational potential by calculating the group in other sample new cases, e.g., for the Monster at all primes.

1. Introduction

In modular representation theory of finite groups, the indecomposable \( kG \)-modules \( M \) that upon restriction to a Sylow \( p \)-subgroup \( S \) split as the trivial module \( k \) plus a free \( kS \)-module are basic yet somewhat mysterious objects. Such modules form a group \( T_k(G, S) \) under tensor product, discarding projective \( kG \)-summands, with neutral element \( k \) and \( M^* \) the inverse of \( M \) (see [2.1] for details). It is also denoted \( K(G) \) in the literature. It contains the one-dimensional characters \( \text{Hom}(G, k^\times) \) as a subgroup, but has been observed to sometimes also contain exotic elements. The group \( T_k(G, S) \) is an important subgroup of the larger group of all so-called endotrivial modules \( T_k(G) \), i.e., \( kG \)-modules \( M \) where \( M^* \otimes M \cong k \oplus P \), for \( P \) a projective \( kG \)-module; namely \( T_k(G, S) = \ker( T_k(G) \to T_k(S) ) \), the kernel of the restriction to \( S \). The group of endotrivial modules has a categorical interpretation as \( T_k(G) \cong \text{Pic}(\text{StMod}_{kG}) \), the Picard group of the stable module category. Such modules occur in many parts of representation theory, e.g., as source modules; see the surveys [The07, Car12, Car17] and the papers quoted below.

Classifying endotrivial modules has been a long-running quest, which has been reduced to calculating \( T_k(G, S) \), through a series of fundamental papers: The group \( T_k(S) \) was determined in celebrated works of Dade [Dad78a, Dad78b], Alperin [Alp01], and Carlson–Thévenaz [CT04, CT05]. From this, Carlson–Mazza–Nakano–Thévenaz [CMN06, MT07, CMT13] worked out the image of the restriction \( T_k(G) \to T_k(S) \), at least as an abstract abelian group, and showed that the restriction is split onto its image, except in well-understood cases with cyclic Sylow \( p \)-subgroup; see Remark 3.18. Subsequently there has been an intense interest in calculating \( T_k(G, S) \), with contributors Balmer, Carlson, Lassueur, Malle, Mazza, Nakano, Navarro, Robinson, Thévenaz, and others.

In this paper we give an elementary and computable homological description of the group \( T_k(G, S) \), as the first cohomology group of the orbit category on non-trivial \( p \)-subgroups of \( G \), with constant coefficients in \( k^\times \), for any finite group \( G \) and any field \( k \) of characteristic \( p \) dividing the order of \( G \) (not assumed algebraically closed). Using homological methods, adapted from mod \( p \) homology decompositions (but now with \( k^\times \)-coefficients, hence “prime to \( p \)”), we deduce a range of structural and computational results on \( T_k(G, S) \), with answers expressed in terms of standard \( p \)-local group theory: We write \( T_k(G, S) \) as an inverse limit of homomorphisms from normalizers of chains of \( p \)-subgroups to \( k^\times \), answering the main conjecture of Carlson–Thévenaz [CT15, Ques. 5.5] in the positive.

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Theorem A.  

For the harder primes 3, 5, 7, 11, and 13 which had been left open in the literature \cite{LM15b}; see \cite{1.5}

Lastly we provide consequences of these results for specific classes of groups, e.g., finite groups of Lie type and sporadic groups, obtaining new computations as well as recover and simplify many old ones in the vast literature. As an example we try out one of our formulas on the Monster sporadic simple group, and easily calculate \( T \) for the harder primes 3, 5, 7, 11, and 13 which had been left open in the literature \cite{LM15b}; see \cite{1.5}

Our proof of the identification of \( T_k(G,S) \) is direct and self-contained, and provides “geometric” models for the module generator in \( T_k(G,S) \) corresponding to a 1–cocycle: it is the class in the stable module category represented by the unreduced Steinberg complex of \( G \) twisted by the 1–cocycle. It has the further conceptual interpretation as the homotopy left Kan extension of the 1–cocycle from the orbit category on non-trivial \( p \)–subgroups, to all \( p \)–subgroups. Our identification was originally inspired by a characterization due to Balmer of \( T_k(G,S) \) in terms of what he dubs “weak homomorphisms” \cite{Bal13}, and we also indicate another argument of how to deduce the identification using these.

Let us now describe our work in detail. Call a \( kG \)–module \( \text{Sylow-trivial} \) if it upon restriction to \( kS \) splits as the trivial module plus a projective \( kS \)–module. Hence \( T_k(G,S) \) identifies with the group of equivalence classes of \( \text{Sylow-trivial} \) modules, identifying two if they become isomorphic after discarding projective \( kG \)–summands; each equivalence class contains a unique indecomposable representative, up to isomorphism (see Proposition \ref{prop:equivalence}). Let \( \mathcal{O}_p^G(G) \) denote the orbit category of \( G \) with objects \( G/P \), for \( P \) a non-trivial \( p \)–subgroup, and morphisms \( G \)–maps. The following is our main identification of \( T_k(G,S) \).

**Theorem A.** Fix a finite group \( G \) and \( k \) a field of characteristic \( p \) dividing the order of \( G \). The group \( T_k(G,S) \) is described via the following isomorphism of abelian groups

\[
\Phi : T_k(G,S) \stackrel{\sim}{\longrightarrow} H^1(\mathcal{O}_p^G(G); k^\times) \tag{1.1}
\]

The map \( \Phi \) takes \( M \in T_k(G,S) \) to the functor \( \varphi : \mathcal{O}_p^G(G) \to k^\times \) defined by first taking zeroth Tate cohomology

\[
k_{\varphi} : G/P \mapsto \tilde{H}^0(P; M) = M^P / (\sum_{g \in P} g) M,
\]

a functor from \( \mathcal{O}_p^G(G) \) to the groupoid of one-dimensional \( k \)–modules and isomorphisms, and then identifying the target with the group \( k^\times \), by picking \( k_{\varphi} (G/S) \) as basepoint.

The inverse to \( \Phi \) is given by the unaugmented “twisted Steinberg complex”

\[
\Phi^{-1}(\varphi) = C_* (|\mathcal{S}_p(G)|; k_{\varphi}) \in D^b(kG)/D_{\text{perf}}(kG) \rightleftarrows \text{stmod}_{kG}
\]

where \( k_{\varphi} \) is the \( G \)–twisted coefficient system with values \( k \) induced by \( \varphi : \mathcal{O}_p^G(G) \to k^\times \).

In fact we establish in Theorem 3.10 a more general correspondence between \( kG \)–modules that split as a sum of trivial and projective modules upon restriction to \( S \), and representations of the fundamental group \( \pi_1(\mathcal{O}_p^G(G)) \), refining parts of the Green correspondence.

That \( \tilde{H}^0(P;M) \) is one-dimensional follows by definition of \( \text{Sylow-trivial} \), since \( P \) is subconjugate to \( S \). Also, we used the equivalence of categories \( \text{stmod}_{kG} \rightleftarrows D^b(kG)/D_{\text{perf}}(kG) \) between the finitely generated stable module category and the bounded derived category modulo perfect complexes recalled in \cite{2.5} \( G \)–twisted coefficient systems on the \( p \)–subgroup complex \( |\mathcal{S}_p(G)| \), the nerve of the poset of non-trivial \( p \)–subgroups of \( G \), is recalled in \cite{2.4.1}

The chain complex \( C_* (|\mathcal{S}_p(G)|; k_{\varphi}) \) can be interpreted as the value on \( G/e \) of the homotopy left Kan extension of \( \varphi \) along \( \mathcal{O}_p^G(G)^{\text{op}} \to \mathcal{O}_p^G(G)^{\text{op}}, \) i.e., to the opposite orbit category on all \( p \)–subgroups, which has model the homotopy colimit in chain complexes \( \text{hocollim}_{P \in \mathcal{S}_p(G)^{\text{op}}} k_{\varphi} \); see Remark 3.7. When \( \varphi = 1 \), the complex is the Steinberg complex without the \( k \)–augmentation in degree \(-1\), and the fact
that $\Phi^{-1}(k) \cong k$ is equivalent to projectivity of the augmented Steinberg complex, proved by Quillen [Qui78, 4.5] and Webb [Web91].

We now embark in putting the model for $T_k(G,S)$ of Theorem 4.9 to use, expressing $H^1(\Theta_p(G); k^\infty)$ in terms of $p$–local information about the group. We start with some elementary observations: By standard algebraic topology (recalled in [2,3])

$$H^1(\Theta_p^*(G); k^\infty) \cong \text{Rep}(\Theta_p^*(G), k^\infty) \cong \text{Hom}(\pi_1(\Theta_p^*(G)), k^\infty) \cong \text{Hom}(H_1(\Theta_p^*(G)), k^\infty)$$  \hspace{1cm} (1.1)

where $\text{Rep}$ means isomorphism classes of functors, viewing $k^\infty$ as a category with one object. Let

$$G_0 = \langle N_G(Q) | 1 < Q \leq S \rangle$$  \hspace{1cm} (1.2)

also called the 1–generated core $\Gamma_{S,1}(G)$ by group theorists, which, if proper in $G$, is the smallest strongly $p$–embedded subgroup containing $S$; see Remark [7,15]. Recall that a Frattini argument implies that we have surjections $N_G(S)/S \twoheadrightarrow (G_0)_{p'} \twoheadrightarrow G_{p'}$, where we throughout the paper adopt the convention that a subscript $p'$ means factoring out the normal subgroup generated by elements of finite $p$–power order. (Hence, $G_{p'} = G/\text{O}^{p'}(G)$, when $G$ is finite, and $M_{p'} = M/\text{Tors}_p(M)$ when $M$ is abelian.) An application of Alperin’s fusion theorem [Alp67, §] shows that $\Theta_p^*(G)$ and $\Theta_p^*(G_0)$ are equivalent categories and the Frattini surjections above refine to

$$N_G(S)/S \twoheadrightarrow \pi_1(\Theta_p^*(G)) \twoheadrightarrow (G_0)_{p'} \twoheadrightarrow G_{p'}$$  \hspace{1cm} (1.3)

displaying $\pi_1(\Theta_p^*(G))$ as a finite $p'$–group (see Proposition 4.1). Via Theorem A this encodes the classical bounds on $T_k(G,S)$ (cf. [CMN06, Prop. 2.6], [MT07, Lem. 2.7], and Proposition 3.2):

$$\text{Hom}(G, k^\infty) \leq \text{Hom}(G_0, k^\infty) \leq T_k(G,S) \leq \text{Hom}(N_G(S)/S, k^\infty)$$  \hspace{1cm} (1.4)

using that $k^\infty$ does not contain $p$–torsion.

We describe the precise kernel of $N_G(S)/S \twoheadrightarrow \pi_1(\Theta_p^*(G))$ in terms of $p$–local group theory in Theorem 1.4–it already directly implies a list of structural properties of $T_k(G,S)$ via Theorem A and (1.1); see e.g., Corollary 4.13.

In the rest of the introduction we summarize our further descriptions of $\pi_1(\Theta_p^*(G))$ and its abelianization $H_1(\Theta_p^*(G))$, each highlighting different structural properties. We divide this into 5 subsections:

1.1 Subgroup categories, 1.2 Decompositions, 1.3 The Carlson–Thévenaz conjecture, 1.4 Fusion systems, and 1.5 Computations.

1.1. Descriptions in terms of subgroup complexes. Let $\mathcal{F}_p(G)$ denote the transport category of $G$ with objects the non-trivial $p$–subgroups of $S$, and $\text{Hom}_\mathcal{F}(P,Q) = \{g \in G | gP \leq Q \}$. We have a functor $\mathcal{F}_p(G) \to \Theta_p^*(G)$ sending $g$ to the $G$–map $G/P \to G/Q$ specified by $eP \mapsto g^{-1}Q$. Since $\Theta_p^*(G)$ is a quotient of $\mathcal{F}_p(G)$ by automorphisms in $p$–subgroups $Q$,

$$\pi_1(\mathcal{F}_p(G))_{p'} \cong \pi_1(\Theta_p^*(G))$$  \hspace{1cm} (1.5)

(by Thomason’s theorem [Tho79, Thm. 1.2], where $X_{BG} = EG \times_G X$ denotes the Borel construction; see also Remark 2.2. In particular we have a fibration sequence

$$|S_p(G)| \to |S_p(G)|_{hG} \to BG_0$$  \hspace{1cm} (1.7)

where we use

$$|S_p(G)| = G \times_{G_0} |S_p(G_0)|$$  \hspace{1cm} (1.8)

with $|S_p(G_0)|$ connected, as observed by Quillen [Qui78, §5] [Gro02, Prop. 5.8]; see Proposition 7.14. (In particular $|S_p(G)|$ is connected if and only if $G = G_0$.) On fundamental groups it induces an exact sequence

$$1 \to \pi_1(S_p(G)) \to \pi_1(\mathcal{F}_p(G)) \to G_0 \to 1$$  \hspace{1cm} (1.9)

displaying $\pi_1(\mathcal{F}_p(G))$ as an in general infinite group, with $G_0$ as quotient. By [1.5] and Theorem A 1–dimensional characters of $\pi_1(\mathcal{F}_p(G))$ also parametrize Sylow-trivial modules for $G$, i.e.,

$$T_k(G,S) \cong \text{Hom}(\pi_1(\mathcal{F}_p(G)), k^\infty)$$  \hspace{1cm} (1.10)
The low-degree homology sequence of the group extension \((1.9)\) produces an exact sequence

\[ H_2(G_0)_{p'} \xrightarrow{\partial} (H_1(\mathcal{S}_p(G_0)))_{p'} \rightarrow H_1(\mathcal{G}_p^*(G)) \rightarrow H_1(G_0)_{p'} \rightarrow 0 \]  
\((1.11)\)

where the subscript \(G_0\) denotes coinvariants. In cohomology, with Theorem A this yields:

**Theorem B** (Subgroup complex sequence). Let \(G\) be a finite group, \(k\) a field of characteristic \(p\), and \(G_0 \leq G\) as in \((1.2)\). We have an exact sequence

\[ 0 \rightarrow \text{Hom}(G_0,k^\times) \rightarrow T_k(G,S) \rightarrow H^1(\mathcal{S}_p(G_0);k^\times) \xrightarrow{\partial} H^2(G_0;k^\times) \]

where superscript \(G_0\) means invariants. In particular if \((H_1(\mathcal{S}_p(G_0))_{p'}) = 0\), then \(T_k(G,S) \cong \text{Hom}(G_0,k^\times)\). If \(\mathcal{S}_p(G)\) is simply connected, then \(T_k(G,S) \cong \text{Hom}(G,k^\times)\).

In words, \(T_k(G,S)\) is an extension of \(\text{Hom}(G_0,k^\times)\) by a “truly exotic” part, described as the kernel of the boundary map \(\partial: H^1(\mathcal{S}_p(G_0);k^\times) \rightarrow H^2(G_0;k^\times)\); see also Remark 4.6. The group \(H^2(G_0;k^\times)\) identifies with the \(p'\)-part of the Schur multiplier of \(G_0\) if \(k\) is algebraically closed.

There is already an extensive literature on when \(|\mathcal{S}_p(G)|\) is simply connected, see e.g., [Smi11, §9]. It is known to hold generically for symmetric groups, finite groups of Lie type at the characteristic, as well as some finite group of Lie type away from the characteristic and certain sporadic groups. It is conjectured to hold for many more; see also (1.5).

The description of \(T_k(G,S)\) as 1-dimensional representations of \(\pi_1(\mathcal{S}_p(G))\) in \((1.10)\) combined with manipulations with subgroup complexes also enables us to see precisely how \(T_k(G,S)\) behaves under for instance passage to \(p'\)-index subgroups or \(p'\)-central extensions (see Corollaries 4.15 and 4.17). Furthermore the groups \(\pi_1(\mathcal{G}_p^*(G))\) and \(\pi_1(\mathcal{S}_p(G))\) only depend on very few of the \(p\)-subgroups of \(G\), as we analyze in detail in the appendix Section 7—see in particular Theorems 7.11 and 7.13.

Before moving to homology decompositions we record the following corollary of Theorem A by \((1.10)\) and \((1.6)\) above:

**Corollary C.** For any finite group \(G\) and \(k\) any field of characteristic \(p\),

\[ T_k(G,S) \cong H^1(|\mathcal{S}_p(G)|_{hG};k^\times) \]

In particular if \(H_1(\mathcal{S}_p(G)|_{hG})_{p'} \xrightarrow{\cong} H_1(G)_{p'}\) then \(T_k(G,S) \cong \text{Hom}(G,k^\times)\).

From this perspective, exotic Sylow-trivial modules parametrize the failure of the collection of non-trivial \(p\)-subgroups to be ‘\(\mathcal{H}_1(\mathcal{S}_p(G)|_{hG})\)–ample’ in the spirit of Dwyer [Dwy97, 1.3]; see Remark 4.7 and Theorem 4.34. The corollary also lets us deduce a very recent result of Balmer [Bal18]; see Remark 4.8.

1.2. **Homology decomposition descriptions.** We now use homology decomposition techniques to get formulas for \(T_k(G,S)\). These techniques have a long history for providing results about mod \(p\) group cohomology; see e.g., [Dwy97, Gro02, GS06] and their references, but here we are interested in coefficients in \(k^\times\), an abelian group with no \(p\)-torsion. We can however still describe the low-degree \(p'\)-homology of \(|\mathcal{S}_p(G)|_{hG}\), by examining the bottom corner of spectral sequences, even if they do not collapse. More precisely, given a arbitrary collection \(\mathcal{C}\) of subgroups (i.e., a set of subgroups closed under conjugation), there are 3 homology decompositions one usually considers associated to the \(G\)–action on \(|\mathcal{C}|\): the subgroup decomposition, the normalizer decomposition, and the centralizer decomposition. The subgroup decomposition does not provide new information, if \(\mathcal{C}\) is a collection of \(p\)-subgroups, but the two others do. Let us start with the normalizer decomposition.

**Theorem D** (Normalizer decomposition). Let \(G\) be a finite group, \(k\) a field of characteristic \(p\), and \(\mathcal{C} \subseteq \mathcal{S}_p(G)\) a subcollection such that the inclusion is a \(G\)–homotopy equivalence, e.g., \(\mathcal{C}\) the collection of non-trivial \(p\)-radical subgroups or of non-trivial elementary abelian \(p\)-subgroups (see [1.3]). Then

\[ T_k(G,S) \cong \lim_{\mathcal{C} \subseteq \mathcal{S}_p(G)} \text{Hom}(N_G(P_0 < \cdots < P_n),k^\times) \]

inside \(\text{Hom}(N_G(S)/S,k^\times)\), with limit taken over conjugacy classes of chains in \(\mathcal{C}\) ordered by refinement. Explicitly:

\[ T_k(G,S) \cong \ker \left( \oplus_{[P]} \text{Hom}(N_G(P),k^\times) \rightarrow \oplus_{[P<Q]} \text{Hom}(N_G(P) \cap N_G(Q),k^\times) \right) \cong \text{Hom}(N_G(S),k^\times) \]
Different possibilities for $\mathcal{C}$ are tabulated in the appendix Theorem 7.5 see also [GS06 Thm. 1.1]. Section 7 along with Proposition 5.3 provides a precise analysis of which subgroups are needed to make the conclusion of Theorem D hold. We note that the data going into calculating the right hand side, normalizers of chains of say $p$–radical subgroups, or elementary abelian $p$–subgroups, has been tabulated for a large number of groups, and this relates to a host of problems in local group theory and representation theory, such as the classification of finite simple groups and conjectures of Alperin, McKay, Dade, etc. To obtain Theorem D from Theorem A the only extra input, apart from the isotropy spectral sequence, is a result of Symonds [Sym98], formerly known as Webb’s conjecture, which we provide as a short proof of in the appendix Proposition 7.3 that appears to be new.

We now explain the centralizer decomposition, which ties into fusion systems. Let $\mathcal{F}_C(G)$ denote the restricted $p$–fusion system of $G$ with objects $P \in \mathcal{C}$ and $\text{Hom}_{\mathcal{F}_C(G)}(P, Q) = \text{Hom}_{\mathcal{F}_C(G)}(P, Q)/C_G(P)$ i.e., monomorphisms induced by $G$–conjugation, writing $\mathcal{F}_C^*(G)$ when $\mathcal{C} = S_p(G)$.

**Theorem E** (Centralizer decomposition). For $G$ a finite group and $k$ a field of characteristic $p$, we have an exact sequence

$$0 \to H^1(\mathcal{F}_C^*(G); k^\times) \to T_k(G, S) \to \lim_{V \in \mathcal{F}_C^*(G)} \text{Hom}(C_G(V), k^\times) \to H^2(\mathcal{F}_C^*(G); k^\times)$$

where $\mathcal{A}_p^2$ denotes the collection of elementary abelian $p$–subgroups of rank one or two.

In particular, if $H_1(C_G(x)) \neq 0$ for all elements $x$ of order $p$, and $H_1(N_G(S)/S)$ is generated by elements in $N_G(S)$ that commute with some non-trivial element in $S$, then $T_k(G, S) = 0$.

This breaks $T_k(G, S)$ up into parts depending on the underlying fusion system and parts calculated from centralizers. It may be illuminating to note that it specializes to the sequence in cohomology with $k^\times$ coefficients induced by $1 \to C_G(V) \to N_G(V) \to N_G(V)/C_G(V) \to 1$ in the very special case where $G$ has a unique non-trivial elementary abelian $p$–subgroup $V$, up to conjugacy; see Corollary 4.13 [2].

In particular $H^1(\mathcal{F}_C^*(G); k^\times)$ is a subgroup of $T_k(G, S)$ depending only on the fusion system. We describe how to calculate cohomology of $\mathcal{F}_C^*(G)$ in Sections 4.3 and 7.3, the first cohomology group is in fact zero in many, but not all, cases. The assumptions of the ‘in particular’ are often satisfied for the sporadic groups, e.g., the Monster.

### 1.3. The Carlson–Thévenaz conjecture

**Theorem D** implies the Carlson–Thévenaz conjecture, which predicts an algorithm for calculating $T_k(G, S)$ from $p$–local information, essentially by a change of language, taking $\mathcal{C} = S_p(G)$. Set $A^p(G) = O^p(G)/G[G, G]$, the smallest normal subgroup of $G$ such that the quotient is an abelian $p'$–group.

**Theorem F** (The Carlson–Thévenaz conjecture [CT15 Ques. 5.5]). Let $G$ be a finite group with non-trivial Sylow $p$–subgroup $S$, and define $\rho^p(S) \leq N_G(S)$ (depending on $G$ and $S$) via the following definition (cf. [CMN14 Prop. 5.7] [CT15 §4]):

$$\rho^1(Q) = A^p(N_G(Q)), \quad \rho^p(Q) = \langle N_G(Q) \cap \rho^{p-1}(R) \mid 1 < R \leq S \rangle \supseteq \rho^{p-1}(Q).$$

Then $\rho^p(S)$ for any field $k$ of characteristic $p$,

$$T_k(G, S) \cong \text{Hom}(N_G(S)/\rho^p(S), k^\times)$$

for any $r$ at least either the maximal rank of an elementary abelian $p$–subgroup plus one, or the number of groups in the longest proper chain of non-trivial $p$–radical subgroups.

Hence by Theorem A for any field $k$ of characteristic $p$,

$$\rho^p(S) \cong \text{Hom}(N_G(S)/\rho^p(S), k^\times)$$

This in fact strengthens the Carlson–Thévenaz conjecture, by providing a rather manageable bound on $r$. (The original conjecture was only a prediction about the union $\rho^\infty(S)$, and also had an algebraically closed assumption on $k$.) Theorem F is well adapted to implementation on a computer, and indeed Carlson has already made one such implementation calculating $\rho^p(S)$; to use this for proofs, a theoretical bound on when the $\rho^p(S)$ stabilize is obviously also necessary.

As already noted, the inverse limit in Theorem D identifies with a subset of $\text{Hom}(N_G(S), k^\times)$. One may naively ask if the limit could simply be described as the elements in $\text{Hom}(N_G(S), k^\times)$ whose restriction to $\text{Hom}(N_G(P) \cap N_G(S), k^\times)$ was zero on $A^p(N_G(P)) \cap N_G(S)$ for all $P \in \mathcal{C}/G$, an obvious necessary condition to lie in the limit. In other words, if we, in the language of Theorem F, could always take $r = 2$. Computer calculations announced in [CT15], say this is not the case for $G_2(5)$.
when \( p = 3 \). The main theorem of that paper \cite[Thm. 5.1]{CT15} however show that this naïve guess is true when \( S \) is abelian. As a corollary of Theorem \[D\], we can also generalize this:

**Corollary G.** If all non-trivial \( p \)-radical subgroups in \( G \) with \( P < S \) are normal in \( S \), then

\[
T_k(G, S) \cong \ker \left( \text{Hom}(N_G(S), k^\times) \to \bigoplus_{[P]} \text{Hom}(N_G(S) \cap A^p(N_G(P)), k^\times) \right)
\]

where \([P]\) runs through \( N_G(S)\)-conjugacy classes of non-trivial \( p \)-radical subgroups with \( P < S \). In particular in the notation of Theorem \[F\]

\[
T_k(G, S) \cong \text{Hom}(N_G(S)/P^2(S), k^\times)
\]

More generally, if only each \( N_G(S)\)-conjugacy class of pairs \([P \leq Q]\), with \( P, Q < S \) both \( p \)-radical in \( G \), satisfy \( N_G(P \leq Q \leq S)A^p(N_G(P \leq Q)) = N_G(P \leq Q) \), then the same conclusion holds.

The last part of the corollary provides a strengthening of Carlson–Thévenaz’s more technical \cite[Thm. 7.1]{CT15}, which instead of abelian assumes that \( N_G(S) \) controls \( p \)-fusion along with extra conditions; see Remark \[5.12\]. Corollary \[G\] however moves beyond these cases with limited fusion, and e.g., also holds for finite groups of Lie type in characteristic \( p \). To illustrate the failure in general we calculate \( G_2(5) \) at \( p = 3 \) in Proposition \[6.3\] using Theorem \[D\], Remark \[5.10\] contains a more detailed discussion of bounds on \( r \).

### 1.4. Further relations to fusion systems.

Recall that a \( p \)-subgroup \( P \) is said to be centric if \( Z(P) = C_G(P) \) and \( p \)-centric if \( Z(P) \) is a Sylow \( p \)-subgroup in \( C_G(P) \); we denote by a superscript \( c \) full subcategories with objects the \( p \)-centric subgroups. Subgroups of \( \pi_1(F^c) \) parametrize sub-fusion systems of \( p' \)-index of a fusion system \( F \) by \cite[§5.1]{BCG+07}, and calculating \( \pi_1(F^c) \) is of current interest within \( p \)-local group theory—see e.g., \cite{AOV12}, \cite[Rui07], and \cite[Ch. 16]{Asc11}; the condition that \( \pi_1(F^c) = 1 \) is one of the conditions for a fusion system to be reduced \cite[Def. 2.1]{AOV12}. We here state a rough, but yet useful, relation between Sylow-trivial modules and \( \pi_1(F^c) \)—much more precise information is given in the paper proper.

**Theorem H.** We have the following commutative diagram of monomorphisms

\[
\begin{array}{ccc}
T_k(G, S) & \xrightarrow{\text{Hom}(\pi_1(\mathcal{C}_p^c(G)), k^\times)} & \text{Hom}(N_G(S)/S, k^\times) \\
\downarrow & & \downarrow \\
\text{Hom}(\pi_1(\mathcal{F}_p^c(G)), k^\times) & \xrightarrow{\text{Hom}(\pi_1(\mathcal{F}_p(G)), k^\times)} & \text{Hom}(N_G(S)/S\mu_G(S), k^\times)
\end{array}
\]

In particular if all \( p \)-centric \( p \)-radical subgroups are centric then \( \pi_1(\mathcal{C}_p^c(G)) \cong \pi_1(\mathcal{F}_p^c(G)) \) and

\[
\text{Hom}(\pi_1(\mathcal{F}_p^c(G)), k^\times) \leq T_k(G, S) \leq \text{Hom}(\pi_1(\mathcal{F}_p^c(G)), k^\times)
\]

The condition that \( p \)-radical \( p \)-centric subgroups are centric is satisfied for finite groups of Lie type at the characteristic, but also holds e.g., for many sporadic groups. In general the inclusions in the diagram of Theorem \[H\] may all be strict: for \( \text{GL}_n(F_q) \) and \( p \) not dividing \( q \), the main theorem in Ruiz \cite{Rui07} states that \( \pi_1(\mathcal{F}_p^c(\text{GL}_n(F_q))) \cong \mathbb{Z}/e \) for \( e \) the multiplicative order of \( q \) mod \( p \) and \( n \geq ep \), whereas \( \pi_1(\mathcal{C}_p^c(\text{GL}_n(F_q))) = 1 \) when the \( p \)-rank of \( \text{GL}_n(F_q) \) is at least \( 3 \) by \cite[Thm. 12.4]{L1} combined with \cite[Thm. 7.1]{Qui78} \cite[Thm. A]{Das95}. Again, the paper proper contains much more information about the relationship between these invariants, where much of the analysis is carried out for \( \mathcal{C} \) and \( \mathcal{F}_c \), with \( \mathcal{C} \) an arbitrary collection of \( p \)-subgroups.

### 1.5. Computational results.

We have already described how the results of this paper can be used to obtain a range of structural and computational results about \( T_k(G, S) \). To further illustrate the computational potential we will in Section \[6\] go through different classes of groups: symmetric, groups of Lie type, sporadic, \( p \)-solvable, and others, obtaining new results, and reproving a range of old results. We briefly summarize this:

For sporadic groups \( |S_p(G)| \) is sometimes known to agree with a building, where simply connectivity has been studied extensively (see \cite[§9]{Sm11}). However it is often easier to apply Theorems \[D, E, H\] or \[4.9\] directly, in particular since the necessary \( p \)-local data has already been tabulated, due to interest arising from counting conjectures in modular representation theory. We demonstrate this in
by showing that $T_k(G, S) = 0$ for $G$ the Monster and $k$ a field of characteristic $p = 3, 5, 7, 11$, or 13, the primes left open in the recent paper [LM15b]; see Theorem 6.1. It should be possible to fill in the remaining gaps in the existing sporadic group computations using similar arguments, though this is outside the scope of the present paper. (This has subsequently been carried out by David Craven.)

For finite groups of Lie type in characteristic $p$ the $p$-subgroup complex is just the Tits building $\mathbb{Q}i\mathbb{U}$, which is simply connected when the rank is at least 3, again recovering results of Carlson–Mazza–Nakano [CMN08]. For finite groups of Lie type in arbitrary characteristic, the $p$-subgroup complex is also believed to generically be a wedge of high dimensional spheres, which would imply that generically there were no exotic Sylow-trivial modules by Theorem B. It has been verified in a number of cases [Das95, Das98, Das00], and this way, we recover very recent results of Carlson–Mazza–Nakano for the general linear group for any characteristic [CMN14, CMN16], again using Theorem B and for symplectic groups we get the following new result.

**Theorem I.** Let $G = \text{Sp}_{2n}(q)$, and $k$ a field of characteristic $p$. If the multiplicative order of $q$ mod $p$ is odd, and $G$ has an elementary abelian $p$-subgroup of rank 3, then $T_k(G, S) = 0$.

See §6.2. In joint work in progress with Carlson, Mazza, and Nakano, we determine the group of endotrivial modules for all finite groups of Lie type using the methods of this paper combined with the "$\mathbb{Q}F_p$-local" approach to finite groups of Lie type.

For symmetric and alternating groups, the $p$-subgroup complex is known to be simply connected (except known small exceptions) [Kso03, Kso04], so we recover results of Carlson–Hemmer–Mazza–Nakano [CMN09, CHM10], using an inductive argument, that at one inductive step indirectly rely on the classification of finite simple groups. We link this inductive step to stronger conjectures by Quillen and Aschbacher about the connectivity of the $p$-subgroup complex for $p$-solvable groups; see §6.4.

As explained above, a calculation of $\pi_1(\mathcal{O}^p_1(G))$ and $H_1(\mathcal{O}^p_1(G))$ for all finite simple groups should be within reach. To pass to arbitrary finite groups on may then hope to use the structural properties of $\pi_1(\mathcal{O}^p_1(G))$ given in this paper (in particular §§6.1, 6.2) to reduce the question of the existence of exotic Sylow-trivial modules to the simple case, using the generalized Fitting subgroup $F^*(G)$ from finite group theory. We note in this connection that Aschbacher [Asc93] has, in a certain sense, reduced the question of simple connectivity of $|\mathcal{S}_p(G)|$ to simple groups, modulo his conjecture for $p$-solvable groups alluded to above. Based on available data, it may be that $\pi_1(\mathcal{O}^p_1(G)) \cong (G_0)^\pi$ when the $p$–rank is at least three? This would imply $T_k(G, S) \cong \text{Hom}(G_0, k^\times)$ under that assumption. See Remarks 5.10, 7.15 and Section 6 for more information.

