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Higher limits via subgroup complexes

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1. Introduction

Abelian group-valued functors on the orbit category of a finite group G are ubiquitous both in group theory and in much of homotopy theory. In this paper we give a new finite model for $\lim^*_{\mathbf{D}} F$, the higher derived functors of the inverse limit functor, for a general functor $F : \mathbf{D}^{\mathrm{op}} \to \mathbf{Z}_{(p)}$- mod, where $\mathbf{D}$ is the p-orbit category of a finite group G (consisting of transitive G-sets with isotropy groups, p-groups and equivariant maps between them), or one of several other commonly studied categories associated to G.

In homotopy theory such functors arise in connection with homology decompositions of the classifying space BG of G, where $\mathbf{D}$ serves as indexing category for the decomposition. The groups $\lim^*_{\mathbf{D}} F$ then occur in the E_2-term of several fundamental spectral sequences such as the Bousfield-Kan homotopy (or cohomology) spectral sequence of a homotopy limit (or colimit)

$$E_2^{i,j} = \lim_{\mathbf{D}}{}^{-i} \pi_j(X(d)) \Rightarrow \pi_{i+j}(\mathrm{holim}_{\mathbf{D}}\, X(d)),$$

where X is a functor $\mathbf{D}^{\mathrm{op}} \to \mathbf{Spaces}_*$ ([18] or unpointed spaces [17]). In applications, $\mathrm{holim}_{\mathbf{D}}\, X(d)$ will be a space of significant topological interest such as the space of maps from BG to some other given space. It is hence of great importance to be able to calculate these higher limits for calculating maps between classifying spaces [45], [47], [39], finding homology decompositions [44], [30], [32], and settling uniqueness issues [55].

In group theory, F is typically a Mackey functor such as a group cohomology. Here one can often show that the higher limits vanish for $* > 0$ for general reasons (cf. [44], [81] and Section 8). In this seemingly trivial case the general formulas for $\lim^*_{\mathbf{D}} F$ in fact specialize to give interesting expressions, in the form of finite exact sequences, for the 'group of stable elements' $\lim^0_{\mathbf{D}} F$, which is equal, or closely related, to $F(G)$.

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By a collection we mean a set $\mathcal{C}$ of subgroups in G, closed under conjugation in G. We can view a collection as a poset under inclusion of subgroups, and the nerve $|\mathcal{C}|$ of this poset is what is called the *subgroup complex* associated to the collection $\mathcal{C}$. Obviously it is a *finite* (ordered) G-simplicial complex. Subgroup complexes play an important role in finite group theory, where they are thought of as generalized buildings (or geometries) of G.

Let $\mathbf{O}_\mathcal{C}$ denote the $\mathcal{C}$-orbit category, that is, the full subcategory of the orbit category $\mathbf{O}(G)$ with objects the transitive G-sets with isotropy groups in $\mathcal{C}$. This category arises in connection with the 'subgroup homology decomposition' [45], [30], [32]. (We will also consider the (opposite) $\mathcal{C}$-conjugacy category $\mathbf{A}_\mathcal{C}^{\mathrm{op}}$, associated with the 'centralizer decomposition', and the orbit simplex category $(\operatorname{sd}\mathcal{C})/G$ associated with the 'normalizer decomposition'.)

We show that for many collections $\mathcal{C}$, such as the collection of all nontrivial p-subgroups, higher limits $\lim^*_{\mathbf{D}} F$ over any of the three types of categories can be interpreted as Bredon equivariant cohomology of $|\mathcal{C}|$ with values in a G-local coefficient system (or presheaf) constructed from F. Here a G-local coefficient system is understood in the sense of Ronan-Smith [62], [63], [10], which is close to the original definition of Bredon [20], but is more general than what is usually called a (Bredon) coefficient system, and this is a key generalization which the result depends upon. This result gives a finite model for the higher limits, breaking the complexity of the orbit category up into two parts, separating the poset structure from the G-conjugation, which can in some sense be viewed as a topological analog of Alperin's fusion theorem [2]. Computationally the model is a large improvement over earlier Bredon cohomology models for the higher limits over $\mathbf{O}_\mathcal{C}^{\mathrm{op}}$ and $\mathbf{A}_\mathcal{C}$ which have been infinite [51], [46], [45], [81]. It furthermore reveals a close connection between higher limits and modern finite group theory, and gives an easy systematic way to see which subgroups can be omitted from $\mathcal{C}$ without changing the higher limits.

We now give a more detailed description of our results in the case of $\mathbf{O}_\mathcal{C}^{\mathrm{op}}$. (We obtain similar results for $\mathbf{A}_\mathcal{C}$ and $((\operatorname{sd}\mathcal{C})/G)^{\mathrm{op}}$.) In order to do so, we first need to introduce some notation. Let $\mathcal{S}_p(G)$ and $\mathcal{B}_p(G)$ denote the collection of *nontrivial* p-subgroups and p-radical subgroups respectively. Recall that a *p-radical subgroup* of G is a p-subgroup P with the property that $O_p(NP/P) = e$. (Here, $O_p(\cdot)$ means largest normal p-subgroup.) We use the notation $\mathcal{S}_p^e(G)$ etc. to denote the same posets as above, but now not excluding the trivial subgroup e.

Let $\bar{\mathrm{d}}\,\mathcal{C}$ denote the opposite category of the simplex category of $|\mathcal{C}|$. The objects are the simplices of $|\mathcal{C}|$, i.e., chains of subgroups $H_0 \leq \cdots \leq H_n$, with $H_i \in \mathcal{C}$, and the morphisms are generated by the opposite of face and degeneracy maps. Let $(\bar{\mathrm{d}}\,\mathcal{C})_G$ be the category that has the same objects as

$\bar{\mathrm{d}}\,\mathcal{C}$ but where a morphism $\sigma \to \sigma'$ is a pair $(f : g\sigma \to \sigma', g)$, where f is a morphism in $\bar{\mathrm{d}}\,\mathcal{C}$ and $g \in G$, and morphisms are composed via $(f, g) \circ (f', g') = (f \circ (gf'), gg')$ (see Section 2.1). By a G-local coefficient system (over $\mathbf{Z}_{(p)}$) on $|\mathcal{C}|$ we mean a (covariant) functor $\mathcal{F} : (\bar{\mathrm{d}}\,\mathcal{C})_G \to \mathbf{Z}_{(p)}$- mod. For a functor on $\mathbf{O}(G)$ and its subcategories we write $F(H)$ for F evaluated on G/H.

Given a functor $F : \mathbf{O}_{\mathcal{C}}^{\mathrm{op}} \to \mathbf{Z}_{(p)}$- mod we equip $|\mathcal{C}|$ with the G-local coefficient system $\mathcal{F}$ given by precomposing it with the canonical map $(\bar{\mathrm{d}}\,\mathcal{C})_G \to (\mathcal{C}^{\mathrm{op}})_G \to \mathbf{O}_{\mathcal{C}}^{\mathrm{op}}$ (see Remark 2.2). On objects, we have $\mathcal{F}(H_0 \leq \cdots \leq H_n) = F(H_0)$.

Theorem 1.1. *Let $\mathcal{C}$ be any collection of p-subgroups of a finite group G, closed under passage to p-radical overgroups. Let $F : \mathbf{O}_{\mathcal{C}}^{\mathrm{op}} \to \mathbf{Z}_{(p)}$- mod be any functor, and let $\mathcal{F}$ be the G-local coefficient system on $|\mathcal{C}|$ given by precomposing F with $(\bar{\mathrm{d}}\,\mathcal{C})_G \to (\mathcal{C}^{\mathrm{op}})_G \to \mathbf{O}_{\mathcal{C}}^{\mathrm{op}}$. Then*

$$\lim_{\mathbf{O}_{\mathcal{C}}}{}^{*} F \cong H_G^*(|\mathcal{C}|; \mathcal{F}).$$

Furthermore, for any collection $\mathcal{C}'$ satisfying $\mathcal{C} \cap \mathcal{B}_p^e(G) \subseteq \mathcal{C}' \subseteq \mathcal{C}$, the inclusion $|\mathcal{C}'| \to |\mathcal{C}|$ induces an isomorphism $H_G^*(|\mathcal{C}|; \mathcal{F}) \xrightarrow{\cong} H_G^*(|\mathcal{C}'|; \mathcal{F})$, where $\mathcal{F}$ is a coefficient system on $|\mathcal{C}'|$ by restriction.

If F is a functor concentrated on conjugates of a single subgroup P, then

$$\lim_{\mathbf{O}_{\mathcal{C}}}{}^{i} F \cong H^{i-1}(\mathrm{Hom}_{NP/P}(\mathrm{St}_*(NP/P), F(P))).$$

Here $H_G^*(|\mathcal{C}|; \mathcal{F})$ denotes Bredon cohomology $H_G^*(|\mathcal{C}|; \mathcal{F}) = H(C^*(|\mathcal{C}|; \mathcal{F})^G)$ (see Definition 2.3), which of course is zero above the dimension of $|\mathcal{C}|$, and $\mathrm{St}_*(W)$ denotes the Steinberg complex of W, the reduced normalized chain complex $\tilde{C}_*(|\mathcal{S}_p(W)|; \mathbf{Z}_{(p)})$. The result that one only needs the p-radical subgroups for calculating the higher limits is due to Jackowski-McClure-Oliver [45]. They used a Bredon cohomology model ([51], [46], and Prop. 2.10) but the space was infinite, and hence hard to use directly for calculations.

By the work of Webb ([79], [80], and see Theorem 5.1) it is known that the Steinberg complex over the p-adic integers $\mathbf{Z}_p$ is G-chain homotopic to a (unique minimal) direct summand which is a chain complex of projective $\mathbf{Z}_pG$-modules; we denote this by $\mathbb{S}\mathrm{t}_*(G)$. This chain complex has been recognized by group theorists as a worthy generalization of the Steinberg module to arbitrary finite groups [5], [79], [80] and we study its properties in Section 5. (If G is a finite group of Lie type of characteristic p then $\mathbb{S}\mathrm{t}_*(G)$ is just the ordinary Steinberg module concentrated in degree one less than the semisimple rank of G.) Theorem 1.1 combined with (mostly straightforward) properties of the Steinberg complex give significant generalizations of the properties of higher limits over functors concentrated on one conjugacy class, so called *atomic* functors, found in [45].

In homotopy theory, the functors which occur in practice often have the property that the centralizer CP of P acts trivially on $F(P)$ (i.e., the action of NP/P on $F(P)$ factors through $NP/PC(P)$). In this case one can remove additional subgroups from the collection. Let $\mathcal{D}_p(G)$ be the collection consisting of the subgroups which are *p-centric* (i.e., the center ZP of P is a Sylow p-subgroup in CP), and such that $O_p(NP/PC(P)) = e$. These subgroups are necessarily p-radical subgroups and, like the p-radical groups, play a role in Alperin's theory of weights for arbitrary finite groups, where they are the p-radical subgroups for the principal block [4, §3]. For this reason we call the subgroups in $\mathcal{D}_p(G)$ the *principal* p-radical subgroups.

THEOREM 1.2. *Let $\mathcal{C}$ be any collection of p-subgroups of a finite group G, closed under passage to p-radical overgroups. Suppose that F is a functor $\mathbf{O}_{\mathcal{C}}^{\mathrm{op}} \to \mathbf{Z}_{(p)}\text{-mod}$ such that for all $P \in \mathcal{C}$, CP acts trivially on $F(P)$. Then, for any collection $\mathcal{C}'$ satisfying $\mathcal{C} \cap \mathcal{D}_p(G) \subseteq \mathcal{C}' \subseteq \mathcal{C}$,*

$$\lim_{\mathbf{O}_{\mathcal{C}}}{}^{*} F \cong \lim_{\mathbf{O}_{\mathcal{C}'}}{}^{*} F \cong H_G^*(|\mathcal{C}'|;\mathcal{F}),$$

where $\mathcal{F}$ is the G-local coefficient system on $|\mathcal{C}'|$ induced via $(\bar{\mathrm{d}}\,\mathcal{C}')_G \to (\bar{\mathrm{d}}\,\mathcal{C})_G \to (\mathcal{C}^{\mathrm{op}})_G \to \mathbf{O}_{\mathcal{C}}^{\mathrm{op}}$.

If F is a functor concentrated on conjugates of a single subgroup $P \in \mathcal{C} \cap \mathcal{D}_p(G)$, then

$$\lim_{\mathbf{O}_{\mathcal{C}}}{}^{i} F \cong H^{i-1}(\mathrm{Hom}_{NP/PC(P)}(\mathrm{St}_*(NP/PC(P)), F(P))).$$

The formulas for higher limits of functors concentrated on one conjugacy class easily show that the questions of minimality of the collections $\mathcal{C} \cap \mathcal{B}_p^e(G)$ in Theorem 1.1 and $\mathcal{C} \cap \mathcal{D}_p(G)$ in Theorem 1.2 both, in a certain sense, are equivalent to a version of Quillen's conjecture on the contractibility of $|\mathcal{S}_p(W)|$ (see Remarks 3.9 and 3.12).

The more explicit results in Theorems 1.1 and 1.2 for atomic functors provide a computation of the E_1-term of a spectral sequence converging to the higher limits of a general functor F obtained by filtering F in such a way that the associated quotients are atomic functors. By a G-invariant height function ht on a collection $\mathcal{C}$ we mean a strictly increasing map of posets $\mathcal{C}/G \to \mathbf{Z}$.

THEOREM 1.3. *Let $\mathcal{C}$ be any collection of p-subgroups of a finite group G, closed under passage to p-radical overgroups. For any functor $F : \mathbf{O}_{\mathcal{C}}^{\mathrm{op}} \to \mathbf{Z}_{(p)}\text{-mod}$ there is a spectral sequence*

$$\begin{aligned} E_1^{i,j} &= \bigoplus_{[P]\in \mathcal{C}'/G,\mathrm{ht}(P)=-i} H^{i+j-1}(\mathrm{Hom}_{NP/P}(\mathrm{St}_*(NP/P), F(P))) \\ &\Rightarrow \lim_{\mathbf{O}_{\mathcal{C}}}{}^{i+j} F \end{aligned}$$

where $\mathcal{C}' = \mathcal{C} \cap \mathcal{B}_p^e(G)$ *and* ht *is some chosen* G*-invariant height function on* $\mathcal{C}'$. *The differentials are cohomological differentials* $d_n : E_n^{i,j} \to E_n^{i+n,j-(n-1)}$.

If for all $P \in \mathcal{C}$, CP *acts trivially on* $F(P)$, *then* $\mathcal{C}'$ *may be replaced by* $\mathcal{C} \cap \mathcal{D}_p(G)$ *and* NP/P *by* $NP/PC(P)$.