**Further vistas.** In addition to the structural and computational consequences described so far, it is natural to wonder about further representation theoretic significance of the finite $p'$-groups $\pi_1(\mathcal{O}^p_1(G))$, $\pi_1(\mathcal{O}^p_1(G))$, $\pi_1(\mathcal{F}_p^c)$, and $\pi_1(\mathcal{F}_p^c)$, and the higher homotopy and homology groups, yet to be found? In a similar vein one may wonder if the method for constructing representations in Theorems A and 3.10 of this paper, when applied to more general coefficient systems on $|\mathcal{S}_p(G)|$ (see §3.4), could enable one to describe a larger slice of the stable module category, potentially shedding light on standard counting conjectures and their derived generalizations? Remarks 3.16, 3.18, 3.20, 4.11, and 4.38 give some hints in these directions. Further afield, one may look for a contribution of the homotopy type of the orbit or fusion category on collections of $p$–subgroups for problems involving any other ‘$p$–local’ symmetric monoidal (infinity) category that depend on $G$, whether in algebra or topology, similar to the one discovered here for StMod$_G$ (see also [Mat16] for infinity categorical considerations).

**Organization of the paper.** Section 2 states conventions and introduce the categories and constructions needed for the main results, providing a fair amount of detail in the hope of making the paper accessible to both group representation theorists and topologists. Section 3 proves Theorem A only relying on the recollections in Section 2. In Section 4 we prove a number of results on $\pi_1(\mathcal{O}^p_1(G))$ and $\pi_1(\mathcal{F}_p^c(G))$ and among other things deduce the consequences described in §§1.1, 1.4. Section 5 proves the decompositions and the Carlson–Thévenaz conjecture, as stated in §§1.2, 1.3. Section 6
goes through the computational consequences. Finally the appendix Section \[ \] presents a number of results about changing the collection of subgroups, which are used throughout Sections \[ \] and should also be of independent interest.

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2. Notation and preliminaries

2.1. Conventions. In this paper \( G \) will always be an arbitrary finite group and \( p \) an arbitrary prime dividing the order of \( G \) (to avoid having to make special statements in the trivial case where this is not so). We use the notation \( S \) for its Sylow \( p \)-subgroup and set \( G_0 = \langle N_C(Q) \mid 1 < Q \leq S \rangle \), which when \( G_0 < G \) is the smallest strongly \( p \)-embedded subgroup of \( G \) containing \( S \); see also the introduction \[ ] and \[ ] By \( k \) we will always mean a field of characteristic \( p \), where \( p \) divides \( |G| \), but subject to no further restrictions, like algebraically closed. Note that the units \( k^\times \) cannot have \( p \)-torsion, as the Frobenius map is injective, and it will be uniquely \( p \)-divisible if \( k \) is perfect; see also \[ ]. Our \( kG \)-modules will not be assumed finitely generated, though everything could also be phrased inside the smaller category of finitely generated modules with the same result. Tensor products are over \( k \).

As stated in the introduction we use the term Sylow-trivial for our basic objects: \( kG \)-modules that when restricted to a Sylow \( p \)-subgroup \( S \) split as the trivial module direct sum a projective module. Note that the tensor product of two Sylow-trivial modules is again Sylow-trivial, as the tensor product of a projective \( kS \)-module with any \( kS \)-module is again projective. Also if \( M \downarrow_S \cong k \oplus (\text{proj}) \), then the cokernel of the unit \( kG \)-map \( k \to M^* \otimes M \) is a projective \( kG \)-module, since as \( kS \)-modules, \( M^* \otimes M \cong k^* \otimes k \oplus (\text{proj}) \), and the map \( k \to k^* \otimes k \cong k \) the identity, so the cokernel is projective as a \( kS \)-module, and hence also as a \( kG \)-module. Thus \( M^* \) is indeed an inverse to \( M \) in \( T_k(G,S) \), and \( T_k(G,S) \leq T_k(G) \). (See Proposition \[ ] for why each equivalence class has a unique indecomposable representative, up to isomorphism.)

By a collection of \( p \)-subgroups \( \mathcal{C} \), we mean a set of \( p \)-subgroups of \( G \), closed under conjugation. We use standard notation for various specific collections of \( p \)-subgroups, like non-trivial elementary abelian \( p \)-subgroups \( A_p(G) \), non-trivial \( p \)-radical subgroups \( B_p(G) \), etc, which we also recall in the appendix Section \[ ] We use standard group theoretic notation, with the addition of \( -|_p \) and \( A^p(-) \) which were defined in the introduction. By a space we will for convenience mean a simplicial set, \( | \cdot | \) denotes the nerve functor from categories to simplicial sets, and homotopy equivalence means homotopy equivalence after geometric realization; group theorists not familiar with simplicial sets are largely free to think of them as simplical complexes, or topological spaces, and can find a lucid introduction in [DH10].

2.2. Categorical constructions. Define the transport category \( \mathcal{T}(G) \) as the category with objects all subgroups of \( G \) and morphisms \( \text{Mor}(P,Q) = \{ g \in G \mid gP \leq Q \} \), i.e., the Grothendieck construction, or transport category, of the left conjugation action of \( G \) on the poset of all subgroups (see also Remark \[ ]). We have a quotient functor \( \mathcal{T}(G) \to \mathcal{O}(G) \) which on objects assigns \( G/H \) to \( H \) and assigns to \( (g,^gH \leq K) \) the \( G \)-map that sends the trivial coset \( eH \) to \( g^{-1}K \); see also e.g., [DS87, I.10].
Denote by \( \mathcal{O}_C(G) \) and \( \mathcal{R}_C(G) \) the full subcategories with objects \( G/H \) and \( H \), respectively, for \( H \in \mathcal{C} \), and we continue to use the notation \( \mathcal{O}_C^p(G) \) and \( \mathcal{R}_C^p(G) \) for these categories when \( \mathcal{C} = S_p(G) \) is the collection of all non-trivial \( p \)-subgroups. We also introduce the fusion category \( \mathcal{F}_C(G) \) and fusion-orbit category \( \mathcal{F}_C(G) \) both with objects \( P \in \mathcal{C} \) and morphisms

\[
\text{Hom}_{\mathcal{F}_C(G)}(P, Q) = \text{Hom}_{\mathcal{R}_C(G)}(P, Q)/CG(P) \quad \text{and} \quad \text{Hom}_{\mathcal{F}_C(G)}(P, Q) = Q \setminus \text{Hom}_{\mathcal{R}_C(G)}(P, Q)/CG(P)
\]

respectively, i.e., monomorphisms induced by conjugation in \( G \) and ditto modulo conjugation in the target. (The fusion-orbit category is called the exterior quotient \( \mathcal{F} \) of \( \mathcal{F}_C(G) \) by Puig \[Pui06\], and is also sometimes denoted \( \mathcal{O}(\mathcal{F}) \).) All four categories hence have object-set identifiable with \( \mathcal{C} \), and morphisms related via quotients

\[
\mathcal{F}_C(G) \longrightarrow \mathcal{F}_C(G) \leftarrow \mathcal{F}_C(G)
\]

### 2.3. Low dimensional cohomology and homotopy of categories.

The cohomology of a small category \( D \) with constant coefficients in an abelian group \( A \) is defined as the cohomology of the simplicial set \( |D| \) with constant coefficients \( A \). In particular note that \( H^1(D; A) \) identifies with functors \( D \to A \), up to natural isomorphism of functors, where \( A \) is viewed as a category with one object. Indeed, a 1–cocycle is a function \( F: \text{Mor}(D) \to A \) such that \( F(\beta \circ \alpha) = F(\beta) + F(\alpha) \) (hence \( F(\text{id}) = 0 \)), in other words a functor \( F: \text{Mor}(D) \to A \); and a 1–coboundary is a 1–cocycle of the form \( F(\alpha) = g(\text{cod}(\alpha)) - g(\text{dom}(\alpha)) \), for a function \( g: \text{Ob}(D) \to A \), i.e., a functor that admits a natural transformation (hence isomorphism) to the zero functor. Furthermore \( H^1(D; A) \overset{\cong}{\to} \text{Hom}(H_1(D), A) \), where \( H_1(D) \) is the abelian group of cycles of morphisms in \( D \), modulo the equivalence relation coming from composition, a special case of the universal coefficients theorem \[Hat02\, Thm. 3.2 \]. (See also e.g., \[Web07\].)

Homotopy groups are defined as \( \pi_i(D, d) = \pi_i(|D|, d) \), and also have a categorical description for \( i = 1 \): the fundamental groupoid \( \pi(\text{Mor}(D)) \) identifies as \( D[\text{Mor}(D)^{−1}] \) the category obtained by formally inverting all morphisms in \( D \) (left adjoint to the inclusion functor from groupoids to categories), providing a canonical isomorphism \( \pi_1(|D|, d) \cong \text{Aut}_{D[\text{Mor}(D)^{−1}]}(d) \). (See also e.g., \[Qui73\, §1 \].)

Assume that \( D \) is connected, i.e., that all objects can be connected by a long zig-zag of morphisms. The universal properties gives us isomorphisms

\[
\text{Hom}(\pi_1(D, d), A) \cong H^1(\pi_1(D, d); A) \cong H^1(D[\text{Mor}(D)^{−1}]; A) \cong H^1(D; A)
\]

and one also sees directly that the canonical map \( \pi_1(D, d) \to H_1(D) \), viewing loops as 1–cycles, is abelianization, a special case of the Hurewicz theorem \[Hat02\, Thm. 2A.1 \].

It is often convenient to have a concrete way of writing elements in \( \pi_1(D, d) \). For this, pick for each object \( x \in D \) a path \( \iota_x \) from \( x \) to \( d \), producing a functor \( D \to \pi_1(D, d) \) via \( (\iota_x : x \to y) \mapsto \iota_y \circ \varphi \circ \iota_x^{-1} \). This is induces a functor \( \omega: D[\text{Mor}(D)^{−1}] \to \pi_1(D, d) \), which is manifestly an inverse equivalence of categories to the inclusion, and we can think of elements of \( \pi_1(D, d) \) as long zig-zags of morphisms in \( D \) in this way. In particular for \( D = \mathcal{O}_p^\alpha(G) \), we can, up to equivalence of categories, take objects \( G/Q \) for \( Q \leq S \), and basepoint \( G/S \), so that we have canonical maps \( i_{G/Q} : G/Q \to G/S \); note also that \( \omega(G/P \to G/Q) = 1 \) for the map induced by \( P \leq Q \), allowing us to effectively ignore inclusions; Similarly for the other standard categories from \[GJ99\, I.2\, DK83\, 5.1 \]. We will often suppress the basepoint from the notation, and the above considerations show why we can do this without ambiguity.

### 2.4. Coefficient systems.

We recall the definition of \( G \)-local coefficient systems on a \( G \)-space \( X \) from \[Gro02\, §2.1 \], which is slightly more general than what is usually considered in topology (but similar to definitions used in group theory \[RS86\, §1 \]). Recall that to a space \( X \) we can associate the category of simplices \( \Delta X \) (aka the simplex, or division, category \( \Delta \downarrow X \)), with objects all simplices, and morphisms given by iterated face and degeneracy maps \[GJ99\, I.2\, DK83\, 5.1 \]. For a ground ring, a general (homological) \( G \)-local coefficient system on a \( G \)-space \( X \) is just a functor \( A: (\Delta X)_{\text{G}} \to R\text{-mod} \), to \( R \)-modules, where \( (\Delta X)_{\text{G}} \) is the associated transport category (or Grothendieck construction) of the left \( G \)-action on \( \Delta X \); it has objects the simplices of \( X \) and morphisms from \( \sigma \) to \( \tau \) consists of pairs \( (g, f: g\sigma \to \tau) \) where \( g \in G \) and \( f \) is a face or degeneracy map.
The chain complex $C_*(X; A)$, with $C_n(X; A) = \oplus A(\sigma)$, and the standard simplicial differential, is a chain complex of $kG$-modules via the induced $G$-action, $A((g, \id))$; see e.g., [Gro02 §2] (where contravariant, i.e., cohomological, coefficient systems are considered). More generally we can consider $C_*(X; A)$ as a functor from $\Theta(G)^{op}$ to chain complexes via $G/H \mapsto C_*(X^H; A^H)$, with $A^H(\sigma) = A(\sigma)^H$, and $(G/H \xrightarrow{h} G/H') \mapsto C_*(X^{H'}; A^{H'}) \xrightarrow{A(h, id)} C_*(X^H; A^H)$, with orbit category notation as explained in [2.2] in particular $N_G(H)/H$ acts on $C_*(X^H; A^H)$, which is the standard action when $H$ is trivial.

Three special kinds of such coefficient systems play a special role: One is $G$–categorical coefficient systems where $X$ is the nerve of a small $G$–category $\mathcal{D}$ and $A$ is induced by a functor $\mathcal{D}_G \to \mathcal{R}$-mod, via $(\Delta|\mathcal{D}|)_G \to \mathcal{D}_G \to \mathcal{R}$-mod, where $\Delta|\mathcal{D}|$ is given by $(d_0 \to \cdots \to d_n) \mapsto d_0$. Another is Bredon $G$–isotropy coefficient systems, in which $A$ is induced by a functor $\Theta(G)$ to $\mathcal{R}$-mod via the functor $(\Delta X)_G \to \Theta(G)$, given by $\sigma \to G/G_\sigma$ and $(g, f: \sigma \to \tau) \mapsto (G/G_\sigma \xrightarrow{g^{-1}} G/G_\sigma \xrightarrow{f} G/G_\tau)$. A third kind is a $G$–twisted coefficient system, in which $A$: $(\Delta X)_G \to \mathcal{R}$-mod is assumed to send all morphisms to isomorphisms. (A twisted coefficient system is also sometimes called invertible; compare [Qui78 §7], [Ben91b 6.2.3].) By the universal property, such coefficient systems factor through the fundamental groupoid of the category $(\Delta X)_G$, which is equivalent to the fundamental groupoid of $X_{hG}$ by Thomason’s theorem (see [Tho79 Thm. 1.2] and Remark 2.2) and the homotopy equivalence $|\Delta X| \simeq X$ [[GJ99] IV.5.1]. In particular if $X_{hG}$ is connected, specifying a $G$–twisted coefficient system is equivalent to specifying a $\pi_1(X_{hG}, x)$–module $M$, for some choice of basepoint $x \in X_{hG}$. Note also that if $X = |\mathcal{D}|$, then specifying a $G$–twisted coefficient system is the same as giving a functor on the fundamental groupoid of $\mathcal{D}_G$, i.e., on $\pi_1(\mathcal{D}_G)$ if $\mathcal{D}_G$ is connected, again by Thomason’s theorem; in particular any twisted coefficient system is categorical in this case. By the above description, $G$–isotropy coefficient systems are $G$–homotopy invariants of $X$ and $G$–twisted coefficient systems are even $hG$–homotopy invariants of $X$, i.e., invariant under $G$–maps which are homotopy equivalences; none of this is true for arbitrary $G$–local coefficient systems.

2.4.1. Coefficient systems on subgroup complexes. In our special case of interest $X = |\mathcal{C}|$, for $\mathcal{C}$ a collection, specifying a $G$–categorical coefficient system on $\mathcal{C}$ is hence the same as specifying a functor $\mathcal{R}_G(G) \to \mathcal{R}$-mod, and a $G$–twisted coefficient system is the same as specifying a functor $\mathcal{R}_G(G) \to \mathcal{R}$-mod sending all morphisms to isomorphisms. Furthermore our $\mathcal{C}$ will usually be such that $\mathcal{R}_G(G)$ is a connected category (e.g., if $\mathcal{C}$ is a collection of $p$–subgroups containing $S$), and hence a $G$–twisted coefficient system can be identified with a $k\pi_1(\mathcal{R}_G(G), P)$–module, for $P \in \mathcal{C}$. In particular, a one-dimensional $k\pi_1(\mathcal{R}_G(G), P)$-module is just a homomorphism $\pi_1(\mathcal{R}_G(G), P) \to k^\times$. So, for $\varphi$: $\mathcal{R}_G(G) \to k^\times$, we consider the corresponding functor $\mathcal{C}_\varphi$ from $\mathcal{R}_G^p(G)$ to one dimensional $k$–vector spaces and isomorphisms, and equip $|\mathcal{S}_p(G)|$ with the canonical $G$–twisted coefficient induced by $k\varphi$, via the composite

$$\Delta|\mathcal{S}_p(G)| \to \mathcal{S}_p(G) \to \mathcal{R}_G(G) \to \mathcal{R}_G^p(G)$$

Explicitly, $\Delta|\mathcal{S}_p(G)| \to \mathcal{S}_p(G)$ sends $(P_0 \leq \cdots \leq P_n) \mapsto P_0$ and $\mathcal{S}_p(G) \to \mathcal{R}_G^p(G)$ sends $(q P \leq P', g) \mapsto (G/P \xrightarrow{g^{-1}} G/P')$. Note that, if one ignores the group action, this is a twisted coefficient system in the ordinary (non-equivariant) sense, depending on the fundamental groupoid of $|\mathcal{S}_p(G)|$.

Remark 2.1 (On op’s and inverses). Since op’s and inverses are a common source of light confusion, we make a few remarks about their presence in the formulas: A group $G$ viewed as a category with one object is isomorphic as a category to $G^{op}$ via the map $g \mapsto g^{-1}$. In particular $\mathcal{F}(G)$ is isomorphic to the category with morphism set $\text{Mor}^{\text{op}}(P, Q) = \{ g \in G | P^g \leq Q \}$, which is the Grothendieck construction $\mathcal{S}_p(G)_{\text{op}}$. Redefining the transport category this way would get rid of the inverse appearing in the formula for the projection map $\mathcal{F}(G) \to \Theta(G)$; alternatively one could reparametrize the orbit category, the choice made e.g., in [AKO11 III.5.1]. Notice also that when we are considering functors to an abelian group such as $k^\times$, viewed as a category with one object, covariant functors naturally equals contravariant functors. (The identification using the isomorphism between $G$ and $G^{op}$ produces the automorphism given by “pointwise inverse”.)

Remark 2.2 (Thomason’s theorem for Borel constructions). We have used at various points that $|\mathcal{D}_G| \cong |\mathcal{D}_{hG}$ for a small category $\mathcal{D}$ with an action of $G$, where $\mathcal{D}_G$ is the Grothendieck construction
and $|D|_{hG}$ the Borel construction, e.g., to conclude that $|\mathcal{R}| = |C|_G \cong |C|_{hG}$ as in (1.6). As mentioned, this is a special case of Thomason’s theorem [Tho79, Thm. 1.2], but since it in this special case is elementary let us just recall the proof: The Grothendieck construction $D_G$ has objects the objects of $D$ and morphisms from $x$ to $y$ given by a pair $(g, f : gx \rightarrow y)$, where $x, y \in \text{Ob}(D)$. Define the category $EG$ to be the category with objects the elements of $G$ and a unique morphism between all elements, so that $|EG| = EG$, the universal free contractible $G$–space. Our group $G$ acts freely on the product category $EG \times D$, on objects given by $g \cdot (h, x) = (h g^{-1}, g x)$. The quotient $EG \times_G D$ identifies with $D_G$. Since the nerve functor commutes with products and free $G$–actions we have identifications $|D|_{hG} = EG \times_G |D| \cong |EG \times_G D| = |D_G|$ as wanted.

2.5. The Buchweitz–Rickard equivalence. The last thing we recall is the equivalence $$\text{stmod}_{kG} \cong \mathcal{D}^b(kG)/\mathcal{D}^\text{perf}(kG),$$ employed in Theorem A. As written, it is an equivalence between the stable module category of finitely generated $kG$–modules, and the bounded derived category of finitely generated $kG$–modules $\mathcal{D}^b(kG)$, modulo perfect complexes $\mathcal{D}^\text{perf}(kG)$ (i.e., complexes quasi-isomorphic to a bounded complex of finitely generated projective modules), and comes from [Ric89, Thm. 2.1] and [Buc, Thm. 4.4.1]. The map in the one direction is obvious, viewing a module as a chain complex concentrated in degree zero. Let us also explain the map in the other direction, following [Ric89, Pf. of Thm. 2.1], which is the one used in Theorem A. Represent the object in $\mathcal{D}^b(kG)$ by a bounded below complex $P_\ast$, of finitely generated projectives. In $\mathcal{D}^b(kG)/\mathcal{D}^\text{perf}(kG)$ this complex is equivalent to its truncation $P_r = \cdots \rightarrow P_{r+1} \rightarrow P_r \rightarrow 0 \rightarrow \cdots$, where we take $r$ to be the degree of the top non-trivial homology class. This complex has homology only in degree $r$, and we take the module to be the $-r$th Heller shift of of the $r$th homology group, i.e., $\Omega^{-r}(P_r/\text{im}(d_{r+1}))$, which is well defined as an object of $\text{stmod}_{kG}$. (Recall that the inverse Heller shift $\Omega^{-1}(M)$ is the cokernel of the map from $M$ to its injective hull, the “suspension” in the triangulated structure.)

3. Proof of Theorem A

In this section we prove Theorem A just using the preparations from the preceding section. We start with some elementary facts about Sylow-trivial modules, including dealing with the finite-dimensionality issue once and for all.

Proposition 3.1. Any Sylow-trivial $kG$–module $M$ is of the form $N \oplus P$, where $N$ is an indecomposable direct summand of $k[G/S]$ and $P$ is projective, and $N$ is determined uniquely up to isomorphism, from the equivalence class of $M$ given by adding projectives. If $O_p(G) \neq 1$, then any indecomposable Sylow-trivial $kG$–module is one-dimensional.

Proof. This is well known, and follows from [BBC09, Thm. 2.1] and [MT07, Lem. 2.6], but since it is among the few “classical” representation theory facts used, we give a direct proof. Let $M$ be our Sylow-trivial module, and recall that $M$ is a direct summand of $M \downarrow^G_S \downarrow^S_S$, since the composite of the $kG$–map $M \rightarrow M \downarrow^G_S \downarrow^S_S M, m \mapsto \sum g_i \in G/S g_i \otimes g_i^{-1} m$, and the $kG$–map $M \downarrow^S_S \rightarrow M, m \mapsto g m$, is multiplication by $|G : S|$, which is a unit in $k$. (See also e.g., [Ben91a, Cor. 3.6.10], where the blanket finitely generated assumption is not being used.) By assumption $M \downarrow^S_S \cong k \oplus \text{proj}$, so $M$ is a summand of $k \uparrow^G_S \oplus \text{proj}$. As explained in [Ric97, Lem. 3.1], any $kG$–module, also infinite dimensional, can be written as a direct sum of a projective module and a module without projective summands, and the non-projective part is unique up to isomorphism; furthermore, by [Ric97, Lem. 3.2], this decomposition respects direct sums. This shows that $M \cong N \oplus P$, with $N$ a non-projective direct summand of $k \uparrow^G_S$ which is determined, up to isomorphism, by the equivalence class of $M$. Furthermore, $N$ has to be indecomposable, since otherwise it cannot be Sylow-trivial.

Now assume $O_p(G) \neq 1$. Since $M$ is a direct summand of $k \uparrow^G_S$ by the first part, then $M \downarrow^S_S$ is a direct summand of $k \uparrow^G_S \downarrow^S_S \cong \oplus_{g \in S \backslash G/S} \downarrow^S_S \uparrow^G_S$, which does not contain any projective summands, since $S \cap S^g \geq O_p(G) \neq 1$. Hence $M \downarrow^G_S \cong k$ as wanted. \hfill \Box

Let us also give the following well known special case of the Green correspondence [Ben91a, Thm. 3.12.2] (see also [CMN06, Prop. 2.6(a)]), used for injectivity of the map $\Phi$ of Theorem A.
Proposition 3.2. Let $M$ be a $kG$-module such that $M \downarrow_{N_G(S)} \cong k \oplus (\text{proj})$. Then $M \cong k \oplus (\text{proj})$. In particular $T_k(G, S) \leq T_k(N_G(S), S) \cong \text{Hom}(N_G(S)/S, k^\times)$.

Proof. As mentioned this is a special case of the Green correspondence, but let us extract a direct argument: By Proposition 3.1 it is enough to see that if $M$ is indecomposable, then $M \cong k$. So, set $N = N_G(S)$, and assume that $M$ is an indecomposable $kG$-module such that $M \downarrow_{N} \cong k \oplus (\text{proj})$. As in Proposition 3.1 $M$ will be a summand of $M \downarrow_{N} \cong k \oplus (\text{proj})$, and hence a summand of $k \uparrow_{N} \cong k \oplus L$, for $L$ a complement of $k(\sum_{g \in G/N \setminus gN})$. But $L \uparrow_{S} \cong \oplus_{g \in S \setminus G/N \setminus k} \uparrow_{S \setminus N}$, and in particular it does not contain $k$ as a direct summand. Hence $M$ has to be a direct summand of $k$. Also $T_k(G, S) \leq T_k(N_G(S), S)$ and $T_k(N_G(S), S) \cong \text{Hom}(N_G(S)/S, k^\times)$ by Proposition 3.1.

We can already now prove a part of Theorem A.

Proposition 3.3. For any Sylow-trivial module $M$, $G/P \mapsto \hat{H}^0(P; M) = M^P/\langle \sum g \in P \rangle M$ defines a functor from $\mathcal{O}_\mathfrak{p}(G)^{\text{op}}$ to one-dimensional $k$-modules and isomorphisms, which we can identify with a functor $\mathcal{O}_\mathfrak{p}(G) \mapsto k^\times$. The assignment that sends a Sylow-trivial module $M$ to the above functor defines an injective group homomorphism $\Phi : T_k(G, S) \mapsto H^1(\mathcal{O}_\mathfrak{p}(G); k^\times)$.

Proof. It is clear that $\hat{H}^0(P; M) = M^P/\langle \sum g \in P \rangle M$ is one-dimensional, since $M \downarrow_P \cong k \oplus (\text{proj})$ by assumption, and the construction kills the projective part. Varying $P$, it furthermore defines a functor on the $p$-orbit category, that sends $G/P \mapsto G/P'$ to the morphism $M^{P'}/\langle \sum g \in P' \rangle M \mapsto M^P/\langle \sum g \in P \rangle M$ given by multiplication by $g$. Picking a basepoint, say $M^S/\langle \sum g \in S \rangle M$, identifies this with a functor $\mathcal{O}_\mathfrak{p}(G) \mapsto k^\times$. (See also Remark 2.1 for a discussion of variance.)

It is also clear that $T_k(G, S) \mapsto H^1(\mathcal{O}_\mathfrak{p}(G); k^\times)$ is a group homomorphism, where the group structure on the right is pointwise multiplication. It is injective by the last part of Proposition 3.2 since if $\Phi(M)$ is the identity, then the action of $N_G(S)$ on $\hat{H}^0(S; M)$ is in particular trivial.

Remark 3.4. For the modules we consider in Theorem A $\hat{H}^0(P; M) \cong M^P/\langle \sum_{Q \leq P} M^Q \rangle$, the Brauer quotient, and this may be another way to view this construction.

The next proposition gives surjectivity of $\Phi$.

Proposition 3.5. Let $\varphi : \pi_1(\mathcal{O}_\mathfrak{p}(G)) \mapsto k^\times$ be a homomorphism and equip $|S_p(G)|$ with the corresponding $G$-twisted coefficient system $k_\varphi$, as in [2.4.7]. Then for any non-trivial $p$-subgroup $P$ we have the following zig-zag of equivalences in $D^b(kN_G(P))/D^\text{perf}(kN_G(P))$:

$$\mathcal{O}_\mathfrak{p}(G/P) \mapsto C_*|S_p(G)|; k_\varphi \mapsto C_*|S_p(G)|; k_\varphi$$

In particular $C_*(|S_p(G)|; k_\varphi)$ gives a Sylow-trivial module $M$ via $\text{stmod}_k \sim \rightarrow D^b(kG)/D^\text{perf}(kG)$ (cf. [2.5] and $\Phi(M) = \varphi$).

Proof. First recall that for a non-trivial $p$-subgroup $P$, $|S_p(G)|$ is $N_G(P)$-equivariantly contractible by Quillen’s argument: $|S_p(G)|^P = |S_p(G)|^P$, which is contractible via a contracting homotopy induced by $Q \leq PQ \leq P$; see [Qui78] 1.3 [GS06] Rec. 2.1. Hence the left map is an equivalence in $D^b(kN_G(P))$, by the $N_G(P)$-equivariances $|\{P\}| \rightarrow |S_p(G)|^P \rightarrow |\{P\}|$ and the fact that $G$-twisted coefficient systems are $G$-equivariant in $D^b(kN_G(P))$.

That the right map is an equivalence in $D^b(kN_G(P))/D^\text{perf}(kN_G(P))$ is a adaption of Quillen’s proof that the (reduced) Steinberg complex is isomorphic to a finite complex of projectives [Qui78] Prop 4.1 and 4.5 (see also [Web91] Thm. 2.7.4 and [Sym08]): Choose a Sylow $p$-subgroup $S'$ of $N_G(P)$, and note that it is enough to prove that the map is an equivalence in $D^b(kS'/D^\text{perf}(kS')$, since projectivity is detected on a Sylow $p$-subgroup. Also since $|S_p(G)|^S' \rightarrow |S_p(G)|^P$ is an $S'$-equivariant equivalence, it is enough to prove the statement with $P$ replaced by $S'$. Now, set $\Delta = |S_p(G)|$ and let $\Delta_\alpha$ denote the singular set under the $S'$-action. By definition we have an exact sequence of chain complexes

$$0 \rightarrow C_*(\Delta_\alpha, \Delta_\alpha; k_\varphi) \rightarrow C_*(\Delta, \Delta; k_\varphi) \rightarrow C_*(\Delta, \Delta_\alpha; k_\varphi) \rightarrow 0$$

As observed in [Qui78] Prop. 4.1 (though only stated for $P$ the Sylow $p$-subgroup), the singular set $\Delta_\alpha$ is contractible (to see this, one can also note that $\Delta_\alpha$ is covered by the contractible subcomplexes $\Delta Q$, for $1 \neq Q \leq S'$, all of whose intersections are also contractible; see [Seg68] §4 or [LD08]...
Hence $\Delta^S \rightarrow \Delta_s$ is a homotopy equivalence, so $C_*(\Delta_s, \Delta^S; k_\varphi)$ is acyclic (see again \textbf{[2.4]} for generalities about coefficient systems). Therefore $f$ is an equivalence in $D^b(kS^t)$. By definition $C_*(\Delta, \Delta^S; k_\varphi)$ is a complex of free $kS^t$–modules, so $C_*(\Delta, \Delta^S; k_\varphi)$ is in $D_{\text{perf}}^b(kS^t)$ as wanted, establishing the first part of the proposition.

To see the claim about Sylow-trivial, it is enough to prove that $C_*(|S_p(G); k_\varphi)$ is equivalent to the trivial module $k$ in $D^b(kS)/D_{\text{perf}}^b(kS)$, since the equivalence $\text{stmod}_{kG} \rightarrow D^b(kG)/D_{\text{perf}}^b(kG)$ is compatible with restriction. However this follows from the first part taking $P = S$.

We note that also $\Phi(C_*(|S_p(G); k_\varphi)) = \varphi$. Namely, the identification of $C_*(|S_p(G); k_\varphi)$ with $k_\varphi$ in $\text{stmod}_{kN_G}(Q)$ above is compatible with restriction and conjugation, so defines an isomorphism of functors on $\mathcal{O}_S^*(G)$ (see also \textbf{[2.4]}), which shows that $\Phi(C_*(|S_p(G); k_\varphi)) = \varphi$ in $H^1(\mathcal{O}_S^*(G); k^\times)$ as wanted. (Alternatively, by Proposition \textbf{4.1} it is enough to see that the two functors agree as $kN_G(S)$–modules when evaluated on $G/S$, which follows by the first part.)

\textbf{Proof of Theorem \textbf{A}} By Proposition \textbf{3.3} $\Phi$ is a group monomorphism. Proposition \textbf{3.5} shows that $C_*(|S_p(G); k_\varphi)$ does define a Sylow-trivial module via the equivalence of categories $\text{stmod}_{kG} \cong D^b(kG)/D_{\text{perf}}^b(kG)$, and this assignment is a right inverse to $\Phi$. So $\Phi$ is surjective as well. \hfill \square

\textbf{Remark 3.6} (Sylow-trivial modules for groups with a strongly $p$–embedded subgroup). Since $C_*(|S_p(G); k_\varphi) \cong C_*(|S_p(G_0); k_\varphi) \uparrow^G_{G_0}$

by \textbf{[1.8]}, Theorem \textbf{A} also encodes the well known bijection between Sylow-trivial $kG_0$–modules and Sylow-trivial $kG$–modules, modulo projectives, given by induction \textbf{[MT07]} Lem. 2.7(2).

\textbf{Remark 3.7} (Kan extensions and homotopy Kan extensions). As noted in the introduction, we have natural identifications of $kG$–chain complexes

$$C_*(|S_p(G); k_\varphi) \cong \text{hocolim}_{P \in \mathcal{S}_p(G)} k_\varphi \cong \text{hocolim}_{\eta \downarrow G/e} k_\varphi \cong (\mathbb{L}\text{Kan}_\eta k_\varphi)(G/e) \quad (3.1)$$

The homotopy colimits are taken in $k$–chain complexes, and the left identification of $kG$–chain complexes is by the standard model for the homotopy colimit \textbf{[Hir03]} Ch. 18.1.1. The third $kG$–chain complex is the formula for the value at $G/e$ of the homotopy left Kan extension $\mathbb{L}\text{Kan}_\eta k_\varphi : \mathcal{O}_p(G)^{\text{op}} \rightarrow k$–(chain complexes) of $k_\varphi$ along $\eta : \mathcal{O}_p(G)^{\text{op}} \rightarrow \mathcal{O}_p(G)^{\text{op}}$; this is equivalent in $D^b(kG)$ to the homotopy colimit over $\mathcal{S}_p(G)^{\text{op}}$, as the overcategory $\eta \downarrow G/e$ admits a canonical $G$–equivariant functor to $\mathcal{S}_p(G)^{\text{op}}$, which is an equivalence of categories; see \textbf{(7.2)}.

The naïve guess for the inverse map in Theorem \textbf{A} might have been the non-derived version $\text{colim}_{P \in \mathcal{S}_p(G)} k_\varphi$, with again identifies with the ordinary left Kan extension of $k_\varphi$ along $\eta$, evaluated at $G/e$. This identifies with the 0th homology group of $C_*(|S_p(G); k_\varphi)$, by the definition of homology. However, $\text{colim}_{P \in \mathcal{S}_p(G)} k_\varphi(G_0/P)$ is a module of dimension at most one (since $\mathcal{S}_p(G_0)$ is connected by \textbf{[1.8]}), which, if non-trivial, extends the action of $N_{G_0}(S)$ on $k_\varphi(G_0/S)$ to $G_0$. Hence $\text{colim}_{P \in \mathcal{S}_p(G)} k_\varphi(G/P)$ has to be zero unless $\varphi$ corresponds to a Sylow-trivial module induced from a 1-dimensional module on $G_0$. It was in fact this viewpoint that led us to the formula in Theorem \textbf{A}.

Using the above remarks, Theorem \textbf{A} gives a more explicit model for the Sylow-trivial module when $|S_p(G)|$ is $G$–homotopy equivalent to one-dimensional complex—this in fact appears to cover all currently known examples of exotic Sylow-trivial modules! Recall the Heller shift $\Omega$ from \textbf{[2.5]}

\textbf{Corollary 3.8.} Suppose that $G$ is a finite group such that $|S_p(G)|$ is $G$–homotopy equivalent to a one-dimensional complex (e.g., if $G$ has $p$–rank at most 2, or at most one proper inclusion between $p$–radicals), and suppose $\varphi \in \text{Hom}(\pi_1(\mathcal{O}_p(G)), k^\times)$ is not in the subgroup $\text{Hom}(G_0, k^\times)$, cf., \textbf{[1.3]}. Then the corresponding Sylow-trivial module is given as $\Omega^{-1}(H_1(|S_p(G_0); k_\varphi)) \uparrow^G_{G_0}$

\textbf{Proof.} As explained in Remark \textbf{3.7} $H_0(|S_p(G_0); k_\varphi)$ is trivial, if $\varphi$ does not come from a homomorphism $\text{Hom}(G_0, k^\times)$. Hence $C_*(|S_p(G_0); k_\varphi)$ is quasi-isomorphic over $kG_0$ to $H_1(|S_p(G_0); k_\varphi)$, viewed as a chain complex concentrated in degree 1. The corollary now follows from Theorem \textbf{A} and Remark \textbf{3.6} see also \textbf{[2.5]}. \hfill \square
Remark 3.9. If $C$ is a 1-dimensional collection in $G_0$ that is $G_0$-homotopy equivalent to $|\mathcal{S}_p(G_0)|$, then by definition
\[
H_1((\mathcal{S}_p(G_0); k_{\varphi}) \cong \ker \left( \oplus_{[P<Q] \in |\mathcal{C}_1/G_0} k_{\varphi}(P) \uparrow_{G_0/G_0}^{G_0} \oplus_{[P] \in |\mathcal{C}_0/G_0} k_{\varphi}(P) \uparrow_{G_0/G_0}^{G_0} \right)
\]
as $kG_0$-modules, where $k_{\varphi}(P)$ is the 1-dimensional $N_G(P)$-module given by $N_G(P) \to \pi_1(\mathcal{O}_p^*(G)) \to k^\times$. Often $C$ can be chosen so that $|\mathcal{C}_1/G_0|$ is small, maybe even a single element; see §6.3 for an example. (The relationship between collections in $G$ and $G_0$ is explained in §7.5.)

Lastly we remark that $\Omega^{-1}M$ is isomorphic to $\Omega^{-1}k \otimes M$, modulo projectives, so in terms of finding generators for the group of endotrivial modules, $\Omega^{-1}M$ works as well as $M$.

One may wonder what happens if one applies the map $\Phi^{-1}(-) = C_*(\mathcal{S}_p(G); -)$ of Theorem A to an arbitrary $k\pi_1(\mathcal{O}_p^*(G))$-module (semi-simple since $\pi_1(\mathcal{O}_p^*(G))$ is a $p'$-group by (1.3)). The proof of Theorem A in fact shows that $\Phi$ will still be a left inverse, and one can identify the image. The following more precise theorem generalizes Theorem A and arose as response to questions by Radha Kessar (Remark 3.15 below) and David Craven. Call a $kG$-module $M$ Sylow-semi-simple if $M \downarrow S \cong k^r \oplus (kS)^s$ for non-negative integers $r,s$.

Theorem 3.10 (Classification of "Sylow-semi-simple" modules). Restriction followed by discarding projective summands induces a bijection from isomorphism classes of Sylow-semi-simple $kG$-modules to $k\pi_1(\mathcal{O}_p^*(G))$-modules with trivial action of $\ker(N_G(S) \to \pi_1(\mathcal{O}_p^*(G)))$.

In this bijection $M \mapsto N$ with $M \downarrow N_G(S) \cong N \oplus (\text{proj})$, indecomposable modules correspond to simple modules, and it restricts to the bijection between Sylow-trivial modules and 1-dimensional $k\pi_1(\mathcal{O}_p^*(G))$-modules of Theorem A. The bijection is a restriction of the Green correspondence.

The forward map in the bijection identifies with the functor $\Phi$ sending a $kG$-module $M$ to the $k\pi_1(\mathcal{O}_p^*(G))$-module corresponding to the $kG$-module $M$ and restriction to the connected groupoid of $k$-vector spaces and isomorphisms, given by $G/P \mapsto H^0(P; M)$, as in Theorem A.

The inverse map is described as assigning to a $k\pi_1(\mathcal{O}_p^*(G))$-module $N$, the element $C_*(\mathcal{S}_p(G); N) \in D^b(kG)/D^{\text{perf}}(kG)$, with coefficient system on $\mathcal{S}_p(G)$ as in Theorem A, and identifying this with a unique $kG$-module $M$ without projective summands, via the Buchweitz–Rickard equivalence of 2.5.

Proof. The argument is a small modification of the proof of Theorem A. Suppose that $M$ is a $kG$-module as above. Then it is clear that $\Phi$, via the functor $G/P \mapsto H^0(P; M)$, produces a $k\pi_1(\mathcal{O}_p^*(G))$-module, which when inflated along $N_G(S) \to \pi_1(\mathcal{O}_p^*(G))$ agrees with the module $\mathcal{C}_*(\mathcal{S}_p(G); N)$ of the decomposition $M \downarrow N_G(S) \cong N \oplus P$. By the Green correspondence [Ben91a, Thm. 3.12.2] there is a bijection between indecomposable $kG$-modules with vertex $S$ and simple $N_G(S)/S$-modules, given by restriction and disposing summands not with vertex $S$. In particular the map in the theorem is injective and it will follow that indecomposable modules correspond to simple modules, once we have seen that it is surjective.

For surjectivity, suppose $N$ is a $k\pi_1(\mathcal{O}_p^*(G))$-module, and let $M$ be the $kG$-module without projective summands corresponding to $\mathcal{C}_*(\mathcal{S}_p(G); N)$. Observe that the argument given in the first half of Proposition 3.5 still gives equivalences in $\mathcal{D}_p(S)$:
\[
N \leftarrow \mathcal{C}_*(\mathcal{S}_p(G); S; N) \rightarrow \mathcal{C}_*(\mathcal{S}_p(G); N)
\]
which again implies that $N$ and $M \downarrow N_G(S)$ are isomorphic after throwing away projective summands, by the Buchweitz–Rickard equivalence 2.5. \qed

Remark 3.11. Subsequent sections describe many ways of computing $\ker(N_G(S) \to \pi_1(\mathcal{O}_p^*(G)))$; in particular Theorem 1.9 gives a group theoretic description in terms of generators and relations.

Remark 3.12. As noted, the map in Theorem 3.10 is a restriction of the Green correspondence [Ben91a, Thm. 3.12.2] which provides a bijection $M \mapsto N$ between indecomposable $kG$-modules $M$ that split $M \downarrow N_G(S) \cong N \oplus N'$ where $N$ is simple and $N'$ is a sum of indecomposable modules with vertex a proper subgroup of $S$, and all simple $kN_G(S)/S$-modules.

Remark 3.13. Since the restriction map preserves direct sum and tensor product, Theorem 3.10 in fact gives an isomorphism of semi-rings between Sylow-semi-simple modules without projective
summands and finitely generated \( \pi_1(\mathcal{O}_p^*(G)) \)-modules, where the tensor product on Sylow-semi-simple modules means tensoring and discarding projective summands. Sylow-trivial modules constitute the units in the semi-ring of Sylow-semi-simple modules.

**Corollary 3.14.** Suppose \( M \) is a \( kG \)-modules that arises from the correspondence of Theorem \( \ref{thm:correspondence} \), with \( N \) an absolutely simple \( k\mathcal{O}_p^*(G) \)-module. Then \( \text{End}(M, M) \cong k \oplus M' \) as \( kG \)-modules, where \( M' \) is a \( kG \)-module without \( k \) in its socle.

**Proof.** If \( N \) is absolutely simple, its dimension divides \( |\pi_1(\mathcal{O}_p^*(G))| \) (see \cite{Ser79} §6.5 Cor. 2), and is in particular prime to \( p \), since \( \pi_1(\mathcal{O}_p^*(G)) \) is a \( p' \)-group by \( \text{(I.3)} \). Hence the dimension of \( M \) is prime to \( p \), since the dimension of \( M \) is congruent to the dimension of \( N \) modulo \( |S| \), by Theorem \( \ref{thm:correspondence} \). In particular \( k \to \text{End}(M) \xrightarrow{\text{Tr}_G} k \) is an isomorphism, so \( k \) splits off \( \text{End}(M) \). To see that \( k \) is not in the socle of \( M' \) note that

\[
\text{Hom}_G(M, M) \subseteq \text{Hom}_{\mathcal{O}_p^*(S)}(M, M) \cong \text{Hom}_{\mathcal{O}_p^*(S)}(N, N) \cong \text{Hom}_{\pi_1(\mathcal{O}_p^*(G))}(N, N) \cong k
\]

since \( N \) is absolutely simple. \( \square \)

**Remark 3.15.** As pointed out to us by Radha Kessar, modules as in Corollary \( \ref{cor:correspondence} \) are interesting since they are candidates for the image of simple modules under self-equivalences of the stable module category. Carlson proved in \cite{Car98} that for \( p \)-groups, modules satisfying \( \text{Hom}_G(N, N) \cong k \) are in fact endotrivial, but for general finite groups the class is bigger, e.g., it contains all simple modules. Its size in general, and the precise image given via Theorem \( \ref{thm:correspondence} \) is at present unclear.

**Remark 3.16** (More general modules from \( p \)-local information). The process of constructing modules from \( p \)-local information via a homotopy left Kan extension as in Theorems \( \ref{thm:homotopy-kan-extension} \) and \( \ref{thm:correspondence} \) should be of interest also for more general families of modules on \( p \)-local subgroup \( N_G(P) \), even when they are not translates of a fixed module on \( N_G(S) \); or, said differently, when one considers more general \( G \)-local coefficient systems on \( |S_p(G)| \), as explained in \( \text{(2.4)} \). For instance it would be worthwhile to understand the work of Wheeler \cite{Whe02} from this point of view (see also \cite{Mat16}). Fundamental counting conjectures in representation theory, such as Alperin’s \cite{Alp87}, predict a relationship between \( kG \)-modules and modules on normalizers of \( p \)-subgroups, and it is not unnatural to expect that the glue between these subgroups provided by the transport and orbit categories should play a role in categorifying, and potentially proving, these conjectures.

**Remark 3.17** (Discrete valuation rings). Note that our model for Sylow-trivial modules can also be lifted to characteristic 0: Suppose that \( (K, R, k) \) is a \( p \)-modular system, with \( R \) a complete rank one discrete valuation ring with residue field \( k \) of characteristic \( p \) \cite{Ben91a} §1.9. Then reduction modulo \( k \) provides an isomorphism on finite roots of unity in \( R \) and \( k \) by Hensel’s lemma. Hence \( H^1(\mathcal{O}_p^*(G); R^\times) \xrightarrow{\text{res}} H^1(\mathcal{O}_p^*(G); k^\times) \) so we can uniquely define the twisted Steinberg complex \( C_*(|S_p(G)|; R_\varphi) \) over \( R \). By the \( RG \)-lattice version of the Buchweitz–Rickard theorem \cite{Pon} Prop. 3.4 it defines an object in the stable module category of \( RG \)-lattices, lifting the \( kG \)-module corresponding to \( \varphi \in H^1(\mathcal{O}_p^*(G); k^\times) \). This is helpful since we can use the lift to exist by virtue of Sylow-trivial modules being trivial source \cite{Ben91a} Cor. 3.11.4(i)].

**Remark 3.18** (The structure of \( T_k(G) \)). For the group of endotrivial modules \( T_k(G) \) in general, we have an exact sequence

\[
0 \to T_k(G, S) \to T_k(G) \xrightarrow{\text{res}^G_L} L \to 0
\]

(3.3) with \( L = \text{im}(T_k(G) \to T_k(S)) \). The torsion-free rank of \( L \) has been determined in \cite{CMN06} §3, extending the work of Alperin \cite{Alp01}. By classification of endotrivial modules for finite \( p \)-groups, the torsion-free rank of \( L \) in general, we can describe explicitly by a case-by-case analysis carried out in \cite{MT07, CMT13} and it turns out that in all those cases the torsion part of \( L \) equals that of \( T_k(S) \). When \( S \) is a semi-dihedral or quaternionic \( 2 \)-group the restriction is furthermore split by \cite{CMT13} Thms. 6.4 and 4.5. When \( S \) is cyclic, \( T_k(\mathbb{Z}/2) = 0 \), and for \( |S| > 2 \), \( T_k(S) \cong \mathbb{Z}/2 \); the 4-fold periodic resolution of the trivial \( \mathbb{F}_3\mathfrak{S}_3 \)-module shows that the restriction is not always split, but the structure of the extension above can in all cases be described explicitly; see \cite{MT07} Thm. 3.2 and Lem. 3.5].
We remark that while $L$ is known as an abstract abelian group for any finite group $G$, explicit generators for the torsion-free part are not; the (then) current state of affairs is summarized in [CMT14]. In light of the present work, it is natural to wonder if the whole group of endotrivial modules $T_k(G)$ admits an orbit or transport-category description extending Theorem [A], and one may ask how the sequence (3.3) relates to the canonical exact sequence

$$0 \to H^1(\mathcal{O}_p^s(G); k^\times) \to T_k(G) \to \lim_{G/P \in \mathcal{O}_p(G)} T_k(P)$$

(3.4)

where by construction $L \subseteq \lim_{G/P \in \mathcal{O}_p(G)} T_k(P)$. Answering this question, describing when elements in the limit lie in $L$, is the subject of ongoing joint work with Tobias Barthel and Joshua Hunt.

### 3.1. Addendum: $A_k(G, S) \cong H^1(\mathcal{O}_p^s(G); k^\times)$

The purpose of this addendum is to relate Balmer’s notion of a weak homomorphism to $H^1(\mathcal{O}_p^s(G); k^\times)$, providing a different proof of Theorem [A]. This version is less direct since it uses the main result of [Bal13], where he identifies $T_k(G, S)$ with a group he calls $A_k(G, S)$ of weak $S$–homomorphisms from $G$ to $k^\times$, but may nevertheless be instructive for readers familiar with that work; it would be interesting to play off the construction of endotrivial modules in Theorem [A] with [Bal13] Constr. 2.5 and Thm. 2.9.

**Proposition 3.19.** For any finite group and field $k$, $A_k(G, S) \cong H^1(\mathcal{O}_p^s(G); k^\times)$, where $A_k(G, S)$ is the group of weak $S$–homomorphisms from $G$ to $k^\times$ of [Bal13] Def. 2.2.

**Proof.** A weak $S$–homomorphism is a map from $G$ to $k^\times$ such that (WH1) $\varphi(g) = 1$ for $g \in S$, (WH2) $\varphi(g) = 1$ when $S \cap S^g = 1$, and (WH3) $\varphi(g)\varphi(h) = \varphi(gh)$ when $S \cap S^g \cap S^{gh} \neq 1$, where as usual $H^g = g^{-1}Hg$. What we shall do here is to prove that $A_k(G, S)$ identifies with $\text{Hom}(\pi_1(\mathcal{O}_p^s(G)), k^\times)$, by observing that there are canonical group homomorphisms in both directions, that we check are well defined and inverses to each other.

Up to equivalence of categories (which does not affect the conclusion of the theorem) we can replace $\mathcal{O}_p^s(G)$ by the equivalent full subcategory with objects $G/P$ for $1 < P \leq S$, for our fixed Sylow $p$–subgroup $S$. We will use this category from now on. Recall furthermore the bijection

$$\text{Hom}(\mathcal{O}_p^s(G), k^\times) \cong \{g \in G | P^g \leq Q\}/Q$$

described in [2.2]. Recall also from [2.3] that we have an isomorphism of functors

$$\text{Rep}(\mathcal{O}_p^s(G), k^\times) \xrightarrow{\sim} H^1(\mathcal{O}_p^s(G); k^\times)$$

where $\text{Rep}$ means isomorphism classes of functions, which is an isomorphism of abelian groups, where the abelian group structure on the left is pointwise multiplication in the target.

We hence are just left with verifying that isomorphism classes of functors $\mathcal{O}_p^s(G) \to k^\times$ agree with the group $A_k(G, S)$ that Balmer introduced: Given $\varphi \in A_k(G, S)$ define $\Phi : \mathcal{O}_p^s(G) \to k^\times$ by sending a morphism $G/Q \xrightarrow{\phi} G/Q'$ to $\varphi(g)$. This is well defined, since replacing $g$ by $gg'$, for $q \in Q$, yields $\varphi(gg) = \varphi(g)\varphi(g') = \varphi(g)$ by (WH3) and (WH1). It is likewise a functor: By (WH1), $\Phi([\text{id}_{G/Q}]) = \varphi(1) = 1$ and given a composite $G/Q \xrightarrow{\phi} G/Q' \xrightarrow{h} G/Q''$ we have $Q^g \leq Q' \leq S$ and $(Q')^h \leq Q'' \leq S$, so

$$S^g \cap S^h \cap S = (S^g \cap S)^h \cap S \geq (Q^g)^h \cap S = (Q^g)^h > 1$$

Hence by (WH3), writing composition in categories from right to left,

$$\Phi([h] \circ [g]) = \Phi([gh]) = \varphi(gh) = \varphi(g)\varphi(h) = \varphi(h)\varphi(g) = \Phi([h])\Phi([g])$$

as wanted. Conversely given a functor $\Phi : \mathcal{O}_p^s(G) \to k^\times$ construct a weak homomorphism $\varphi : G \to k^\times$ as follows: Set

$$\varphi(g) = \begin{cases} \Phi(G/(S^g \cap S)) & \text{if } S \cap S^g \neq 1 \\ 1 & \text{otherwise} \end{cases}$$

It is clear that (WH1) and (WH2) are satisfied. For (WH3) recall that, e.g., by the discussion in [2.3] quotients $G/Q \to G/Q'$ for $Q \leq Q'$ are sent to the identity in $k^\times$. Now suppose that $S \cap S^g \cap S^h \neq 1$
and consider the diagram

\[
\begin{array}{ccc}
G/(ghS \cap gS \cap S) & \xrightarrow{[gh]} & G/(S \cap S^h \cap S^{gh}) \\
\downarrow{[g]} & & \downarrow{[h]} \\
G/(hS \cap S \cap S^g) & & \\
\end{array}
\]

where the top map is the quotient of \(G/(ghS \cap S) \xrightarrow{[gh]} G/(S \cap S^h \cap S^{gh})\), and similarly for the two other maps. Hence applying \(\Phi(-)\) to this diagram, and using that quotients go to the identity we see that \(\varphi(gh) = \Phi([gh]) = \Phi(h)\Phi(g) = \varphi(h)\varphi(g) = \varphi(g)\varphi(h)\) as wanted.

We have hence constructed maps back and forth between \(A_k(G,S)\) and \(\text{Rep}(\Theta^+_{\pi}(G), k^\times)\) which are obviously group homomorphisms, under pointwise multiplication in the target, and inverses to each other, as wanted. \(\Box\)

**Remark 3.20.** Another perspective on “weak homomorphisms” can be given by showing that they correspond to morphisms of partial groups in the sense of Chermak [Che13] from a locality of \(G\) based on all non-trivial subgroups \(p\)-subgroups to \(k^\times\).

### 4. Fundamental groups of orbit and fusion categories

In this section we describe how to calculate and manipulate our basic invariants \(\pi_1(\Theta^+_{\pi}(G))\) and \(\pi_1(\mathcal{F}^+_{\pi}(G))\), and related groups. In §4.1 we establish basic properties of \(\pi_1(\Theta^+_{\pi}(G))\) and establish Theorems B and C in §4.2 we expand on its properties and in §4.3 we carry out a similar analysis for \(\pi_1(\mathcal{F}^+_{\pi}(G))\). In §4.4 we look at higher homotopy groups—these occur naturally in the analysis, even if one is ultimately only interested \(\pi_1\). Some results are stated for an arbitrary collection of \(p\)-groups \(\mathcal{C}\), appealing to the appendix §7 to get minimal hypothesis on \(\mathcal{C}\)—the reader may take \(\mathcal{C} = \mathcal{S}_p(G)\) at first reading. We refer to §7 for much additional information.

The categories discussed were introduced in [2.2]. Also recall that \(H_{p'}\) means the quotient of \(H\) by the subgroup generated by elements of \(p\)-power order (so \(H_{p'} = H/O_{p'}(H)\) when \(H\) is finite).

#### 4.1. Fundamental groups of orbit categories: proof of Theorems B and C

For a fixed Sylow \(p\)-subgroup \(S\) and a collection \(\mathcal{C}\), set \(G_{0,\mathcal{C}} = \langle N_G(Q) | Q \leq S, Q \in \mathcal{C} \rangle\) and \(G_0 = \{Q \in \mathcal{C} | Q \leq G_{0,\mathcal{C}}\}\); when \(\mathcal{C} = \mathcal{S}_p(G)\), \(G_{0,\mathcal{C}} = G_0\) of (1.2). Also recall that a \(p\)-subgroup is said to be \(p\)-essential if \(\mathcal{S}_p(N_G(P)/P)\) is disconnected (hence non-empty); see [1.2] for an elaboration of these definitions.

**Proposition 4.1** (Bounds on \(\pi_1(\Theta_{\mathcal{C}}(G))\)). Let \(\mathcal{C}\) be a collection of \(p\)-subgroups closed under passage to \(p\)-essential and Sylow overgroups. Then \(\Theta_{\mathcal{C}}(G)\) and \(\Theta_{\mathcal{C}_0}(G_{0,\mathcal{C}})\), as well as \(\mathcal{R}_{\mathcal{C}}(G)\) and \(\mathcal{R}_{\mathcal{C}_0}(G_{0,\mathcal{C}})\), are equivalent categories, all connected, and we have a sequence of surjections

\[N_G(S)/S \twoheadrightarrow \pi_1(\Theta_{\mathcal{C}}(G)) \twoheadrightarrow (G_{0,\mathcal{C}})_{p'} \twoheadrightarrow G_{p'}\]

In particular \(\pi_1(\Theta_{\mathcal{C}}(G))\) is a finite \(p'\)-group.

**Proof.** The categories are connected since \(S \in \mathcal{C}\). That \(\Theta_{\mathcal{C}}(G)\) and \(\Theta_{\mathcal{C}_0}(G_{0,\mathcal{C}})\) are equivalent categories follows from Alperin’s fusion theorem: By Sylow’s theorem they are both equivalent to their subcategories with objects \(G/Q\) for \(Q \leq S\) and \(Q \in \mathcal{C}\), for some fixed Sylow \(p\)-subgroup \(S\); Now Alperin’s fusion theorem [Alp67, §3], in the version of Goldschmidt–Miyamoto–Puig [My77 Cor. 1], says that any conjugation \(G/P \xrightarrow{g} G/Q\) with \(P, Q \leq S\) we can write \(g = g_1 \cdots g_n\) where \(g_i \in N_G(P_i)\) with \(P_i \leq S\) \(p\)-essential and \(n \in N_G(S)\). In particular \(g \in G_{0,\mathcal{C}}\). Hence the two subcategories are isomorphic. The same argument applies verbatim to \(\mathcal{R}\). (See also [Gro02, §10] for info on versions of the fusion theorem.)

Furthermore, by the Frattini argument [Gor68 Thm. I.3.7] we have surjections \(N_G(S)/S \twoheadrightarrow (G_{0,\mathcal{C}})_{p'}\) and \(N_G(S)/S \twoheadrightarrow G_{p'}\), so we have established the proposition if we show the surjection \(N_G(S)/S \twoheadrightarrow \pi_1(\Theta_{\mathcal{C}}(G))\). For this, first note that by Theorem 7.11(1) we can without restriction assume that \(\mathcal{C}\) is closed under passage to all \(p\)-subgroups, not just \(p\)-essential and Sylow subgroups. Next, recall the model for \(\pi_1(\Theta_{\mathcal{C}}(G))\) of [2.3]. Take \(G/S\) as basepoint and consider the functor \(\Theta_{\mathcal{C}}(G) \twoheadrightarrow \pi_1(\Theta_{\mathcal{C}}(G))\) given by sending \(G/P \xrightarrow{g} G/Q\), for \(P, Q \leq S\), to the loop given by \(G/S \xrightarrow{g} G/P \xrightarrow{g} G/Q \xrightarrow{g} G/S\).
Inclusions map to the identity and the image of the functor generates \( \pi_1(\mathcal{O}_C(G)) \), since any element can be written as a product of elements of this type. Again by the fusion theorem, \( \pi_1(\mathcal{O}_C(G)) \) is generated by \( N_G(P)/P \) for \( P \leq S, \: P \in \mathcal{C} \) (compare also \cite{BLO03d} Pf. of Prop. 1.12). Furthermore, the fusion theorem in Alperin’s version \cite{Alp67} §3, says that any conjugation \( G/Q \xrightarrow{\theta} G/R \) can be obtained as a sequence of conjugations by \( p \)-power elements in \( N_G(P) \) for \( p \)-groups containing a conjugate of \( Q \), and an element in \( N_G(S) \). However, any element \( x \) in \( N_G(P)/P \) of \( p \)-power order is trivial in the fundamental group, since it will be conjugate to an element in \( S \), which is zero: To see this explicitly, pick \( g \) which conjugates \( \langle x, P \rangle \) into \( S \), then we can consider the diagram

\[
\begin{array}{ccc}
G/P & \xrightarrow{g^{-1}} & G/P \\
\downarrow \hspace{1cm} x & & \hspace{1cm} \downarrow \hspace{1cm} g \cdot x \cdot g^{-1} \\
G/P & \xrightarrow{g^{-1}} & G/P \\
\end{array}
\]

which commutes since \( g \cdot x \cdot g^{-1} \in S \), hence showing that \( G/P \xrightarrow{x} G/P \) maps to the identity in \( \pi_1(\mathcal{O}_C(G)) \). This shows that \( N_G(S)/S \xrightarrow{x} \pi_1(\mathcal{O}_C(G)) \) is surjective as wanted. \( \square \)

Remark 4.2. For \( \mathcal{C} = \mathcal{A}_2(G) \) and \( G = \mathbb{Z}/4, \: \mathcal{O}_C(G) \cong B\mathbb{Z}/2, \) so \( \pi_1(\mathcal{O}_C(G)) \) is not a \( 2' \)-group.

Remark 4.3. The maps in Proposition 4.1 are natural in the collection, so if \( \mathcal{C}' \leq \mathcal{C} \) we have a surjection \( \pi_1(\mathcal{O}_C'(G)) \rightarrow \pi_1(\mathcal{O}_C(G)) \), introducing a natural filtration on \( \pi_1(\mathcal{O}_C'(G)) \).

Remark 4.4. For \( \mathcal{C} \) a collection of \( p \)-subgroups of \( G \), closed under passage to \( p \)-overgroups, and \( N_G(S) \leq H \leq G \), then by Proposition 4.1 we have surjections

\[
N_G(S)/S \rightarrow \pi_1(\mathcal{O}_C(H)) \rightarrow \pi_1(\mathcal{O}_C(G))
\]

for \( \mathcal{C}' \) the elements of \( \mathcal{C} \) that are subgroups of \( H \). Via Theorem A this can be seen as a refinement of the fact that restriction to \( H \) is injective on Sylow-trivial modules, as is usually seen via the Green correspondence \cite[Prop. 2.6(a)]{CMN06}.

Proposition 4.5 (Quotienting out by \( p \)-torsion). For any collection \( \mathcal{C} \) of \( p \)-subgroups, and any basepoint \( P \in \mathcal{C} \),

\[
\pi_1(\mathcal{F}_C(G))_{\mathcal{C}}' \xrightarrow{\cong} \pi_1(\mathcal{O}_C(G))_{\mathcal{C}}' \quad \text{and} \quad \pi_1(\mathcal{F}_C(G))_{\mathcal{C}}' \xrightarrow{\cong} \pi_1(\mathcal{O}_C'(G))_{\mathcal{C}}'
\]

Proof. Note that we include the basepoint \( P \) in the formulation, since without further assumptions on \( \mathcal{C} \) the categories could be disconnected (though this is never the case for the \( \mathcal{C} \) we are interested in); see \S2.3 for more detail. We first prove \( \pi_1(\mathcal{F}_C(G))_{\mathcal{C}}' \xrightarrow{\cong} \pi_1(\mathcal{O}_C(G))_{\mathcal{C}}' \). Note that \( \mathcal{F}_C \) and \( \mathcal{O}_C \) have the same path components, since the quotient functor is a bijection on objects and a surjection on morphisms. Choose for each \( Q \in \mathcal{C} \) which lie in the same path component as \( P \), a preferred path in \( \mathcal{F}_C \) from \( Q \) to \( P \), as explained in \S2.3 which induces a corresponding path in \( \mathcal{O}_C \). Now, the morphisms in \( \mathcal{F}_C(G) \) surject onto the morphisms of \( \mathcal{O}_C(G) \), and if two morphisms in \( \mathcal{F}_C(G) \) are mapped to the same morphism in \( \mathcal{O}_C(G) \), then they differ by an automorphism of \( p \)-power order. Since the morphisms in the category generate the fundamental group, we conclude that \( \pi_1(\mathcal{F}_C(G))_{\mathcal{C}}' \xrightarrow{\cong} \pi_1(\mathcal{O}_C(G))_{\mathcal{C}}' \) as wanted. The case \( \pi_1(\mathcal{F}_C(G))_{\mathcal{C}}' \xrightarrow{\cong} \pi_1(\mathcal{O}_C'(G))_{\mathcal{C}}' \) is identical. \( \square \)

We have now justified all the ingredients in the theorems from the introduction \S1.1

Proof of Theorem B and Corollary C. These two theorems were already established in \S1.1 using Theorem A with forward references to two facts, Propositions 4.1 and 4.5 now justified. \( \square \)

Remark 4.6. The boundary map \( \partial \) in Theorem B pairs an element in \( H^1(S_p(G_0); k^\times)G_0 \) with the extension class in \( H^2(G_0; H_1(S_p(G_0))) \) of \( \langle 1 \rangle \), producing an element in \( H^2(G_0; k^\times) \); see e.g., \cite[Thm. 4]{HS53} or [\cite{Exe91} Thm. 7.3.1]. Dually for the sequence (1.11). The nature of this extension class may deserve closer study.

Remark 4.7. Propositions 4.1 and 4.5 should be compared to the situation at the prime \( p \), where \( H_*(\mathcal{F}_p(G))_{(p)} \xrightarrow{\cong} H_*(G)_{(p)} \) by a classical result of Brown \cite{Bro94} Thm. X.7.3, and the theory of ‘ample collections’ described for which \( \mathcal{C} \) this continues to hold; see \cite[1.3]{Dwy97} and \cite[§9]{Gro02}. In particular we see that \( H_1(\mathcal{F}_p(G)) \xrightarrow{\cong} H_1(G) \) if and only if \( H_1(\mathcal{O}_p'(G)) \xrightarrow{\cong} H_1(G)_{(p)} \).
Remark 4.8 (Equivariant complex line bundles on $S_p(G)$). Via a remark of Totaro [Bal18, Rem. 2.7], Corollary [C] easily implies a very recent theorem of Balmer [Bal18 Thm. 1.1], that identifies $T_k(G, S)$ with the $p'$-torsion part of the group of $G$-equivariant complex line bundles on $|S_p(G)|$, under the assumption that $k$ is algebraically closed. Namely, in this case we have an embedding $\text{Tors}_{p'}(\mathbb{Q}/\mathbb{Z}) \cong \mu_{\infty}(k) \subseteq k^\times$, where $\text{Tors}_{p'}$ means the subgroup of elements of finite order prime to $p$, and $\mu_{\infty}(k)$ are the units of finite order. Hence we have isomorphisms

$$\text{Tors}_{p'}(H^2(|S_p(G)|_{hG}; \mathbb{Z})) \cong H^1(|S_p(G)|_{hG}; \mathbb{Q}/\mathbb{Z}) \cong H^1(|S_p(G)|_{hG}; k^\times)$$

where the first is induced by the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ and the second uses that $H^1(|S_p(G)|_{hG}; k^\times)$ is finite, e.g., by \cite{1.5}. But now the left-hand term identifies with the $p'$-torsion part of the $G$-equivariant complex line bundles on $|S_p(G)|$, as remarked by Totaro [Bal18 Rem. 2.7], and the right-hand side identifies with $T_k(G, S)$ by Corollary [C]. More generally, since Remark 4.7 shows that $H^2(|S_p(G)|_{hG}; \mathbb{Z})$ is a finite abelian group with $p'$-torsion part $H^2(G)_{(p)}$, we can describe all $G$-equivariant complex line bundles on $|S_p(G)|$ as

$$\text{Pic}^G(|S_p(G)|) \cong H^2(G)_{(p)} \oplus T_k(G, S). \quad (4.1)$$

Here $k$ should in fact just be large enough so that the one dimensional representations of $\pi_1(\mathcal{O}_p^*(G))$ does not depend on $k$, e.g., containing all $|N_G(S) : S|$th roots of unity.