We can always normalize a height function such that for a Sylow p-subgroup P, $\mathrm{ht}(P) = 0$, and then this spectral sequence occupies a triangle in the 'eighth octant'. Often the most convenient G-invariant height function on a collection $\mathcal{C}$ is the 'minimal' one given by $\mathrm{ht}(Q) = -\dim |\mathcal{C}_{\geq Q}|$. If G is a finite group of Lie type of characteristic p the poset $\mathcal{B}_p^e(G)$ has a relatively simple structure (it is Cohen-Macaulay). In this case, if we choose the right height function, the E_1-term is concentrated on the horizontal axis, and hence yields a chain complex, with an explicitly described differential, for calculating the higher limits (see Theorem 4.1).

Applications to group cohomology and other Mackey functors. In cases where $\lim^i$ vanishes for $i > 0$, the chain complex defining Bredon cohomology in Theorem 1.1 produces finite exact sequences starting with $\lim^0_{\mathbf{D}} F$. (We get similar exact sequences when treating higher limits over $\mathcal{C}$-conjugacy categories and orbit simplex categories.) For group cohomology this gives exact sequences expressing $H^n(G; M)$ in terms of $H^n(H; M)$ for proper subgroups H of G.

THEOREM 1.4. *Let* G *be a finite group, and let* $H^n(-)$ *denote group cohomology with coefficients in* $\mathbf{Z}_{(p)}$. *There are the following three exact sequences coming from acyclic complexes*

$$\begin{aligned}
0 \to H^n(G) &\to \oplus_{[P] \in \mathcal{D}_p(G)/G} H^n(P)^{NP} \\
&\to \oplus_{[P_0 < P_1] \in |\mathcal{D}_p(G)|_1/G} H^n(P_0)^{NP_0 \cap NP_1} \to \cdots \to 0, \\
0 \to H^n(G) &\to \oplus_{[V] \in \mathcal{A}_p(G)/G} H^n(CV)^{NV} \\
&\to \oplus_{[V_0 < V_1] \in |\mathcal{A}_p(G)|_1/G} H^n(CV_1)^{NV_0 \cap NV_1} \to \cdots \to 0, \\
0 \to H^n(G) &\to \oplus_{[P] \in \mathcal{D}_p(G)/G} H^n(NP) \\
&\to \oplus_{[P_0 < P_1] \in |\mathcal{D}_p(G)|_1/G} H^n(NP_0 \cap NP_1) \to \cdots \to 0.
\end{aligned}$$

Here $\mathcal{A}_p(G)$ denotes the collection of nontrivial elementary abelian p-subgroups, that is subgroups of the form $(\mathbf{Z}/p)^k$, for some $k > 0$, and $|\mathcal{C}|_k/G$ denotes G-conjugacy classes of nondegenerate k-simplices in $|\mathcal{C}|$. The last exact sequence is a version of the celebrated exact sequence of Webb, but obtained for the small collection $\mathcal{D}_p(G)$, whereas the rest appear to be new. We explain these sequences as part of a general framework in Section 8, and also give related exact sequences.

The spectral sequence in Theorem 1.3 can also be used to give a useful description of $\lim^0_{\mathbf{O}_\mathcal{C}} F$, regardless of the vanishing of the higher limits, simply by looking at what happens at the $(0,0)$-spot. In this way we recover expressions for the stable elements, usually proved using various versions of Alperin's fusion theorem, via homotopy theoretic considerations (see Section 10).

Applications to homology decompositions. A homology decomposition is said to be *sharp* [32] if the Bousfield-Kan cohomology spectral sequence of the homotopy colimit collapses at the E_2-term onto the vertical axis. (This can also be formulated in other equivalent ways—see Section 9.) Our general formulas for higher limits can be used to obtain new sharpness results for homology decompositions. It is easy to see that the collection $\mathcal{S}_p^e(G)$ is sharp with respect to both the subgroup, centralizer and normalizer homology decompositions. Theorem 1.2, for F equal to mod p group cohomology, shows that the E_2-term of the cohomology spectral sequence associated to the subgroup decomposition using all p-subgroups remains unchanged when all subgroups not in $\mathcal{D}_p(G)$ are removed. This, together with the analogous result for the normalizer decomposition (Theorem 7.3), yields

THEOREM 1.5. *The collection $\mathcal{D}_p(G)$ of principal p-radical subgroups gives a subgroup and normalizer sharp* mod *p homology decomposition of BG* (*and is in particular ample*).

(See Theorem 9.1 for a more elaborate version of this result.) The subgroup sharpness result improves a result of Dwyer [30], [32] that the collection of p-radical and p-centric subgroups is subgroup sharp. The normalizer sharpness result improves a result of Webb [79] that the collection of p-radical subgroups is normalizer sharp and verifies a suspicion of Smith-Yoshiara [69] that the collection of p-radical and p-centric subgroups should be normalizer sharp.

Related work. This work grew out of a need to calculate higher limits (initially using [45]) in order to compute monoids of vector bundles over classifying spaces completed at a prime p, as a part of a larger program to understand the 'homotopy representation theory' of finite, compact Lie, or even p-compact groups [31], at the prime p. This motivating application of our results will be published separately [39].

The relationships between groups, posets, and spaces which underlie everything in this paper date back to the fundamental work of Brown [21] and Quillen [59]. We assume some familiarity with the methods of [59]. The approach taken in this paper can be seen as a continuation along the lines of the papers of Dwyer [30], [32], which we highly recommend to the reader, although this paper can be read independently of those.

The methods of this paper are also closely related to a 'Lie theoretic' approach to local representation theory of finite groups advocated by Alperin [4], [5] and Webb [79] amongst others, and likewise have ties to early ideas of Puig [58] who examines the basic but yet elusive question of what is a p-local group?

Results relating higher limits over various categories to higher limits over orbit simplex categories have been found independently by J. Słomińska [66]. More precisely, using for instance Proposition 7.1 (to remove a subdivision), one sees that [66, Prop. 2.9] (which is formulated in a more abstract setup) corresponds to Corollary 3.5 which is an important step in the proof of Theorem 1.1. I am grateful to S. Jackowski for informing me of her unpublished work and for providing me with a copy. After the submission of the present paper Jackowski and Słomińska have finished the paper [43] which includes Słomińska's unpublished results.

Organization of the paper. In Section 2 we prove various facts about higher limits and Bredon cohomology needed in the later sections. In Section 3 we prove our general results on higher limits over orbit categories, several times referring to results from Section 5. Section 4 elaborates on these results in the special case where G is a finite group of Lie type of characteristic p. In Section 5 we examine the structure of the Steinberg complex $\mathrm{St}_*(G)$ for an arbitrary finite group G. Sections 6 and 7 treat higher limits over conjugacy categories and orbit simplex categories respectively. In the last three sections we look at special cases of the general results, focusing on the case where the higher limits in fact vanish. In Section 8 we explain how one can obtain exact sequences in the case where all higher limits vanish, and provide some examples when this is satisfied. In Section 9 we explain how the general results lead to new sharpness results on homology decompositions. Finally, in Section 10 we explain how our results relate to the calculation of stable elements and Alperin's fusion theorem.

Notation and conventions. Throughout, G denotes a finite group and p a fixed prime. This paper is written simplicially so that the word space, without any prefix, means a simplicial set. Hence the nerve construction $|\cdot|$ is a functor from categories to simplicial sets. We remark in this connection that for G-categories, taking G-fixpoints commutes with taking nerves. We let R denote an arbitrary commutative unital ring. In most of this paper we however work over $\mathbf{Z}_{(p)}$, and we adopt the convention that unless otherwise indicated, $\mathbf{Z}_{(p)}$ is taken as the ground ring. We denote the normalizer and centralizer of H in G by NH and CH respectively, and write WH for the Weyl group NH/H. The word 'functor' without any prefix means a covariant functor. The suspension ΣX of a G-space X should be understood as $\mathrm{Cone}X/X$, where $\mathrm{Cone}X$ here is the *unreduced* cone on X. This is a pointed G-space, with G-basepoint $[X]$.

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2. Preliminaries

In this section we explain various facts about equivariant cohomology and higher limits. All of the results explained in this section are well-known.

2.1. *Equivariant cohomology and local coefficient systems.* Let X be a space. Define the simplex category (or division [33]) $\mathrm{d}X$ of X to be the category with objects the simplices of X and morphisms generated by face and degeneracy maps. In other words the objects are maps $\sigma : \Delta[n] \to X$, where $\Delta[n]$ is the standard n-simplex, and a morphism $\sigma \to \sigma'$ is given by a commutative triangle

$$\begin{tikzcd} \Delta[n] \arrow[rr] \arrow[dr, "\sigma'"'] & & \Delta[m] \arrow[dl, "\sigma"] \\ & X. & \end{tikzcd}$$

Often we need the opposite category of $\mathrm{d}X$, which we denote $\bar{\mathrm{d}}\,X$. If X is the nerve of a small category $\mathbf{D}$, then we abbreviate $\mathrm{d}|\mathbf{D}|$ to $\mathrm{d}\mathbf{D}$. Define a *local coefficient system* on X (over R) to be a functor $\mathcal{F} : \bar{\mathrm{d}}\,X \to R\text{-}\mathrm{mod}$.

For any local coefficient system $\mathcal{F}$ we can define the simplicial cochain complex $(C^*(X;\mathcal{F}), \delta)$ of X with coefficients in $\mathcal{F}$ by setting

$$C^n(X;\mathcal{F}) = \prod_{\sigma \in X_n} \mathcal{F}(\sigma) \text{ and } (\delta f)(\sigma) = \sum_i (-1)^i \mathcal{F}(d_i^{\mathrm{op}})(f(d_i\sigma)).$$

We define $H^*(X;\mathcal{F})$ to be the homology of this cochain complex (cf. also [36, p. 42ff]). As usual we may as well use the *normalized* simplicial cochain complex for calculating $H^*(X;\mathcal{F})$ (cf. [82, §8.3]).

Note that we have a natural transformation $\bar{\mathrm{d}}\,\mathbf{D} \to \mathbf{D}$ on objects given by $(x_0 \to \cdots \to x_n) \mapsto x_n$. We also have a natural transformation $\bar{\mathrm{d}}\,\mathbf{D} \to \mathbf{D}^{\mathrm{op}}$ on objects given by $(x_0 \to \cdots \to x_n) \mapsto x_0$. If $F : \mathbf{D} \to R\text{-}\mathrm{mod}$ is a covariant (resp. contravariant) functor then F induces a local coefficient system $\mathcal{F}$ on $|\mathbf{D}|$ via $\bar{\mathrm{d}}\,\mathbf{D} \to \mathbf{D}$ (resp. $\bar{\mathrm{d}}\,\mathbf{D} \to \mathbf{D}^{\mathrm{op}}$). In both the co- and contravariant case we denote the cohomology groups by $H^*(\mathbf{D}; F)$, i.e., we omit indicating taking nerves, and indicate that the local coefficient system comes from a functor on $\mathbf{D}$ by keeping the notation F instead of $\mathcal{F}$.

We now want to define Bredon equivariant cohomology. For our purposes it is most natural to use a notion of a G-local coefficient system which is more general than the one given in e.g. [20], but which agrees with the usage in group theory (cf. [10, Ch. 7]). Roughly speaking, we want to allow different elements of G to induce different automorphisms of $C^*(X;\mathcal{F})$ even though they act the same way on X. For this purpose we introduce the following construction. Let $\mathbf{D}$ be a small G-category, and let α be the functor $\alpha : G \to \mathbf{Cat}$ which defines this G-category. (G is viewed as a category with one object, and $\mathbf{Cat}$ denotes the category of small categories.) Note that if $\mathbf{D}$ is a G-category, then so is $\mathbf{D}^{\mathrm{op}}$, via $\mathrm{op} : \mathbf{Cat} \to \mathbf{Cat}$. Set $\mathbf{D}_G$ equal to the Grothendieck construction of α [76], i.e., the category which has the same objects as $\mathbf{D}$ but where the morphisms $x \to y$ in $\mathbf{D}_G$ are given by pairs $(f : gx \to y, g)$, where f is a morphism in $\mathbf{D}$ and $g \in G$, which compose via $(f', g') \circ (f, g) = (f' \circ (g'f), g'g)$. Note that this can be thought of as a sort of semi-direct product of $\mathbf{D}$ with G, and just amounts to enlarging the morphisms in $\mathbf{D}$ with the elements of G in the freest possible way.

Remark 2.2. Let $\mathbf{D} = \mathcal{C}$ for some collection $\mathcal{C}$. The categories $\mathcal{C}_G$, $\mathbf{O}_{\mathcal{C}}^{\mathrm{op}}$ and $\mathbf{A}_{\mathcal{C}}$ can all be viewed as having the same objects, namely the objects of $\mathcal{C}$. The morphisms from H to H' in $\mathcal{C}_G$ can be identified with the elements of $g \in G$ such that $gHg^{-1} \leq H'$. The morphisms from H' to H in $\mathbf{O}_{\mathcal{C}}^{\mathrm{op}}$ may be identified with the elements $[g] \in G/H'$ such that $gH'g^{-1} \geq H$ and the morphisms from H to H' in $\mathbf{A}_{\mathcal{C}}$ may be identified with the elements $[g] \in G/CH$ such that $gHg^{-1} \leq H'$.

We have canonical functors $\mathcal{C}_G \to \mathbf{A}_{\mathcal{C}}$ and $(\mathcal{C}^{\mathrm{op}})_G \to \mathbf{O}_{\mathcal{C}}^{\mathrm{op}}$, which in the above picture take g to $[g]$. Note furthermore that we have a functor $\alpha : \mathbf{A}_{\mathcal{C}} \to \mathbf{O}(G)^{\mathrm{op}}$ on objects given by $H \mapsto G/CH$. In practice, our functors on $\mathbf{A}_{\mathcal{C}}$ will be constructed in this way from functors on $\mathbf{O}(G)^{\mathrm{op}}$.

Definition 2.3. Let X be a G-space. We define a G-local coefficient system on X to be a functor $\mathcal{F} : (\bar{\mathrm{d}}\, X)_G \to R\text{-}\mathrm{mod}$. Define a G-action on $C^*(X;\mathcal{F})$, by letting g act by $(gf)(\sigma) = \mathcal{F}((1_{g\sigma}, g))(f(g^{-1}))$. Define the Bredon cohomology of X with coefficients in $\mathcal{F}$ as $H^*_G(X;\mathcal{F}) = H(C^*_G(X;\mathcal{F}))$, where $C^*_G(X;\mathcal{F}) = C^*(X;\mathcal{F})^G$.

We may replace $C^*(X;\mathcal{F})$ by the corresponding normalized chain complex for the purpose of calculations. If M is an RG-module, then we denote cohomology with values in the 'constant' coefficient system $\sigma \mapsto M$ by $H^*_G(X;M)$.

Note also that to any G-local coefficient system $\mathcal{F}$ we may construct a new coefficient system $\tilde{\mathcal{F}}$ given by $\sigma \mapsto \mathcal{F}(\sigma)^{G_\sigma}$, where G_σ denotes the isotropy subgroup of σ. We have a natural transformation $\tilde{\mathcal{F}} \to \mathcal{F}$ which satisfies $H^*_G(X;\tilde{\mathcal{F}}) \xrightarrow{\cong} H^*_G(X;\mathcal{F})$, and $\tilde{\tilde{\mathcal{F}}} \xrightarrow{=} \tilde{\mathcal{F}}$. (E.g., the constant coefficient system

M corresponds to the generic coefficient system $H^0(-;M)$ in this manner.) Hence, for the purpose of Bredon cohomology, one could (as in [20]) without loss of generality have restricted attention to G-local coefficient systems which satisfied $\mathcal{F} = \tilde{\mathcal{F}}$. However, in practice making this restriction seems slightly unnatural and moreover the more general definition corresponds to the generality needed when studying homology representations (see [62], [10, Ch. 7]).

A *generic coefficient system* is a functor $\mathfrak{F} : \mathbf{O}(G)^{\mathrm{op}} \to R\text{-}\mathrm{mod}$. Note that for any G-space X we have a canonical (covariant) functor $\theta : (\bar{\mathrm{d}}\, X)_G \to \mathbf{O}(G)^{\mathrm{op}}$. On objects this is given by $\sigma \mapsto G_\sigma$, and on morphisms it is given by $(f : g\sigma \to \sigma', g) \mapsto (gG_{\sigma'}g^{-1} \geq G_\sigma, g)$, in the setting of Remark 2.2. A generic coefficient system induces functorially a G-local coefficient system $\mathcal{F}$ on any G-space X when $\mathcal{F} = \mathfrak{F}\theta$. (In much of the literature generic coefficient systems are just called coefficient systems, or Bredon coefficient systems.) We call a G-local coefficient system, which is induced by some generic coefficient system, an *isotropy G-local coefficient system*. (Bredon calls this a *simple coefficient system* on X.) (Note that any functor $\mathfrak{F} : \mathbf{O}_{\mathcal{C}}^{\mathrm{op}} \to R\text{-}\mathrm{mod}$ can be extended (nonuniquely) to a functor $\mathfrak{F}' : \mathbf{O}(G)^{\mathrm{op}} \to R\text{-}\mathrm{mod}$ (see [48, X3.4]), so the exact domain of definition is not a major issue.) When there is no danger of confusion we will not distinguish between a generic coefficient system and the isotropy G-local coefficient system it induces. However, to avoid confusion we have chosen not to give meaning to the term coefficient system without any prefix.

In general, cohomology with values in a G-local coefficient system on X is of course not a G-homotopy invariant of X. However, G-equivariant Bredon cohomology with values in a generic coefficient system does define a G-homotopy invariant cohomology theory on G-spaces (see e.g. [20]).

Recall that a Mackey functor is a generic coefficient system, but defined on a category with more morphisms than the orbit category, where one also has a 'transfer' map (cf. e.g., [50], [81]). We actually only directly need the Mackey functor $H^n(-;M)$ where the transfer map is the usual transfer map in group cohomology. This is a so-called cohomological Mackey functor, meaning that restriction from one subgroup to a smaller subgroup followed by the transfer back equals multiplication by the index.

Given a G-space X with a G-local coefficient system $\mathcal{F}$, we define Borel equivariant cohomology as $H^*_{hG}(X;\mathcal{F}) = H^*_G(EG \times X;\mathcal{F})$, where $EG \times X$ is given the diagonal G-action and coefficient system pulled back via the projection map $EG \times X \to X$.

Given a G-category $\mathbf{D}$ and a functor $F : \mathbf{D}_G \to R\text{-}\mathrm{mod}$ (or $F : (\mathbf{D}^{\mathrm{op}})_G \to R\text{-}\mathrm{mod}$), we naturally get a G-local coefficient system on $|\mathbf{D}|$ via $\bar{\mathrm{d}}\,\mathbf{D} \to \mathbf{D}$ (or $\bar{\mathrm{d}}\,\mathbf{D} \to \mathbf{D}^{\mathrm{op}}$).

Remark 2.4. In most of the paper we work over $\mathbf{Z}_{(p)}$, for the reason that we need multiplication by the index of a Sylow p-subgroup to be invertible in R. (Any result stated over $\mathbf{Z}_{(p)}$ holds verbatim over any $\mathbf{Z}_{(p)}$-algebra.) As a technical point however, when dealing with the Steinberg complex, we need to work over $\mathbf{Z}_p$ instead, in order to appeal to Webb's Theorem 5.1. One can translate results back to $\mathbf{Z}_{(p)}$ in the following way: $\mathbf{Z}_p$ is faithfully flat over $\mathbf{Z}_{(p)}$, so that a map of $\mathbf{Z}_{(p)}$-modules is an isomorphism if and only if it is an isomorphism after tensoring with $\mathbf{Z}_p$. Furthermore if X is of finite type, then one checks that $H^*_G(X;\mathcal{F}) \otimes_{\mathbf{Z}_{(p)}} \mathbf{Z}_p \cong H^*_G(X;\mathcal{F} \otimes_{\mathbf{Z}_{(p)}} \mathbf{Z}_p)$, and the same statement is of course true non-equivariantly. Hence, when showing certain maps are isomorphisms, we can without problems work over $\mathbf{Z}_p$.

2.5. *Higher limits.* Let $R\mathbf{D}$-mod denote the category of functors $\mathbf{D} \to R$-mod. By $\lim^*_{\mathbf{D}}$ we mean the right derived functors of the inverse limit functor $\lim_{\mathbf{D}} : R\mathbf{D}\text{-mod} \to R\text{-mod}$. We have the following fundamental identification.

Proposition 2.6 (cf. [35, 3.3]). *Let $F : \mathbf{D} \to R$-mod be any functor. Then*

$$\lim_{\mathbf{D}}{}^* F \cong \operatorname{Ext}^*_{R\mathbf{D}}(R, F) \cong H^*(\mathbf{D}; F).$$

Proof. First note that we can write $\lim^0_{\mathbf{D}} F = \operatorname{Hom}_{R\mathbf{D}}(R, F)$ so $\lim^*_{\mathbf{D}} F = \operatorname{Ext}^*_{R\mathbf{D}}(R, F)$. Let $\mathbf{D} \downarrow d$ denote the over-category of $d \in \mathbf{D}$ and consider the functor $C_*(|\mathbf{D} \downarrow -|; R) : \mathbf{D} \to R$- (chain complexes), which to each $d \in \mathbf{D}$ associates the simplicial chain complex of $|\mathbf{D} \downarrow d|$. One easily checks that the degree n of the cochain complex $\operatorname{Hom}_{R\mathbf{D}}(C_*(|\mathbf{D} \downarrow -|; R), F)$ equals $\prod_{x_0 \to \cdots \to x_n} F(x_n)$ with boundary map coming from the simplicial structure on $|\mathbf{D}|$. In particular we see that $\operatorname{Hom}_{R\mathbf{D}}(C_*(|\mathbf{D} \downarrow -|; R), -)$ preserves exactness, so that $C_n(|\mathbf{D} \downarrow -|; R)$ is projective in $R\mathbf{D}$-mod. But since $\mathbf{D} \downarrow d$ has an initial object, and therefore has contractible nerve, $C_*(|\mathbf{D} \downarrow -|; R)$ defines a projective resolution of the constant functor R in $R\mathbf{D}$-mod. This shows $\lim^*_{\mathbf{D}} F$ equals $H^*(\mathbf{D}; F)$. □

Remark 2.7. The cohomology $H^*(\mathbf{D}; F)$ can be viewed as sheaf cohomology of a certain sheaf Φ on $\operatorname{Top}(|\mathbf{D}|)$, the topological space associated to $|\mathbf{D}|$, constructed from F. In this remark we sketch this construction. Suppose for concreteness that F is contravariant. For each open set $U \subseteq \operatorname{Top}(|\mathbf{D}|)$ let

$$\Phi(U) = \{r \in \prod_{\sigma \in |\mathbf{D}|, U \cap \sigma \neq \emptyset} F(\sigma) | F(\phi)(r_{\sigma'}) = r_\sigma \text{ for all } \phi : \sigma \to \sigma' \text{ in } \mathrm{d}\mathbf{D}\}.$$

It is easy to see that this defines a sheaf on $\operatorname{Top}(|\mathbf{D}|)$, and if $y \in \operatorname{Top}(|\mathbf{D}|)$, then the stalk at y is equal to $F(\sigma)$, where σ is the unique nondegenerate simplex such that y lies in the open cell associated to σ. (An open zero cell is a point.)

We now sketch how to see that $H^*(\mathbf{D}; F) = H^*(\mathrm{Top}(|\mathbf{D}|); \Phi)$. We will restrict ourselves to the case where $\mathbf{D}$ is a poset—the general case can be obtained (rather messily) by passing to a double subdivision. If $\mathbf{D}$ is a poset $\mathrm{Top}(|\mathbf{D}|)$ is an ordered simplicial complex. For $\sigma \in |\mathbf{D}|$, let $\mathrm{star}(\sigma) \subseteq \mathrm{Top}(|\mathbf{D}|)$ be the open star of σ, i.e., the union of all open cells whose boundary contains σ. Note that $\mathrm{star}(\sigma)$ is open and contractible, with $\Phi(\mathrm{star}(\sigma)) = F(\sigma)$. Furthermore the Čech complex associated to the open covering given by $\mathrm{star}(\sigma)$, $\sigma \in |\mathbf{D}|$, equals the cochain complex considered in the proof of Proposition 2.6. Since the Čech complex calculates sheaf cohomology [36], we are done.

2.8. *Some auxiliary G-categories.* Let $\iota : \mathbf{O}_\mathcal{C} \to \mathbf{O}_{\mathcal{C}\cup\{e\}}$ be the inclusion functor, and consider the undercategory $E\mathbf{O}_\mathcal{C} = G/e \downarrow \iota$ (see [48, p. 46]). Note that $E\mathbf{O}_\mathcal{C}$ has a G-action in the obvious way, on objects $g \cdot f = gf$. (Alternatively, we may view $E\mathbf{O}_\mathcal{C}$ as the category with objects pairs (H, x) where $H \in \mathcal{C}$ and $x \in G/H$, and with a unique morphism $(H, x) \to (H', x')$ if there exists a G-map $G/H \to G/H'$ sending x to x'. The G-action in this picture is given by $g \cdot (H, x) = (H, gx)$.) Likewise, set $E\mathbf{A}_\mathcal{C} = \iota \downarrow G$, where $\iota : \mathbf{A}_\mathcal{C} \to \mathbf{A}_{\mathcal{C}\cup\{G\}}$ is the inclusion. We define a G action via $g \cdot i = c_g i$, where $(i : H \to G) \in E\mathbf{A}_\mathcal{C}$ and c_g denotes conjugation by g. (Note that the category $E\mathbf{O}_\mathcal{C}$ is isomorphic to the category denoted $\mathbf{X}^\beta_\mathcal{C}$ in [32], and $E\mathbf{A}_\mathcal{C}$ is equivariantly equivalent to the category denoted $\mathbf{X}^\alpha_\mathcal{C}$ in [32].)

Remark 2.9. For any collection $\mathcal{C}$ we have G-equivariant functors

$$E\mathbf{O}_\mathcal{C} \to \mathcal{C} \leftarrow E\mathbf{A}_\mathcal{C}$$

given by $(f : G \to G/H) \mapsto G_f$ and $(i : H \to G) \mapsto i(H)$ respectively. These are (in general non-equivariant) equivalences of categories with maps the other way being given by $H \mapsto (\mathrm{proj} : G \to G/H)$ and $H \mapsto (\mathrm{incl} : H \to G)$ respectively. (This was a key observation made by Dwyer [30], [32].)

Note that any functor $F : \mathbf{O}_\mathcal{C} \to R\text{-}\mathrm{mod}$ induces a G-local coefficient system on $|E\mathbf{O}_\mathcal{C}|$ via $(\bar{\mathrm{d}}\, E\mathbf{O}_\mathcal{C})_G \to (\bar{\mathrm{d}}\, \mathcal{C})_G \to (\mathcal{C}^{\mathrm{op}})_G \to \mathbf{O}^{\mathrm{op}}_\mathcal{C}$, and is hence a pullback of the coefficient system on $|\mathcal{C}|$. This coincides with the isotropy G-local coefficient system induced by F viewed as a generic coefficient system, and is on objects given by $(G \xrightarrow{f} G/H_0 \to \cdots \to G/H_n) \mapsto F(G_f)$. We keep the notation F for this G-local coefficient system on $|E\mathbf{O}_\mathcal{C}|$. Likewise we get a G-local coefficient system on $|E\mathbf{A}_\mathcal{C}|$ via $(\bar{\mathrm{d}}\, E\mathbf{A}_\mathcal{C})_G \to (\bar{\mathrm{d}}\, \mathcal{C})_G \to \mathcal{C}_G \to \mathbf{A}_\mathcal{C}$, on objects given by $(H_0 \to \cdots H_n \xrightarrow{i} G) \mapsto F(i(H_n))$, and we also keep the notation F for this coefficient system. (Note that if F is of the form $F = \mathfrak{F}\alpha$, where α is the canonical functor $\alpha : \mathbf{A}_\mathcal{C} \to \mathbf{O}^{\mathrm{op}}_\mathcal{C}$ and $\mathfrak{F} : \mathbf{O}(G)^{\mathrm{op}} \to R\text{-}\mathrm{mod}$ is some functor, then this is the isotropy G-local coefficient system induced by $\mathfrak{F}$.)

Proposition 2.10. *There are natural isomorphisms* $|E\mathbf{O}_{\mathcal{C}}|/G = |\mathbf{O}_{\mathcal{C}}|$ *and* $|E\mathbf{A}_{\mathcal{C}}|/G = |\mathbf{A}_{\mathcal{C}}|$. *Furthermore* $H^*(\mathbf{O}_{\mathcal{C}}; F) = H^*_G(|E\mathbf{O}_{\mathcal{C}}|; F)$ *and* $H^*(\mathbf{A}_{\mathcal{C}}; F) = H^*_G(|E\mathbf{A}_{\mathcal{C}}|; F)$.

Proof. The first two statements follow by inspection of the simplices. But by the definition of the coefficient systems and the first two statements, we now get that $C^*(|E\mathbf{O}_{\mathcal{C}}|; F)^G = C^*(\mathbf{O}_{\mathcal{C}}; F)$ and $C^*(|E\mathbf{A}_{\mathcal{C}}|; F)^G = C^*(\mathbf{A}_{\mathcal{C}}; F)$. Hence the last two statements follow by the definition of Bredon cohomology. $\square$

3. Higher limits over $\mathcal{C}$-orbit categories

In this section we examine higher limits over $\mathcal{C}$-orbit categories.