4.2. Fundamental groups of orbit categories: further structural results. The structure of the group $\pi_1(\mathcal{O}_p^*(G))$ can be described in purely group theoretic description.

Theorem 4.9 (Group theoretic description of $\pi_1(\mathcal{O}_p^*(G))$). Let $K$ be the subgroup of $N_G(S)$ generated by elements $g \in N_G(S)$ such that there exists nontrivial subgroups $1 < Q_0, \ldots, Q_r \leq S$ and a factorization $g = x_1 \cdots x_r$ in $G$, where $x_i \in O^p(G(Q_i))$ and $Q_0^x \cdots x_i \leq Q_{i+1}$ for $i \geq 0$. Then

$$N_G(S)/K \xrightarrow{\sim} \pi_1(\mathcal{O}_p^*(G))$$

In particular $N_G(S) \cap O^p(G(P)) \leq \ker (N_G(S) \to \pi_1(\mathcal{O}_p^*(G)))$ for $1 < P \leq S$.

Proof. First note that it is clear that any such element $g$ goes to zero under the canonical map $N_G(S) \to \pi_1(\mathcal{O}_p^*(G))$. Namely in $\pi_1(\mathcal{O}_p^*(G))$ we can decompose $g = x_1 \cdots x_r$, where $x_i$ is viewed as an element in $N_G(Q_i)/Q_i$, and suppressing inclusions from the notation. But $x_i$ is zero in $\pi_1(\mathcal{O}_p^*(G))$ as it is a product of $p'$-power elements, and $\pi_1(\mathcal{O}_p^*(G))$ is a $p'$-group by Proposition 4.1. Hence $K \leq \ker (N_G(S) \to \pi_1(\mathcal{O}_p^*(G)))$ and the ‘in particular’ also follows.

We are left with seeing that $K$ is all of the kernel. We will do this, by using Alperin’s fusion theorem to construct a well-defined functor $\mathcal{O}_p^*(G) \to N_G(S)/K$; the universal property of the fundamental group (see \cite{2.3}) hence produces a homomorphism $\pi_1(\mathcal{O}_p^*(G)) \to N_G(S)/K$, which will clearly be an inverse. Recall that $\mathcal{O}_p^*$ is equivalent to a category with objects $G/P$ for $P \leq S$, and consider $G/P \xrightarrow{\theta} G/P^p$, a morphism in $\mathcal{O}_p^*(G)$, with $P, P^p \leq S$; where $g$ is only well-defined as a representative of $gP^p$. Now, by Alperin’s fusion theorem [Alp67, \S 3], we can factorize $g$ as $g = x_1 \cdots x_r n$ where $P = Q_0, x_i \in N_G(Q_i)$ is an element of $p'$-power order, and $Q_0^{x_i} \cdots x_i^{x_i} \leq Q_{i+1}$ for $i \geq 0$, and $n \in N_G(S)$. We claim that the map sending $g \to n$ on isomorphisms, and sends inclusions to the identity gives a well defined functor $\pi_1(\mathcal{O}_p^*(G)) \to N_G(S)/K$. If $g = x_1 \cdots x_r n = y_1 \cdots y_m$ then $mn^{-1} = y_1^{-1} \cdots y_r^{-1} x_1 \cdots x_r$, where the right-hand side lies in $K$, so $n$ equals $m$ in the quotient; also note that changing $g$ by a different coset representative will not change this image. Likewise the map is a functor since if we have a composite $G/P \xrightarrow{\theta} G/P^p \xrightarrow{h} G/P^h$ with $g = x_1 \cdots x_r n$ and $h = y_1 \cdots y_m$, then $gh = x_1 \cdots x_r n y_1 n^{-1} y_2 \cdots y_m n^{-1} y_m n^{-1} nm$ and hence is sent to $nm$ as wanted. By the universal property of the fundamental group we hence get an induced group homomorphism $\pi_1(\mathcal{O}_p^*(G)) \to N_G(S)/K$, which is clearly both a left and a right inverse. \qed

Remark 4.10. Let us briefly discuss the assumptions in Theorem 4.9. First note that the relationship between the subgroups $Q_i$ is an important part of the statement, i.e., we cannot just consider the bigger subgroup $N_G(S) \cap (O^p(G(P)))$ as examples such as $G_7$ at $p = 3$ show. Second, one could ask if the elements $x_i$ could be assumed to lie in $N_G(S)$, i.e., if the subgroups of the ‘in particular’ generate the kernel. This often holds but fails e.g., for $G_2(5)$ at $p = 3$, as we shall analyze in connection
with the Carlson–Thévenaz conjecture; it is a translate of the fact that \( \rho_2 \neq \rho_3 \). Finally we remark that one cannot just assume that all \( Q_i \) are \( p \)-essential, i.e., \( N_G(Q_i) \) cannot be replaced by \( O^p(N_G(Q_i)) \) in the Goldschmidt–Miyamoto–Puig \cite[Cor. 1]{Miy77} version of the fusion theorem; \( G_2(5) \) at \( p = 3 \) is again a counterexample by Example \ref{BCG+07}. Which subgroups are needed is analyzed in detail in \cite{FO} in terms of homotopy properties of the collection \( \mathcal{C} \).

**Remark 4.11.** The subcategories of \( \mathcal{F}_p(G) \) and \( \mathcal{O}^*_{p}(G) \) obtained as preimages of subgroups of \( \pi_1(\mathcal{O}^*_{p}(G)) \) can be thought of as subcategories “of \( p' \)-index analogous to the results in \cite[\S 5]{BCG+07} on fusion system and linking systems, and the above proof may be compared to \cite[Pt. of Prop. 5.2]{BCG+07}. Furthermore \( \mathcal{F}_p(G) \) is an example of an abstract transporter system as defined in \cite[Def. 3.1]{OV07}, so in light of \cite[1.10]{LM15}, it would be interesting to further understand the group theoretic significance of these subcategories.

Let us describe the relationship between \( \pi_1(\mathcal{O}^*_{p}(G)) \) and \( G_{p'} = G/O^p(G) \) in simple cases.

**Corollary 4.12.** If \( |\mathcal{S}_p(G)| \) is simply connected then \( \pi_1(\mathcal{F}_p(G)) \xrightarrow{\simeq} G \) and \( \pi_1(\mathcal{O}^*_{p}(G)) \xrightarrow{\simeq} G_{p'} \). If just \( |\mathcal{S}_p(G_0)| \) is simply connected, \( \pi_1(\mathcal{F}_p(G_0)) \xrightarrow{\simeq} G_{p'} \) and \( \pi_1(\mathcal{O}^*_{p}(G_0)) \xrightarrow{\simeq} (G_0)_{p'} \).

**Proof.** By \cite{1.9} we have an exact sequence \( 1 \to \pi_1(\mathcal{S}_p(G_0)) \xrightarrow{\simeq} \pi_1(\mathcal{F}_p(G)) \to G \to 1 \). This gives the statements about \( \mathcal{F}_p(G) \) and \( \mathcal{F}_p(G_0) \) and the statements about \( \pi_1(\mathcal{O}^*_{p}(G)) \) and \( \pi_1(\mathcal{O}^*_{p}(G_0)) \) now follows using Propositions \ref{4.1} and \ref{4.5}. \( \Box \)

The next corollary recovers and extend “classical” facts about \( T_k(G, S) \), by Carlson, Mazza, Nakano, and Thévenaz, via Theorem A (See Remark \ref{4.14} below for some historical references.)

**Corollary 4.13** (Basic calculations of \( T_k(G, S) \) via \( \pi_1(\mathcal{O}^*_{p}(G)) \)).

1. If for all \( g \in G \), \( S \cap S^g \neq 1 \) (e.g., if \( O_p(G) \neq 1 \)) then \( \pi_1(\mathcal{O}^*_{p}(G)) \xrightarrow{\simeq} G_{p'} \) and hence \( T_k(G, S) \cong \text{Hom}(G, k^\times) \).

2. If \( S \) has \( p \)-rank one, then \( \pi_1(\mathcal{O}^*_{p}(G)) \xrightarrow{\simeq} (G_0)_{p'} \), with \( G_0 = N_G(pZ(S)) \), and hence \( T_k(G, S) \cong \text{Hom}(N_G(pZ(S)), k^\times) \), with \( pZ(S) \) the elements of order at most \( p \) in \( Z(S) \).

3. If \( G \) is a trivial intersection (T.I.) group, then \( G_0 = N_G(S) \), \( \pi_1(\mathcal{O}^*_{p}(G)) \cong N_G(S)/S \), and hence \( T_k(G, S) \cong \text{Hom}(N_G(S), k^\times) \).

4. If \( G = H \circ K \), with \( p \parallel |H| \), \( p \parallel |K| \), then \( \pi_1(\mathcal{O}^*_{p}(G)) \xrightarrow{\simeq} G_{p'} \) and \( T_k(G, S) \cong \text{Hom}(G, k^\times) \).

5. If \( G = H \wr \mathfrak{S}_n \) with \( p \parallel |H| \) and \( n \geq 2 \), then \( \pi_1(\mathcal{O}^*_{p}(G)) \xrightarrow{\simeq} G_{p'} \) and \( T_k(G, S) \cong \text{Hom}(G, k^\times) \).

**Proof.** \cite{1}: Suppose \( G/P \xrightarrow{\simeq} G/Q \) goes to \( 1 \in G_{p'} \), then \( g \) can be written as a product in \( G \) of elements of \( p \)-power order. We can hence without loss of generality assume that \( g \) is itself of \( p \)-power order. Further, by changing \( g \) up to conjugacy we can assume that \( Q \leq S \), and it is enough to prove that \( G/S \xrightarrow{\simeq} G/S^g \) is the identity in \( \pi_1(\mathcal{O}^*_{p}(G)) \). However here it factors as \( G/S \twoheadrightarrow G/(S \cap S^g) \twoheadrightarrow G/S^g \), which is the identity in \( \pi_1(\mathcal{O}^*_{p}(G)) \), as \( g \) has \( p \)-power order in \( N_G(S \cap S^g)/S \cap S^g \). \cite{2}: Since \( |\mathcal{S}_p(G)| \) is \( G \)-homotopy equivalent to the collection of non-trivial Sylow-intersections, e.g., by \cite[Thm. 1.1]{GS06}, \( |\mathcal{S}_p(G)| \approx G/N_G(pZ(S)) \) and \( G_0 = N_G(S) \) and the claim follows from \cite{1}. \cite{4}: Write \( S = S' \wr S'' \) for Sylow \( p \)-subgroups \( S' \) and \( S'' \) in \( H \) and \( K \), and notice that \( N_G(S) \cap O^p(G) = N_H(S') \cap O^p(H) \cap N_K(S'') \cap O^p(K) = (N_G(S) \cap O^p(N_G(S'))) \cap (N_G(S) \cap O^p(N_G(S'))) \). Hence the result follows from the “in particular” in Theorem \ref{4.9}. \cite{5}: Suppose \( G = H \wr \mathfrak{S}_n \). Let \( H_i \) denote the \( i \)-th copy of \( H \) and \( S_i = \text{Sylow} \) \( p \)-subgroup, and let \( \Delta \) denote Sylow \( p \)-subgroup of \( H \) embedded diagonally. Since \( \mathfrak{S}_n \leq C_G(\Delta) \), \( O^p(\mathfrak{S}_n) \leq O^p(N_G(\Delta)) \). Likewise since \( H_i \leq C_G(H_j) \) for \( i \neq j \), \( O^p(H_i) \leq O^p(N_G(S_j)) \). But \( N_G(S) \cap O^p(G) = N_G(S) \cap \left( \bigcap_i O^p(H_i) \right) \approx N_G(S) \cap \left( \bigcap_i O^p(H_i) \right) \approx (N_G(S) \cap O^p(H_i)) = (N_G(S) \cap O^p(H_i)) \). So the result again follows from the “in particular” in Theorem \ref{4.9}. \( \Box \)

**Remark 4.14.** The \( T_k(G, S) \) consequence in \cite{1} above is \cite[Lem. 2.6]{MT07}. (Note also that if \( O_p(G) \neq 1 \), \( |\mathcal{S}_p(G)| \) is contractible by \cite[Prop. 2.4]{Qui78}, and compare also to Proposition \ref{4.1}.) For the consequence in \cite{2} see \cite[Lem. 3.5]{MT07} and compare also \cite[\S 6]{CT15}. For \cite{3} see \cite[Prop. 2.8 and Rem. 2.9]{CMN06} and also \cite[\S 3.3]{LM15}. The statement in \cite{4} should be compared to the
Corollary 4.15 (Subgroups of \(p'\)–index). If \(H < G\) is of \(p'\) index, there is a diagram with exact rows

\[
\begin{align*}
1 & \longrightarrow \pi_1(\mathcal{O}_p^*(H)) \longrightarrow \pi_1(\mathcal{O}_p^*(G)) \longrightarrow G/H \longrightarrow 1 \\
1 & \longrightarrow H_p' \longrightarrow G_p' \longrightarrow G/H \longrightarrow 1 \\
\end{align*}
\]

In particular \(\pi_1(\mathcal{O}_p^*(H)) \xrightarrow{\sim} H_p'\) if and only if \(\pi_1(\mathcal{O}_p^*(G)) \xrightarrow{\sim} G_p'\). Also, if \(H_1(\mathcal{O}_p^*(H)) \xrightarrow{\sim} H_1(H)_p'\) then \(H_1(\mathcal{O}_p^*(G)) \xrightarrow{\sim} H_1(G)_p'\), so \(T_k(H, S) \cong \text{Hom}(H, k^\times)\) implies \(T_k(G, S) \cong \text{Hom}(G, k^\times)\).

Proof. Note that \(|\mathcal{S}_p(H)| = |\mathcal{S}_p(G)|\), hence taking Borel construction and using (1.6) produces the following diagram, where the rows are fibration sequences.

\[
\begin{align*}
|\mathcal{T}_p(H)| & \longrightarrow |\mathcal{T}_p(G)| \longrightarrow B(G/H) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
B \mathcal{H} & \longrightarrow BG \longrightarrow B(G/H)
\end{align*}
\]

By looking at the associated long exact sequence of homotopy groups, and modding out by elements of finite \(p\)–power order produces the diagram in the lemma. The ‘in particular’ about \(\pi_1\) follows from the 5-lemma, and the homology claim results from the associated 5–term exact sequence

\[
\begin{align*}
H_2(\pi_1(\mathcal{O}_p^*(G))) & \longrightarrow H_2(G/H) \longrightarrow H_1(\mathcal{O}_p^*(H))_{G/H} \longrightarrow H_1(\mathcal{O}_p^*(G)) \longrightarrow H_1(G/H) \longrightarrow 0 \\
H_2(G)_p' & \longrightarrow H_2(G/H) \longrightarrow (H_1(H)_p')_{G/H} \longrightarrow H_1(G)_p' \longrightarrow H_1(G/H) \longrightarrow 0
\end{align*}
\]

The statement about \(T_k(G, S)\) is using Theorem \([A]\). \(\square\)

Remark 4.16. The converse for \(T_k(G, S)\) in Corollary 4.15 fails for \(G = \text{SL}_2(\mathbb{F}_8) \rtimes C_3\), with \(C_3\) acting via the Frobenius, \(H = \text{SL}_2(\mathbb{F}_8)\), and \(p = 2\), where \(\pi_1(\mathcal{O}_2(G)) = C_7 \rtimes C_3\) and \(\pi_1(\mathcal{O}_2^*(H)) = C_7\) (see also Remark 7.17). This shows the advantage of working with the finer invariant \(\pi_1(\mathcal{O}_p^*(G))\), rather than just \(H_1(\pi_1(\mathcal{O}_p^*(G)))\), and the proof shows precisely how the \(H_1\) statement fails.

Corollary 4.15 says that a full calculation of \(\pi_1(\mathcal{O}_p^*(G))\) not only allows the determination of the Sylow-trivial modules for \(G\) but also that of its normal subgroups of \(p'\) index. We illustrate this in \([6.3]\) with the symmetric and alternating groups, in fact correcting a small mistake in the literature.

Corollary 4.17 (Central \(p'\)–extensions). Suppose \(Z\) is a central \(p'\)–subgroup of \(G\). Then there is a diagram of spaces, with horizontal maps fibration sequences

\[
\begin{align*}
& BZ \longrightarrow |\mathcal{T}_p(G)| \longrightarrow |\mathcal{T}_p(G/Z)| \\
& \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
& BZ \longrightarrow BG_0 \longrightarrow BG_0/Z
\end{align*}
\]

and hence a ladder of exact sequences

\[
\begin{align*}
H_2(\mathcal{O}_p^*(G)) & \longrightarrow H_2(\mathcal{O}_p^*(G/Z)) \stackrel{\partial}{\longrightarrow} Z \longrightarrow H_1(\mathcal{O}_p^*(G)) \longrightarrow H_1(\mathcal{O}_p^*(G/Z)) \longrightarrow 0 \\
H_2(G_0)_p' & \longrightarrow H_2(G_0/Z)_p' \stackrel{\partial'}{\longrightarrow} Z \longrightarrow H_1(G_0)_p' \longrightarrow H_1(G_0/Z)_p' \longrightarrow 0
\end{align*}
\]

In particular the sequence \(0 \rightarrow \text{im}(\partial')/\text{im}(\partial) \rightarrow \ker(\varphi) \rightarrow \ker(\bar{\varphi}) \rightarrow 0\) is exact; and if \(H_2(\mathcal{O}_p^*(G/Z)) \rightarrow H_2(G/Z)_p'\) is surjective, e.g., if \(\bigoplus_{[P] \in \mathcal{S}_p(G)/G} H_2(N_{G/Z}(P))_p' \rightarrow H_2(G/Z)_p'\), then \(\ker(\varphi) \xrightarrow{\sim} \ker(\bar{\varphi})\), i.e., \(G\) and \(G/Z\) have isomorphic groups of “very exotic” Sylow-trivial modules.
Proof. Since $Z$ is central, we have a principal fibration sequence

$$BZ \to (|\mathcal{S}_p(G)| \times EG)/G \to (|\mathcal{S}_p(G)| \times E(G/Z))/G$$

with the canonical action of $BZ$ on $(|\mathcal{S}_p(G)| \times EG)/G$; see e.g., [GJ99, V.3]. Furthermore, the projection map $\mathcal{S}_p(G) \xrightarrow{\sim} \mathcal{S}_p(G/Z)$ is a bijection, so the top fibration sequence of the corollary follows from the rewriting in (1.6). The map to the standard bottom fibration sequence follows as $\bar{\mathcal{J}}_p(G)$ is equivalent to $\mathcal{J}(G)$ by Proposition [4.1]. The ladder of exact sequences in now the 5–term exact sequence in homology of a fibration, and using that we can replace $\mathcal{J}_p$ by $\mathcal{O}_p^*$ when considering $p'$–coefficients by Proposition [4.33] and Theorem [4.34].

\[\square\]

Remark 4.18. Note that the kernel of $\varphi$ is described via (1.11), and this can be used for an alternative derivation of the last part of Corollary [4.17].

Example 4.19. Let us illustrate Corollary [4.17] by the symmetric groups: Recall that by homological stability, $\mathbb{F}_2 \cong H_2(\mathfrak{S}_4) \cong H_2(\mathfrak{S}_{n+4})$ for any $n \geq 0$ [Ker05 Thm. 2], so $H_2(N\mathfrak{S}_n,((1 \cdots p))) \to H_2(\mathfrak{S}_n)$ when $n \geq p + 4$. Hence $T_k(2^{\pm} \mathfrak{S}_n, S) \xrightarrow{\sim} T_k(\mathfrak{S}_n, S)$ for $n \geq p + 4$ and $k$ of odd characteristic, by Corollary [4.17] where $2^{\pm} \mathfrak{S}_n$ denotes the two double covers of $\mathfrak{S}_n$, following ATLAS notation. This was first proved in [LM15a Thm. B(2)] (under an algebraically closed assumption), by lifting to characteristic zero and examining the list of possible characters.

Remark 4.20. In [LT17 Thm. 1.1] a reduction of the problem describing Sylow-trivial modules for arbitrary $p'$–extensions to that of central $p'$–extensions, at least as far as obtaining an upper bound on Sylow-trivial modules of the extension. The proof relies, through reference to earlier results, on the classification of finite simple groups. It would be interesting to rework and extend this result in light of the methods of the present paper.

4.3. Fundamental groups of fusion categories. We now analyze $\pi_1(\mathcal{F}_p^*G)$ and related categories, and use this to prove Theorem [11] and set the stage for later calculations involving the centralizer decomposition, Theorem [E]. We can bound $\pi_1(\mathcal{F}_p^*G)$ analogously to (1.3) for $\pi_1(\mathcal{O}_p^*G)$:

$$N_G(S)/C^{p'}_p(N_G(S)) \to \pi_1(\mathcal{F}_p^*G) \to G_0/C^{p'}pG_0 \to G/C^{p'}G$$

(4.3)

where $C^{p'}_p(G) = \langle x \in G | p \mid |C_G(x)| \rangle$, the group generated by “positive defect” elements. We will establish this in a stronger form below, for which we recall that a $p$–subgroup $P$ is called $p$–centric if $Z(P)$ is a Sylow $p$–subgroup of $C_G(P)$, and it is called $F$–essential if it is not Sylow and $W_0(\mathcal{P}_C(P)/P)$ is a proper subgroup of $W = N_G(P)/P$, with $W_0$ as in (1.2); see [7.4] for more information. Continuing the notation of [4.1] the analog of Proposition [4.1] states:

Proposition 4.21 (Bounds on $\pi_1(\mathcal{F}_C(G))$). Suppose that $\mathcal{C}$ is a collection of $p$–subgroups closed under passage to $F$–essential and Sylow overgroups. Then $\mathcal{F}_C(G)$ is equivalent to $\mathcal{F}_{0C}(G,\mathcal{C})$ and with $\bar{C}_C^p(H) = \langle PC_H(P) | P \leq H, P \in \mathcal{C} \rangle$, for $S \leq H \leq G$, we have canonical surjections

$$N_C(S)/SC_C(S) \to N_C(S)/\bar{C}_C^p(N_C(S)) \xrightarrow{\sim} \pi_1(\mathcal{F}_C(N_C(S))) \to \pi_1(\mathcal{F}_C(G)) \to G_0/C^{p'}C_0 \to \pi_1(\mathcal{F}_C(G))$$

If $\mathcal{C}$ contains all minimal $p$–centric subgroups, then $\pi_1(\mathcal{F}_C(G)) \xrightarrow{\sim} \pi_1(\mathcal{F}_C(G))$, a finite $p'$–group.

Proof. First, Alperin’s fusion theorem in the version of Goldschmidt–Miyamoto–Puig [Miy77 Cor. 1] shows that $\mathcal{F}_C(G)$ and $\mathcal{F}_{0C}(G,\mathcal{C})$ are equivalent categories (see again [Gro02 §10] for more on the assumptions in the fusion theorem, and also Section [4.1]). For the rest of the proof we can hence without loss of generality assume that $G = G_0\mathcal{C}$.

We model $\pi_1(\bar{\mathcal{F}}_C(G))$ analogously to the proof of Proposition [4.1] with generators maps in $\bar{\mathcal{F}}_C(G)$ between subgroups $P, Q$ of $S$ with $P, Q \in \mathcal{C}$, via the functor $\mathcal{F}_C(G) \to \pi_1(\mathcal{F}_C(G))$, taking $S$ as basepoint, cf. again [2.3]. In this notation the right-most epimorphism $\pi_1(\mathcal{F}_C(G)) \to G_0C/C_0^p(G_0C)$ is the map sending $(c_g; \ P \to Q)$ to $\bar{g} \in G_0C/C_0^p(G_0C)$. Since $\pi_1(\mathcal{F}_C(G))$ is a quotient of $\pi_1(\mathcal{O}_C(G))$, we have a surjection $N_C(S) \to \pi_1(\mathcal{F}_C(G))$ by Proposition [4.1] and by the same argument $N_C(S) \to \pi_1(\mathcal{F}_C(N_C(S)))$. It is furthermore clear by definition that $SC_C(S)$ lies in the kernel of both maps.
To finish showing that we have the stated sequence of maps we therefore just have to show that $\overline{C}^p_{\mathcal{C}}(N_G(S))$ is exactly the kernel of $N_G(S) \to \pi_1(\overline{\mathcal{C}}_G(N_G(S)))$. It is in the kernel since if $g \in PC_{N_G(S)}(P)$ then we have a commutative diagram

$$
\begin{array}{ccc}
P & \longrightarrow & S \\
\downarrow{id} & & \downarrow{c_g} \\
P & \longrightarrow & S
\end{array}
$$

in $\overline{\mathcal{C}}_G(G)$ and hence $c_g: S \to S$ represents the identity in $\pi_1(\mathcal{F}_p^*(N_G(S)))$. However, then the kernel has to be exactly $\overline{C}^p_{\mathcal{C}}(N_G(S))$, since taking $G = N_G(S)$, the second and fifth term of the sequence of the proposition agree and the map is the identity.

Assume now that $\mathcal{C}$ contains all minimal $p$–centric subgroups and we will show that $\pi_1(\mathcal{F}_G(G)) \cong \pi_1(\overline{\mathcal{C}}_G(G))$. By Proposition 4.12 we can assume that $\mathcal{C}$ is closed under passage to all $p$–overgroups, and in particular contains all $p$–centric subgroups. By the fusion theorem, again, $\pi_1(\mathcal{F}_G(G))$ is generated by self-maps of elements $P \in \mathcal{C}$, $P \leq S$, so we just need to see that all elements of $p$–power order in $\text{Aut}_F(P) \cong N_G(P)/C_G(P)$ are trivial in $\pi_1(\mathcal{F}_G(G))$. We can without loss of generality assume that $N_S(P)$ is a Sylow $p$–subgroup of $N_G(P)$ (i.e., that $P$ is “fully $G$–normalized” in $S$), $G$–conjugating $P$ if necessary. By conjugation in $\text{Aut}_F(P)$ it is furthermore enough to prove that all elements in the image of $N_S(P)$ in $\text{Aut}_F(P)$ are trivial in $\pi_1(\mathcal{F}_G(G))$. In $\pi_1(\mathcal{F}_G(G))$, such elements are equal to elements of $\text{Inn}(S) \leq \text{Aut}_F(S)$, since inclusions are trivial in $\pi_1(\mathcal{F}_G(G))$. The claim is hence reduced to seeing that $\text{Inn}(S) \leq \text{Aut}_F(S)$ map to the identity in $\pi_1(\mathcal{F}_G(G))$. Now, any element $x \in S$ is $G$–conjugate to an element $x' \in S$ such that $C_S(x')$ is a Sylow $p$–subgroup of $C_G(x')$ (i.e., $x'$ is “fully $G$–centralized” in $S$). Note that $Q = C_S(x')$ is $p$–centric in $G$, since $C_G(Q) \leq C_G(x')$, and $Q$ is obviously $p$–centric in $C_G(x')$, it being a Sylow $p$–subgroup. The image of $x' \in S$ in $\text{Aut}_F(S)$ identifies with the image of $x'$ in $\text{Aut}_F(Q)$, as elements of $\pi_1(\mathcal{F}_G(G))$, again since inclusions map to the identity. But $x'$ goes to the identity in $\text{Aut}_F(Q)$, so $x' \in S$ represents the identity in $\pi_1(\mathcal{F}_G(G))$. Hence so does the $G$–conjugate element $x$, as wanted.

Let us spell out what Proposition 4.21 says for $\mathcal{C} = S_p(G)$, in classical group-theoretic terms, as used in the proof of Theorem E in the next section.

**Corollary 4.22** (A vanishing condition for $\pi_1(\mathcal{F}_p^*(G))$). Let $L$ be a complement to $S$ in $N_G(S)$, and set $L_0 = \langle C_L(x) | x \in S \setminus 1 \rangle \leq L$. Then

$$L/L_0 \cong \pi_1(\mathcal{F}_p^*(N_G(S))) \to \pi_1(\mathcal{F}_p^*(G))$$

In particular if $L$ is generated by elements that commute with at least one non-trivial element in $S$, then $\pi_1(\mathcal{F}_p^*(G)) = 1$. If just $H_1(L)$ is generated by such elements then $H_1(\mathcal{F}_p^*(G)) = 0$.

We also get the following corollary, analogous to Corollary 4.12.

**Corollary 4.23.** If $|S_p(G)|$ is simply connected, e.g., if $O_p(G) \neq 1$, then $\pi_1(\mathcal{F}_p^*(G)) \cong G/C^p(G)$. If just $|S_p(G)|$ is simply connected then $\pi_1(\mathcal{F}_p^*(G)) \cong G_0/C^p(G_0)$.

**Proof.** The second case reduces to the first since $\mathcal{F}_p^*(G_0) \to \mathcal{F}_p^*(G)$ is an equivalence of categories by Proposition 4.21 also using (1.8). Now, consider the diagram obtained also using (1.9)

$$
\begin{array}{ccc}
\pi_1(\mathcal{F}_p(G)) & \longrightarrow & G \\
\downarrow & & \downarrow \\
\pi_1(\mathcal{F}_p^*(G)) & \longrightarrow & G/C^p(G)
\end{array}
$$

We just need to see that any element $x \in G$ which lies in $C^p(G)$ goes to zero in $\pi_1(\mathcal{F}_p^*(G))$ under the zig-zag given by the top isomorphism and the left-hand epimorphism. Assume that $x$ is a generator of $C^p(G)$, i.e., we can find a non-trivial $p$–subgroup $P$ such that $x \in C_G(P)$. Lift $x$ to $x': P \to P$ in $\pi_1(\mathcal{F}_p(G))$, which maps to the identity in $\pi_1(\mathcal{F}_p^*(G))$ as wanted.

$$\square$$
Remark 4.24. Note that Corollary 4.23 can also be applied to subgroups $H$ of our group $G$. And if $N_G(S) \leq H \leq G$ then $\pi_1(F_p^S(H)) \rightarrow \pi_1(F_p^G(\bar{G}))$ by naturality, as in Remark 4.4.

We also state the following corollary, suggested to us by Ellen Henke.

**Corollary 4.25.** If $N_G(S)/SC_G(S)$ is abelian, then $\pi_1(F_p^G(S))$ is cyclic.

**Proof.** Set $H = N_G(S)/SC_G(S)$ and pick an irreducible $H$–submodule $V$ of $pZ(S)$, the elements of order at most $p$ in $Z(S)$. Then $H/C_H(V)$ is cyclic by elementary representation theory [Gor68, Thm. 3.2.3], which implies the claim by Corollary 4.22. \qed

**Remark 4.26.** As hinted above, the quotient $G/Cp^\nu(G)$ is often trivial or a rather small group, and bounding its size is an interesting and so far apparently unexplored group theoretical question about $p'$–actions; we study this in joint work in progress with Geoffrey Robinson.

Analogous to Theorem 4.9 we can also describe $\pi_1(F_p^G(S))$ purely group theoretically.

**Theorem 4.27 (Group theoretic description of $\pi_1(F_p^G(S))$).** Let $K$ be the subgroup of $N_G(S)$ generated by elements $g \in N_G(S)$ such that there exists nontrivial subgroups $1 < Q_0, \ldots, Q_r \leq S$ and a factorization $g = x_1 \cdots x_r$ in $G$, where $x_i \in C^{p^\nu}(N_G(Q_i))$ and $Q_0^+ \cdots Q_i^+ \leq Q_i+1$ for $i \geq 0$. Then

$$N_G(S)/K \cong \pi_1(F_p^G(S))$$

In particular $N_G(S) \cap C^{p^\nu}(N_G(P)) \leq \ker(N_G(S) \rightarrow \pi_1(F_p^G(S)))$ for $1 < P \leq S$.

**Proof.** The proof is analogous to Theorem 4.9. First note that $N_G(Q_i) \rightarrow N_G(Q_i)/C_G(Q_i) \rightarrow \pi_1(F_p^G(Q_i))$ is equal to $N_G(Q_i) \rightarrow \pi_1(F_p^G(N_G(Q_i))) \rightarrow \pi_1(F_p^G(S))$, as the choice of basepoint does not matter (cf. [2.3]). The kernel of $N_G(Q_i) \rightarrow \pi_1(F_p^G(N_G(Q_i)))$ is exactly $C^{p^\nu}(N_G(Q_i))$ by Corollary 4.23, so this shows that $K$ is in the kernel, and as well as the ‘in particular’, as in Theorem 4.9.