Theorem 3.1. *Let* $\mathcal{C}$ *be an arbitrary collection. Let* $F : \mathbf{O}_{\mathcal{C}}^{\mathrm{op}} \to R\text{-}\mathrm{mod}$ *be a functor concentrated on conjugates of a fixed subgroup* $H \in \mathcal{C}$. *Then*

1. $H^*_G(|E\mathbf{O}_{\mathcal{C}}|; F) \cong H^*_{hWH}(|\mathcal{C}_{\geq H}|, |\mathcal{C}_{>H}|; F(H)) \cong \tilde{H}^*_{hWH}(\Sigma|\mathcal{C}_{>H}|; F(H))$.
2. *If* $R = \mathbf{Z}_{(p)}$ *and* $|\mathcal{C}_{>H}|^Q$ *is* $\mathbf{F}_p$*-acyclic for all nontrivial* p*-subgroups* Q *in* WH*, then*

$$\begin{aligned} H^*_G(|E\mathbf{O}_{\mathcal{C}}|; F) &\cong H^*_G(|\mathcal{C}|; \mathcal{F}) \cong H^*_{WH}(|\mathcal{C}_{\geq H}|, |\mathcal{C}_{>H}|; F(H)) \\ &\cong \tilde{H}^*_{WH}(\Sigma|\mathcal{C}_{>H}|; F(H)), \end{aligned}$$

where $\mathcal{F}$ *is the coefficient system obtained from* F *by precomposing with* $(\bar{\mathrm{d}}\,\mathcal{C})_G \to (\mathcal{C}^{\mathrm{op}})_G \to \mathbf{O}_{\mathcal{C}}^{\mathrm{op}}$.

Proof. We first perform some preliminary rewritings. By definition

$$C^n_G(|\mathcal{C}|; \mathcal{F}) = \left(\prod_{H_0 < H_1 < \cdots < H_n} F(H_0) \right)^G .$$

Since F is atomic this equals $(\prod_{H < H_1 < \cdots < H_n} F(H))^{NH}$ by Frobenius reciprocity, so that

$$C^n_G(|\mathcal{C}|; \mathcal{F}) \cong C^n_{NH}(|\mathcal{C}_{\geq H}|, |\mathcal{C}_{>H}|; F(H)).$$

We want to do a similar rewriting with $E\mathbf{O}_{\mathcal{C}}$ in place of $|\mathcal{C}|$. View for a moment $E\mathbf{O}_{\mathcal{C}}$ as being replaced by its (G-equivalent) skeletal subcategory with objects G-maps $G \to G/H'$ where H' runs through a set of conjugacy classes of objects in $\mathcal{C}$. To simplify the notation, set $Y = \cup_{H' \in \mathcal{C}_{>H}} |E\mathbf{O}_{\mathcal{C}}^{H'}|$. As in the first rewriting we have

$$\begin{aligned} C_G^n(|E\mathbf{O}_{\mathcal{C}}|; F) &= \left(\prod_{G \xrightarrow{f} G/H \to G/H_1 \to \cdots \to G/H_n} F(G_f) \right)^G \\ &\cong \left(\prod_{G \xrightarrow{f} G/H \to G/H_1 \to \cdots \to G/H_n, f(e) \in WH} F(H) \right)^{NH} \\ &= C_{NH}^n(|E\mathbf{O}_{\mathcal{C}}^H|, Y; F(H)). \end{aligned}$$

These rewritings are natural, and give us corresponding identifications:

$$\begin{array}{ccc} H_G^*(|\mathcal{C}|; \mathcal{F}) & \cong & H_{WH}^*(|\mathcal{C}_{\geq H}|, |\mathcal{C}_{>H}|; F(H)) \\ \downarrow & & \downarrow \\ H_G^*(|E\mathbf{O}_{\mathcal{C}}|; F) & \cong & H_{WH}^*(|E\mathbf{O}_{\mathcal{C}}^H|, Y; F(H)) \ . \end{array}$$

Note that on the right-hand side the coefficient systems are constant coefficient systems, and hence come from the generic coefficient system $H^0(-; F(H))$.

We now prove 1 of Theorem 3.1. First note that the WH acts freely on the pair $(|E\mathbf{O}_{\mathcal{C}}^H|, Y)$, or said differently, the fixed-point pair under all nontrivial subgroups of WH becomes a trivial pair of the form (X, X). Hence

$$H_{WH}^*(|E\mathbf{O}_{\mathcal{C}}^H|, Y; F(H)) \cong H_{hWH}^*(|E\mathbf{O}_{\mathcal{C}}^H|, Y; F(H))$$

by the defining property of Borel cohomology. This shows the first isomorphism in 1, since the WH-map $(|E\mathbf{O}_{\mathcal{C}}^H|, Y) \to (|\mathcal{C}_{\geq H}|, |\mathcal{C}_{>H}|)$ is a homotopy equivalence of pairs. The second isomorphism follows directly from our definition of the suspension.

We now assume $R = \mathbf{Z}_{(p)}$ and proceed to prove 2. Consider again the homotopy equivalence

$$(|E\mathbf{O}_{\mathcal{C}}^H|, Y) \to (|\mathcal{C}_{\geq H}|, |\mathcal{C}_{>H}|). \tag{3.1}$$

Let P be a Sylow p-subgroup of WH. Since $H^0(-; F(H))$ is a cohomological Mackey functor we will be done, by an application of the transfer, if we can see that this is a P-$\mathbf{Z}_{(p)}$-equivalence if and only if $|\mathcal{C}_{>H}|^Q$ is $\mathbf{F}_p$-acyclic for all nontrivial p-subgroups Q in WH. To this end, we have already observed that the Q-fixed point set of the left-hand side of (3.1) becomes the trivial pair. Also $|\mathcal{C}_{\geq H}|^Q$ is contractible since $\mathcal{C}_{\geq H}^Q$ has the minimal element H. Hence (3.1) is a P-$\mathbf{Z}_{(p)}$-equivalence if and only if $|\mathcal{C}_{>H}|^Q$ is $\mathbf{Z}_{(p)}$-acyclic for all nontrivial p-subgroups Q in WH. Since $|\mathcal{C}_{>H}|^Q$ is of finite type, we may replace $\mathbf{Z}_{(p)}$ by $\mathbf{F}_p$. □

Remark 3.2. By Smith theory (cf. e.g. [50]) it is in Theorem 3.1 enough to verify that $|\mathcal{C}_{>H}|^Q$ is acyclic for all subgroups Q of order p.

Remark 3.3. Note that Theorem 3.1 shows that if F is concentrated on conjugates of a subgroup H, then the equality $H^*_G(|E\mathbf{O}_\mathcal{C}|; F) \cong H^*_G(|\mathcal{C}|; \mathcal{F})$ is equivalent to the equality $\tilde{H}^*_{WH}(\Sigma|\mathcal{C}_{>H}|; F(H)) \cong \tilde{H}^*_{hWH}(\Sigma|\mathcal{C}_{>H}|; F(H))$. The condition in part 2 of Theorem 3.1 just gives an obvious criterion for when relative Bredon and Borel cohomology with values in a cohomological Mackey functor over $\mathbf{Z}_{(p)}$ coincide, namely when the action of a Sylow p-subgroup is free (on the appropriate *pair*). Alternatively the condition can be viewed as ensuring the collapse of the E_2-term, a certain (relative) isotropy spectral sequence, onto the *horizontal* axis (see [32]). Conditions of this type feature prominently in the work of Webb [79], [78], [80], and were also exploited by Dwyer [32].

Remark 3.4. If H is a maximal element in $\mathcal{C}$, and F is concentrated on conjugates of H, then the formula of Theorem 3.1 shows that

$$H^*_G(|E\mathbf{O}_\mathcal{C}|; F) \cong H^*(WH; F(H)).$$

In particular if $\mathcal{C}$ contains a maximal element H which is a non-Sylow p-group, then $p||WH|$, by a standard property of p-groups. Thus, for example, when $F(H) = \mathbf{Z}_{(p)}$ with trivial WH-action, $\lim^*_{\mathbf{O}_\mathcal{C}} F \cong H^*(WH; \mathbf{Z}_{(p)})$ is nonzero in infinitely many degrees. This shows that one does not in general get good vanishing results for $\lim^*_{\mathbf{O}_\mathcal{C}} F$, when $\mathcal{C} = \mathcal{A}_p(G)$, the collection of nontrivial elementary abelian p-groups.

Corollary 3.5. *Let* $F : \mathbf{O}_\mathcal{C}^{\mathrm{op}} \to \mathbf{Z}_{(p)}$-mod *be a functor, and assume that* $|\mathcal{C}_{>H}|^Q$ *is* $\mathbf{F}_p$*-acyclic for all* $H \in \mathcal{C}$ *and for all nontrivial* p*-subgroups* Q *of* WH*. Then* $H^*_G(|\mathcal{C}|; \mathcal{F}) \xrightarrow{\simeq} H^*_G(|E\mathbf{O}_\mathcal{C}|; F)$.

Proof. Theorem 3.1 establishes this corollary for atomic functors. We want to establish the corollary for an arbitrary functor F by induction. Choose a maximal subgroup H in $\mathcal{C}$ on which F is nonzero, and let F' denote the subfunctor of F obtained by setting $F'(H') = 0$ if H' is conjugate to H and $F'(H') = F(H')$ otherwise. We have the following diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^*(|\mathcal{C}|; \mathcal{F}')^G & \longrightarrow & C^*(|\mathcal{C}|; \mathcal{F})^G & \longrightarrow & C^*(|\mathcal{C}|; \mathcal{F}/\mathcal{F}')^G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^*(|E\mathbf{O}_\mathcal{C}|; F')^G & \longrightarrow & C^*(|E\mathbf{O}_\mathcal{C}|; F)^G & \longrightarrow & C^*(|E\mathbf{O}_\mathcal{C}|; F/F')^G & \longrightarrow & 0.
\end{array}$$

(The sequences remain exact after we take G-fixed points, since they are split as modules.) By induction on the number of conjugacy classes of subgroups in $\mathcal{C}/G$ on which the functor is nonzero, we can assume that the statement is true for F', and the statement is also true for F/F', since it is an atomic functor. Hence the right and the left vertical arrows induce isomorphisms on homology; so by the five-lemma this is also true for the middle arrow. $\square$

Before the next proof, recall that for a poset $\mathcal{P}$, and $x \in \mathcal{P}$, $\mathbf{star}_{\mathcal{P}}\, x$ denotes the subposet $\{y \in \mathcal{P} | x \leq y \text{ or } x \geq y\}$ and likewise $\mathbf{link}_{\mathcal{P}}\, x = \{y \in \mathcal{P} | x > y \text{ or } y < x\}$.

Proof of Theorem 1.1. We first prove the first part of the theorem under the assumption that $\mathcal{C}$ is actually closed under passage to all p-overgroups, and will later show that non-p-radical subgroups do not play a role. We already know that $\lim^*_{\mathbf{O}_{\mathcal{C}}} F = H^*_G(|E\mathbf{O}_{\mathcal{C}}|; F)$ from Propositions 2.6 and 2.10. Hence to prove the first part of the theorem, we just have to show that the conditions of Corollary 3.5 apply. By the assumption that $\mathcal{C}$ is closed under passage to p-overgroups, we have that for all $P \in \mathcal{C}$, $\mathcal{C}_{>P} = \mathcal{S}_p(G)_{>P}$. But for all nontrivial p-subgroups Q of WP, $\mathcal{S}_p(G)^Q_{>P}$ has contractible nerve. To see this note that for all $H \in \mathcal{S}_p(G)^Q_{>P}$, identifying Q with its preimage in NP, $QH \in \mathcal{S}_p(G)^Q_{>P}$ so the inequalities $H \leq QH \geq Q$ provide a contracting homotopy (cf. also [59, 1.5],[10, Lemma 6.4.5]). Hence Corollary 3.5 shows that $H^*_G(|E\mathbf{O}_{\mathcal{C}}|; F) \cong H^*_G(|\mathcal{C}|; \mathcal{F})$.

Skipping slightly in the order in which the theorem was stated, we now proceed to prove the atomic case. From Theorem 3.1 we know that $\lim^*_{\mathbf{O}_{\mathcal{C}}} F \cong \tilde{H}^*_{NP}(\Sigma|\mathcal{C}_{>P}|; F(P))$. By assumption $\mathcal{C}_{>P} = \mathcal{S}_p(G)_{>P}$. But the posets $\mathcal{S}_p(G)_{>P}$ and $\mathcal{S}_p(WP)$ have NP-homotopy equivalent nerves, induced by the NP-equivariant maps $H \mapsto (H \cap NP)/P$ and $H/P \mapsto HP$ (cf. [59, Prop. 6.1], [10]), so we can use the space $|\mathcal{S}_p(WP)|$ instead of $|\mathcal{C}_{>P}|$. Writing the definition of Bredon cohomology we now obtain the wanted formula.

The last step in the proof is to see that we can replace the collection $\mathcal{C}$ (which we had assumed to be closed under passage to all p-overgroups) by the smaller collection $\mathcal{C}'$ as in the statement of the theorem, without changing higher limits or Bredon cohomology. By filtering the functor as in the proof of Corollary 3.5, we see that it is enough to prove this when F is an atomic functor, concentrated on conjugates of a subgroup $P \in \mathcal{C}$. If $P \in \mathcal{C} \setminus \mathcal{C}'$ then $\mathcal{S}_p(WP)$ is WP-contractible, a contracting homotopy being given by $H \leq HO_p(WP) \geq O_p(WP)$. Hence $H^*_G(|\mathcal{C}|; F) = 0$, by the formula for atomic functors, which proves the wanted result, since F in this case is zero on $\mathcal{C}'$. So, suppose that $P \in \mathcal{C}'$. By Theorem 3.1 it is enough to prove that $\mathcal{C}_{>P}$ and $\mathcal{C}'_{>P}$ are WP-homotopy equivalent. We will do this by showing that we can successively, in order of increasing size, remove WP-conjugacy classes of subgroups in $\mathcal{C}_{>P} \setminus \mathcal{C}'_{>P}$ from $\mathcal{C}_{>P}$, without changing its WP-homotopy type. Let $\tilde{\mathcal{C}}$ be a WP-subposet of $\mathcal{C}_{>P}$, with $\mathcal{C}'_{>P} \subseteq \tilde{\mathcal{C}} \subseteq \mathcal{C}_{>P}$. Suppose that $P' \in \tilde{\mathcal{C}} \setminus \mathcal{C}'_{>P}$ and that $\tilde{\mathcal{C}}_{>P'} = \mathcal{C}_{>P'}$. Let $\hat{\mathcal{C}}$ be the poset obtained from $\tilde{\mathcal{C}}$ by removing

WP-conjugates of P'. It is easy to check that we have a pushout square, which is a homotopy pushout of NP-space.

$$\begin{array}{ccc} NP \times_{NP\cap NP'} |\,\mathbf{link}_{\tilde{\mathcal{C}}}\, P'| & \longrightarrow & |\hat{\mathcal{C}}| \\ \Big\downarrow & & \Big\downarrow \\ NP \times_{NP\cap NP'} |\,\mathbf{star}_{\tilde{\mathcal{C}}}\, P'| & \longrightarrow & |\tilde{\mathcal{C}}|. \end{array}$$

Obviously $|\,\mathbf{star}_{\tilde{\mathcal{C}}}\, P'|$ is NP'-contractible. Note that $\mathbf{link}_{\tilde{\mathcal{C}}}\, P' = \mathcal{C}_{>P'} \star \tilde{\mathcal{C}}_{<P'}$, where $\star$ denotes the join of posets (corresponding to the join of spaces; see [59]). By the previous manipulations of posets, $|\mathcal{C}_{>P'}|$ is NP' contractible since $P' \in \mathcal{C} \setminus \mathcal{C}'$. Hence $|\,\mathbf{link}_{\tilde{\mathcal{C}}}\, P'|$ is NP'-contractible as well, so the above pushout square shows that $|\hat{\mathcal{C}}| \to |\tilde{\mathcal{C}}|$ is an NP-homotopy equivalence.

An induction now shows that $|\mathcal{C}_{>P}|$ and $|\mathcal{C}'_{>P}|$ are NP-homotopy equivalent as wanted. (Compare also, e.g., [73, Prop 1.7], [30, Thm. 7.1].) □

Remark 3.6. In [45] the higher limits of a functor $F : \mathbf{O}_{\mathcal{S}_p^e(G)} \to \mathbf{Z}_{(p)}$- mod concentrated on the trivial subgroup e with $F(e) = M$ are denoted $\Lambda^*(G; M)$. So in this notation Theorem 1.1 in particular implies that

$$\Lambda^n(G; M) \cong H^{n-1}(\mathrm{Hom}_G(\mathrm{St}_*(G), M)).$$

Remark 3.7. The p-radical subgroups have long been considered in finite group theory, and e.g. appear in Gorenstein's 1968 book, but have more recently obtained a central role in connection with Alperin's weight conjecture [4], [6]. Homotopy properties of the poset of p-radical subgroups were first studied by Bouc [15]. Their pertinence to the homotopy theory of classifying spaces was (independently, and in the more general context of compact Lie groups) discovered by Jackowski-McClure-Oliver [45] who dubbed them p-stubborn subgroups, since they just would not go away.

Remark 3.8. Taking $\mathcal{C} = \mathcal{S}_p^e(G)$ and $P = e$ in the above proof, we see that $\mathcal{S}_p(G)$ and $\mathcal{B}_p(G)$ are G-homotopy equivalent. This is a result of Bouc [15] and Thévenaz-Webb [73].

Remark 3.9. Note that Theorem 1.1 tells us that we can remove a *single* conjugacy class $[H]$ from a collection $\mathcal{C} \cap \mathcal{B}_p^e(G)$ as in Theorem 1.1 and still get the same result for *all* functors F only if $\mathrm{St}_*(WH)$ is contractible. But a strong version of Quillen's conjecture asserts that $O_p(W) = e$ implies that $\mathrm{St}_*(W)$ is noncontractible, so that minimality of the collection $\mathcal{C} \cap \mathcal{B}_p^e(G)$ in the sense above is in fact equivalent to this version of Quillen's conjecture. Note that, by Webb's Theorem 5.1, $\mathrm{St}_*(W)$ is contractible if and only if $\tilde{H}^*(|\mathcal{S}_p(W)|; \mathbf{Z}_{(p)}) = 0$. Quillen's conjecture, in its original form states that $O_p(W) = e$ implies that $|\mathcal{S}_p(W)|$ is non-contractible [59, Conj. 2.9]. Quillen's conjecture has been proven for a very large number of groups in [8], where they showed that $O_p(W) = e$ implies that $\tilde{H}^*(|\mathcal{S}_p(W)|; \mathbf{Z}) \neq 0$ under some restrictions on W and p. (See also [45, p. 206-207].)

Remark 3.10. The *non-equivariant* cohomology of subgroup complexes with values in a G-local coefficient system has been studied previously by, among others, Ronan-Smith [62], [63], [61] (cf. also [10]), as a method for constructing modular representations. It would be interesting to try to exploit the interplay between the equivariant and non-equivariant cohomology.

Remark 3.11. The proof of Theorem 1.1 reveals that the conclusion of the theorem holds not only for collections of p-subgroups but also for all collections $\mathcal{C}$ which are closed under extensions by p-groups. More precisely we need that if $H \triangleleft H' \leq G$ with $H \in \mathcal{C}$ and H'/H a p-group then $H' \in \mathcal{C}$.

Proof of Theorem 1.2. The proof follows the pattern of the last part of the proof of Theorem 1.1, but uses the additional assumptions about F together with information about the structure of the Steinberg complex (for which we refer to Section 5) to remove the additional subgroups. As explained in Remark 2.4 it is enough to prove the statement with $\mathbf{Z}_p$ in place of $\mathbf{Z}_{(p)}$. Hence, for the rest of the proof we adopt the convention that *all* coefficients are taken over $\mathbf{Z}_p$.

By filtering the functor as in the proof of Corollary 3.5, it is enough to prove the statement when F is an atomic functor, concentrated on conjugates of a subgroup $P \in \mathcal{C}$.

Suppose first that $P \in \mathcal{C} \setminus \mathcal{C}'$. Since in this case $H^*_G(|\mathcal{C}'|; \mathcal{F}) = 0$ we need to see that $H^*_G(|\mathcal{C}|; \mathcal{F}) = 0$, i.e., by Theorem 1.1 that $\mathrm{Hom}_{WP}(\mathrm{St}_*(W), F(P))$, or equivalently $\mathrm{Hom}_{WP}(\mathbb{S}\mathrm{t}_*(W), F(P))$ (see Section 5), is acyclic.

If P is non-p-centric, then the kernel of the action of WP on $F(P)$ has order divisible by p and so, by Corollary 5.4, $\mathrm{Hom}_{WP}(\mathbb{S}\mathrm{t}_m(WP), F(P)) = 0$ for all m and $H^*_G(|\mathcal{C}|; \mathcal{F}) = 0$ as wanted.

If P is p-centric then ZP is a Sylow p-subgroup in CP and we can write $CP = ZP \times K$ where $p \nmid |K|$. By Proposition 5.7 we have $\mathrm{St}_*(WP)_K = \mathrm{St}_*(NP/PC(P))$, where $(\cdot)_K$ denotes coinvariants, and hence

$$\mathrm{Hom}_{WP}(\mathrm{St}_*(WP), F(P)) = \mathrm{Hom}_{WP}(\mathrm{St}_*(NP/PCP), F(P)).$$

But $O_p(NP/PCP) \neq e$ since $P \in \mathcal{C} \setminus \mathcal{C}'$ so that $\mathcal{S}_p(NP/PC(P))$ is NP-contractible and therefore $\mathrm{Hom}_{WP}(\mathrm{St}_*(W), F(P))$ is acyclic.

Now suppose that $P \in \mathcal{C}'$. It is, by Theorem 3.1, enough to prove that

$$\tilde{H}^*_{WP}(\Sigma|\mathcal{C}_{>P}|; F(P)) = \tilde{H}^*_{WP}(\Sigma|\mathcal{C}'_{>P}|; F(P)),$$

and that the same statement holds for Borel cohomology instead of Bredon cohomology. As in the proof of Theorem 1.1, we do this by showing that we can successively, in order of increasing size, remove WP-conjugacy classes of subgroups in $\mathcal{C}_{>P} \setminus \mathcal{C}'_{>P}$ from $\mathcal{C}_{>P}$, without changing either the Bredon or Borel equivariant cohomology with coefficients $F(P)$.

Let the notation and assumptions be as in the last part of the proof of Theorem 1.1. The pushout square used there, together with the Mayer-Vietoris sequence in Bredon cohomology, shows that we need to see that $\tilde{H}^*_{NP\cap NP'}(\Sigma|\,\mathbf{link}_{\tilde{\mathcal{C}}} P'|; F(P)) = 0$, and that this also holds for Borel cohomology. Now $\mathbf{link}_{\tilde{\mathcal{C}}} P' = \mathcal{C}_{>P'} \star \tilde{\mathcal{C}}_{<P'}$. In particular for all nontrivial p-subgroups Q of NP' we have that $|(\mathbf{link}_{\tilde{\mathcal{C}}} P')^Q|$ is contractible, since $|\mathcal{C}^Q_{>P'}|$ is contractible. Hence reduced Bredon and Borel cohomologies coincide, and so we can restrict to considering Bredon cohomology (see Remark 3.3).

By the definition of the join

$$\tilde{C}_*(|\,\mathbf{link}_{\tilde{\mathcal{C}}} P'|) \cong \tilde{C}_*(|\mathcal{C}_{>P'}|) \otimes \Sigma\tilde{C}_*(|\tilde{\mathcal{C}}_{<P'}|)$$

where Σ denotes the suspension (i.e., dimension shifting) of chain complexes. This chain complex is again NP' chain homotopy equivalent to $\mathbb{St}_*(WP') \otimes \Sigma\tilde{C}_*(|\tilde{\mathcal{C}}_{<P'}|)$. Assume that P' is non-p-centric. Then we have that

$$\begin{aligned}
&\mathrm{Hom}_{NP\cap NP'}(\mathbb{St}_m(WP') \otimes \tilde{C}_n(|\tilde{\mathcal{C}}_{<P'}|), F(P)) \\
&\qquad \cong \mathrm{Hom}_{NP\cap NP'}(\mathbb{St}_m(WP'), \tilde{C}_n(|\tilde{\mathcal{C}}_{<P'}|) \otimes F(P)) \\
&\qquad \cong \mathrm{Hom}_{NP'}(\mathbb{St}_m(WP'), (\tilde{C}_n(|\tilde{\mathcal{C}}_{<P'}|) \otimes F(P)) \uparrow^{NP'}_{NP\cap NP'}) \\
&\qquad = 0.
\end{aligned}$$

Here the first isomorphism uses the fact that permutation modules are self-dual and the last equality follows from Corollary 5.4, since CP' acts trivially on $(\tilde{C}_n(|\mathcal{C}_{<P'}|) \otimes F(P)) \uparrow^{NP'}_{NP\cap NP'}$ and that P' is non-p-centric. But this is true for all m, n, so we conclude that $\tilde{H}^*_{NP\cap NP'}(\Sigma|\,\mathbf{link}_{\tilde{\mathcal{C}}} P'|; F(P)) = 0$ as wanted.

Now assume that P' is p-centric and write $CP' = ZP' \times K$, where $p \nmid |K|$. We have that $|\,\mathbf{link}_{\tilde{\mathcal{C}}} P'|/K = |\mathcal{C}_{>P'}|/K \star |\tilde{\mathcal{C}}_{<P'}|$. But $|\mathcal{C}_{>P'}|/K$ is NP'-equivalent to $|\mathcal{S}_p(NP'/P'C(P'))|$, by Proposition 5.7, which is NP'-contractible by the assumption that $O_p(NP'/P'C(P')) \neq e$. Therefore

$$\tilde{H}^*_{NP\cap NP'}(\Sigma|\,\mathbf{link}_{\tilde{\mathcal{C}}} P'|; F(P)) = \tilde{H}^*_{NP\cap NP'}(\Sigma|\,\mathbf{link}_{\tilde{\mathcal{C}}} P'|/K; F(P)) = 0,$$

also in this case, which finishes the proof of the theorem. □

We note that the above proof can be simplified somewhat if one is willing to settle for removing *all* non-p-centric subgroups, using the fact that the collection of p-centric subgroups is closed under passage to p-overgroups.

Remark 3.12. As in Remark 3.9 the formula in Theorem 1.2 for the higher limits over atomic functors shows that minimality of the collection $\mathcal{C} \cap \mathcal{D}_p(G)$ for all functors which satisfy that CP acts trivially on $F(P)$ is equivalent to a strong version of Quillen's conjecture.

Example 3.13. The G-homotopy type of $|\mathcal{D}_p(G)|$ is in general different from that of $|\mathcal{S}_p(G)|$. For example if $G = \Sigma_3 \times \mathbf{Z}/2$ then $|\mathcal{S}_2(G)|$ is G-contractible, whereas $|\mathcal{D}_2(G)| \simeq G/(\mathbf{Z}/2 \times \mathbf{Z}/2)$.

Remark 3.14. For a given p-group P and a subgroup $Q \leq P$ we are interested in when we can have $Q \in \mathcal{D}_p(G)$ for some finite group G with Sylow p-subgroup P. The definition of $\mathcal{D}_p(G)$ shows that necessary conditions include

$$C_P(Q) = ZQ \text{ and } N_P(Q)/Q \cap O_p(\mathrm{Out}(Q)) = e,$$

which are conditions expressed just in terms of P and Q. Much more elaborate conditions have been given in [57]. Determining the possible principal p-radical Q's along with the possibilities for the groups $N_G(Q)/QC_G(Q) \leq \mathrm{Out}(Q)$ is in fact equivalent to Brauer's 8th question [19] of determining the possible fusion patterns in a p-group P. (See Section 10 and [58], [57], [75].)

Example 3.15. A concrete example, where $\mathcal{D}_p(G)$ is a proper subcollection of the collection of p-radical and p-centric subgroups, is given by letting G equal the p-nilpotent group $(\mathbf{Z}/2 \times \mathbf{Z}/2) \rtimes \Sigma_3$, where Σ_3 acts on $\mathbf{Z}/2 \times \mathbf{Z}/2$ via $\Sigma_3 \to \mathbf{Z}/2 \xrightarrow{\text{flip}} \mathrm{Aut}(\mathbf{Z}/2 \times \mathbf{Z}/2)$. In this case both the Sylow 2-subgroups and $\mathbf{Z}/2 \times \mathbf{Z}/2$ are 2-radical and 2-centric, but only the Sylow 2-subgroups belong to $\mathcal{D}_2(G)$.