The proof that $K$ is equal to the kernel is also identical to Theorem 4.9; the only difference is that the element $g$ is only well-defined up to an element in $C_G(P)$, but any two such elements map to the same element in $K$ in the current definition, so the inverse is still well-defined. \qed

**Proof of Theorem 7.** Consider the following commutative diagram

$$\begin{array}{cccc}
N_G(S)/SC_G(S) & \longrightarrow & \pi_1(F_p^G(S)) & \longrightarrow \\
\downarrow & & \downarrow & \\
N_G(S)/SC_G(S)S & \longrightarrow & \pi_1(F_p^G(S)) & \longrightarrow \\
\downarrow & & \downarrow & \\
\pi_1(F_p^G(S)) & \cong & \pi_1(F_p^G(S)) & \cong \\
\end{array}$$

The top horizontal maps in (4.4) are epimorphisms by Proposition 4.1, as the collection of $p$–centric subgroups is closed under passage to $p$–overgroups; the surjections between the top and middle row follow by definition, and the properties of the maps in and between the second and third row follow by Proposition 4.21. Applying Hom($-, k^\times$) and Theorem A now gives the first part of Theorem H.

If all $p$–centric $p$–radical subgroups are centric, then $\theta_p^{cr}(G) = F_p^{cr}(G)$, with the superscript denoting centric radicals. Hence $\pi_1(\theta_p^{cr}(G)) \cong \pi_1(F_p^{cr}(G)) \cong \pi_1(F_p^{cr}(G))$ by Theorem 7.11 and Proposition 7.12. Hence we have a canonical factorization

$$\pi_1(F_p^G(S)) \cong \pi_1(F_p^G(S)) \cong \pi_1(F_p^G(S)) \rightarrow \pi_1(F_p^G(S)) \rightarrow \pi_1(F_p^G(S)) \cong \pi_1(F_p^G(S))$$

where the ‘in particular’ again follows by applying Hom($-, k^\times$) and using Theorem A. \qed

**Remark 4.28.** Note that Theorem 7.11 and Proposition 7.12 in fact give weaker conditions on which $p$–centric subgroups need to be centric, in order for the factorization (4.5) to hold.

**Remark 4.29.** The $p'$–quotient groups of $\pi_1(F_c^G)$, were first studied in [BCG+07, §5.1], where they were related to subsystems of the fusion system of $p'$–index. It was remarked to the authors by Aschbacher that the group itself is a finite $p'$–group, see [BCG+07, p. 3839] and [Asc11, Ch. 11]. The direct proof of this fact in Proposition 4.21 can also be adapted to abstract fusion systems.
Let us round off our discussion of $\pi_1(F_C(G))$ for now by giving a few computational examples—see also Section 7 for more information on how it depends on $C$.

**Example 4.30.** For $\varphi$ an automorphism of a finite $p$–group $S$, $\pi_1(F^*_C(S\times\langle\varphi\rangle)) \cong \mathbb{Z}/r$, generated by $\varphi$, with $r$ the smallest natural number where $\varphi^r$ acts with a fixed-point on $S \setminus 1$, by Corollary 4.22.

**Proposition 4.31.** Suppose that $F$ is a fusion system over $S = 3^{1+2}_+$. Then $\pi_1(F^*) = 1$ unless $F = F_3(S)$ for $G = 3^{1+2}_+$, in which case $\pi_1(F^*) = \mathbb{Z}/2$. The sporadic group $J_2$ is the unique finite simple group with 3–fusion system $F_3(3^{1+2}_+ : 8)$, and hence for all other finite simple groups $G$ with Sylow $3$–subgroup $3^{1+2}_+$, $\pi_1(F_3(G)) = 1$.

**Proof.** We have $\text{Out}(3^{1+2}_+) \cong \text{GL}_2(\mathbb{F}_3)$ of order 48, so $L = N_F(S)/S$ is a subgroup of the Sylow 2–subgroup $SD_{16}$, which identifies with the semi-linear automorphisms of $\mathbb{F}_9$, generated by a generator $\sigma$ of $\mathbb{F}_9^*$ and the Frobenius $\tau$, subject to the relations $\sigma^2 = \tau^2 = 1$, and $\sigma\tau\sigma = \tau^3$. Both $\sigma$ and $\tau$ have determinant $-1$ inside $\text{GL}_2(\mathbb{F}_3)$. We claim that $\sigma^k$, for $k$ odd, are the only elements that act fixed-point freely on $3^{1+2}_+$. It is clear that $\sigma$ acts fixed-point freely iff $k$ is odd. Furthermore the elements $\sigma^{2k+1}\tau$ act with a fixed-point, since they act trivially on the center, and likewise $\sigma^{2k}\tau$ acts with a fixed-point, since $\sigma^{2k}(\tau(\sigma^{-k})\tau) = \sigma^{2k}(\sigma^{-3k}) = \sigma^{-k}$. The only two subgroups which contain $\sigma$ are $\langle \sigma \rangle$ and $SD_{16}$. For $L = SD_{16}$ we have $\sigma = \sigma\tau\sigma$, in the notation of Corollary 4.22, so $L = L_0$. For $L = \langle \sigma \rangle$, $L/L_0 \cong \mathbb{Z}/2$ by Example 4.30.

To finish the proof we use that by [RV04] Tables 1.1 and 1.2, $F_3(3^{1+2}_+ : 8)$ is the only fusion system on $3^{1+2}_+$ with $N_F(S)/S \cong \mathbb{Z}/8$, and $J_2$ is the unique finite simple group realizing it. \hfill $\square$

**Example 4.32.** By the ATLAS the centralizers of 3–elements in $J_2$ are $C_{J_2}(3A) = 3 \cdot \text{PSL}_2(9)$ and $C_{J_2}(3B) = 3 \times A_4$; in particular they satisfy $H_1(-,\mathbb{Z}) = 0$. Hence Proposition 4.31 combined with Theorem E gives $T_{F_3}(J_2, S) \cong \text{Hom}(\pi_1(\mathcal{F}_3(3_2)), \mathbb{F}_3^*) \cong \mathbb{Z}/2$ in this case. See [LM15b, 6.4] for a very different derivation of this result.

### 4.4. Higher homotopy groups

We conclude the section by making precise how $|\mathcal{F}_p(G)|$ and $|\mathcal{O}_p^G(G)|$ relate, and showing that all homotopy groups of $|\mathcal{O}_p^G(G)|$ and $|\mathcal{F}_p(G)|$ are finite $p$–groups. A statement about $H_2$ was already used in Corollary 4.17. The arguments are less self-contained than elsewhere, and can be skipped at first reading.

We first establish the homological relation for arbitrary $C$ and $p$ inverted, as in Proposition 4.5.

**Proposition 4.33.** For $C$ any collection of $p$–subgroups of $G$,

$$H_*(\mathcal{F}_C(G)) \otimes \mathbb{Z}\frac{1}{p} \cong H_*(\mathcal{O}_C(G)) \otimes \mathbb{Z}\frac{1}{p}, \quad H_*(\mathcal{F}_C(G)) \otimes \mathbb{Z}\frac{1}{p} \cong H_*(\mathcal{F}_C(G)) \otimes \mathbb{Z}\frac{1}{p},$$

and $$H_*(|C|/G) \otimes \mathbb{Q} \cong H_*(\mathcal{F}_C(G)) \otimes \mathbb{Q} \cong H_*(\mathcal{F}_C(G)) \otimes \mathbb{Q}.$$

**Proof.** The statements follow by a Grothendieck composite functor spectral sequence argument, since the morphisms differ by finite $p$–groups or finite groups. More precisely, [BLO03a, Lem. 1.3] implies that the two first maps are equivalences in homology with $\mathbb{F}_\ell$–coefficients for all primes $\ell \neq p$. Since the spaces are of finite type this implies equivalence in homology with $\mathbb{Z}\langle\ell\rangle$–coefficients, for all primes $\ell \neq p$, and hence an isomorphism in homology with $\mathbb{Z}\langle\ell\rangle$–coefficients, and after tensoring with $\mathbb{Z}\langle\ell\rangle$. The third isomorphism is since the isotropy spectral sequence, Proposition 5.1, for the $G$–action on $|C|$ converging to $|\mathcal{F}_p(G)|$, collapses, and the fourth isomorphism again follows from the proof of [BLO03a, Lem. 1.3], now using that $H_*(C_G(P); \mathbb{Q}) = 0$. \hfill $\square$

**Theorem 4.34.** For $C$ a collection of $p$–subgroups of $G$, closed under passage to $p$–radical overgroups, $|\mathcal{O}_C(G)|$ and $|\mathcal{F}_C(G)|$, and $|\mathcal{O}_p(G)|$ are connected spaces with finite homotopy groups and homology groups, all of order prime to $p$. Furthermore, for $i > 0$, $H_i(\mathcal{F}_C(G))$ is finite, and when $C$ is ample (cf. Remark 4.3), $H_1(\mathcal{F}_C(G)) \cong H_1(\mathcal{O}_C(G)) \oplus H_1(G)$. Also $\pi_i(\mathcal{F}_C(G)) \cong \pi_i(|C|)$ for $i \geq 2$, in general not finite. The space $|\mathcal{O}_p^G(G)|$ is the localization of $|\mathcal{F}_p(G)|$ with respect to the multiplication by $p$ map $w: S^1 \to S^1$, or with respect to $M(\mathbb{Z}/p, 1) \to \ast$, for $M(\mathbb{Z}/p, 1)$ the mapping cone of $w$, with localization in the sense of Anderson, Bousfield and Farjoun [And72, Far96].

Before giving the proof, note that the assumptions on $C$ in Theorem 4.34 are necessary.
Remark 4.35. The fundamental group, or higher homotopy and homology groups of \( \mathcal{T}_C(G) \), \( \mathcal{O}_C(G) \), \( \mathcal{F}_C(G) \), or \( \mathcal{F}_C(G) \) will generally not be finite without assumptions on \( C \). For example if \( G \) is abelian then \( \mathcal{F}_C(G) = C \) and \( |C|/G = |C| \), so examples can be constructed using Proposition 4.33. Taking \( G = (\mathbb{Z}/p)^r \) and \( C \) the collection of proper non-trivial subgroups of \( G \); then \( |C| \) has homotopy type a wedge of spheres (e.g., \( p \) + 1 points and a wedge of \( p^3 \) circles, for \( r = 2, 3 \) respectively).

**Proof of Theorem 4.34.** The spaces are connected since \( C \) contains \( S \). By Theorem 7.5.1 we can throughout, without loss of generality, assume that \( C \) is closed under passage to all \( p \)-overgroups.

We start with \( |\mathcal{O}_C(G)| \): By Proposition 4.1 \( \pi_1(\mathcal{O}_C(G)) \) is a finite \( p \)-group. Let \( X \) denote the universal cover of \( \mathcal{O}_C(G) \), and note that \( H^*(X;\mathbb{Z}/p) \cong H^*(\mathcal{O}_C(G);\mathbb{Z}/p) \), with the canonical twisted coefficient system coming from the fundamental group; see [2.4]. This again equals \( \lim_{\mathcal{F}_C(G)}^*\mathbb{Z}/p \ cong H^*(\mathcal{O}_C(G);\mathbb{Z}/p) \), where \( \pi_1(\mathcal{O}_C(G)) \) being a finite \( p \)-group. Hence the \( E_1 \)-term of the spectral sequence is zero except at \( (0,0) \), and we conclude that \( H^*(X;\mathbb{Z}/p) = 0 \). Now, since \( \pi_1(\mathcal{O}_C(G)) \) is finite, \( X \) is of finite type, and hence \( H_i(X) \) is a finite \( p \)-group for all \( i \). By the Hurewicz theorem modulo the Serre class of finite \( p \)-groups, the same is then true for \( \pi_k(X) \), so \( \pi_k(\mathcal{O}_C(G)) \) is a finite \( p \)-group for \( i \geq 2 \) as wanted.

For \( \mathcal{F}_C(G) \) we need the fact that also \( H^*(\mathcal{F}_C(G);\mathbb{Z}) \cong H^*_C(|\mathcal{C}|/\mathbb{Z}/p) \), for any \( M : \mathcal{F}_C(G)^{op} \to \mathbb{Z}_{(p)}^{op} \) mod, which will then finish the proof with \( M = \mathbb{Z}/p[\pi_1(\mathcal{F}_C(G))] \) as for \( |\mathcal{O}_C(G)| \). The category \( \mathcal{F}_C \) is not treated in [Gro02], but the desired fact follows as in the proof of [Gro02] Thm. 1.1] as we now briefly explain: One observes that \( \mathcal{O}_C \) defined in Thm. 3.2 (also holds for \( E\mathcal{A}_C \), since the pair \( (\{E\mathcal{A}_C\} ; \{E\mathcal{A}_C\}_{>p}) \) is \( S^1 \)-equivalent to \( (\{C_{>p} \} ; \{C_{>p} \}) \), for \( S^1 \) a Sylow \( p \)-subgroup of \( \mathcal{N}_G(P)/P \); and the reason for this is that \( \mathcal{E}_A(\mathcal{A}^{\leq p}) \to \{Q \in \mathcal{C}_{>p} \mid R \leq \mathcal{C}(Q) \} \) is contractible when \( R \) is a non-trivial \( p \)-subgroup of \( \mathcal{N}_G(P)/P \) via the standard contraction \( Q \leq Q \to R \geq P \). By Theorem 7.5.2, \( H^*_C(|\mathcal{C}|/\mathbb{Z}/p) \to H^*_C(G/P), \) so the statement about \( H^*_C(|\mathcal{C}|/\mathbb{Z}/p) \) also follows.

For \( \mathcal{F}_C \), recall that \( |\mathcal{F}_C(G)| \to |\mathcal{C}| \to |\mathcal{O}_C| \to \mathbb{Z}/G \to BG \) shows \( \pi_k(|\mathcal{F}_C(G)|) \cong \pi_k(|\mathcal{C}|) \) for \( i \geq 2 \). Now \( H_i(\mathcal{F}_C) \) is obviously finitely generated, since there are finitely many simplices in each dimension, so by Proposition 4.33 and the first part, \( H_i(\mathcal{F}_C) \) is finite (alternatively, use that, also by Proposition 4.33 \( H_*\mathcal{F}_C(G);G) \cong H_*\mathcal{F}_C(G);G) \) and \( |\mathcal{C}|/G \) is cyclic by Proposition 7.3. Being ample means that \( H_i(\mathcal{F}_C(G);G) \to H_i(\mathcal{F}_C(G);G) \to H_i(\mathcal{O}_C(G)) \) follows using Proposition 4.33 again.

Now, for localization statement: Note that \( L_f(\mathcal{F}_C(G);G) \cong \mathcal{F}_C(G) \), for a finite \( p \)-group \( P \), where \( L_f \) is localization with respect to \( f \) : \( \mathbb{Z}/p \to \mathbb{Z}/p \times \mathbb{Z}/p \), this can be seen directly by observing that we can construct a contractible space from \( \mathcal{B}_P \) by successively attaching spaces \( M(\mathbb{Z}/p,n) \) for \( n > 1 \) killing homotopy groups (or appeal to e.g., [Cas93] Thm. 4.4). Also recall that \( H_\mathcal{F}_C(\mathcal{F}_C(G);G) \to H_\mathcal{F}_C(\mathcal{F}_C(G);G) \) finishes the proof with \( \mathcal{F}_C(G) \). (See also [CP93] 3.4) \( \square \)

**Corollary 4.36.** For any characteristic \( p \) field \( k \), \( H^i(\mathcal{F}_p(G);k^\times) \cong H^i(\mathcal{O}_p(G);k^\times) \oplus H^i(\mathcal{O}_p(G);k^\times) \) for \( i > 0 \). If \( k \) is perfect, then \( k^\times \) is uniquely \( p \)-divisible, and \( H^i(\mathcal{F}_p(G);k^\times) \cong H^i(\mathcal{F}_p(G);k^\times) \).
Proof. The first claim is a consequence of Theorem 4.34, the Universal Coefficient Theorem, and the five-lemma, using that $S_p(G)$ is ample, cf. Remark 4.7. That $k$ is perfect means that the Frobenius map $(-)^p: k^\times \to k^\times$ is not only injective, but also surjective, i.e., $k^\times$ is uniquely $p$–divisible, and hence $H^i(G; k^\times)(p) = 0$, for $i > 0$ by an application of the transfer. \qed

Remark 4.35. Any finite field or any algebraically closed field is of course perfect. For any field of characteristic $p$ we have $H^1(G; k^\times)(p) = 0$, as $k^\times$ is $p$–torsion free, but the higher groups are non-trivial in general for non-perfect fields. E.g., if $k = \mathbb{F}_p(x)$, rational functions in one variable over $\mathbb{F}_p$, the units $k^\times$ is isomorphic to $\mathbb{F}_p^\times \times \mathbb{Z}^{(\infty)}$, as an abelian group, with a basis for the torsion-free part given by monic irreducible polynomials, so $H^i(G; k^\times)(p) \cong (H^i(G; \mathbb{Z}(p)))^{(\infty)}$ in that case.

Remark 4.36 (Interpretation of $H^2(\mathcal{O}_p^*(G); k^\times)$). The group $H^2(\mathcal{O}_p^*(G); k^\times)$ may be through of as a “$p$–local Schur multiplier”, analogous to $H^2(G; k^\times)$. One may also ask if it also has a representation theoretic interpretation, as a suitable Brauer group? Note in this connection that $H^2(\mathcal{F}; k^\times)$ occurs in connection with the so-called gluing problem for blocks, see [Lin04, Lin05, Lin09].

5. Homology decompositions and the Carlson–Thévenaz conjecture

In this section we establish the results about homology decompositions stated in the introduction, and show how they imply the Carlson–Thévenaz conjecture. The key tool is the isotropy spectral sequence, recalled below. Applied to the space $|C|$ this gives us the normalizer decomposition (Theorem [D]). For the centralizer decomposition (Theorem [E]) we instead use the space $|EA_C|$, where $EA_C$ is the overcategory $\mathcal{C} \downarrow G$ for $\mathcal{C} \leq G: A_C \to A_{C \cup G}$; see [7.1] for details. (There is also a third decomposition, the subgroup decomposition, based on a space $|EC\setminus G|$, but since the isotropy subgroups are $p$–groups, it does not provide us with new information when taking coefficients prime to $p$.) We will work in both homology and cohomology—these are essentially equivalent, but from a practical viewpoint it may feel more convenient to work in homology, only mapping into $k^\times$ at the end, so we give both versions.

5.1. Homology decompositions. Let $H^G_i(X; \mathfrak{F})$ denote Bredon homology equipped with an isotropy coefficient system $\mathfrak{F}$; see §2.4.

Proposition 5.1 (The isotropy spectral sequence). Let $G$ be a finite group, $X$ a $G$–space, and $A$ an abelian group. We have a homological isotropy spectral sequence for the action of $G$ on $X$

$$ E^2_{i,j} = H^G_i(X; H_j(-; A)) \Rightarrow H_{i+j}(X_{hG}; A) $$

The bottom right-hand corner produces an exact sequence

$$ H_2(X_{hG}; A) \to H_2(X/G; A) \to H_0^G(X; H_1(-; A)) \to H_1(X_{hG}; A) \to H_1(X/G; A) \to 0 $$

The dual spectral sequence in cohomology produces

$$ 0 \to H^1(X/G; A) \to H^1(X_{hG}; A) \to H^0_G(X; H^1(-; A)) \to H^2(X/G; A) \to H^2(X_{hG}; A) $$

If $H_1(X/G; A) = H_2(X/G; A) = 0$ then this degenerates to $H_1(X_{hG}; A) \cong H^G_0(X; H^1(-; A))$.

Dually if $H^1(X/G; A) = H^2(X/G; A) = 0$ then $H^1(X_{hG}; A) \cong H^1_G(X; H^1(-; A))$. By definition

$$ H^G_0(X; H_1(-; A)) = \text{coker} \left( \oplus_{x \in X_{hG}} H_1(G_x; A) \xrightarrow{d_{\alpha}-d_{\alpha}} \oplus_{x \in X_{hG}} H_1(G_x; A) \right) $$

for $X_i$ the non-degenerate $i$–simplices, and dually for cohomology.

Proof. As explained in standard references such as [Dwy98] §2.3 [Bro94] VII(5.3)], the (homology) isotropy spectral sequence is constructed as the spectral sequence of the double complex $C_*(EG) \otimes_G C_*(X; A)$, filtered via the skeletal filtration of $X$. Hence $E^1_{i,j} = H_j(G; C_*(X; A))$, and the $E^2$–term is obtained by taking homology induced by the differential on $C_*(X; A)$. The stated properties now follow from the definitions. \qed

We would like to alternatively view the $H^G_0$ in Proposition 5.1 as a colimit, so we also recall the general principle behind this: Recall from (2.4) that a general (covariant) coefficient system on $X$ is just a functor $\Delta X \to R$–mod, where $\Delta X$ is the category of simplices.
It is convenient to say that a space is complex-like if every non-degenerate simplex $\Delta[n] \to X$ is an injection on sets, i.e., if it “looks like” an ordered simplicial complex [Tho80 p. 311]. For a complex-like space, the subdivision category $sd X$ is the full subcategory of $\Delta X$ on the non-degenerate simplices; it has a unique morphism $\sigma \to \tau$ if $\tau$ can be obtained from $\sigma$ via face maps, and no other morphisms, see [DK83 §5]. The following classical proposition gives the relationship we need, stated also for higher homology for clarity:

**Proposition 5.2.** Let $D$ be a small category, and $R$ a commutative ring.

1. For any functor $F: \mathbf{D} \to \mathbf{R}$-mod, $\text{colim}^D F \cong H_*(|D|; R)$, where $F$ is the coefficient system induced via $\Delta|D| \to \mathbf{D}$, $(d_0 \to \cdots \to d_n) \mapsto d_0$.

2. For any functor $F: \mathbf{D}^{op} \to \mathbf{R}$-mod, $\text{colim}_{\mathbf{D}}^{op} F \cong H_*([-|D|]; R)$, where $F$ is the coefficient system induced via $\Delta|D| \to \mathbf{D}^{op}$, $(d_0 \to \cdots \to d_n) \mapsto d_n$.

3. Suppose $X$ is a complex-like space. For any functor $F: \text{sd} X \to \mathbf{R}$-mod, $\text{colim}_{\text{sd} X} F \cong H_*(X; R)$, where $F$ is induced from $F$ via $\Delta X \to \text{sd} X$, the functor sending all degeneracies to identities; see [DK83 §5].

**Proof.** We shall only need non-derived $*=0$ part of these statements, which follows easily by writing down the definitions (for the last point also using cofinality), which we invite the reader to do. For [1] and [2], in the general case, see [GZ67, App. II.3.3], and also [Gro92 Prop. 2.6]. (The point is that both sides can be seen as homology of $C_\bullet([−→ D]) \otimes \mathbf{D} F$ respectively $F \otimes \mathbf{D} C_\bullet([D \downarrow →])$.) For [3], notice that both sides can be seen as the homology of $C_\bullet|\sigma| \otimes_{\text{sd} X} F$, where $|\sigma|$ is the $n$–simplex defined by the vertices of $\sigma$ (by assumption distinct); it is a contravariant functor on sd $X$ by to $(\sigma \to \tau)$ assigning the map induced by the unique face inclusion of $\tau$ in $\sigma$ (i.e., the extra structure on sd $X$ allows us to “avoid a subdivision”; see also [Gro92 Prop. 7.1]).

**Proposition 5.3.** Let $C$ be a collection such that $H_1(|C|/G)_{p'} = H_2(|C|/G)_{p''} = 0$, e.g., a collection $G$–homotopy equivalent to a collection closed under passage to $p$–overgroups. Then

$$H_1(\mathcal{O}_C(G))_{p'} \cong \text{coker}(d_0 - d_1 : \oplus_{P < Q} H_1(N_G(P < Q))_{p'} \to \oplus_{P} H_1(N_G(P))_{p'}) \cong \text{colim}_{P < \cdots < P_n} H_1(N_G(P_0 < \cdots < P_n))_{p'}$$

where the colim is over $G$–conjugacy classes of strict chains in $C$, ordered by reverse refinement.

**Proof.** First note that Symonds’ theorem, Proposition 7.3, implies that $H_1(|C|/G) = H_2(|C|/G) = 0$ if $C$ is $G$–homotopy equivalent to a collection closed under passage to $p$–overgroups. If $H_1(|C|/G)_{p'} = H_2(|C|/G)_{p''} = 0$ then the isotropy spectral sequence, Proposition 5.1, applied to the $G$–space $|C|$ with $A = Z_p^G|C|$, gives $H_1(|C|/hG; Z_p^G|C|) \cong H_1^G(|C|; H_1(−; Z_p^G|C|))$. Since the right-hand side is obviously finite so is the left-hand side, hence taking coefficients in $Z_p|C|$ is the same as applying $(-)_{p'}$. Furthermore $H_1(\mathcal{O}_C(G))_{p'} \cong H_1(\mathcal{J}_C(G))_{p'} \cong H_1(\mathcal{J}_C|\mathcal{J}_C|/G)_{p'}$ by Proposition 4.5 and Remark 2.2. The first formula now follows by definition of $H_1^G$. The second rewriting as a colimit over conjugacy classes of strict chains follows from Proposition 5.2,3, noting that $|C|/G$ is complex-like.

**Proof of Theorem 7.1.** By Theorem A and (1.1), $T_k(G, S) \cong \text{Hom}(H_1(\mathcal{O}_p^G), k_\times)$. Combining this with Proposition 5.3 for $C = S^p(G)$ gives the wanted expression, using that Hom$(-, k_\times)$ is left exact. Again, it is a subset of Hom$(N_G(S)/\mathcal{J}_C, k_\times)$ by Proposition 4.1.

We now prove the centralizer version.

**Proposition 5.4.** For any collection $C$ of $p'$–subgroups of $G$ there is an exact sequence

$$H_2(\mathcal{F}_C(G); Z_p^G|C|) \to \text{colim}_{P \in \mathcal{F}_C(G)_p'} H_1(C_G(P); Z_p^G|C|) \to H_1(\mathcal{O}_C(G); Z_p^G|C|) \to H_1(\mathcal{F}_C(G); Z_p^G|C|) \to 0$$

**Proof.** Consider the isotropy spectral sequence, Proposition 5.1, in low degrees, for the $G$–space $|E\mathcal{A}_C|$ introduced in the beginning of this section (and in more detail in 7.1):

$$H_2(|E\mathcal{A}_C|/G; Z_p^G|C|) \to H_0^G(|E\mathcal{A}_C|/G; H_1(\mathcal{O}_p^G); Z_p^G|C|) \to H_1(|E\mathcal{A}_C|/hG; Z_p^G|C|) \to H_1(|E\mathcal{A}_C|/G; Z_p^G|C|) \to 0$$

We want to identify this sequence with the sequence of the proposition. As remarked in 7.1 $|E\mathcal{A}_C|/G = |\mathcal{F}_C|$, and the $G$–map $|E\mathcal{A}_C| \to |C|$ is a homotopy equivalence, and induces $|E\mathcal{A}_C|/hG \cong |\mathcal{J}_C|/hG$. Also by Proposition 4.5 (or 4.33) and Remark 2.2 $H_1(|\mathcal{J}_C|/hG; Z_p^G|C|) \cong H_1(\mathcal{O}_C(G); Z_p^G|C|)$, as
used before. So, after inverting \( p \), the first, third and fourth term identify as stated. For the second term, notice that the stabilizer of an \( n \)-simplex \( i: V_0 \to V_1 \to \cdots \to V_n \to G \) is \( C_G(i(V_0)) \). Hence Proposition \[5.2[3] \] also identifies \( H^2_C([EA_C]; H_1(-; \mathbb{Z}^{\mathbb{Q}})) \) with the colimit as stated.

\[ \square \]

**Remark 5.5.** To calculate the colimit in Proposition \[5.4 \] one only needs a cofinal subcategory of \( T_C(G)^{op} \); see e.g., [Mac71, XI.3]. E.g., if \( C \) is a collection of non-trivial \( p \)-subgroups, containing the elementary abelian \( p \)-groups of rank one or two, then we can replace \( C \) by \( \mathbb{A}_p^2(G) \).

**Proof of Theorem \[7] :** The main statement and proof is dual to Proposition \[5.4 \] using the cohomological isotropy spectral sequence instead, with coefficients in \( k^x \), and \( C \) the collection of all non-trivial \( p \)-subgroups. In the limit we can restrict to elementary abelian \( p \)-subgroups of rank one or two, since this subcategory is cofinal; see Remark \[5.5 \]. For the ‘in particular’ part, note that since the centralizers of elements \( x \) of order \( p \) are assumed to satisfy \( H_1(C_G(x)); p' = 0 \) (i.e., are “\( p' \)-perfect”), the inverse limit is obviously zero since the values are zero. The assumptions on the action of \( N_G(S) \) implies that \( H^1(F^e_2(G); k^x) = 0 \) by Corollary \[4.22 \].

\[ \square \]

**Remark 5.6 (Isotropy versus Bousfield–Kan spectral sequence).** The above arguments in terms of the isotropy spectral sequence can equivalently be recast in terms of the Bousfield–Kan spectral sequence of a homotopy colimit. As explained e.g., by Dwyer [Dwy98, §3] [Dwy97, §3.3], the isotropy spectral sequence identifies with the Bousfield–Kan spectral sequence associated to the homotopy colimit

\[ |C|_{hG} \simeq \hocolim_{\sigma \in |C|/G} EG \times_G G/G_{\sigma} \]

It is also possible to work with \( \mathcal{O}_C(G) \) directly, instead of passing via \( |C|_{hG} \), since by Slo91 Cor. 2.18 the orbit category admits a normalizer decomposition

\[ |\mathcal{O}_C(G)| \simeq \hocolim_{\{P_0 < \cdots < P_n\} \in |C|/G} BN_G(P_0 < \cdots < P_n)/P_0 \]

(5.1)

(Where \( BN_G(P_0 < \cdots < P_n)/P_0 \) has to be interpreted as \( E(G/P_0) \times_G \cdots \times_G E(G/P_n) \), for \( E(G/P_0) \) the translation groupoid of the \( G \)-set \( G/P_0 \), in order to get a strict functor to spaces). The associated spectral sequence for this homotopy colimit, can also be obtained in a more low-tech way as the Leray spectral sequence of the projection map \( |\mathcal{O}_C(G)| = |E\mathcal{O}_C|/G \to |C|/G \).

5.2. **The Carlson–Thévenaz conjecture.** We will now prove the results in \[1,3 \] and in particular deduce Theorem \[F \] from Theorem \[D \]. Via Theorem \[A \] this will amount to spelling out how the colimit appearing in Proposition \[5.3 \] can be calculated. Since these are rather general facts about colimits we separate it out as a couple of propositions. The first proposition gives a step-by-step way of calculating colimits over the orbit simplex category \( sd |C|/G \), which will then in the second proposition be related to the colimit occurring in the Carlson–Thévenaz conjecture.

**Proposition 5.7.** Let \( C \) be a collection of subgroups of \( G \), and \( F: \ sd |C|/G \to R \)-mod a functor. For any \( P \in C \) define an increasing filtration on \( F(P) \) by setting \( F_1(P) = 0 \) and letting

\[ F_r(P) = (F(d_P)(F(d_Q))^{-1}(F_{r-1}(Q))) \quad |Q \in C \text{ with } Q \leq P \text{ or } P \leq Q \]

with \( d_P, d_Q \) the boundary maps from \([P \leq Q]\) (or \([Q \leq P]\)) to \([P]\) and \([Q]\) respectively. If \( T \in C \) fulfills \( \dim |C_{\leq T}| = \dim |C|, \) then \( F_r(T) = \ker (F(T) \to \hocolim_{sd |C|/G} F) \) when \( r \geq \dim |C| + 1 \).

**Proof.** Recall that by Proposition \[5.2[3] \] the colimit can be viewed as the quotient of \( \oplus_{[P]} F(P) \), where we identify elements that are images of the same element via a zig-zag

\[ F(Q) \leftarrow F(Q < Q') \to F(Q') \]

(1)

for various \([Q < Q']\). It is hence clear that any \( F_r(T) \) maps to zero in the colimit.