Proof of Theorem 1.3. Let F_i be the subfunctor of F obtained by setting $F(P) = 0$ for $\mathrm{ht}(P) > i$, and let $\mathcal{F}_i$ be the corresponding G-local coefficient system. This induces an increasing filtration $\{C^*(\mathcal{C}'; \mathcal{F}_i)^G\}_i$ on $C^*(\mathcal{C}'; \mathcal{F})^G$. The spectral sequence is now just the spectral sequence associated to a filtered chain complex [82]. Since we will need it in a while, let us write up explicitly the differential d^1 in the E^1-term:

$$C^k(|\mathcal{C}'|; \mathcal{F}_{i+1}/\mathcal{F}_i)^G \xrightarrow{d^1} C^{k+1}(|\mathcal{C}'|; \mathcal{F}_i/\mathcal{F}_{i-1})^G.$$

More precisely, given

$$\begin{aligned} \alpha \in {} & C^k(|\mathcal{C}'|; \mathcal{F}_{i+1}/\mathcal{F}_i)^G \\ & = \bigoplus_{[P'] \in \mathcal{C}'/G; \mathrm{ht}(P') = i+1} \mathrm{Hom}_{NP'}(C_k(|\mathcal{C}'_{\geq P'}|, |\mathcal{C}'_{> P'}|), F(P')), \end{aligned}$$

we get by conjugation a unique element (also denoted α), which lives in the same sum, but is now taken over all P', and not just conjugacy classes. We can describe $d^1(\alpha)$ as the following composition:

$$\begin{aligned} d^1(\alpha) : C_{k+1}(|\mathcal{C}'_{\geq P}|, |\mathcal{C}'_{>P}|) &\xrightarrow{\partial} \bigoplus_{P<P';\mathrm{ht}(P')=\mathrm{ht}(P)+1} C_k(|\mathcal{C}'_{\geq P'}|, |\mathcal{C}'_{>P'}|) \\ &\xrightarrow{\alpha} \bigoplus_{P<P';\mathrm{ht}(P')=\mathrm{ht}(P)+1} F(P') \overset{F(P<P')}{\rightarrow} F(P) \end{aligned}$$

where ∂ is the boundary map associated to the triple

$$(|\mathcal{C}'_{\geq P}|, |\mathcal{C}'_{\geq P,\mathrm{ht}\geq \mathrm{ht}(P)+1}|, |\mathcal{C}'_{\geq P,\mathrm{ht}\geq \mathrm{ht}(P)+2}|).$$

That we can replace $\mathcal{C} \cap \mathcal{B}_p^e(G)$ by $\mathcal{C} \cap \mathcal{D}_p(G)$ and NP/P by $NP/PC(P)$ if CP acts trivially on $F(P)$ for all $P \in \mathcal{C}$, follows from Theorem 1.2. $\square$

4. The case of a finite group of Lie type

The purpose of this section is to examine the spectral sequence in Theorem 1.3 in the special case of a finite group of Lie type G of characteristic p, where it degenerates into a chain complex.

We first need to recall some standard Lie notation. To a finite group of Lie type G of characteristic p, we can associate a split BN-pair of characteristic p (cf. e.g. [25, p. 50], [27]), which we will always assume to be fixed. Let S be the set of simple reflections of the Weyl group of the BN-pair. (Beware that this Weyl group need not coincide with the Weyl group of the ambient algebraic group if the Frobenius map is twisted.) Any parabolic subgroup in G is conjugate to exactly one 'standard' parabolic subgroup, and the standard parabolic subgroups in G are in one-to-one correspondence with subsets $J \subseteq S$, and are denoted P_J correspondingly. One defines U_J as the largest normal p-subgroup in P_J. Recall that $N_G(U_J) = P_J$ and $P_J = U_J \rtimes L_J$, where L_J again is a finite group of Lie type, with a split BN-pair of rank $|J|$. To any finite group of Lie type G of characteristic p we can associate the Steinberg module St_G (see [42], [27]). One of the many definitions of St_G is $\mathrm{St}_G = H_{r-1}(\Delta; \mathbf{Z}_{(p)}) = H_{r-1}(\Delta; \mathbf{Z}) \otimes \mathbf{Z}_{(p)}$, where Δ is the Tits building of the BN-pair of G and r is the rank of the BN pair. It is a projective indecomposable module of dimension $|G|_p$, whose reduction mod p is also simple.

A theorem of Borel and Tits [11], [22] says that any p-radical subgroup is conjugate to exactly one U_J, and it easily follows (cf. [73] and Remark 4.3) that $\mathcal{B}_p^e(G)$ is equal to the opposite poset of the poset of parabolic subgroups in G. Hence the NP/P which occur in Theorem 1.3 will be of the form L_J and, e.g., by Webb's Theorem 5.1, we may replace $\mathrm{St}_*(L_J)$ with St_{L_J} in degree $|J| - 1$ (cf. Remark 2.4). When the height function $\mathrm{ht}(U_I) = -|I|$ is chosen, Theorem 1.3 now takes the following simple form.

THEOREM 4.1. *Let G be a finite group of Lie type of characteristic p, $\mathcal{C}$ be a collection of p-subgroups closed under passage to p-radical overgroups, and $F : \mathbf{O}_{\mathcal{C}}^{\mathrm{op}} \to \mathbf{Z}_{(p)}\text{-}\mathrm{mod}$ any functor. Then $\lim^*_{\mathbf{O}_{\mathcal{C}}} F$ equals the homology of a cochain complex $(C^*(F), \delta)$ with*

$$C^i(F) = \bigoplus_{J \subseteq S; |J|=i; U_J \in \mathcal{C}} \mathrm{Hom}_{L_J}(\mathrm{St}_{L_J}, F(U_J)).$$

We now proceed to examine the boundary map in Theorem 4.1, which will be the content of Theorem 4.7. It turns out that special properties of finite groups of Lie type make this boundary map very tractable.

Let $R\mathbf{O}(G)\text{-}\mathrm{mod}$ denote the category of functors $\mathbf{O}(G)^{\mathrm{op}} \to R\text{-}\mathrm{mod}$. For any subgroup H of G, we have a functor $R\mathbf{O}(G)\text{-}\mathrm{mod} \to RWH\text{-}\mathrm{mod}$ given by evaluation $F \mapsto F(H)$. This functor has a right adjoint given by associating to a WH-module M the functor $E_{H,M}$ defined as

$$E_{H,M}(H') = (\oplus_{\mathrm{Mor}_G(G/H, G/H')} M)^{WH}.$$

Here WH acts on M in the prescribed way, and on $\mathrm{Mor}_G(G/H, G/H')$ via $g \cdot \alpha = \alpha \circ g^{-1}$. ($\mathrm{Mor}_G$ denotes morphisms in the category of left G-sets.) In particular, for any functor F we get a natural transformation $F \to E_{H,F(H)}$ induced by the unit of this adjunction. (See [38] for a careful explanation of the role of such adjunctions.)

For finite groups of Lie type, this right adjoint has a particularly simple form.

LEMMA 4.2. *If G is a finite group of Lie type of characteristic p, and U_I, $U_{J'}$ are the unipotent radicals of standard parabolic subgroups P_I, $P_{I'}$, then*

$$E_{U_I,M}(U_{I'}) = \begin{cases} 0 & \text{if } I' \not\subseteq I \\ M^{U_{I'}} & \text{if } I' \subseteq I. \end{cases}$$

Proof. As usual we have

$$\mathrm{Mor}_G(G/U_I, G/U_{I'}) = \{g \in G | g^{-1} U_I g \leq U_{I'}\}/U_{I'}.$$

We claim that

$$\{g \in G | g^{-1} U_I g \leq U_{I'}\}/U_{I'} = \begin{cases} P_I/U_{I'} & \text{if } I' \subseteq I \\ \emptyset & \text{otherwise} \end{cases}. \tag{4.1}$$

To show this we follow standard BN-pair notation as in [27, §64, 65, 69] (except we use U_I for their V_I), and assume some familiarity with this material. If $g \in P_I$, it is clear that $g^{-1} U_I g \subseteq U_{I'}$ if and only if $I' \subseteq I$, and so we suppose $g \notin P_I$. Using the Bruhat decomposition [27, Thm. 65.4] we can write $g = b\dot{w}b'$, where $\dot{w}$ is a lift of a $w \in W \setminus W_I$, and $b, b' \in B$. Hence we can without restriction assume $g = \dot{w}$, where w has an expression as a reduced word of simple reflections, starting with s_α, for some $\alpha \in S \setminus I$. But then $w^{-1}\alpha$ is a negative root, by the theory of root systems [27, Thm. 64.16(vii)]. This means that $g^{-1}U_I g \not\subseteq U_\emptyset$ [27, 69.5], and the claim is justified.

We now see that

$$E_{U_I,M}(U_{I'}) = (\oplus_{P_I/U_{I'}} M)^{P_I/U_I} = \mathrm{Hom}_{P_I}(\mathbf{Z}_{(p)}, M \uparrow_{U_{I'}}^{P_I}) = M^{U_{I'}}. \qquad \square$$

Remark 4.3. The above lemma can be used to show that the poset of nontrivial p-radical subgroups in G is naturally G-equivalent to the opposite poset of the poset of parabolic subgroups in G. Namely, by the Borel-Tits theorem [11] there is a one-to-one correspondence between the objects being given by taking normalizer and $O_p(\cdot)$ respectively. Now (4.1) and [27, Thm. 65.19] combine to show that this actually is an equivalence of *posets*. (This fact is of course well known, although a complete proof does not seem to be in the literature.)

Remark 4.4. The proof of Lemma 4.2 easily adapts to give a proof of another related fact. For G a finite group of Lie type and $\mathcal{C}$ the collection of p-subgroups which contain a conjugate subgroup of U_I, the category $\mathbf{O}_{\mathcal{C}\cap\mathcal{B}_p^e(G)}$ is equivalent to $\mathbf{O}_{\mathcal{B}_p^e(L_I)}$. Note that L_I has smaller semisimple rank than G if $I \neq S$, which allows for proofs by induction. One can for instance use this to give a different inductive proof of Theorem 4.1 and its addendum Theorem 4.7, which instead of subgroup complexes only relies on knowledge of the irreducible representations of G.

LEMMA 4.5. $(\mathrm{St}_{\mathbf{Z}_{(p)}G})^{U_J} \cong \mathrm{St}_{\mathbf{Z}_{(p)}L_J}$ *as* $\mathbf{Z}_{(p)}L_J$*-modules.*

Proof. The corresponding statement over $\mathbf{F}_p$ instead of $\mathbf{Z}_{(p)}$ is a consequence of a general theorem of S. Smith about U_J-fixed points of irreducible modules of finite groups of Lie type (cf. [68], [23]). We now show how to get the statement over $\mathbf{Z}_{(p)}$ from that over $\mathbf{F}_p$, in a slightly indirect way. Note that $\mathrm{Ext}^1_{U_J}(\mathrm{St}_{\mathbf{Z}_{(p)}G}, \mathbf{Z}_{(p)}) = 0$ since $\mathrm{St}_{\mathbf{Z}_{(p)}G}$ is projective. Hence we have an exact sequence

$$0 \to \mathrm{Hom}_{U_J}(\mathrm{St}_{\mathbf{Z}_{(p)}G}, \mathbf{Z}_{(p)}) \xrightarrow{p} \mathrm{Hom}_{U_J}(\mathrm{St}_{\mathbf{Z}_{(p)}G}, \mathbf{Z}_{(p)}) \to \mathrm{Hom}_{U_J}(\mathrm{St}_{\mathbf{Z}_{(p)}G}, \mathbf{F}_p) \to 0$$

which shows that $(\mathrm{St}_{\mathbf{Z}_{(p)}G})^{U_J} \otimes_{\mathbf{Z}_{(p)}} \mathbf{F}_p \cong (\mathrm{St}_{\mathbf{F}_pG})^{U_J}$, using the fact that $\mathrm{St}_{\mathbf{Z}_{(p)}G}$ is self-dual.

Hence the two expressions in the lemma are isomorphic after reduction mod p. Since $\mathrm{St}_{\mathbf{Z}_{(p)}L_J}$ is $\mathbf{Z}_{(p)}L_J$-projective this isomorphism can be lifted over to $\mathbf{Z}_{(p)}L_J$, where it has to be an isomorphism by Nakayama's lemma. $\square$

Lemma 4.6. *Assume $J' \subset J$, $|J| = |J'| + 1$, and let $F = E_{U_J, \mathrm{St}_{L_J}}$. Then the restriction and projection of the boundary map $\delta : C^{|J'|}(F) \to C^{|J|}(F)$ onto the corresponding factors J' and J,*

$$\delta : \mathrm{Hom}_{L_{J'}}(\mathrm{St}_{L_{J'}}, (\mathrm{St}_{L_J})^{U_{J'}}) \to \mathrm{Hom}_{L_J}(\mathrm{St}_{L_J}, \mathrm{St}_{L_J})$$

is an isomorphism.

Proof. First note that by Lemma 4.5 we know that $(\mathrm{St}_{L_J})^{U_{J'}} \cong \mathrm{St}_{L_{J'}}$. By Nakayama's lemma it is enough to prove the stated isomorphism after reducing both sides modulo p. Furthermore, one immediately checks, as in Lemma 4.5, that the appropriate Ext^1 vanishes so that we may commute $- \otimes_{\mathbf{Z}_{(p)}} \mathbf{F}_p$ and Hom. We have hence reduced the question of showing the isomorphism to a question over $\mathbf{F}_p$, and we therefore make the notational convention that in the rest of this proof, everything will be taken over $\mathbf{F}_p$ instead of $\mathbf{Z}_{(p)}$ which we suppress from the notation.

Since $\mathrm{St}_{L_{J'}}$ is absolutely irreducible over $\mathbf{F}_p$, $\mathrm{Hom}_{L_{J'}}(\mathrm{St}_{L_{J'}}, \mathrm{St}_{L_{J'}}) = \mathbf{F}_p$ (cf. [16, §13.4]), and likewise for J. Hence to show the isomorphism, it is enough to show that the identity map of $\mathrm{St}_{L_{J'}}$ is carried to a nonzero map under δ.