Conversely assume that \( x \in F(T) \) maps to zero in the colimit. We need to see that \( x \) can be be connected to zero via at most \( \dim |C| \) zig-zags as above. By Proposition \[5.2[3] \] being zero in the colimit means that we can express \( x \) as the image under the boundary map \( d \) of an element \( y \in \oplus_{[P < Q]} F([P < Q]) \), where the sum is over \( G \)-conjugacy classes of pairs \( P < Q \) of subgroups in \( C \). Define the height of \( P \in C \) as \( \text{ht}_C(P) = \dim |C_{\leq P}| \), and write \( y = \sum_{i=0}^{\dim |C| - 1} y_i \) where \( y_i \) are those terms of \( y \) where \( P \) has height \( i \). Note that \( d(\sum_{i=0}^{\dim |C| - 1} y_i) \) is zero on the summands corresponding to subgroups of height \( s \) or less, for \( s \leq \dim |C| - 1 \), since \( x \) is only non-zero on the \([T]\) coordinate.
We claim that $d_1(\sum_{i=0}^s y_i)$ is connected to zero by $s$ zig-zags. For $s = 0$ this is clear, since $d_1 y_0 = 0$, as $x$ is non-zero only on the $[T]$ coordinate, and only $d_1 y_0$ contributes to the part of $x = dy$ corresponding to height zero subgroups. Suppose by induction it is true for $s - 1$. Note that $d_1 y_s = d_0(y_0 + \cdots + y_{s-1})$ since $x$ is zero on the height $s$ part as $s \leq \dim |C| - 1$, and these are the only terms contributing to the height $s$ part of $x$. Hence

$$d_1(y_0 + \cdots + y_s) = d_1(y_0 + \cdots + y_{s-1}) + d_1 y_s = d_1(y_0 + \cdots + y_{s-1}) + d_0(y_0 + \cdots + y_{s-1})$$

Since $d_0(y_0 + \cdots + y_{s-1})$ is connected to $d_1(y_0 + \cdots + y_{s-1})$ via one zig-zag, we conclude by the induction assumption that $d_1(y_0 + \cdots + y_s)$ is connected via $s$ zig-zags to zero as claimed. For $s = \dim |C| - 1$, we then get that $x = d_1 y - d_1 y = d_0(y_0 + \cdots + y_s) - d_1(y_0 + \cdots + y_s)$ is generated by elements connected via a zig-zag of length $s + 1 = \dim |C|$ to zero, as wanted. \hfill $\square$

The next lemma shows how to relate the above colimit to one where we do not demand inclusions between the subgroups.

**Lemma 5.8.** For any functor $H: \mathcal{O}(G) \to R$-mod, and $\mathcal{C}$ a collection of $p$-subgroups, consider $F: \sd |C|/G \to R$-mod given by $F([P_0 \leq \cdots \leq P_n]) = H(N_G(P_0) \cap \cdots \cap N_G(P_n))$. For $P \in \mathcal{C}$, set $F_1'(P) = 0$ and define by induction $F_r'(P) \leq F(P)$ as $F_r'(P) = \langle d_P(d_Q^{-1}(F_{r-1}'(Q))) \rangle [P, Q \in \mathcal{C}_{\leq S}]$, where $d_P : H(N_G(P) \cap N_G(Q)) \to H(N_G(P))$. Then, with $F_r$ as in Proposition 5.7, $F_r(P) \leq F_r'(P)$ and, if $\mathcal{C}$ is closed under passage to $p$-overgroups, $F_r'(P) \leq F_{2r-1}(P)$.

**Proof.** First note that in the definition of $F_r$ we can without loss of generality demand that $P, Q \leq S$, since this will hold up to conjugacy of chains. Now, it is clear that $F_r(P) \leq F_r'(P)$, since we are allowing more subgroups in the definition of $F_r'$ by not demanding inclusion between $P$ and $Q$.

For the other inclusion, assume that $\mathcal{C}$ is closed under passage to $p$-overgroups. one-step zig-zag between arbitrary subgroups of $S$ is related to a two-step zig-zag between groups included in each other as follows:

$$H(N_G(P < PQ)) \longrightarrow H(N_G(P))$$

$$\downarrow$$

$$H(N_G(P) \cap N_G(Q)) \longrightarrow H(N_G(PQ))$$

$$\downarrow$$

$$H(N_G(Q < PQ)) \longrightarrow H(N_G(Q))$$

The claim now follows by induction: Then group $F_r'(P)$ consists of elements in $F_r(P)$ generated by $d_P(d_Q^{-1}(F_{r-1}'(Q)))$ for various $Q$. But (1) shows that this image is generated in two steps from $F_{r-1}'(Q)$, going via $PQ \in \mathcal{C}$; since by induction $F_{r-1}'(P) \leq F_{2r-3}(P)$, this shows that $d_P(d_Q^{-1}(F_{r-1}'(Q))) \leq F_{2r-3+1}(P) = F_{2r-1}(P)$ as wanted. \hfill $\square$

Note that taking $H = H_1(\cdot)_{\rho'}$ and $\mathcal{C} = S_p(G)$ we have $F_r'(S) = \rho'(S)$, as appearing in the Carlson–Thévenaz conjecture, Theorem $[F]$. We can now state.

**Theorem 5.9.** Let $\mathcal{C}$ be a collection of $p$-subgroups containing a Sylow subgroup $S$, for which $H_2([C]/G)_{\rho'} = 0$ and $H_1(\mathcal{O}_G(G))_{\rho'} \overset{\sim}{\longrightarrow} H_1(\mathcal{O}_G^p(G))$ (e.g., $\mathcal{C} = S_p(G), B_p(G)$, or $A_p(G) \cup [S]$). Then

$$H_1(\mathcal{O}_G(G))_{\rho'} \overset{\sim}{\longrightarrow} \text{colim}_{[C]/G} H_1(N_G(P_0 < \cdots < P_n))_{\rho'} \overset{\sim}{\longrightarrow} F(S)/F_r(S) \cong N_G(S)/\rho'(S)$$

for $r \geq \dim \mathcal{C} + 1$, with $F(-) = H_1(N_G(-))_{\rho'}$, and $F_r$ and $\rho'$ defined in Proposition 5.7 and Theorem $[F]$. Hence, for such $r$, $T_k(G, S) \cong \text{Hom}(N_G(S)/\rho'(S), k^\times)$ by Theorem $[A]$.

**Proof.** As in the proof of Proposition 5.3, Proposition 5.1 provides a diagram

$$H_2([C]/G)_{\rho'} \longrightarrow \text{colim}_{[C]/G} H_1(N_G(P_0 < \cdots < P_n))_{\rho'} \longrightarrow H_1(\mathcal{O}_G(G))_{\rho'}$$

$$\downarrow$$

$$H_1(\mathcal{O}_G^p(G))$$
This is the more technical assumption given in the last part of the corollary. Now, suppose that $N \in \mathcal{C}$. Hence if $H_2(\mathcal{C}/G)_{p'} = 0$, and $H_1(\mathcal{C}/G)_{p'} \xrightarrow{\sim} H_1(\mathcal{C}/G)_{p'}$ then $\text{colim}_{\mathcal{C}/G} H_1(N_G(P_0 < \cdots < P_n))_{p'} \xrightarrow{\sim} H_1(N_G(P_0 < \cdots < P_n))_{p'}$. Note that the assumptions are satisfied for $S_p(G)$ by Proposition 7.3 and hence also for $B_p(G)$ by Theorem 7.5, and $A_p(G) \cup [S]$ by Theorem 7.5 and Lemma 7.1.

We are left with showing that the kernel of the second map in the composite $N_G(S) \to H_1(N_G(S))_{p'} \to \text{colim}_{\mathcal{C}/G} H_1(N_G(P_0 < \cdots < P_n))_{p'}$ equals $F_r(S)$ and the kernel of the whole composite equals $\rho(S)$, for $r \geq \dim \mathcal{C} + 1$. That it equals $F_r(S)$ follows from Proposition 5.7 and hence $F_r(P) = F_r(P)$ for $r \geq \dim \mathcal{C} + 1$ by Lemma 5.8. Furthermore these groups are independent of $\mathcal{C}$ with the above assumptions, so taking $\mathcal{C} = S_p(G)$, we see that the kernel the whole composite equals $\rho(S)$ as wanted. □

**Proof of Theorem 5.9.** The theorem is a slightly less elaborate version of Theorem 5.9. □

**Remark 5.10** (The bound $r$ in Theorem 5.9 and sparsity of Sylow-trivial modules). In §7 we provide a detailed analysis of how to find collections $\mathcal{C}$ satisfy the assumptions Theorem 5.9 hence get other bounds on $r$ in the Carlson–Thévenaz conjecture, Theorem 5.9. By Theorem 7.11 and Proposition 7.3 we take $\mathcal{C}$ to be the smallest collection closed under passage to $p$–radical overgroups and containing all the $p$–subgroup $P$ where $N_G(P)/P$ admits an exotic Sylow-trivial module (the $p$–endo-essential subgroups of 7.4, a subset of the $p$–subgroups with $S_p(G)/P$ not simply connected). For a finite group of Lie type of characteristic $p$ this subcollection of $B_p(G)$ identifies with unipotent radicals of parabolic subgroups of rank at most one. As far as we know, this poset could have a uniform dimension bound in general, independent of the finite group $G$. Dually, Theorem 7.13 says that the 2–dimensional collection of rank one and two elementary abelian $p$–subgroups and $S$, satisfies that $H_1(\mathcal{C}/G)_{p'} \xrightarrow{\sim} H_1(\mathcal{C}/G)_{p'}$, but here it is in general not clear that $H_1(\mathcal{C}/G)_{p'} = 0$. The group $G_2(5)$ at $p = 3$ is an example where both bounds in Theorem 5.9 give $r = 3$ and $r = 2$ does not work; see the discussion before Proposition 6.3. We do not know of an example where one cannot take $r = 3$. Indeed, much stronger: To the best of our knowledge, in all finite groups where $T_k(G,S)$ has been calculated, either $\rho(S) = N_{A^p(G)}(S)$ (and hence $T_k(G,S) = \text{Hom}(G,k^\times)$) or $S_p(G)$ is $G$–homotopy equivalent to a 1–dimensional complex (and in the latter case even $rk_p(G) \leq 2$ in the known cases when $G = G_0$). The results of this paper indicates that the the answer to this question in general may have links to many facets of $p$–local finite group theory; see also §§7.4, 7.5.

**Proof of Corollary 7.** Start by noticing that if all $p$–radical subgroups are normal in $S$, then $S$ is a Sylow $p$–subgroup in $N_G(P \leq Q)$, so by the Frattini argument

$$N_G(P \leq Q \leq S)A^p(N_G(P \leq Q)) = N_G(P \leq Q).$$

(*)

This is the more technical assumption given in the last part of the corollary. Now, suppose that $\varphi \in \ker \left( \text{Hom}(N_G(S),k^\times) \to \oplus_{[P]} \text{Hom}(N_G(S) \cap A^p(N_G(P),k^\times)) \right)$

We need to see that this can be extended to an element in the inverse limit. By definition of the limit, we have to specify compatible elements in $\text{Hom}(N_G(P),k^\times)$ for all $p$–radical subgroups in $P$, where compatibility means that they agree on the intersection $N_G(P \leq Q)$. For each chain $P \leq Q < S$ we can consider the restriction $\varphi|_{N_G(P \leq Q \leq S)}$, which, by choice of $\varphi$, is zero on $N_G(P \leq Q < S) \cap A^p(N_G(Q))$. Furthermore, by (*)

$$N_G(P \leq Q \leq S)/N_G(P \leq Q \leq S) \cap A^p(N_G(P \leq Q)) \xrightarrow{\sim} N_G(P \leq Q \leq S)A^p(N_G(P \leq Q))/A^p(N_G(P \leq Q)) \cong N_G(P \leq Q)/A^p(N_G(P \leq Q)),$$

so we get unique elements in $\text{Hom}(N_G(Q),k^\times)$ and $\text{Hom}(N_G(P \leq Q),k^\times)$ for all $P \leq Q$. 

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All these elements were induced by restriction of the map on $N_G(S)$ so they are obviously compatible as is illustrated in the following diagram:

\[ \begin{array}{ccc}
N_G(P \leq Q \leq S) & \rightarrow & N_G(Q) \\
N_G(P \leq Q \leq S) & \rightarrow & N_G(Q \leq S) \\
N_G(P \leq Q) & \rightarrow & N_G(P) \\
\end{array} \]

Note that the map on $N_G(P \leq Q)$ obtained by restriction from either $N_G(P)$ or $N_G(Q)$ agree, since they are induced by the same map on $N_G(P \leq Q \leq S)$ via the isomorphism above. 

We record the dual description of $H_1(\mathcal{E}_p^*(G))$, which also comes out of the above proof:

**Proposition 5.11.** Under the assumptions of Corollary $[\text{CIT}_15$, Thm. 7.1],
\[ H_1(\mathcal{E}_p^*(G)) \cong \text{coker} \left( \bigoplus_{[p]} H_1(N_G(S) \cap A^P(N_G(P)))_{p'} \rightarrow H_1(N_G(S))_{p'} \right), \]
where the sum runs over $N_G(S)$-conjugacy classes of non-trivial $p$-radical subgroups $P < S$. 

**Remark 5.12** (A strong version of $[\text{CIT}_15$, Thm. 7.1]). Suppose that $N_G(S)$ controls $p$-fusion in $G$, and that for each nontrivial $p$-radical subgroup $Q \leq S$
\[ (N_G(S) \cap C_G(Q))A^P(C_G(Q)) = C_G(Q) \]
Then the general assumption $[\text{CIT}_15$, Thm. 7.1$]$ of Corollary $[\text{CIT}_15$, 7.1$]$ is satisfied. Namely $\geq$ is clear and for $\leq$ we note:
\[ N_G(P < Q) = N_G(P < Q < S)C_G(Q) \leq N_G(P < Q < S)A^P(C_G(Q)) \leq N_G(P < Q < S)A^P(N_G(P < Q)) \]
Here the equality is by control of fusion and first inclusion by assumption. This provides a slightly stronger version of $[\text{CIT}_15$, Thm. 7.1$]$, where the condition is only checked on $p$-radical subgroups.

6. Computations

By Theorem $[\text{A}]$ calculating $T_k(G, S)$ amounts to calculating $H_1(\mathcal{E}_p^*(G))$, and we have developed a number of theorems and tools for this in the preceding sections. Formulas such as Theorems $[\text{D}]$ and $[\text{E}]$, make it computable for individual groups, since the input data has often already been tabulated, e.g., in connection with inductive approaches to the Alperin and McKay conjectures. Similarly Theorem $[\text{E}]$ allows us to tap into the large preexisting literature on the fundamental group of subgroup complexes, which has been studied in topological combinatorics, due to its relationship to other combinatorial problems, as well as in finite group theory, where it is related to uniqueness question of a group given its $p$–local structure, and the classification of finite simple groups. Expanding on the summary in $[\text{L}_13$, §6$]$ we will in this section go through different classes of groups, and show how the strategy translates into explicit computations. We only pick some low-hanging fruit, but with a recipe for how to continue.

6.1. Sporadic groups. We complete the general discussion from $[\text{L}_13$, §6$]$ by using Theorem $[\text{E}]$ to determine Sylow-trivial modules for the Monster finite simple group, as a computational example:

**Theorem 6.1.** Let $G = M$ be the Monster sporadic group, and $k$ any field of characteristic $p$ dividing $|G|$. Then
\[ T_k(G, S) \cong \begin{cases} 
0 & \text{for } p \leq 13 \\
\text{Hom}(N_G(S)/S, k^\times) & \text{for } p > 13
\end{cases} \]

The case $p = 2$ is clear since $N_G(S) = S$, and if $p > 13$, $S$ is cyclic so the formula is standard, Corollary $[\text{L}_13$, §2$]$, (with values tabulated in $[\text{LM}_15$, Table 5$]$). We prove the remaining cases below, which were left open in the recent paper $[\text{LM}_15$, Table 3$]$, using our formulas:
Proof of Theorem 6.1 for $p = 3, 5, 7, 11, 13$. There is some choice in methods, since several of our theorems can be used. For primes $p = 3, 5, 7, 11$ the easiest is probably to observe that in all cases $\text{colim}^0_{V \in \mathcal{F}_p \mathcal{A}_p} H_1(C_G(V)) = 0$ and $H_1(F_1^p(G)) = 0$, and then appeal to the centralizer decomposition Theorem 8 or more precisely the homological Proposition 5.4 and Theorem A. The vanishing statements will follow by a coup d'œil at the standard data about the Monster from [W1888 and AW10] (correcting Yos05), and the ATLAS; in fact even the $H_1$'s that appear in centralizer colimit vanish, and $H_1(F_1^p(G)) = 0$ by the vanishing criterion of Corollary 4.22.

In detail:

For $p = 3$: According to the ATLAS there are 3 conjugacy classes of subgroups of order 3, with the following centralizers: $C_G(3A) = 3 \times F_{24}^d$, $C_G(3B) = 3_+^{1+12} 2 S u z$, $C_G(3C) = 3 \times T h$. All of these have zero $H_1(-,3)$, since $F_{24}^d$, $2 S u z$ and $T h$ are perfect. Hence trivially colim = 0. We want to use Corollary 4.22 to see that also $\pi_1(F_1^p(G)) = 1$. By [AW10] Table 2] $N_G(S) = S : (2^2 \times S D_{16})$ a subgroup of $N_G(3A^3) = 3^{3+2+1+1+4} (L_3(3) \times S D_{16})$. Hence $S D_{16}$ acts trivially on $3A^3$ and $2^2$ acts as the diagonal matrices in $S L_Q(F_3)$ and is generated by elements that fix a non-trivial element in $3A^3$. We conclude by Corollary 4.22 that $\pi_1(F_1^p(G)) = 1$.

For $p = 5$: There are two conjugacy classes of subgroups of order 5 with centralizers $C_G(5A) = 5 \times H N$ and $C_G(5B) = 5_+^{1+16} : 4 J 2$, which have zero $H_1(-,5)$, since $H N$ and $4 J 2$ are perfect. Hence colim = 0. For $\pi_1(F_2^5(G)) = 1$, note that $N_G(S) = S : (8_3 \times 4^2)$ inside $N_G(5B^2) = 5_+^{2+1+1+1} (8_3 \times G L_2(5))$. Hence $8_3$ acts trivially on $5B^2$, and $4^2$ is generated by elements which act with a non-trivial fixed-point on $5B^2$, so the conclusion again follows by Corollary 4.22.

For $p = 7$: There are 2 conjugacy classes of subgroups of order 7 with centralizers $C_G(7A) = 7 \times H e$ and $C_G(7B) = 7^{1+4} : 2 S 7$, both of which have vanishing $H_1(-,7)$, so colim = 0. By [W1888, Thm. 7] and [AW10, Table 1], $N_G(S) = S : 6^2$ inside $N_G(7B^2) = 7^{2+1+1+2} : G L_2(7)$, so again we can use Corollary 4.22.

For $p = 11$: We have just one conjugacy class of subgroups of order 11 with $C_G(11A) = 11 \times M_{12}$, which satisfy $H_1(C_G(11A))_{11} = 0$. By [AW10, Table 1], $N_G(S) = N_G(11A^2) = 11^2 : (5 \times 2 A_5)$. We want to see that $H_1(N_G(S)/S)$ is generated by elements which commute with a non-trivial element in $S$, so that we can apply Corollary 4.22. For this we describe the action more explicitly: Note that $(5 \times 2 A_5) \cap S L_2(11)$ (by the classification of maximal subgroups of $P S L_2(11)$, say), so $(5 \times 2 A_5) \cap S L_2(11) = 2 A_5$. Furthermore the 5-factor has to lie in the center of $G L_2(11)$, since it commutes with $2 A_5$, and otherwise the action of $2 A_5$ on $11^2$ would be reducible. In matrices we can hence write a generator of the 5-factor as $\text{diag}(\alpha, \alpha)$, where $\alpha$ is a primitive 5th root of unity in $F_1^{11}$. However since $2 A_5$ is a subgroup of $S L_2(11)$ and has order divisible by 5, it contains up to conjugacy in $G L_2(11)$ the element $\text{diag}(\alpha, \alpha^{-1})$. Hence $\text{diag}(\alpha^2, 1) \in N_G(S)/S \leq G L_2(11)$ generates $H_1(N_G(S)/S)$ and centralizes a non-trivial element is $S$. We conclude that $H_1(F_1^{11}(G)) = 0$ as wanted by Corollary 4.22.

For $p = 13$: We use Theorem $\text{[H]}$ by [AW10, Table 1] all $p$–centric $p$–radicals are centric, so the assumptions of the last part of that theorem are satisfied, and by the same reference $N_G(13B) = 13^{1+2} : (3 \times 4 S_4)$. But $\pi_1(F_1^{13}(M)) = 1$, since otherwise there would by [BCG+07, Thm. 5.4] need to exist a subsystem of index 3 or 2, but by [RV04, Thm. 1.1] no such subsystems exist. Hence Theorem $\text{[H]}$ implies that $T_k(G, S) = 0$ as well. \(\square\)

(The remaining sporadic groups left open in [LM15b, Table 5] has subsequently also been dealt with by David Craven using our methods, and in fact he redoes all sporadic groups this way.)

6.2. Finite groups of Lie type. The $p$–subgroup complex of a finite group of Lie type $G$ at the characteristic is simply connected if the Lie rank is at least 3, since it is homotopy equivalent to the Tits building [Qui78, §3]. Theorem $\text{[B]}$ hence implies that there are no exotic Sylow-trivial modules in that case, a result originally found in [CMN06]. (This is also true in rank two, by a small direct computation, using Theorem $\text{[A]}$ and [Sta68, Thm. 12.6(b)],)

Away from the characteristic the $p$–subgroup complex is also expected to be simply connected, if $G$ is “large enough”, but this is not known in general. Stronger yet, the $p$–subgroup complex appears often to be Cohen–Macaulay [Qui78, Thm. 12.4] [Das98] [Das00] [Smi11, §9.4]. Partial results in this direction imply by Theorem $\text{[D]}$ that there are no exotic Sylow-trivial modules in those cases. We give
two examples of this, for $GL_n(q)$ and $Sp_{2n}(q)$. The $GL_n(q)$ case also follows from very recent work of Carlson–Mazza–Nakano, while the $Sp_n(q)$ case is new.

**Theorem 6.2 (CMN14, CMN16).** Let $G = GL_n(\mathbb{F}_q)$ with $q$ prime to the characteristic $p$ of $k$. If $G$ has an elementary abelian $p$–subgroup of rank 3, then $T_k(G, S) \cong \text{Hom}(G, k^\times)$.

**Proof.** It was proved by Quillen [Qui78, Thm. 12.4], that $|S_p(G)|$ is simply connected (and in fact Cohen-Macaulay) under the stated assumptions if $p \nmid q - 1$, and when $p \nmid q - 1$ it is shown in [Das95, Thm. A]. Hence the result follows from Theorem B.

**Theorem I.** Let $G = Sp_{2n}(q)$, and $k$ a field of characteristic $p$. If the multiplicative order of $q$ mod $p$ is odd, and $G$ has an elementary abelian $p$–subgroup of rank 3, then $T_k(G, S) = 0$.

**Proof of Theorem I.** The main theorem in [Das98] states that $|S_p(G)|$ is simply connected under the assumptions, so the result follows from Theorem B.

That the $p$–rank of $G$ is at least 3 is not enough to ensure simple connectivity in general. At the characteristic $SL_2(\mathbb{F}_{p^r})$ or $SL_3(\mathbb{F}_{p^r})$ shows this, and even away from the characteristic it is not true as $U_4(3) = O_5(3)$ has 2–rank 4, but the 2–subgroup complex is not simply connected; see [Smi11, Ex. 9.3.11]. In this case $H_1(\mathcal{O}_3^+(G)) \cong H_1(G(2)_2) = 0$ though, just by virtue of the Sylow 2–subgroup being its own normalizer. See e.g., [Asc93, Section 9.3] for more on simply connectivity.

As mentioned in the introduction, in joint work in progress with Carlson, Mazza, and Nakano we classify all Sylow-trivial modules for finite groups of Lie type using the more detailed theorems of this paper together with the “$\Phi$–local” approach to finite groups of Lie type. Here we will just do one more example, namely $G_2(5)$ at $p = 3$, which is one of the borderline cases where $H_1(\mathcal{O}_3^+(G)) = 0$, but $\mathcal{A}_p(G)$ is one-dimensional and not simply connected. It is furthermore interesting since Carlson–Thévenaz observed via computer that $ρ^2(S) \neq ρ^3(S) = ρ^\infty(S) = N_G(S)$ in [CT15]. As required, the assumptions of Corollary C are not satisfied, as all $p$–subgroups turn out to be $p$–radical, but one subgroup of order 3 is not normal in $S$. The group can however be easily calculated using either Theorem D (with $C$ either $B_3(G)$ or $A_3(G)$) or Theorem E (using Proposition 4.31)—we choose the first, slightly more cumbersome, method, to see exactly how $ρ^2(S) \neq ρ^3(S) = ρ^\infty(S) = N_G(S)$.

**Proposition 6.3.** For $k$ of characteristic 3, $T_k(G_2(5), S) = 0$.

**Proof.** Looking at the $\text{ATLAS}$, we see that $N_G(S) \cong 3^{1+2} : SD_{16}$ and that there are 2 conjugacy classes of elements of order 3, where $3A$ is a central element in $3^{1+2}$ and $3B$ is a non-conjugate non-trivial element. In fact $N_G(S)$ controls 3–fusion in $G$, as otherwise $3A$ and $3B$ would need to be conjugate (see e.g., [RV04, Lem. 4.1]). We hence have 3 conjugacy classes of proper, non-trivial subgroups $\langle 3A \rangle$, $\langle 3B \rangle$, and $V = \langle 3A, 3B \rangle$ and furthermore conjugacy classes of chains in this case coincide with chains of conjugacy classes. (The subgroups all turn out to be 3–radical, but $\langle 3B \rangle$ is not normal in $S$, so the assumptions of Corollary C are not satisfied.)

We want to see that any element in $H_1(N(G(S)))_3$ is equivalent to zero in the colimit; we do this by considering $H_1(\cdot)_3$ of the following subdiagram:

$$
\begin{array}{cccc}
N_G(3B < V < S) & N_G(V < S) & N_G(3B < S) & N_G(3A < S) \\
N_G(3B < V) & N_G(V < S) & N_G(3B < S) & N_G(3A < S) \\
N_G(3B) & N_G(V) & N_G(3B) & N_G(3A) \\
\end{array}
$$

Note that $N_G(3B < V) = N(3B) \cap N(3A)$, by the description of fusion in $S$. By looking at the order of centralizers of elements and the list of maximal subgroups, as described by the $\text{ATLAS}$, we see that $N_G(V)$ and $N_G(S)$ are contained in $N_G(3A) \cong 3 : U_3(5) : 2$, and that $N_G(3B)$ is contained in the maximal subgroup $N_G(2A) \cong 2 : (\mathfrak{A}_5 \times \mathfrak{A}_5).2 = 2 : A(\mathcal{S}_5 \times \mathcal{S}_5)$, the index 2 subgroup of $2\mathcal{S}_5 \circ 2\mathcal{S}_5$ of even permutations, for $2\mathcal{S}_5$ the central extension with Sylow 2–subgroup $Q_{16}$. Recall also that
$SD_{16} = \langle \sigma, \tau | \sigma^8 = \tau^2 = 1, \tau \sigma \tau = \sigma^3 \rangle \leq \Out(3^{1+2}_+) \cong \GL_2(\mathbb{F}_3)$, with $\sigma^4$ identifying with the non-trivial central element in $\Out(3^{1+2}_+)$ which acts as $-1$ on $3^{1+2}_+/Z(3^{1+2}_+)$ and trivially on $Z(3^{1+2}_+)$. In our model of $3^{1+2}_+$, $\sigma^4$ will hence conjugate $3B$ to $-3B$ and commute with $3A$, and we can furthermore choose the generator $\tau$ so it commutes with $3B$ and conjugates $3A$ to $-3A$. Inside $2 \cdot A(\mathcal{S}_5 \times \mathcal{S}_5)$ we can hence represent $3B$ as $(123)$, $3A$ as $(1'2'3')$, $\sigma^4$ as $(12)(4'5')$ and $\tau$ as $(45)(1'2')$. Hence $N_G(3B) = 2 \cdot A(\mathcal{S}_3 \times \mathcal{S}_2 \times \mathcal{S}_5)$ and $N_G(3B < V) = 2 \cdot A(\mathcal{S}_3 \times \mathcal{S}_2 \times \mathcal{S}_3 \times \mathcal{S}_2)$.

Working inside these groups the diagram identifies as follows:

$$V : (2 \times 2) \quad 3^{1+2} : (2 \times 2) \quad V : (2 \times 2) \quad 3^{1+2} : SD_{16} \quad 3 \cdot U_3(5) : 2$$

(6.1)

$$2 \cdot A(\mathcal{S}_3 \times \mathcal{S}_2 \times (\mathcal{S}_3 \times \mathcal{S}_2)) \quad N_G(V) \quad 3 \cdot U_3(5) : 2$$

The double headed arrows are surjections by a Frattini argument, and we conclude from this that $H_1(N_G(V)) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, since $N_G(V)/C_G(V) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, by the $G$–fusion in $V$. The indicated generators follow since we know how $\sigma^4$ and $\tau$ acts on $3A$ and $3B$ (and hence $V$). The right-hand part of the diagram immediately reveals that $\bar{\sigma}$ is zero in the colimit. The element $\bar{\tau}$ is also zero in the colimit: Namely, consider the element $(1'2')(4'5') \in N_G(3B < V)$. This maps to zero in $H_1(N_G(3B))$, by the above description. But in $H_1(N_G(V))$ it maps to the same element as $\tau$, since it acts the same way on $V$. Hence $\bar{\tau}$ represents the zero element in the colimit as wanted.

6.3. Symmetric groups. The Sylow-trivial modules for the symmetric groups are understood via representation theoretic methods by the work of Carlson–Hemmer–Mazza–Nakano [CMN03] [CHM10] (see also [LM15a] for extensions). Let us point out that simply connectivity of $\mathcal{S}_p(G)$ and our work directly implies their results, at least in the generic case.

Theorem 6.4. If $p$ is odd, and $3p + 2 \leq n < p^2$ or $n \geq p^2 + p$ then $T_k(G, S) \cong \Hom(\mathcal{S}_n, k^\times) \cong \mathbb{Z}/2$.

Proof. In [Kso04] Thm. 0.1] (building on [Kso03] and [Bou02]) it is proved that $\mathcal{A}_p(G)$, and hence $\mathcal{S}_p(G)$, is simply connected if and only if $n$ is in the above range, when $p$ is odd. So the result follows from Theorem [3].

When $p = 2$, $T_k(\mathcal{S}_n, S) = 0$ since $N_{\mathcal{S}_n}(S) = S$ for all $n$ [Wei25 Cor. 2]. It is an interesting exercise to fill in the left-out cases, where $p$ is odd and $n$ is small relative to $p$, using the methods of this paper. Let us just quickly do this calculation where $n = 2p + b$, for $p$ odd and $0 < b < p$, where $T_k(\mathcal{S}_n, S) = \mathbb{Z}/2 \times \mathbb{Z}/2$, as an illustration (one can in fact also do the general case directly this way, without much more effort). The case is of interest e.g., since $n = 7 = 2 \cdot 3 + 1$ is the smallest case where $\mathcal{S}_p(G)$ is connected but not simply connected, and where the naïve guess [Car12] §5 that groups without strongly $p$–embedded subgroup should have no exotic Sylow-trivial modules fails: Pick $S = \langle (12 \ldots p), ((p + 1) \ldots 2p) \rangle$; there is just one $N_G(3)$–conjugacy classes non-trivial 3–radical subgroups, represented by $A = \langle (12 \ldots p) \rangle$. By Proposition 5.3 $H_1(\mathcal{S}_p^*(G))$ equals the colimit
of $H_1(-)_p$ applied to the diagram $N_G(A) \leftarrow N_G(A < S) \to N_G(S)$, or

$$C_p \times C_{p-1} \times \mathfrak{S}_{p+b} \leftarrow (C_p \times C_{p-1})^2 \times \mathfrak{S}_b \to (C_p \times C_{p-1}) \times C_2 \times \mathfrak{S}_b,$$

which is seen to be $\mathbb{Z}/2 \times \mathbb{Z}/2$. Hence by Theorem $\text{[A]}$ $T_k(\mathfrak{S}_{2p+b}, S) = \text{Hom}(H_1(\mathfrak{S}_p^*(\mathfrak{S}_{2p+b})), k^\times) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ as wanted. Alternatively one may use the centralizer decomposition, using that $|\mathcal{F}_p^*(G)|$ is contractible, since it is one-dimensional with $\pi_1(\mathcal{F}_p^*(G)) = 1$ by Corollary $\text{[4.22]}$.

To come full circle in the above example, we can also use Proposition $\text{[7.4]}$ (or Theorem $\text{[4.9]}$ directly) to determine $\pi_1(\mathfrak{S}_p^*(\mathfrak{S}_{2p+b}))$ by describing the effect of adding the subgroup $A$—this reveals that

$$\pi_1(\mathfrak{S}_p^*(\mathfrak{S}_{2p+b})) \cong C_{p-1} \times C_2 \times \mathfrak{S}_b/(C_{p-1} \wr (C_2 \times \mathfrak{S}_b)) \cap (1 \times \mathfrak{A}_{p+b}) \cong \begin{cases} D_8 & \text{if } b = 1 \\ C_2 \times C_2 & \text{if } 1 < b < p \end{cases}$$

(6.3) still with $p$ odd. For $b = 0$, $G_0 = \mathfrak{S}_p \wr C_2$, and we likewise get $\pi_1(\mathfrak{S}_p^*(\mathfrak{S}_{2p+b})) \cong D_8$.

Notice that by Corollary $\text{[4.15]}$ the full $\pi_1$ calculation for $\mathfrak{S}_n$ enables us to get the corresponding result for $\mathfrak{A}_n$; let us dwell on this for a moment since it both illustrates some of the preceding formulas, and lets us correct a small mistake in the literature: Notice that, in the above notation, the generator of a $C_{p-1}$ and the wreathing $C_2$ correspond to odd elements inside $\mathfrak{S}_{2p+b}$. Hence their product is even and defines an element in $\ker(\pi_1(\mathfrak{S}_p^*(\mathfrak{S}_{2p+b})) \to \mathbb{Z}/2)$, which is of order 4 when $b = 0, 1$. So by Corollary $\text{[4.15]}$

$$\pi_1(\mathfrak{S}_p^*(\mathfrak{A}_{2p+b})) \cong \begin{cases} C_4 & \text{if } b = 0, 1 \\ C_2 & \text{if } 1 < b < p \end{cases} \quad (6.4)$$

still with $p$ odd. In particular

$$T_k(\mathfrak{A}_{2p+b}, S) \cong \text{Hom}(\pi_1(\mathfrak{S}_p^*(\mathfrak{A}_{2p+b})), k^\times) \cong \begin{cases} \mathbb{Z}/4 & \text{if } b = 0, 1 \\ \mathbb{Z}/2 & \text{if } 1 < b < p \end{cases} \quad (6.5)$$

when $p$ is odd and $k$ has a 4th root of unity. The formula $\text{[6.5]}$ for $p > 3$ and $b = 0, 1$ corrects $\text{[CMN09, Thm. 1.2(c)]}$, which states $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ (the mistake being that the constructed modules are not all self-dual).

Notice also that Corollary $\text{[3.8]}$ gives a description of the exotic generator as $\Omega^{-1} H_1(\mathcal{S}_p(\mathfrak{S}_{2p+b}); k^\times)$, with $H_1(\mathcal{S}_p(\mathfrak{S}_{2p+b}); k^\times) \cong \ker(\kappa_1^G \oplus \kappa_1^G |_{N_G(V < S)} \to \kappa_1^G |_{N_G(V)} \oplus \kappa_1^G |_{N_G(S)})$. E.g., taking $n = 7$ and $p = 3$, $H_1(\mathcal{S}_3(\mathfrak{S}_7); k^\times)$ is of dimension $35 = 140 - (35 + 70)$ (cf. also $\text{[Bon92, 5.6]}$). This agrees on the level of dimensions with the description in $\text{[CMN09, Prop. 8.3]}$ as the Young module $Y(4, 3)$, which is of dimension 28 with projective cover of dimension $28 + 35 = 63$, as explained to us by Anne Henke. For untwisted coefficients, the representation given by the top homology group of the $p$-subgroup complex has been studied in some generality by Shareshian–Wachs $\text{[Sha04, SW09]}$.

**6.4. $p$–solvable groups.** When $G$ is a $p$–solvable group, it is proved in $\text{[NR12]}$, building on $\text{[CMT11]}$, that $T_k(G, S) \cong \text{Hom}(G_0, k^\times)$, at least when $k$ is algebraically closed, and $G = G_0$ when the $p$–rank is two or more by $\text{[Go70, Thm. 2.2]}$. The proof in $\text{[NR12, CMT11]}$ reduces to the case $G = AH$, where $A$ is an elementary abelian $p$–group, and $H$ is a normal $p$–group, and appeals to the classification of the finite simple groups in the proof of the last statement, albeit in a mild way. By Theorem $\text{[A]}$ the statement is equivalent to the isomorphism $H^1(\mathfrak{S}_p^*(G); k^\times) \cong \text{Hom}(G_0, k^\times)$. We will not reprove this isomorphism here, but would like to make two remarks:

First, we have the following result.

**Theorem 6.5.** Suppose that $G = AH$ with $H$ a normal $p'$–group and $A$ a non-trivial elementary abelian $p$–group. Then $T_k(G, S) \cong \mathbf{lim}^0_{V \in \mathcal{F}_{p}^*(A)} \text{Hom}(C_H(V), k^\times)$.

**Proof.** Note that $\mathcal{F}_{p}^*(A) \cong \mathcal{F}_{p}^*(G)$, and in particular $|\mathcal{F}_{p}^*(G)|$ is contractible. so the result follows from Theorem $\text{[E]}$ also using that $(C_H(V))_{p'} \cong (C_G(V))_{p'}$. $\Box$

It would obviously be interesting to have an identification of the right-hand side with $\text{Hom}(G, k^\times)$, when $A$ has rank at least 2, via a proof which did not use the classification of finite simple groups.

Second, again $T_k(G, S) = \text{Hom}(G_0, k^\times)$ would follow from the simply connectivity of $\mathcal{S}_p(G)$, by Theorem $\text{[E]}. When G = AH$ as above, Quillen $\text{[Qui78, Prob. 12.3]}$ conjectures that $\mathcal{S}_p(G)$ should in fact be Cohen–Macaulay, and in particular simply connected when $A$ has $p$–rank at least 3, and
proved this when \( G \) is actually solvable \([Qui78, \text{ Thm. 11.2(i)}]\). Aschbacher also conjectured the simple connectivity in \([Asc93]\), and reduced the claim to where \( H = F^\ast(G) \) is a direct product of simple components being permuted transitively by \( A \), see also \([Smi11, \S 9.3]\)–Aschbacher uses the commuting complex \( K_p(G) \), but this is \( G \)-homotopy equivalent to \( S_p(G) \); see e.g., \([Gro02, \text{ p. 431}]\) or \([Smi11, \S 9.3]\).

(The related \([Qui78, \text{ Thm. 11.2(ii)}]\), giving non-zero homology in the top dimension, has in fact been proved this when \( G \) is not of the form \( G = AH \), the complex \( S_p(G) \) need not be simply connected even for large \( p \)-rank; see \([PW00, \text{ Ex. 5.1}]\), so a direct proof in the \( p \)-solvable case would need one of our more precise theorems. (The homology of \( S_p(G) \) when \( G \) is solvable is described in \([PW00, \text{ Prop. 4.2}]\), and when \( G \) is \( p \)-solvable, the \( p \)-essential subgroups are those \( p \)-radical subgroups \( P \) where \( N_G(P)/P \) has \( p \)-rank one by \([Pui76, \text{ Cor. to Prop. II.4}]\).) As noted in \([1.5]\), one may hope to get vanishing results for \( \pi_1(\mathcal{G}^p(G)) \) for any finite group \( G \) by reducing to simple groups, by suitably generalizing the \( p \)-solvable case.

7. Appendix: Varying the collection \( \mathcal{C} \) of subgroups

In this appendix we describe how the homotopy and homology of \( \mathcal{G}_C \) and the other standard categories of this paper behave under changing the collection \( \mathcal{C} \). The results are often extensions of results from mod \( p \) homology decompositions \([Dwy97, Gro02, GS06]\), but now working integrally or away from \( p \), and also examining low-dimensional behavior. The results are referred to throughout the paper when moving to collections smaller than \( \mathcal{S}_p(G) \). We start in \([7.1]\) by recalling various \( G \)-categories associated to \( \mathcal{C} \). In \([7.2]\) we explain how the homotopy type of our categories change upon removing a given subgroup (postponing parts of the proof to \([7.6]\)). We then use it in \([7.3]\) to show that the homotopy types agree for various collections, and in \([7.4]\) to analyze the low-dimensional homotopy type, when more subgroups are removed. Finally \([7.5]\) describes the set of components of \( \mathcal{C} \) and its relation to group theoretic notions of strongly \( p \)-embedded subgroups.

7.1. \( G \)-categories associated to collections of subgroups. Recall the categories \( \mathcal{G}_C(G), \mathcal{F}_C(G) \) and \( \mathcal{F}_C(G) \) from \([2.2]\). We now introduce “fattened up” versions of \( \mathcal{C} \) related to these, also used in e.g., \([Dwy97, Gro02]\) – they will be preordered sets, meaning a category with at most one map between any two objects; preordered sets are equivalent as categories to posets, but usually not equivariantly so. Let \( \mathcal{E}(\mathcal{C}) \) be the category of “pointed \( G \)-sets”, i.e., the \( G \)-category with objects \( (G/P, x) \) for \( P \in \mathcal{C} \), and \( x \in G/P \), and morphisms maps of pointed \( G \)-sets. The group \( G \) acts on objects by \( g \cdot (G/P, x) = (G/P, gx) \). Similarly \( \mathcal{E}(\mathcal{A}) \) is the category with objects monomorphisms \( i : P \to G \) induced by conjugation in \( G \), with \( P \in \mathcal{C} \), and morphisms \( i \to i' \) given by the group homomorphisms \( \varphi : P \to P' \), induced by conjugation in \( G \), such that \( i = i' \varphi \); see also e.g., \([Gro02, \S 2.8]\). \( \mathcal{E}(\mathcal{A}) \) would most logically be called \( \mathcal{F}_C \) in the current notation, but we keep the traditional name, to avoid confusion with other standard terminology.) By inspection of simplices, as already observed in e.g., \([Gro02, \text{ Prop. 2.10}]\),

\[
|\mathcal{E}(\mathcal{C})|/G = |\mathcal{G}_C(G)| \quad \text{and} \quad |\mathcal{E}(\mathcal{A})|/G = |\mathcal{F}_C(G)|
\]  

(7.1)

We also introduce the \( \mathcal{F}_C \)-version \( \mathcal{E}(\mathcal{A}) \): the objects are monomorphisms \( i : P \to G \) modulo conjugation in \( P \), for \( P \in \mathcal{C} \); the morphisms are group homomorphisms \( \varphi : P \to P' \), modulo conjugation in \( P \), such that \( i \) is \( P \)-conjugate to \( i' \varphi \). Here similarly \( |\mathcal{E}(\mathcal{A})|/G = |\mathcal{F}_C(G)| \).

Note also the \( G \)-equivariant functors

\[
\mathcal{E}(\mathcal{C}) \to \mathcal{C} \leftarrow \mathcal{E}(\mathcal{A}) \leftarrow \mathcal{E}(\mathcal{A})
\]  

(7.2)

given by \( (G/P, x) \mapsto G_x \) and \( (i : Q \to G) \mapsto i(Q) \). These are all equivalences of categories but not in general \( G \)-equivalences (non-equivariant functors the other way are given by \( P \mapsto (G/P, e) \) and \( P \mapsto (P \to G) \)). We can use the maps to \( \mathcal{C} \) to describe the fixed-points, as one checks that we have equivalences of categories

\[
\mathcal{E}(\mathcal{C})_H \to \mathcal{C}_{\geq H}, \quad \mathcal{E}(\mathcal{A})_C \to \mathcal{C}_{\leq \mathcal{C}(G \mathcal{H})}, \quad \text{and} \quad \mathcal{E}(\mathcal{A})_H \to \{Q \in \mathcal{C} | H \leq C_G(Q)Q\}
\]

(7.3)

(See also \([GS06, \S 1]\).) We record the following relationship:

**Lemma 7.1.** Let \( G \) be a finite group, and consider collections of \( p \)-subgroups \( \mathcal{C}' \leq \mathcal{C} \).

(a) If \( |\mathcal{C}'| \searrow |\mathcal{C}| \), then \( |\mathcal{G}(\mathcal{C})| \searrow |\mathcal{G}(\mathcal{C})| \) and \( \pi_1(\mathcal{G}_C^p(G))_{\mathcal{C}'} \cong \pi_1(\mathcal{G}_C^p(G))_{\mathcal{C}'} \).
(b) If $|E\theta_C| \to |E\theta_C'|$ is a $G$–homotopy equivalence then $|\theta_C'(G)| \sim |\theta_C(G)|$.

(c) If $|EA_C| \to |EA_C'|$ is a $G$–homotopy equivalence then $|F_C'(G)| \sim |F_C(G)|$.

(d) If $|EA_C| \to |EA_C'|$ is a $G$–homotopy equivalence then $|\bar{F}_C'(G)| \sim |\bar{F}_C(G)|$.

Proof. For (a) consider the diagram

\[
\begin{array}{c}
|\mathcal{C}_C'(G)| \xrightarrow{\sim} |C'_G| \\
\downarrow \downarrow \downarrow \\
|\mathcal{C}_C(G)| \xrightarrow{\sim} |C_G| \\
\end{array}
\]

The horizontal equivalences are by Remark 2.2 and the vertical equivalences follow since $|\mathcal{C}'| \to |\mathcal{C}|$ is a $G$–equivariant map, assumed to be a homotopy equivalence, and hence induces a homotopy equivalence between Borel constructions. Now the statement about fundamental groups follow from the first claim, using Proposition 4.5. Point (b) follows from (7.1), since a $G$–homotopy equivalence induces a homotopy equivalence on orbits $|\theta_C'(G)| = |E\theta_C'/G| \sim |E\theta_C'|/G = |\theta_C(G)|$. Likewise (c) and (d) follow from the similar observations for $|F_C(G)|$ and $|\bar{F}_C(G)|$.

7.2. Removing a single conjugacy class.

The next result describes the effect of removing a conjugacy class of subgroups, called “pruning” in \cite{Dwy98, Dwy}; see also \cite{GS06, Lem. 2.5}. Denote by $(E\theta_C)'_P$ the full subcategory of pairs $(G/Q,x)$ such that there exists a non-isomorphism $(G/Q,x) \to (G/P,e)$, i.e., $G_x > P$, and similarly for $(E\theta_C)_P$ and $EA_C$ and $EA_C$.

**Proposition 7.2** (Pruning collections). Let $\mathcal{C}'$ be a collection of $p$–subgroups obtained from a collection $\mathcal{C}$ by removing all $G$–conjugates of a $p$–subgroup $P \in \mathcal{C}$. We have the following five $G$–homotopy pushout squares, with corresponding quotient homotopy pushout squares:

\[\begin{align*}
G \times_N (|\mathcal{C}_{<P}| \star |\mathcal{C}_{>P}|) & \to |\mathcal{C}'| \\
G/N & \to |\mathcal{C}| \\
G \times_N (EN \times (|\mathcal{C}_{<P}| \star |\mathcal{C}_{>P}|)) & \to EG \times |\mathcal{C}'| \\
G \times_N EN & \to EG \times |\mathcal{C}| \\
G \times_N (EW \times ((|E\theta_C|_{<P}| \star |E\theta_C|_{>P}|)) & \to |E\theta_C'| \\
G \times_N EW & \to |E\theta_C'| \\
G \times_N (EW' \times ((|EA_C|_{<P}| \star |(EA_C)_{>P}|)) & \to |EA_C'| \\
G \times_N EW' & \to |EA_C'| \\
G \times_N (EW'' \times ((|EA_C|_{<P}| \star |(EA_C)_{>P}|)) & \to |\bar{F}_C'| \\
G \times_N EW'' & \to |\bar{F}_C'| \\
\end{align*}\]

\[\begin{align*}
(|\mathcal{C}_{<P}|/P \star |\mathcal{C}_{>P}|)/W & \to |\mathcal{C}'|/G \\
|\mathcal{C}_{<P}|/P \star |\mathcal{C}_{>P}| & \to |\mathcal{C}'|/G \\
|\mathcal{C}_{<P}|/P \star |\mathcal{C}_{>P}| & \to |\mathcal{C}|/G \\
|\mathcal{C}_{<P}|/P \star |\mathcal{C}_{>P}| & \to |\mathcal{C}|/G \\
|\mathcal{C}_{<P}|/P \star |\mathcal{C}_{>P}| & \to |\mathcal{C}|/G \\
\end{align*}\]

where $C = C_G(P)$, $N = N_G(P)$, $W = N/P$, $W' = N/C$, and $W'' = N/PC$.

The proof of Proposition 7.2 is not hard, but is postponed to \S 7.6 to not interrupt the flow. We move on to draw consequences, starting with Symonds’ theorem, ‘Webb’s conjecture’ [Web87, Conj. 4.2], on the contractibility of the orbit space of the $p$–subgroup complex, used in the proofs of Theorems D and E.
Proposition 7.3 ([Sym98]). If \( G \) is a finite group and \( \mathcal{C} \) a non-empty collection of \( p \)-subgroups, closed under passage to \( p \)-radical overgroups, then \( |\mathcal{C}|/G \) is contractible.

Proof. First note that by Lemma 7.10 we may, without loss of generality, assume that \( \mathcal{C} \) is closed under passage to all \( p \)-overgroups. If \( \mathcal{C} \) consists of just all Sylow (i.e., maximal) \( p \)-subgroups, then the claim amounts exactly to the part of Sylow’s theorem saying that these subgroups are all conjugate. We will prove the claim in general by seeing that contractibility is preserved under adding conjugacy classes of subgroups in order of decreasing size, using Proposition 7.2(1b) by an induction on the size of the group: Suppose the claim is true for all groups \( G \) of strictly smaller order, and that \( \mathcal{C} \) is obtained from \( \mathcal{C'} \) by adding \( G \)-conjugates of a subgroup \( P \) this way. We can also assume that \( P \) is non-trivial, since otherwise the claim is clear, as \( \mathcal{C} \) is equivariantly contractible to the trivial subgroup in that case.

Now consider the pushout square of Proposition 7.2(1b): The top left-hand corner becomes \( |\mathcal{C}_{>P}|/W \), for \( W = N_G(P)/P \). But \( |\mathcal{C}_{>P}| \) has a \( W \)-equivariant deformation retraction onto \( |\mathcal{S}_p(W)| \) via \( Q \mapsto N_Q(P)/P \), as observed by Quillen [Qui78, Prop. 6.1] (see also Lemma 7.10). But \( |\mathcal{S}_p(W)|/W \) is contractible by our induction hypothesis, observing that \( W \) is of smaller order since \( P \) is non-trivial and that \( \mathcal{S}_p(W) \) is non-empty as we have already added the Sylow \( p \)-subgroups. Hence \( |\mathcal{C}'|/G \to |\mathcal{C}|/G \) is a homotopy equivalence, and we conclude that the claim is true for all collections \( \mathcal{C} \) as in the proposition, proving the claim for \( G \).

Let us spell out what Proposition 7.2(3b) says when the subgroup \( P \) we remove is minimal.

Proposition 7.4 (The effect of adding or removing a minimal subgroup). Suppose that \( \mathcal{C} \) is a collection of \( p \)-subgroups in a finite group \( G \), closed under passage to \( p \)-radical overgroups, and that \( \mathcal{C}' \) is obtained from \( \mathcal{C} \) by removing the conjugacy class of a minimal \( p \)-subgroup \( P \in \mathcal{C} \). Set \( W = N_G(P)/P \). Then \( |\mathcal{O}_C(G)| \) can be described via a homotopy pushout square

\[
\begin{array}{ccc}
|T_p(W)| & \longrightarrow & |\mathcal{O}_{C'}(G)| \\
\downarrow & & \downarrow \\
BW & \longrightarrow & |\mathcal{O}_C(G)|
\end{array}
\]

Here the top horizontal map identifies with the one induced by the composite \( T_p(W) \to \mathcal{O}_C^*(W) \to \mathcal{O}_C(G) \) where the first map is the natural one from §2.2 and the second sends a finite \( W \)-set \( X \) to \( G/P \times_W X \) (assuming without loss of generality that \( \mathcal{C} \) is closed under all overgroups). In particular

1. \( \pi_1(\mathcal{O}_C(G)) \cong \pi_1(\mathcal{O}_{C'}(G))/\text{im}(K) \), with \( K = \ker(\pi_1(\mathcal{O}_p^*(W)) \to W_{p'}) \) and the overline denoting normal closure.
2. \( H_1(\mathcal{O}_C(G)) \cong H_1(\mathcal{O}_{C'}(G))/\text{im}(K) \), with \( K = \ker(H_1(\mathcal{O}_p^*(W)) \to H_1(W)) \).

Hence

3. If \( \pi_1(\mathcal{O}_p^*(W)) \xrightarrow{\cong} W_{p'} \) then \( \pi_1(\mathcal{O}_C(G)) \xrightarrow{\cong} \pi_1(\mathcal{O}_C(G)) \).
4. If \( H_1(\mathcal{O}_p^*(W)) \xrightarrow{\cong} H_1(W) \) then \( H_1(\mathcal{O}_C(G)) \xrightarrow{\cong} H_1(\mathcal{O}_C(G)) \).

Proof. The homotopy pushout square is a special case of Proposition 7.2(3b), since the left-hand corner identifies with \( |T_p(W)| \) via the homotopy equivalences \( |\mathcal{C}_{>P}|/hW \cong \mathcal{S}_p(W)/hW \cong |T_p(W)| \), using Lemma 7.10 and (1.6). Now (1) is a consequence of van Kampen’s theorem [Hat02 §1.2]: By Proposition 4.1 both fundamental groups on the right-hand side of the pushout square are finite \( p' \)-groups, so van Kampen’s theorem and (1.5) produces a pushout of groups

\[
\begin{array}{ccc}
\pi_1(\mathcal{O}_p^*(W)) & \longrightarrow & \pi_1(\mathcal{O}_{C'}(G)) \\
\downarrow & & \downarrow \\
W_{p'} & \longrightarrow & \pi_1(\mathcal{O}_C(G))
\end{array}
\]

with vertical maps surjective, and (1) follows. Point (2) follows similarly, but using the Mayer–Vietoris sequence instead. Points (3) and (4) are now obvious. \( \square \)
7.3. Varying the collection without changing homotopy types. We now see how we can vary our collection $\mathcal{C}$ without changing the homotopy type of associated categories. The omnibus Theorem 7.5 below is mainly a translation of "classical" homotopy equivalences [GS06 Thm. 1.1], using the elementary Lemma 7.1, see also [CMIT15] for $\mathcal{F}$, and [GS06] for historical references.

As usual let $S_p(G)$ and $A_p(G)$ denote non-trivial $p$–groups and non-trivial elementary abelian $p$–subgroups $V \cong (\mathbb{Z}/p)^r$ respectively. The collection $B_p(G)$ is the collection of non-trivial $p$–radical subgroups, i.e., non-trivial $p$–subgroups $P$ such that $O_p(N_G(P)) = P$. Furthermore, let $C_p$ denote the collection of $p$–centric subgroup i.e., the $p$–subgroups $P$ such that $Z(P)$ is a Sylow $p$–subgroup of $C_G(P)$ (implying $C_G(P) = Z(P) \times O_p'(C_G(P))$ by Schur–Zassenhaus), and let $D_p(G)$ denote the collection of "principal $p$–radical", or "$\mathcal{F}$–centric $\mathcal{F}$–radical" subgroups, i.e., those $p$–centric subgroups such $N_G(P)/PC_G(P)$ does not contain a non-trivial normal $p$–subgroup, a subcollection of $C_p \cap B_p$, see [Gro02] Ex. 3.15]. Here and elsewhere the superscript $e$ means that we do not exclude the trivial subgroup.

**Theorem 7.5.** Let $G$ be a finite group and $\mathcal{C}$ a collection of $p$–subgroups.

1. Suppose $\mathcal{C}$ is closed under passage to $p$–radical overgroups. Then $|\mathcal{C} \cap B_p(G)| \sim |\mathcal{C}|$ and $|E\mathcal{G}_{\mathcal{C} \cap B_p(G)}(G)| \sim |E\mathcal{G}_{\mathcal{C}}|$ are $G$–homotopy equivalences and consequently

$$|\mathcal{F}_{\mathcal{C} \cap B_p(G)}(G)| \sim |\mathcal{F}_{\mathcal{C}}(G)|$$

If also $\mathcal{C} \subseteq C_p$, then $|E\mathcal{A}_{\mathcal{C} \cap B_p(G)}| \sim |E\mathcal{A}_{\mathcal{C}}|$ is a $G$–homotopy equivalence and likewise

$$|\mathcal{F}_{\mathcal{C} \cap D_p(G)}(G)| \sim |\mathcal{F}_{\mathcal{C}}(G)|.$$

2. Suppose $\mathcal{C}$ is closed under passage to non-trivial elementary abelian subgroups. Then

$$|\mathcal{C} \cap A_p(G)| \sim |\mathcal{C}|$$

and

$$|E\mathcal{A}_{\mathcal{C} \cap A_p(G)}(G)| \sim |E\mathcal{A}_{\mathcal{C}}(G)| \sim |E\mathcal{A}_{\mathcal{C}}|$$

are $G$–homotopy equivalences. Consequently $|\mathcal{F}_{\mathcal{C} \cap A_p(G)}(G)| \sim |\mathcal{F}_{\mathcal{C}}(G)|$ and

$$|\mathcal{F}_{\mathcal{C}}(G)| \sim |\mathcal{F}_{\mathcal{C} \cap A_p(G)}(G)| = |\mathcal{F}_{\mathcal{C} \cap A_p(G)}(G)| \sim |\mathcal{F}_{\mathcal{C}}(G)|$$

Proof. To prove (1), we first make a reduction: It suffices to prove that the categories $\mathcal{C}$, $E\mathcal{G}_{\mathcal{C}}$, or $E\mathcal{A}_{\mathcal{C}}$ are $G$–homotopy equivalent to the categories indexed by a collection $\mathcal{C}^\prime$ obtained from $\mathcal{C}$, by adding a single conjugacy class $[P]$ to $\mathcal{C}$, where $P$ is a non-$p$–radical subgroup with the property that $\mathcal{C}$ contains all larger $p$–groups superconjugate to $P$. Namely, with this we can add conjugacy classes of non-$p$–radical subgroups in order of decreasing size, to produce a $G$–homotopy equivalences from the categories indexed on $\mathcal{C}$ to categories indexed on a larger collection, closed under passage to all $p$–overgroups, obtained by adding conjugacy classes of non-$p$–radical subgroups to $\mathcal{C}$; we can then apply this to both $\mathcal{C}$ and $\mathcal{C} \cap B_p(G)$, showing that the categories are both $G$–homotopy equivalent to categories on a common larger collection, closed under passage to all $p$–overgroups.

We can now start to prove (1). Assume that $\mathcal{C}$ and $\mathcal{C}^\prime$ are as above, so that $\mathcal{C}^\prime \subseteq \mathcal{C}$ and $\hat{\mathcal{C}}^\prime \subseteq \hat{\mathcal{C}}$. This poset is $N_{\mathcal{G}(P)}(\mathcal{P})$–contractible via the standard contraction

$$Q \geq N_{\mathcal{Q}}(\mathcal{P}) \leq N_{\mathcal{Q}}(\mathcal{P})O_{\mathcal{P}}(N_{\mathcal{G}(P)}) \geq O_{\mathcal{P}}(N_{\mathcal{G}(P)})$$

of Quillen [Qui78] and Bouc [Bou84]. Hence by Proposition 7.2(1a)(3a) (or [GS06 Lem. 2.5(1)(3)]) $|\mathcal{C}^\prime| \sim |\mathcal{C}|$ and $|E\mathcal{G}_{\mathcal{C}^\prime}| \sim |E\mathcal{G}_{\mathcal{C}}|$ are $G$–homotopy equivalences, and the homotopy equivalences on $\mathcal{F}$ and $\mathcal{G}$ follow from this and Lemma 7.1, (1) (or Proposition 7.2(2b)(3b) directly).

To prove the $G$–homotopy equivalence $|E\mathcal{A}_{\mathcal{C} \cap B_p(G)}(G)| \sim |E\mathcal{A}_{\mathcal{C}}|$ in the last part of (1), it is, by Proposition 7.2(5a), enough to see that $(E\mathcal{A}_{\mathcal{C}})^\mathcal{P}$ is $PC(G)$–contractible, with $P$ and $\mathcal{C}$ as in the inductive setup explained above. Note that $P$ acts trivially by definition. By equation (7.3), $(E\mathcal{A}_{\mathcal{C}})^\mathcal{P}$ is equivalent to $\{Q \in C_p | H \leq QC_G(Q)\}$. By assumption $\mathcal{C}$ consists of $p$–centric subgroups, so $C_G(P) = Z(P) \times K$ for a $p$–group $K = O_p'(C_G(P))$. Since $K$ and $O_p(N_{\mathcal{G}(P)})$ normalize each other, and are of coprime order, they commute. The same is true for $K$ and $N_{\mathcal{Q}}(\mathcal{P})$, by the same argument, where $K$ normalizes $N_{\mathcal{Q}}(\mathcal{P}) = Q \cap N_{\mathcal{G}(P)}$ since $K \leq QC_G(Q)$. Hence for any $H \leq PC_G(P) = PK$ the standard contraction (7.5) above induces a contraction of $\{Q \in C_p | H \leq QC_G(Q)\}$ as wanted.

The statement about $\mathcal{F}$ with $\mathcal{C}_p(G) \cap B_p(G)$ instead of $D_p(G)$ follows the $E\mathcal{A}$ statement together with Lemma 7.1(4). To get the stronger statement with $D_p(G)$ we argue more carefully,
showing that |(EAC(C(G)) > P) / C(G) is contractible and using Proposition 7.2(5b): First note that
|EAC(NC(C(G)) > P) / C(G)| is a NC(C(G))–homotopy equivalence, since intersecting with NC(C(G))
gives a deformation retract in the other direction, as for subgroup complexes [Qui78 Prop. 6.1];
then note that |EAC(NC(C(G)) > P) / K = |EAC(NC(C(G)) / K)| > P), using as above that if K normalizes a
p–subgroup, then it centralizes it (see also [Gro02 Prop. 5.7]); but, by (7.2), |EAC(NC(C(G)) > P)| is
homotopy equivalent to |SP NC(C(G)) / PK)|, which is contractible, as P is not in Dp(G) by assumption.

Point [2]: First notice that by the same argument as in [1], comparing C and C ∩ C(C(G)) to a
larger collection closed under passage to non-trivial p–subgroups, it is enough to prove that we have a
G–homotopy equivalences between the categories on C and the categories C, where C is obtained from
C by a adding a conjugacy class of elementary abelian p–subgroups [P], such that C contains all
proper non-trivial subgroups of conjugates of P.

For the homotopy G–homotopy equivalence |C| → |C| it is by Proposition 7.2(1a) enough to see
that |C| is N(C(G))–equivariantly contractible. If C contains the trivial subgroup, then the claim is
obvious, so we can assume that this is not the case and C = SP NC(G) < P. This is N(C(G))–equivariantly
contractible by the standard contraction of Quillen Q ≤ QΦ(P) ≥ Φ(P), where Φ(P) is the Frattini
subgroup, generated by commutators and pth powers and QΦ(P) < P since P is not elementary
abelian. For the G–homotopy equivalence |EAC(C)| → |EAC(C)| also to be able to see that C (C–equivariantly)
contractible, by Proposition 7.2(5a) (or [Gro02 Lem. 2.5(2)]), which we just proved above. The G–homotopy equivalence |EAC(C)| → |EAC(C)| also follows by a variant of this argument: By Proposition 7.2(5a), it is enough to prove that (EAC SP NC(G) < P) is PC(G(G))–equivariantly contractible (the case where C contains the trivial subgroup is again obvious). Note that C(G) acts trivially so it
suffices to see that the H–fixed-points are contractible for H ≤ P. But (EAC SP NC(G) < P) is equivalent to
{Q ∈ SP NC(G) < P | H ≤ QC(C(G))} by (7.3), which has the slightly modified contraction Q ≤ QΦ(P) ∩
Z(P)) ≥ Φ(P) ∩ Z(P), recalling that Φ(P) ∩ Z(P) 6= 1, since any non-trivial normal subgroup of a
p–group always intersects the center non-trivially. The corresponding statements about F, F, and F
follow from Lemma 7.1.

Remark 7.6. Theorem 7.5(1) does not hold for F, since |Fp0(S)/S| ∼= B/S(Z/S)) (compare also Proposition 4.21). Theorem 7.5(2) is not true for C, since |C(C4)| ∼= C/0/2. In the last part of Theorem 7.5(1) it may be that the assumption C ⊆ C is relaxed. We suspect that
|F| ∼= |F| also when C is closed under passage to p–radical overgroups, and C ⊆ C, generalizing the statement on fundamental groups from Proposition 4.21. Also it may be that EAC(C → EAC(C) is a
G–homotopy equivalence when C ⊆ C; the corresponding statement on π1 is Proposition 4.21 and
there is a related suspicion about Euler characteristics in [LM12 Rem 8.11].

7.4. Varying the collection without changing the low dimensional homotopy type. In this
subsection we continue to study applications of Proposition 7.2 but now focusing on the low
dimensional homotopy type. For this, we recall some basic definitions from algebraic topology. A space is
called i–connected, for i ≥ 0, if πi(X) = 0 for 0 ≤ j ≤ i; a non-empty space is called (−1)–connected,
and a contractible space is (0)–connected. In particular 1–connected means simply connected and 0–
connected means contractible, which all includes non-empty since π0(0) = 0. More generally a map
\( f: X \to Y \) is called an (i + 1)–equivalence, for i ≥ 1, if πi(X) → πi(Y) is surjective and for all
x ∈ X, \( πi(X, x) → πi(Y, f(x)) \) is an isomorphism for 0 ≤ j ≤ i and surjective for j = i + 1. Hence
X is (i + 1)–equivalent if \( X \to pt \) is an (i + 1)–equivalence; see also [DD08 Ch. 6] or [Hat02 Ch. 4]. We
record the behavior of connectivity under joins.

Lemma 7.7 (Connectivity of joins). The join of an n–connected space with an m–connected space is
(n + m + 2)–connected. Furthermore:

1. If X and Y are non-empty G–spaces then, as RG–modules,
\[ H_1(X \star Y; R) \cong \tilde{H}_0(X; R) \otimes_R \tilde{H}_0(Y; R) \cong \ker(R[\pi_0(X) \times \pi_0(Y)] \to R\pi_0(X) \times R\pi_0(Y)) \]
where \( \tilde{H}_0 \) denotes reduced homology, the kernel of the augmentation map.
2. If H is connected and H(X; R) = 0 then H(X \star Y; R) = 0.
3. If X is a connected G–space and H_1(XhG; R) \sim H_1(BG; R) or H_1(X; R) = 0, then the same
holds with X replaced by X \star Y, for any G–space Y.
In particular note that the join of a connected space with a non-empty space is simply connected.