The proof of Theorem 1.3 gives a concrete description of δ. Since the homology modules of the chain complexes appearing in the proof of Theorem 1.3 are projective, we can replace the chain complexes with their homology. Hence the restriction and projection of the boundary map $\delta : C^{|J'|}(F) \to C^{|J|}(F)$ onto the corresponding factors J' and J is given as follows:

$$\delta(1) : \mathrm{St}_{L_J} \to \mathrm{St}_{L_{J'}} \uparrow_{P_{J'}}^{P_J} \xrightarrow{1} \mathrm{St}_{L_{J'}} \uparrow_{P_{J'}}^{P_J} \to \mathrm{St}_{L_J}.$$

The first map is injective, by the long exact sequence in homology inducing it. The last map is surjective, since it is seen to be a nontrivial map of P_J-modules by its definition. But it is well known that $\mathrm{St}_{L_{J'}} \uparrow_{P_{J'}}^{P_J}$ contains a *unique* summand of St_{L_J} (cf. [27, Thm. 67.10]), and so we conclude that $\delta(1)$ is in fact nontrivial, and hence is an isomorphism. $\square$

Theorem 4.7 (Addendum to Theorem 4.1). *Assume $J' \subset J$, $|J| = |J'| + 1$, and let e be a central idempotent in $\mathbf{Z}_{(p)} L_J$ corresponding to the Steinberg block* [27]. *The restriction and projection of the boundary map $\delta : C^{|J'|}(F) \to C^{|J|}(F)$ onto the corresponding factors J' and J are given as the top horizontal composite in the following diagram:*

$$\begin{tikzcd}
\delta : \mathrm{Hom}_{L_{J'}}(\mathrm{St}_{L_{J'}}, F(U_{J'})) \arrow[r] & \mathrm{Hom}_{L_{J'}}(\mathrm{St}_{L_{J'}}, F(U_J)^{U_{J'}}) \arrow[r, "\delta'"] \arrow[d, two heads] & \mathrm{Hom}_{L_J}(\mathrm{St}_{L_J}, F(U_J)) \arrow[d, two heads, "\simeq"] \\
 & \mathrm{Hom}_{L_{J'}}(\mathrm{St}_{L_{J'}}, (eF(U_J))^{U_{J'}}) \arrow[r, "\delta''", "\simeq"'] & \mathrm{Hom}_{L_J}(\mathrm{St}_{L_J}, eF(U_J))
\end{tikzcd}$$

Here the top left horizontal map is induced by the map in $\mathbf{O}_{\mathcal{C}'}$ *(landing in the invariants), and* δ' *and* δ'' *are the boundary maps in the chain complex for* $E_{U_J,F(U_J)}$ *and* $E_{U_J,eF(U_J)}$ *respectively.*

Proof. The diagram is commutative, since it is induced by the natural transformations of functors $F \to E_{U_J,F(U_J)} \to E_{U_J,eF(U_J)}$. That δ'' is an isomorphism is the content of Lemma 4.6. □

5. The Steinberg complex

In this section we analyze the Steinberg complex $\mathrm{St}_*(G) = \tilde{C}_*(|\mathcal{S}_p(G)|; \mathbf{Z}_{(p)})$. Many of the results in this section concerning the Steinberg complex are known to group theorists, and are mainly due to Webb [79], [78], [80] (see also [10], [64], [12], [13], [14]). However we find it worthwhile to include them, since they are important for calculations with the results of the previous section. Here we shall mainly work over the p-adic integers $\mathbf{Z}_p$. The main reason for this is the following theorem.

THEOREM 5.1 (Webb [80]). *The Steinberg complex over* $\mathbf{Z}_p$ *has the following structure*

$$\mathrm{St}_*(G; \mathbf{Z}_p) = P_* \oplus D_*$$

as $\mathbf{Z}_pG$*-modules, where* P_* *is a complex of* $\mathbf{Z}_pG$*-projective modules, and* D_* *is a* $\mathbf{Z}_pG$*-split acyclic complex.*

(Unfortunately, the proof given in [80] is not very clear on how to pass from the corresponding statement over $\mathbf{F}_p$, which is first proved, to the statement over $\mathbf{Z}_p$; the problem is that although all morphisms lift, the lifting is not functorial—this was pointed out to me by P. Symonds, who also explained how to overcome it [71].) In the category of bounded chain complexes over $\mathbf{Z}_pG$ the Krull-Schmidt theorem holds, so that there exists a unique minimal such P_* (unique up to *isomorphism* of chain complexes) (see [10, 5.17.1]). We denote such a minimal P_* by $\mathbb{S}\mathrm{t}_*(G)$—it is arguably really $\mathbb{S}\mathrm{t}_*(G)$ which deserves the name the Steinberg complex [5].

We now proceed to give restrictions on which projective modules can occur in $\mathbb{S}\mathrm{t}_*(G)$. Since in any degree $\mathrm{St}_*(G; \mathbf{Z}_p)$ is a sum of permutation modules, we first need a method of seeing which projective modules can occur as summands of a given permutation module. Recall that there is a bijection between projective modules over $\mathbf{Z}_p$ and $\mathbf{F}_p$ (cf. [65, 14.4]), and so we can restrict our attention to fields. Our method is the following lemma (with $T = k$), which is hinted at in [79, Prop. 5.3]. (See also [60].)

Lemma 5.2. *Let k be a field of characteristic p, and assume that H is a subgroup of a finite group G. Let S be a simple kG-module and let T be a simple kH-module. Denote the projective covers P_S and P_T. Then $[\mathrm{End}_G S : k]$ times the multiplicity of P_S in $T \uparrow_H^G$ equals $[\mathrm{End}_H T : k]$ times the multiplicity of P_T in $S \downarrow_H^G$.*

Proof. Let $\Omega^0 M$ denote the largest submodule of a kG-module M, which does not contain any projective summands (unique only up to isomorphism). There is a decomposition $M = \Omega^0 M \oplus P$, where P is projective. Note that the multiplicity of P_T in $S \downarrow_H^G$ is equal to the $\mathrm{End}_H T$-dimension of $\mathrm{coker}(\mathrm{Hom}_H(T, \Omega^0(S \downarrow_H^G)) \hookrightarrow \mathrm{Hom}_H(T, S \downarrow_H^G))$. Likewise the multiplicity of P_S in $T \uparrow_H^G$ equals the $\mathrm{End}_G(S)$-dimension of $\mathrm{coker}(\mathrm{Hom}_G(\Omega^0(T \uparrow_H^G), S) \hookrightarrow \mathrm{Hom}_G(T \uparrow_H^G, S))$. So in order to prove the lemma, we just need to justify that $\mathrm{Hom}_H(T, \Omega^0(S \downarrow_H^G)) \cong \mathrm{Hom}_G(\Omega^0(T \uparrow_H^G), S)$. We will show this by showing that we may replace Hom in both expressions with $\underline{\mathrm{Hom}}$, i.e., Hom in the stable module category, where the statement will follow by Frobenius reciprocity in the stable module category (cf. [3, p. 74]). Assume $\alpha : \Omega^0(k \uparrow_H^G) \to S$ is a kG-map which factors through a projective module, and hence through P_S.

$$
\begin{tikzcd}
 & P_S \arrow[d, two heads] \\
\Omega^0(T \uparrow_H^G) \arrow[ur, "\tilde{\alpha}"] \arrow[r, "\alpha"] & S
\end{tikzcd}
$$

If α is nontrivial then $\Omega^0(T \uparrow_H^G) \to P_S$ has to be surjective, since it is surjective on the head, contradicting that $\Omega^0(T \uparrow_H^G)$ does not contain any projective summands. Hence α is trivial and $\mathrm{Hom}_G(\Omega^0(T \uparrow_H^G), S) \cong \underline{\mathrm{Hom}}_G(T \uparrow_H^G, S)$. The proof that $\mathrm{Hom}_H(T, \Omega^0(S \downarrow_H^G)) \cong \underline{\mathrm{Hom}}_H(T, S \downarrow_H^G)$ is dual. $\square$

By [80] the conclusion of Webb's Theorem 5.1 does not change if we replace $\mathrm{St}_*(G; \mathbf{Z}_p)$ by a G-homotopy equivalent chain complex. Hence $\mathbb{S}\mathrm{t}_*(G)$ has to be a direct summand of the reduced normalized cellular chain complex of any space X G-homotopy equivalent to $|\mathcal{S}_p(G)|$. Lemma 5.2 says that if the space X has 'large' isotropy groups, then the simple modules S, corresponding to projective modules P_S appearing in $\mathbb{S}\mathrm{t}_*(G)$, also have to be 'large'. Note that a nondegenerate n-simplex $\sigma = (P_0 < \cdots < P_n) \in |\mathcal{C}|_n$ has stabilizer $G_\sigma = \cap_{i=0}^n NP_i$.

There are many spaces which are known to be G-homotopy equivalent to $|\mathcal{S}_p(G)|$. This is true for both $|\mathcal{B}_p(G)|$ and $|\mathcal{A}_p(G)|$ ([15], [59], [73]; for proofs see also the proofs of Theorem 1.1 and 6.1). If G is a finite group of Lie type of characteristic p then $|\mathcal{B}_p(G)|$ is just the subdivision of the building Δ of G (see Remark 4.3), and in particular they are G-homotopy equivalent. Furthermore $|\mathcal{A}_p(G)|$ is G-homotopy equivalent to the smaller complex given by the nerve of the collection of elementary abelian p-subgroups V satisfying that V equals the elements of order p in the center of CV [10, §6.6]. The so-called *commuting complex* $K_p(G)$ of Alperin [5], [7], which has n-simplices $(n+1)$-tuples of pairwise commuting subgroups of order p and is also G-equivalent to $|\mathcal{S}_p(G)|$ [74, p. 187]. (One way to see this is to note that the category with objects, arbitrary tuples of pairwise commuting subgroups of order p viewed as a poset under inclusion, is equivariantly equivalent to $\mathcal{A}_p(G)$; functors back and forth are given by associating to an n-tuple the elementary abelian p-subgroup it generates, and to an elementary abelian p-group associating the tuple of all rank one p-subgroups.)

Fix a Sylow p-subgroup P of G and define the following subgroup:

$$B = \cap_{Q \in \mathcal{B}_p(G)_{\le P}} NQ.$$

By Sylow's theorem it is uniquely defined up to conjugacy. It lies between CP and NP and so $p||B|$ if $p||G|$. If G is a finite group of Lie type of characteristic p then $B = NP$ and B is a Borel subgroup.

Proposition 5.3. *Let P_S be the projective module over $\mathbf{Z}_pG$ corresponding to the projective cover over $\mathbf{F}_pG$ of a simple $\mathbf{F}_pG$-module S. Assume P_S appears as a summand in $\mathbb{S}\mathrm{t}_m(G)$. Then we can say the following about S:*

1. *No element of order p in G acts trivially on S.*
2. *The module $S \downarrow_B^G$ contains the projective cover $P_{\mathbf{F}_p}$ of the trivial $\mathbf{F}_pB$-module $\mathbf{F}_p$ as a direct summand. In particular* $\dim_{\mathbf{F}_p} S \ge |B|_p$.
3. *There exists an elementary abelian p-subgroup V of G with* $\operatorname{rk} V \ge m+1$ *such that $S \downarrow_{CV}^G$ contains the projective cover $P_{\mathbf{F}_p}$ of the trivial CV-module $\mathbf{F}_p$ as a direct summand. In particular* $\dim_{\mathbf{F}_p} S \ge |CV|_p \ge p^{m+1}$.

Proof. To see the first claim let $K = \ker(G \to \mathrm{Aut}(S))$ and assume that P_S is a summand of $\mathbf{Z}_p[G/G_\sigma]$ for some $\sigma \in |\mathcal{S}_p(G)|$. Pick a Sylow p-subgroup P containing the chain of p-subgroups defining σ. Since K is normal in G, $P \cap K$ is a Sylow p-subgroup of K. Now suppose there exists an element of order p which acts trivially on S, so that $P \cap K \ne e$. Then, since P and $P \cap K$ are p-groups, we can find a nontrivial element $g \in P \cap K$ which is fixed under the conjugation action of P. This means that $\langle g \rangle \subseteq G_\sigma$. By Lemma 5.2, $S \downarrow_{G_\sigma}^G$ contains a summand of the projective $\mathbf{F}_pG_\sigma$-module $P_{\mathbf{F}_p}$. But then the further restriction $S \downarrow_{\langle g \rangle}^G$ has to contain a projective module as well. Hence it cannot be the trivial module, which is a contradiction since $g \in K$.

To see the second claim we choose to view the Steinberg complex as the reduced normalized chain complex of $|\mathcal{B}_p(G)|$. If P_S appears as a direct summand in $\tilde{C}_*(|\mathcal{B}_p(G)|; \mathbf{Z}_p)$, then P_S appears as a direct summand of $\mathbf{Z}_{(p)}[G/G_\sigma]$, for some $\sigma \in |\mathcal{B}_p(G)|$. Hence by Lemma 5.2, $S \downarrow^G_{G_\sigma}$ contains the projective cover $P_{\mathbf{F}_p}$ of the trivial $\mathbf{F}_p G_\sigma$-module $\mathbf{F}_p$ as a direct summand; so by Sylow's theorem G_σ contains a conjugate of B, and $S \downarrow^G_B$ has to contain a copy of the projective cover of the trivial $\mathbf{F}_p B$-module as well. It is now obvious that $\dim_{\mathbf{F}_p} S \geq |B|_p$, since a projective $\mathbf{F}_p B$-module has dimension divisible by $|B|_p$.

To see the last claim we choose to view the Steinberg complex as the reduced normalized chain complex of $|\mathcal{A}_p(G)|$. Observe that if $\sigma = (V_0 < \cdots < V_m)$ is a nondegenerate m-simplex in $|\mathcal{A}_p(G)|$ then $CV_m \leq G_\sigma$ and V_m has at least rank $m+1$. Hence the last claim follows from Lemma 5.2 as well. □

Note that (1) in the above proposition, combined with Theorem 1.1, has [45, Prop. 5.5] as a consequence (compare also [30, Thm. 6.3]). Let us for reference write a consequence of Proposition 5.3(1) and (3) out explicitly—the part coming from Proposition 5.3(3) is new.

Corollary 5.4. *Let G be a finite group and let M be a $\mathbf{Z}_p G$-module. Assume there exists a filtration $0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_m = M$ of M such that for each i either*

1. $\ker(G \to \mathrm{Aut}(N_i/N_{i-1}))$ *has order divisible by p, or*

2. N_i/N_{i-1} *is generated over $\mathbf{Z}_p$ by strictly less that p^k elements,*

then $\mathrm{Hom}_G(\mathbb{S}\mathrm{t}_{k-1}(G), M) = 0$.

In particular, by Theorem 1.1, *if F is a functor $F : \mathbf{O}^{\mathrm{op}}_{\mathcal{S}^e_p(G)} \to \mathbf{Z}_{(p)}\text{-}\mathrm{mod}$ concentrated on the trivial subgroup e with $F(e) = M$, where M satisfies the above condition, then* $\lim^k_{\mathbf{O}_{\mathcal{S}^e_p(G)}} F = 0$.

Example 5.5. If G is finite group of Lie type in characteristic p, then the Steinberg module St_G is the *only* simple module S such that $S \downarrow^G_B$ contains a summand of $P_{\mathbf{F}_p}$, agreeing nicely with our knowledge that the Steinberg complex in this case is chain homotopy equivalent to the Steinberg module concentrated in degree one less than the semi-simple rank of G.

Example 5.6. The structure of the Steinberg complex for the symmetric group Σ_n is unknown for large n, and appears to be quite complicated. See [12], [13] for low dimensional calculations. Note for instance that the modules which occur do not seem to be limited to any particular set of blocks of $\mathbf{F}_p\Sigma_n$. It would be nice to have a better understanding of this case. (See [6] for a list of the p-radical subgroups in Σ_n.)

The next observation (which has also been used by group theorists [8, p. 479]) gives, when combined with Theorem 1.1, a refinement of [45, Prop. 6.1(iii)].