Proof of Lemma 7.10. By cellular approximation [Hat02 §4.1] the initial connectivity claim reduces to checking it for spheres where it is clear. (More generally, the join identifies with the suspension of the smash $\Sigma(X \wedge Y)$, after picking a basepoint.) (1) follows from the Mayer–Vietoris sequence, identifying the join as the homotopy pushout of the diagram $X \leftarrow X \times Y \rightarrow Y$. (2) follows from (1) when $Y$ is non-empty, and is also true when $Y = \emptyset$, since in that case $X \ast Y = X$. For (3) also observe that it is true by assumption if $Y$ is empty; if $Y$ is non-empty, then $X \ast Y$ is simply connected by the first part, and the claim follows from the low dimensional homology sequence of the fibration $X \ast Y \rightarrow (X \ast Y)_{hG} \rightarrow BG$. 

We have already encountered classical group theoretic properties of a $p$-subgroup $P$ that can be expressed in terms of $S_p(N_G(P)/P)$: $P$ is a Sylow subgroup iff $S_p(N_G(P)/P)$ is empty and it is $p$-essential iff $S_p(N_G(P)/P)$ has more than one connected component. Expanding on this, we call a $p$-subgroup $P$ for $\pi_{\leq i}$-essential if $S_p(N_G(P)/P)$ is not $i$-connected. Hence the $\pi_{\leq 1}$-essential subgroups are the Sylow subgroups and the $\pi_{\leq 0}$-essential subgroups are the subgroups that are either a Sylow subgroup or $p$-essential. We furthermore call $P$ endo-essential if it is $\pi_{\leq 0}$-essential or if $H_1(\Theta_p^\ast(N_G(P)/P)) \rightarrow H_1(N_G(P)/P)'$ is not injective. (When $P$ is not an endo-essential subgroup, $H_1(\Theta_p^\ast(N_G(P)/P)) \xrightarrow{\sim} H_1(N_G(P)/P)'$, by (1.3) and $H_1(\Theta_p(N_G(P)/P)) \xrightarrow{\sim} H_1(N_G(P)/P)$ by Remark 4.7, see also Remark 7.17.) We have the following inclusions:

\[
\{ \text{Sylow} \} \subseteq \{ \pi_{\leq 0}\text{-ess.} \} \subseteq \{ \text{endo-ess.} \} \subseteq \{ \pi_{\leq 1}\text{-ess.} \} \subseteq \{ \pi_{\leq 2}\text{-ess.} \} \subseteq \cdots \subseteq \{ p\text{-radical} \}
\]  

(7.6)

Here the third inclusion is a consequence of (1.5) and (1.6) and the last inclusion is an elementary observation of Quillen [Qui78 Prop. 2.4]. Recall than an $i$-connected space of dimension $i$ is contractible by elementary algebraic topology (see e.g., [Hat02 Exc. 4.12]), so the filtration in (7.6) is finite. Furthermore, a famous conjecture of Quillen [Qui78 Conj. 2.9] predicts that if $P$ is $p$-radical then $S_p(N_G(P)/P)$ is non-contractible, i.e., by the previous remark, that $P$ is $\pi_{\leq i}$-essential for $i$ the dimension of $S_p(N_G(P)/P)$, and this is known to be true for many groups $N_G(P)/P$ (see e.g., [AS93]).

**Example 7.8.** For finite groups of Lie type in characteristic $p$, the $\pi_{\leq i}$-essential subgroups are the unipotent radical of parabolic subgroups of rank at most $(i+1)$, and the endo-essential subgroups here agree with the collection of $\pi_{\leq 0}$-essential subgroup; see [6.2]

**Example 7.9.** For $G = G_2(5)$ at $p = 3$, discussed in Proposition 6.3 the subgroup $\langle 3A \rangle$ is endo-essential but not $\pi_{\leq 0}$-essential. We furthermore see that for the collection $\mathcal{C}$ of non-trivial $p$-subgroups except $\langle 3A \rangle$, $\pi_1(\Theta(G)) \cong C_2$ whereas $\pi_1(\Theta_p^\ast(G)) = 1$.

At several points in the paper we need to know that the $N_G(P)$-equivariant homotopy type of $S_p(G)_P$ is unchanged by removing non-$p$-radical overgroups. This is proved below.

**Lemma 7.10.** Let $\mathcal{C}$ be a collection of $p$-groups in $G$. If $\mathcal{C}$ is closed under passage to $p$-radical overgroups, then $\mathcal{C}_{>P} \rightarrow S_p(G)_P \xleftarrow{\sim} S_p(N_G(P)/P)$ is a zig-zag of $N_G(P)$-homotopy equivalences.

If $\mathcal{C}$ is just closed under passage to $\pi_{\leq i}$-essential overgroups, then $i$ is an $(i+1)$-equivalence.

**Proof of Lemma 7.10.** To see the claims, we add $N_G(P)$-conjugacy classes of $p$-subgroups to $\mathcal{C}_{>P}$ in order of decreasing size to reach $S_p(G)_P$, similar to the proof of Theorem 7.5[1], and also explained in [Gro02 Pf. of Thm. 1.2]: If $\mathcal{C}$ is obtained from $\mathcal{C}$ by adding a $N_G(Q)$-conjugacy class of a $p$-subgroup, maximal with the property of not being in $\mathcal{C}$, then we have a $N_G(Q)$-homotopy pushout square analogous to Proposition 7.2[1a]:

\[
\begin{array}{c}
N_G(P) \times_{N_G(P<Q)} (|S_p(G)_Q| \ast |C_{>P,Q}|) \\
\downarrow \quad \downarrow
\end{array}
\]

\[
\begin{array}{c}
N_G(P)/N_G(P<Q) \\
\downarrow \quad \downarrow
\end{array}
\]

\[
\begin{array}{c}
|C_{>P}| \\
\downarrow
\end{array}
\]

Now, recall that $S_p(G)_Q$ is $N_G(Q)$-homotopy equivalent to $S_p(N_G(Q)/Q)$ via the equivariant deformation retraction $P \rightarrow N_P(Q)/Q$, as already observed by Quillen [Qui78 Prop. 6.1]. Hence if $Q$ is non-$p$-radical, so that $S_p(N_G(Q)/Q)$ is $N_G(Q)$-contractible, the homotopy pushout square shows that
\[|C_{>P}| \rightarrow |\tilde{C}_{>P}|\] is a \(N_G(P)\)-homotopy equivalence. Likewise, if \(Q\) is just not \(\pi_{\leq 1}\)-essential, then the left vertical map in the square above is an \((i+1)\)-equivalence, and hence so is the right-hand vertical map by Blakers-Massey’s excision theorem (see e.g., [D08 Prop. 6.4.2]). \(\square\)

The following theorem states precisely which subgroups can be removed without changing fundamental groups, using the formulas of \([7,2]\).

**Theorem 7.11** (Propagating fundamental groups, \(p\)-overgroup-closed version). Let \(\mathcal{C}\) be a collection of \(p\)-subgroups in \(G\).

1. Suppose that \(\mathcal{C}\) is closed under passage to \(\pi_{\leq 1}\)-essential overgroups, then \(\pi_1(\mathcal{O}_C(G)) \xrightarrow{\sim} \pi_1(\mathcal{O}_{C'}(G))\) if \(C'\) is obtained from \(C\) by removing conjugacy classes of \(p\)-subgroups, and if minimal in \(C\) not \(p\leq 1\)-essential.

2. Suppose that \(\mathcal{C}\) is closed under passage to endo-essential overgroups. Then \(H_1(\mathcal{O}_C(G)) \xrightarrow{\sim} H_1(\mathcal{O}_{C'}(G))\) if \(C'\) is any collection \(\mathcal{C} \cap \{\text{endo-essential subgroups}\} \subseteq C\).

**Proof of Theorem 7.11** [1]: We argue that we can add subgroups of \(C \setminus C'\) in decreasing order of size without changing the fundamental groups, using Proposition \([7,2]\). Assume \(C\) is obtained from \(C'\) by adding a single conjugacy class of subgroups \([P]\). By Lemma \([7,10]\), \(C_{>P}\) is connected, and if \(P\) is minimal in \(C\) even simply connected. Furthermore, if \(P\) is non-minimal, \(|E\mathcal{O}_C|/P\) is non-empty. Therefore, in either case, \(|E\mathcal{O}_C|/P\times[C_{>P}]\) is simply connected by connectivity properties of joins, Lemma \([7,7]\). Hence \((|E\mathcal{O}_C|/P\times[C_{>P}]|_B\mathcal{W} \rightarrow B\mathcal{W}\) induces an isomorphism on fundamental groups, so \(\pi_1(\mathcal{O}_C) \xrightarrow{\sim} \pi_1(\mathcal{O}_{C'})\) by Proposition \([7,2,3b]\) and van Kampen’s theorem. That \(\pi_1(\mathcal{O}_{C'}(G)) \xrightarrow{\sim} \pi_1(\mathcal{O}_{C}(G))\) follows similarly from Proposition \([7,2,2b]\). The general case now follows by downward induction. \([2]\): The proof is as in \([1]\), except that when \(P\) is minimal in \(C\) we appeal to the Mayer–Vietoris theorem instead of van Kampen to conclude that \(H_1(\mathcal{O}_{C'}(G)) \xrightarrow{\sim}\) \(H_1(\mathcal{O}_C(G))\). \(\square\)

We also prove the corresponding result for fundamental groups of fusion systems. Recall that a non-Sylow \(p\)-subgroup \(P\) is called \(F\)-essential if \(W_0(\mathcal{P}G(P)/P)\) is a proper subgroup of \(W = N_G(P)/P\) (with \(W_0\) as in \([1,2]\)), a subcollection of the \(p\)-essential subgroups. The \(F\)-essential subgroups can also be described as the \(F\)-centric subgroups such that \(N_G(P)/\mathcal{P}G(P)\) contains a strongly \(p\)-embedded subgroup (see e.g., [Pui70 Cor. III.2], and compare also [AKO11 Def. I.3.2]).

**Proposition 7.12.** Suppose that \(\mathcal{C}\) is closed under passage to \(F\)-essential and Sylow \(p\)-overgroups, and that \(C'\) is a collection obtained from \(C\) by removing conjugacy classes of \(p\)-subgroups, which are neither Sylow \(p\)-subgroups, \(F\)-essential, or minimal in \(C\). Then \(\pi_1(\mathcal{F}_C) \xrightarrow{\sim} \pi_1(\mathcal{F}_{C'})\).

**Proof.** It is enough to show that for both \(\mathcal{F}_C\) and \(\mathcal{F}_{C'}\) we get the same fundamental group as \(\mathcal{F}_{\tilde{C}}\), for \(\tilde{C}\) the minimal collection containing \(\mathcal{C}\) and closed under all \(p\)-overgroups. And for that, it is enough to show that the fundamental group of \(\mathcal{F}_{\tilde{C}}\) does not change when removing a \(p\)-subgroup from \(\tilde{C}\) which is neither minimal nor \(F\)-essential nor a Sylow \(p\)-subgroup (by removing subgroups in order of increasing size). This will be a consequence of Proposition \([7,2,4b]\), if we see that \((|(E\mathcal{A}_C)_{>P}|/C_G(P))\) is connected. For this note, as in \([7,2]\), that \(|(E\mathcal{A}_C)_{>P}| \rightarrow |C_{>P}|\) is an \(N_G(P)\)-equivariant map, which is a bijection on components, and hence \(|(E\mathcal{A}_C)_{>P}|/C_G(P) \rightarrow |C_{>P}|/C_G(P)\) is also a bijection on components. By assumption \((C_{>P})\) is \(\mathcal{S}_p(\mathcal{N}_G(P)/P)\) by Lemma \([7,10]\). So the claim follows from \([1,8]\) and the definition of being non-\(F\)-essential. \(\square\)

**Theorem 7.13** (Propagating fundamental groups, subgroup-closed version). Let \(\mathcal{C}\) be a collection of \(p\)-subgroups closed under passage to non-trivial elementary abelian subgroups. Then \(\pi_1(\mathcal{O}_C(G)) \xrightarrow{\sim} \pi_1(\mathcal{O}_{C'}(G))\), \(\pi_1(\mathcal{O}_C(G)_{>P} \xrightarrow{\sim} \pi_1(\mathcal{O}_{C'}(G)_{>P})\), and \(\pi_1(\mathcal{F}_C(G)) \xrightarrow{\sim} \pi_1(\mathcal{F}_{C'}(G))\) for \(C'\) the subgroups in \(C\) which are either maximal, or elementary abelian of rank at most two. The same statements hold taking \(C'\) the elementary abelian subgroups of \(C\) of rank at most 3.

**Proof.** First, the claim for \(\mathcal{O}\) follows from \(\mathcal{F}\) by Proposition \([4,5]\) and the claim for \(\mathcal{F}\) from \(\mathcal{F}\) by Theorem \([7,5,2]\). Second, the claims are obvious if \(e \in C\), so we can assume that this is not the case. Now to prove the claim for \(\mathcal{F}\) and \(\mathcal{F}\) it is, as usual, enough to compare \(\mathcal{C}\) and \(\mathcal{C}'\) to a collection \(\tilde{C}\) obtained from \(\mathcal{C}\) by adding non-elementary abelian \(p\)-groups, to make it closed under passage to all non-trivial \(p\)-subgroups. By removing subgroups from \(\tilde{C}\) in order of decreasing size, we just have to...
see that the fundamental groups do not change by removing a non-trivial subgroup $P$ which is not elementary abelian of rank 1 or 2, and if elementary abelian of rank 3 not maximal in $\mathcal{C}$. However, this all follows from Proposition 7.2(2b) and van Kampen’s theorem: If $P$ is not elementary abelian then $\tilde{\mathcal{C}}_{<P} = \mathcal{S}_P(G)_{<P}$ is contractible as seen in the proof of Theorem 7.5(2); if $P$ is elementary abelian of rank at least 4, then $\mathcal{S}_P(G)_{<P}$ is simply connected as it is homotopy equivalent to a simply connected Tits building; and if $P$ is elementary abelian of rank 3 and not maximal in $\mathcal{C}$ then $\tilde{\mathcal{C}}_{<P}$ is connected and is joined with a non-empty space giving something simply connected, cf. Lemma 7.7.

7.5. Connected components of $\mathcal{C}$. We now describe the set of connected components of $\mathcal{C}$ in more detail, generalizing the description for $\mathcal{S}_P(G)$ due to Quillen, already explained in the introduction (1.8)—we need this in Section 4 to get the results in their optimal form, and it also has group theoretic significance, see Remark 7.16. For a fixed Sylow $p$–subgroup $S$ in $G$, set $G_{0,C} = \langle N_G(Q) | Q \leq S, Q \in \mathcal{C} \rangle$ and $\mathcal{C}_0 = \{ Q \in \mathcal{C} | Q \leq G_{0,C} \}$, a collection in $G_{0,C}$.

Proposition 7.14 (Connected components of $\mathcal{C}$). Let $\mathcal{C}$ be a collection of $p$–subgroups in $G$, closed under passage to Sylow and $p$–essential overgroups. Then $G \times_{G_{0,C}} |\mathcal{C}| \xrightarrow{\sim} |\mathcal{C}|$ is a $G$–equivariant isomorphism of simplicial sets, via $(g, P_0 \leq \cdots \leq P_n) \mapsto (gP_0 \leq \cdots \leq gP_n)$, and $|\mathcal{C}_0|$ is connected. On the set of components

$$G/G_{0,C} \xrightarrow{\sim} \pi_0(\mathcal{C}) \xrightarrow{\sim} \pi_0(|\mathcal{C}|) \xrightarrow{\sim} G/G_{0,C'}$$

with $\mathcal{C}'$ the Sylow or $p$–essential subgroups of $\mathcal{C}$.

Furthermore for a subgroup $H \leq G$ the following conditions are equivalent:

1. $G_{0,C}$ is subconjugate to $H$.
2. $p \mid |G : H|$ and if, for any $g \in G$, $H \cap gH$ contains an element of $\mathcal{C}$, then $g \in H$.

In particular $G_{0,C}$ is characterized as a minimal subgroup satisfying (2), containing $S$.

Proof. Let us first show that the two conditions are equivalent. Suppose that (1) is satisfied. We can without loss of generality assume that $G_{0,C} \leq H$, since the condition on $H$ is conjugation invariant. Suppose now that $Q \leq H \cap gH$, and $Q \in \mathcal{C}$; we want to show that $g \in H$. By Sylow’s theorem in $H$, using that $p \mid |G : H|$ we can upon changing $g$ by an element in $H$ assume that $Q \leq S$. By assumption $x^{-1}Q \leq H$, and hence we can, again by Sylow’s theorem find $h \in H$ so that $hx^{-1}Q \leq S$. Alperin’s fusion theorem, in the version of Goldschmidt–Miyamoto–Puig [ Miy77, Cor. 1] (see also [Gro02, §10]), now says that we can find $p$–essential subgroups $P_1, \ldots, P_r$, and elements $g_i \in N_G(P_i)$, and $n \in N_G(S)$, with $Q \leq P_1$ and $g^{-n}Q \leq P_{i+1} \leq S$ for $i \leq r - 1$, such that $hx^{-1} = ng_r \cdots g_1$. However since the right-hand side is in $G_{0,C}$ by assumption, this shows that $x \in H$ as wanted, so (2) holds.

Now suppose that $H$ is a subgroup satisfying (2). By Sylow’s theorem, we can change $H$ up to conjugation so that $S \leq H$. We want to show that $G_{0,C} \leq H$. In fact we will prove the stronger statement that $G_{0,C}$, the stabilizer of $[S] \in \pi_0(\mathcal{C})$, is contained in $H$. As it is clear that $G_{0,C} \leq G_S$, this will show (2) ⇒ (1), and furthermore establish that $G_S = G_{0,C}$ as claimed in the first part of the theorem. Hence suppose that $g \in G_S$, so that $gS$ and $S$ lies in the same component. By definition there exists a sequence of Sylow $p$–subgroups $S_0, \ldots, S_r$, so that $S = S_0, S_r = gS$ such that $S_i(\cap S_i+1)$ contains an element of $\mathcal{C}$. Choose $g_i \in G$ such that $g_i S_i = S_{i+1}$, so that $g_{r-1} \cdots g_0 S = gS$. If $S_i \leq H$, then $S_{i+1} \leq H$ and $g_i \in H$ by our assumption, so by induction we conclude that $g_{r-1} \cdots g_0 \in H$, and hence also $g \in H$ since the two elements differ by an element of $N_G(S) \leq H$.

That $G \times_{G_{0,C}} |\mathcal{C}_0| \xrightarrow{\sim} |\mathcal{C}|$ now follows: The map is surjective since for $P_0 \leq \cdots \leq P_n \in |\mathcal{C}|$ we can, by Sylow’s theorem, find $g \in G$ so $gP_n \leq S$, and hence $(gP_0 \leq \cdots \leq gP_n) \in |\mathcal{C}_0|$. Furthermore if $(gP_0 \leq \cdots \leq gP_n) = (gP'_0 \leq \cdots \leq gP'_n)$ then $g^{-1}gP_0 = P'_0 \leq G_{0,C}$, and likewise $P_0 \leq G_{0,C}$ by definition, i.e., $P_0$ and $P'_0$ lie in the same component so $g^{-1}g \in G_S = G_{0,C}$, by the first part. In other words $gG_{0,C} = g'G_{0,C}$ as wanted.

The last claim we need to justify is that $\pi_0(\mathcal{C'}) \xrightarrow{\sim} \pi_0(|\mathcal{C}|)$ which follows from Lemma 7.10 with $P = e$, except the degenerate case where $e \in \mathcal{C}$; but here it is also true: this is clear if also $e \in \mathcal{C}'$, since both spaces are contractible, and if $e \notin \mathcal{C}'$ then Proposition 7.2(1a) still says that we can add $e$ to $\mathcal{C}'$ without changing the number of components, which hence has to be one.

□

Remark 7.15 (Groups with a strongly $p$–embedded subgroup). To use the theorems in this paper, it is useful to know when $G_0 = \langle N_G(Q) | 1 < Q \leq S \rangle$ is proper in $G$, i.e., in group theoretic language,
when $G$ contains a strongly $p$-embedded subgroup. The answer to this question forms an important chapter in the classification of finite simple groups. The following is a theorem of Bender when $p = 2$ \[\text{[Ben71]}\], and only known as a consequence of the classification when $p$ is odd: Either $rk_r(G) = 1$ and $G_0 = N_G(pZ(S))$, with $G_0 < G$ exactly when $O_p(G) = 1$, or $rk_r(G) \geq 2$, $O_{p'}(G) \leq G_0$, $G = G/O_{p'}(G)$ has a unique minimal normal non-abelian simple subgroup $K = F^*(G)$, and $G/K \leq \mathrm{Out}(K)$; the group $K$ is either a finite group of Lie type of rank $1$ (possibly twisted) in defining characteristic, $A_{2p}$ $(p \geq 3)$, $(L_3(4), 3)$, $(M_{11}, 3)$, $(F_{22}, 5)$, $(Mc, 5)$, $(F_4(2)', 5)$, or $(J_4, 11)$. See \[\text{[GLS98 7.6.1, 7.6.2]}\] and also \[\text{[Qui78 5]}\] for more details.

**Remark 7.16.** As noticed, when taking $\mathcal{C} = S_p(G)$, Proposition \[7.14\] is a strong version of Quillen’s \[\text{[Qui78 Prop. 5.2]}\]. When taking $\mathcal{C}$ to be the collection of $p$-subgroups of $p$-rank at least $k$, $G_{0, \mathcal{C}} = \Gamma_{k, S}(G)$, the $k$-generated $p$-core from finite group theory, and one also makes geometric and extends a standard characterization of it \[\text{[Asc93 (6.2)]}\] for more details.

**Remark 7.17.** By examining the list in Remark \[7.15\] one sees that very often when $G_0 < G$, $H_1(G_0)/H' \rightarrow H_1(G)/H'$ is not injective, providing exotic Sylow-trivial modules via \[\text{(1.4)}\]; take for instance $G = \mathrm{SL}_2(F_{p^r})$ with $p^r \neq 2$. But it may also be injective: Let $p = 2$ and consider $K = \mathrm{SL}_2(F_{2^r})$, $r > 1$ odd, and let $C_r$ act on $K$ via field automorphisms; set $G = K \times C_r$. Then $K_0$ consists of upper triangular matrices of determinant one, and $G_0 = H_0 \times C_r$. Hence $H_1(G_0)/H' \cong H_1(G)/H' \cong C_r$. Other examples may be constructed along these lines, though perhaps limited to small primes. (We are grateful to Ron Solomon for consultations on these points.)

### 7.6. Proof of Proposition \[7.2\]

We now prove Proposition \[7.2\] via general observations about links in preordered sets, an abstraction of observations in \[\text{[Dwy98, GS06]}\]. Let $\mathcal{X}$ be a preordered set, i.e., a small category with at most one morphism between any two objects. Note that our spaces $\mathcal{E}\mathcal{O}_\mathcal{C}$, etc., are all examples of such. Define $\mathbf{star}_\mathcal{X}(\bar{x})$ as the full subcategory of $\mathcal{X}$ of objects which admit a morphism to or from $x$, and $\mathbf{link}_\mathcal{X}(\bar{x})$ as the subcategory that admit a non-isomorphism to or from $x$. Furthermore note that

$\mathbf{star}_\mathcal{X}(\bar{x}) = \mathcal{X}_{<\bar{x}} \star \mathcal{X}_{\geq \bar{x}}$ and $\mathbf{link}_\mathcal{X}(\bar{x}) = \mathcal{X}_{\geq \bar{x}} \star \mathcal{X}_{> \bar{x}}$

where $\mathcal{X}_{< \bar{x}}$ denotes the subcategory of $\mathcal{X}$ on elements smaller than and not isomorphic to $x$, and $\mathcal{X}_{\geq \bar{x}}$ is the full subcategory of $\mathcal{X}$ of objects isomorphic to $x$. The symbol $\star$ means the join of preordered set, introducing a unique non-isomorphism from objects on the left to objects on the right, realizing to the join of spaces. If $\mathcal{X}$ has a $G$-action, these are all $G_{\bar{x}}$-subcategories, where $G_{\bar{x}}$ is the stabilizer of $\bar{x}$ as a set, and furthermore $\mathbf{star}_\mathcal{X}(\bar{x})$ is $G_{\bar{x}}$-contractible to $x$ (but generally not $G_{\bar{x}}$-contractible).

**Proposition 7.18.** Suppose $\mathcal{X}$ is a preordered set equipped with a $G$-action such that isomorphic objects are $G$-conjugate, and let $\mathcal{X}'$ denote the subcategory of $\mathcal{X}$ obtained by removing all $G$-conjugates of an element $x$.

1. There is a pushout square of $G$-spaces, which is also a homotopy pushout square of $G$-spaces:

\[
\begin{array}{c}
G \times_{G_x} |\mathbf{link}_\mathcal{X}(\bar{x})| \ar[r] \ar[d] & |\mathcal{X}'| \ar[d] \\
G \times_{G_x} |\mathbf{star}_\mathcal{X}(\bar{x})| \ar[r] & |\mathcal{X}|
\end{array}
\]

where $\bar{x}$ denotes the subcategory of elements isomorphic to $x$ and $G_{\bar{x}}$ its stabilizer as a set.

2. If in addition the stabilizer $G_x$ of any point $x \in \mathcal{X}$ is a normal subgroup in the stabilizer of its isomorphism class $\bar{x}$, collapsing $E(G_{\bar{x}}/G_x)$ again induces a pushout and homotopy pushout square of $G$-spaces

\[
\begin{array}{c}
G \times_{G_x} E(G_{\bar{x}}/G_x) \times |\mathbf{link}_\mathcal{X}(\bar{x})| \ar[r] \ar[d] & |\mathcal{X}'| \ar[d] \\
G \times_{G_x} E(G_{\bar{x}}/G_x) \times |\mathbf{star}_\mathcal{X}(\bar{x})| \ar[r] & |\mathcal{X}|
\end{array}
\]

and it remains a homotopy pushout of $G$-spaces after replacing $|\mathbf{star}_\mathcal{X}(\bar{x})|$ by a point.
In particular, under these assumptions:

(3) On $G$–orbits there is a homotopy pushout square

$$
\begin{array}{ccc}
((|\mathcal{X}_{<\bar{x}}| \star |\mathcal{X}_{>\bar{x}}|)/G_x)_{G_x/G_x} & \longrightarrow & |\mathcal{X}'|/G \\
\downarrow & & \downarrow \\
B(G_x/G_x) & \longrightarrow & |\mathcal{X}|/G
\end{array}
$$

(4) If $|\mathcal{X}_{<\bar{x}}| \star |\mathcal{X}_{>\bar{x}}|$ is $G_x$–contractible, then $|\mathcal{X}'| \to |\mathcal{X}|$ is a $G$–homotopy equivalence and $|\mathcal{X}'|/G \to |\mathcal{X}|/G$ is a homotopy equivalence.

(5) If $(|\mathcal{X}_{<\bar{x}}| \star |\mathcal{X}_{>\bar{x}}|)/G_x$ is connected then $\pi_1((|\mathcal{X}'|)/G) \to \pi_1(|\mathcal{X}|/G)$ is surjective.

(6) If $(|\mathcal{X}_{<\bar{x}}| \star |\mathcal{X}_{>\bar{x}}|)/G_x$ is simply connected then $\pi_1((|\mathcal{X}'|)/G) \cong \pi_1(|\mathcal{X}|/G)$, for all basepoints.

(7) $H_1(|\mathcal{X}'|/G_x) \cong H_1^{G_x}(|\mathcal{X}_{<\bar{x}}| \star |\mathcal{X}_{>\bar{x}}|)/G_x, pt; A)$, for any abelian group $A$. In particular if $(|\mathcal{X}_{<\bar{x}}| \star |\mathcal{X}_{>\bar{x}}|)/G_x$ is connected and $H_1(|\mathcal{X}_{<\bar{x}}| \star |\mathcal{X}_{>\bar{x}}|)/G_x; A)_{G_x/G_x} = 0$, then $H_1((|\mathcal{X}'|)/G_x; A) \cong H_1(|\mathcal{X}|/G; A)$.

Proof. For (1), notice that it is a pushout of $G$–spaces by the fact that $|\mathcal{X}'|$ and the image of $G \times G_x \smallsetminus \text{star}_\mathcal{X}(\bar{x})$ cover $|\mathcal{X}|$, and the part of $G \times G_x \smallsetminus \text{star}_\mathcal{X}(\bar{x})$ that maps to $\mathcal{X}'$ is the subspace $G \times G_x \smallsetminus \text{link}_\mathcal{X}(\bar{x})$, just by the definitions and that isomorphic elements are $G$–conjugate. It is homotopy pushout of $G$–spaces since $G \times G_x \smallsetminus \text{link}_\mathcal{X}(\bar{x}) \to G \times G_x \smallsetminus \text{star}_\mathcal{X}(\bar{x})$ is an injective map of $G$–spaces.

For (2) it is enough to see that the projection square

$$
\begin{array}{ccc}
G \times G_x E(G_x/G_x) \times |\text{link}_\mathcal{X}(\bar{x})| & \longrightarrow & G \times G_x \times |\text{link}_\mathcal{X}(\bar{x})| \\
\downarrow & & \downarrow \\
G \times G_x E(G_x/G_x) \times |\text{star}_\mathcal{X}(\bar{x})| & \longrightarrow & G \times G_x \times |\text{star}_\mathcal{X}(\bar{x})|
\end{array}
$$

is a pushout and homotopy pushout of $G_x$–spaces, which we do by checking on all fixed-points for $H \leq G_x$. For $H \leq G_x E(G_x/G_x)H$ is contractible and the claim is clear. (Note, we use that $G_x/G_x$ is a group, not just a coset, so the isotropy does not depend on the chosen point $x \in \bar{x}$.) For $H \nleq G_x$, the spaces on the left are empty, and $\text{link}_\mathcal{X}(\bar{x})^H = \text{star}_\mathcal{X}(\bar{x})^H$, so it is also a pushout and homotopy pushout square in that case. For the purposes of having a homotopy pushout square of $G$–spaces, we can replace $\text{star}_\mathcal{X}(\bar{x})$ by a point, since $\text{star}_\mathcal{X}(\bar{x})$ is $G_x$–contractible, and hence $E(G_x/G_x) \times |\text{star}_\mathcal{X}(\bar{x})| \to E(G_x/G_x)$ a $G_x$–homotopy equivalence.

For (3), note that since $G \times G_x E(G_x/G_x) \times |\text{link}_\mathcal{X}(\bar{x})| \to G \times G_x E(G_x/G_x) \times |\text{star}_\mathcal{X}(\bar{x})|$ is a cofibration of $G$–spaces, passing to $G$–orbits in (2) produces the homotopy pushout square in (3).

Likewise (4) is a consequence of (2) under the stated assumption $|\text{link}_\mathcal{X}(\bar{x})| \to |\text{star}_\mathcal{X}(\bar{x})|$ is a $G_x$–equivalence. Points (5) and (6) are consequences of (3) combined with van Kampen’s theorem, since under the stated assumptions $\pi_1((|\mathcal{X}_{<\bar{x}}| \star |\mathcal{X}_{>\bar{x}}|)/G_x) \to \pi_1(B(G_x/G_x))$ is an epimorphism and isomorphism respectively. Finally, the first claim in (7) follows by (3) again, combined with the long exact sequence in homology. The ‘in particular’ also follows from (3), now together with the Mayer–Vietoris sequence: the connected assumption implies that the boundary map in the Mayer–Vietoris sequence to $H_0$ is zero, and the $H_1$ assumption gives that the left-hand vertical map is an isomorphism on $H_1$; together this yields that the right-hand vertical map is an isomorphism on $H_1$ as wanted. □

Proof of Proposition 7.2. This will be applications of Proposition 7.18 (1a) is Proposition 7.18 with $\mathcal{X} = \mathcal{C}$, noting that in this case $\mathcal{X}$ is a poset and $G_x = G_x$. (1b) is Proposition 7.18. For (2a) cross the diagram from (1a) with $EG$, noting that $EG$ as an $N$–space is equivalent to $EN$; for (2b)
take $G$-orbits and note that $|C|_{hG}$ identifies with $|\mathcal{C}|$ by Remark 2.2. (Alternatively, more directly take $X = E \mathcal{C} = e \downarrow \iota$, the undercategory for $\iota: \mathcal{C} \to \mathcal{C}_{\langle e \rangle}$.) For (3a) and (3b) take $X = E G_{Q}$, so for $x=(G/Q,y), G_{x} = G_{y}$, and $G_{x} = N_{G}(G_{y});$ the diagrams now follow from Proposition 7.18(2–3) using that $|(E \mathcal{O}_{G})_{\mathfrak{P}}| \to |\mathcal{C}_{\mathfrak{P}}|$ is a $P$-homotopy equivalence by (7.3). For (4a-b) and (5a-b) take $X$ to be $E A_{C}$ and $E A_{C}$ respectively and again use Proposition 7.18(2–3) (where for $E A_{C}, G_{x} = C_{G}(i(Q))$, and $G_{x} = N_{G}(i(Q)), x = (i: Q \to G),$ and similarly for $E A_{C}$), together with the simplification that $|(E A_{C})_{\mathfrak{P}}| \to |\mathcal{C}_{\mathfrak{P}}|$ is a $C_{G}(P)$-equivalence by (7.3).

References


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