PROPOSITION 5.7. *Let K be a normal subgroup in G of order prime to p. Then $|\mathcal{S}_p(G)|/K = |\mathcal{S}_p(G/K)|$, and in particular* $\mathrm{St}_*(G/K) = \mathrm{St}_*(G)_K$, *where $(-)_K$ denotes coinvariants.*

Proof. To see the first claim we have to see that if $(P_0 \leq \cdots \leq P_n)$ and $(P_0' \leq \cdots \leq P_n')$ are two chains of subgroups in G such that $P_iK = P_i'K$ for all i, then the two chains are conjugate by a single $k \in K$. Since K is normal in G then by Sylow's theorem, $PK = P'K$ if and only if we can find $k \in K$ such that $P' = kPk^{-1}$. In our two chains above we can hence assume that $P_n = P_n'$. But then P_i, P_i' are subgroups of P_n conjugate by an element in K, so that $P_i = P_i'$ (since if $g \in P_n, k \in K$ and $kgk^{-1} \in P_n$ then $kgk^{-1}g^{-1} \in K \cap P_n = e$). The second claim is a trivial consequence of the first. □

Although the homology groups of $\mathrm{St}_*(G)$ need not in general be projective [5], [64] this is however true for H_0, as shown by Quillen [59, Prop. 5.2] and Webb [78, pf. of Thm. E]. Fix a Sylow p-subgroup P of G and let

$$\tilde{G} = \text{ subgroup of } G \text{ generated by } N_G(Q),\ Q \in \mathcal{S}_p(G)_{\leq P}.$$

(Using that $\mathcal{B}_p(G)$ and $\mathcal{S}_p(G)$ are G-equivalent and arguing as in [59, Prop. 5.2], one sees that we may replace $\mathcal{S}_p(G)$ by $\mathcal{B}_p(G)$ in the above definition of $\tilde{G}$.)

PROPOSITION 5.8 (Quillen [59], Webb [78]). $|\mathcal{S}_p(G)| = G \times_{\tilde{G}} |\mathcal{S}_p(\tilde{G})|$, *and $\mathcal{S}_p(\tilde{G})$ is connected. Furthermore* $H_0(\mathrm{St}_*(G)) = \ker(\mathbf{Z}_{(p)}[G/\tilde{G}] \xrightarrow{\text{aug}} \mathbf{Z}_{(p)})$ *is a projective $\mathbf{Z}_{(p)}G$-module and* $\mathrm{St}_*(G) \cong H_0(\mathrm{St}_*(G)) \oplus \mathrm{St}_*(\tilde{G}) \uparrow_{\tilde{G}}^{G}$.

Remark 5.9. Groups G such that $\mathcal{S}_p(G)$ is disconnected are sometimes called p-isolated [37]. Groups such that $\mathcal{S}_p(G)$ is disconnected or empty 'have a strongly p-embedded subgroup'. (See e.g., [59, Prop. 5.2].) Such groups have been 'classified'—see [7, (6.2)].

Example 5.10. If G is a finite group of Lie type with $G \neq \tilde{G}$ then G has semisimple rank 1. In this case there is, up to conjugacy, only one nontrivial p-radical subgroup namely the Sylow p-subgroup and $\tilde{G} = B$, the Borel subgroup. The expression in Proposition 5.8 for $H_0(\mathrm{St}_*(G))$ is just the standard alternating sum formula (cf. [27, Thm. 66.35]) for the Steinberg module.

COROLLARY 5.11. *Let $\mathcal{C}$ be a collection of p-subgroups closed under passage to p-overgroups and let $F : \mathbf{O}_{\mathcal{C}}^{\mathrm{op}} \to \mathbf{Z}_{(p)}$*-mod *be a functor concentrated on conjugates of a subgroup P. Then* $\lim^1_{\mathbf{O}_{\mathcal{C}}} F = \mathrm{Hom}_{WP}(H_0(\mathrm{St}_*(WP)), F(P))$ *and* $\lim^2_{\mathbf{O}_{\mathcal{C}}} F = \mathrm{Hom}_{WP}(H_1(\mathrm{St}_*(WP)), F(P))$.

Proof. The statement for $\lim^1$ follows directly from Theorem 1.1 combined with Proposition 5.8 above. The $\lim^2$ statement also follows from these two results, by the fact that $\operatorname{Hom}_G(-,M)$ is left exact. $\square$

Note that this corollary has [45, Prop. 6.2] as a consequence.

Remark 5.12. If a projective module P_S corresponding to a simple $\mathbf{F}_pG$-module S appears in $\mathbb{S}\mathrm{t}_0(G)$ then by Lemma 5.2, $S\downarrow^G_{\tilde G}$ contains a summand of the projective $\mathbf{F}_p\tilde G$ module $P_{\mathbf{F}_p}$, and $\dim_{\mathbf{F}_p} S \geq |G|_p$, a much stronger estimate than Proposition 5.3.

More precisely, using Frobenius reciprocity in the stable module category, as in the proof of Lemma 5.2, one sees that for any $\mathbf{F}_pG$-module M

$$\begin{aligned}\dim_{\mathbf{F}_p}\operatorname{Hom}_G(\mathbb{S}\mathrm{t}_0(G;\mathbf{F}_p),M) &= (\text{mult. of } P_{\mathbf{F}_p,\tilde G} \text{ in } M\downarrow^G_{\tilde G})\\ &\quad - (\text{mult. of } P_{\mathbf{F}_p,G} \text{ in } M),\end{aligned}$$

where $P_{\mathbf{F}_p,\tilde G}$ and $P_{\mathbf{F}_p,G}$ denote the projective covers of $\mathbf{F}_p$ over $\mathbf{F}_p\tilde G$ and $\mathbf{F}_pG$ respectively.

Example 5.13. Suppose that g is an element of order p in G and that P_S occurs in $\mathbb{S}\mathrm{t}_0(G)$. Then $-\sum_{k=1}^{p-1} g^k$ does not act as the identity on S, strengthening Proposition 5.3, Part 1. To see this, note that we may assume $g\in\tilde G$, so that $\mathbf{F}_p\langle g\rangle$ will be a summand of $S\downarrow_{\langle g\rangle}$, and hence $1+\sum_{k=1}^{p-1} g^k = (g-1)^{p-1}$ does not act as zero on $S\downarrow_{\langle g\rangle}$.

Example 5.14. Let $G=\Sigma_{2p}$ and suppose that $p\geq 5$. Note that $\tilde G = \Sigma_p \wr \mathbf{Z}/2$ and that $\tilde\Sigma_p = \mathbf{Z}/p \rtimes \mathbf{Z}/(p-1)$. Let

$$M = \mathbb{S}\mathrm{t}_0(\Sigma_p) = \ker(\mathbf{Z}_p[\Sigma_p/(\mathbf{Z}/p\rtimes\mathbf{Z}/(p-1))] \xrightarrow{\text{aug}} \mathbf{Z}_p).$$

Note also that $\mathcal{S}_p(\tilde G) = \mathcal{S}_p(\Sigma_p)\star\mathcal{S}_p(\Sigma_p)$, so that $\mathbb{S}\mathrm{t}_*(\tilde G) = \mathbb{S}\mathrm{t}_1(\tilde G) = M\otimes_{\mathbf{Z}_p} M$, with the obvious $\Sigma_p\wr\mathbf{Z}/2$ action. Combination of these facts describes the Steinberg complex of Σ_{2p}. Now, $\mathbb{S}\mathrm{t}_0(G) = \ker(\mathbf{Z}_p[\Sigma_{2p}/(\Sigma_p\wr\mathbf{Z}/2)] \xrightarrow{\text{aug}} \mathbf{Z}_p)$, $\mathbb{S}\mathrm{t}_1(G) = (M\otimes_{\mathbf{Z}_p} M)\uparrow^{\Sigma_{2p}}_{\Sigma_p\wr\mathbf{Z}/2}$, and the differential is zero.

For later reference we also record the following useful fact, which follows directly from the definition of $\mathbb{S}\mathrm{t}_*(G)$ as a suitably minimal summand of a complex of permutation modules.

PROPOSITION 5.15. *The complex $\mathbb{S}\mathrm{t}_*(G)$ is dimensionally self-dual, i.e., $\mathbb{S}\mathrm{t}_n(G)\cong \operatorname{Hom}_{\mathbf{Z}_p}(\mathbb{S}\mathrm{t}_n(G),\mathbf{Z}_p)$ as $\mathbf{Z}_pG$-modules.*

6. Higher limits over $\mathcal{C}$-conjugacy categories

The $\mathcal{C}$-conjugacy category $\mathbf{A}_\mathcal{C}$ is the category with objects subgroups $H \in \mathcal{C}$ and morphisms homomorphisms $\varphi : H \to H'$ for which there exists $g \in G$ such that $\varphi = g(\cdot)g^{-1}$. This category occurs for example when one works with the *centralizer* homology decomposition (cf. [44], [30], [32]). The $\mathcal{S}_p^e(G)$-conjugacy category is often called the *Frobenius category* of G and the $\mathcal{A}_p(G)$-conjugacy category is often called the *Quillen category* of G.

Given a functor $F : \mathbf{A}_\mathcal{C} \to \mathbf{Z}_{(p)}\text{-}\mathrm{mod}$ we equip $|\mathcal{C}|$ with the G-local coefficient system $\mathcal{F}$ given by precomposition with the canonical map $(\bar{\mathrm{d}}\,\mathcal{C})_G \to \mathcal{C}_G \to \mathbf{A}_\mathcal{C}$ (see Remark 2.2). On objects we have $\mathcal{F}(H_0 \le \cdots \le H_n) = F(H_n)$.

For higher limits of functors over $\mathbf{A}_\mathcal{C}$ we obtain results dual to the ones for $\mathbf{O}_\mathcal{C}$.

Theorem 6.1. *Let $\mathcal{C}$ be any collection of nontrivial p-groups closed under passage to nontrivial elementary abelian subgroups, and let $F : \mathbf{A}_\mathcal{C} \to \mathbf{Z}_{(p)}\text{-}\mathrm{mod}$ be any functor. Then*

$$\lim_{\mathbf{A}_\mathcal{C}}{}^* F \cong H^*_G(|\mathcal{C}|;\mathcal{F}),$$

where $\mathcal{F}$ is the G-local coefficient system on $|\mathcal{C}|$ induced via $(\bar{\mathrm{d}}\,\mathcal{C})_G \to \mathcal{C}_G \to \mathbf{A}_\mathcal{C}$.

Furthermore for any collection $\mathcal{C}'$ satisfying $\mathcal{C} \cap \mathcal{A}_p(G) \subseteq \mathcal{C}' \subseteq \mathcal{C}$, the inclusion $|\mathcal{C}'| \to |\mathcal{C}|$ induces an isomorphism $H^*_G(|\mathcal{C}|;\mathcal{F}) \xrightarrow{\cong} H^*_G(|\mathcal{C}'|;\mathcal{F})$.

If F is a functor concentrated on a single conjugacy class of subgroups with representative V, then

$$\lim_{\mathbf{A}_\mathcal{C}}{}^i F = \begin{cases} \mathrm{Hom}_{NV/CV}(\mathrm{St}_{\mathrm{GL}(V)}, F(V)) & \text{if } V \in \mathcal{A}_p(G) \text{ and } i = \mathrm{rk}\, V - 1 \\ 0 & \text{otherwise.} \end{cases}$$

We prove this result later in this section. (To see the duality to the case of orbit categories we note that $\tilde{C}_*(|\mathcal{S}_p(G)_{<V}|)$ is chain homotopy equivalent to $\mathrm{St}_{\mathrm{GL}(V)}$ in degree $\mathrm{rk}\, V - 2$ if V is elementary abelian and is contractible otherwise.) The formula for the higher limits for an atomic functor is due to Oliver [56], and using the rank filtration on $\mathcal{A}_p(G)$ we recover his chain complex, arguing as in the proof of Theorem 1.3.

Theorem 6.2 ([56]). *Under the assumptions and notation of Theorem 6.1, $\lim^*_{\mathbf{A}_\mathcal{C}} F$ equals the homology of a cochain complex $(C^*(F), \delta)$ with*

$$C^i(F) = \bigoplus_{[V] \in \mathcal{C} \cap \mathcal{A}_p(G)/G;\, \mathrm{rk}\, V = i+1} \mathrm{Hom}_{NV/CV}(\mathrm{St}_{\mathrm{GL}(V)}, F(V)).$$

The boundary map can be described as follows. Given

$$\alpha \in \mathrm{Hom}_{NV/CV}(\mathrm{St}_{\mathrm{GL}(V)}, F(V)),$$

then the projection of $\delta(\alpha)$ *onto* $\mathrm{Hom}_{NU/U}(\mathrm{St}_{\mathrm{GL}(U)}, F(U))$ *where* $V \subset U$ *and* $\dim_{\mathbf{F}_p} U = \dim_{\mathbf{F}_p} V + 1$ *is given by the composite*

$$\begin{aligned}\mathrm{St}_{\mathrm{GL}(U)} &\to \bigoplus_{V' \subset U; [V'] = [V] \in \mathcal{A}_p(G)/G} \mathrm{St}_{\mathrm{GL}(V')} \\ &\xrightarrow{\alpha} \bigoplus_{V' \subset U; [V'] = [V] \in \mathcal{A}_p(G)/G} F(V') \to F(U).\end{aligned}$$

Before proving Theorem 6.1 we establish some results.

THEOREM 6.3. *Let* $F : \mathbf{A}_{\mathcal{C}} \to R\text{-}\mathrm{mod}$ *be an atomic functor concentrated on conjugates of* $H \in \mathcal{C}$. *Then*

1. $$\begin{aligned}H^*_G(|E\mathbf{A}_{\mathcal{C}}|; F) &\cong H^*_{h(NH/CH)}(|\mathcal{C}_{\le H}|, |\mathcal{C}_{<H}|; F(H)) \\ &\cong \tilde{H}^*_{h(NH/CH)}(\Sigma|\mathcal{C}_{<H}|; F(H)).\end{aligned}$$

2. *If* $R = \mathbf{Z}_{(p)}$ *and* $|\mathcal{C}_{<H}|^Q$ *is* $\mathbf{F}_p$*-acyclic for all nontrivial* p*-subgroups in* NH/CH, *then*
$$\begin{aligned}H^*_G(|E\mathbf{A}_{\mathcal{C}}|; F) \cong H^*_G(|\mathcal{C}|; \mathcal{F}) &\cong H^*_{NH/CH}(|\mathcal{C}_{\le H}|, |\mathcal{C}_{<H}|; F(H)) \\ &\cong \tilde{H}^*_{NH/CH}(\Sigma|\mathcal{C}_{<H}|; F(H)).\end{aligned}$$

Proof. The proof is dual to the proof of Theorem 3.1, and so we will be brief. Writing out the definitions one sees that:

$$C^n_G(|E\mathbf{A}_{\mathcal{C}}|; F) \cong C^n_{NH}(|E\mathbf{A}_{\mathcal{C}\le H}|, |E\mathbf{A}_{\mathcal{C}<H}|; F(H))$$

where $E\mathbf{A}_{\mathcal{C}\le H}$ (resp. $E\mathbf{A}_{\mathcal{C}<H}$) denotes the full subcategory of $E\mathbf{A}_{\mathcal{C}}$ with objects the objects $f : H' \to G$ in $E\mathbf{A}_{\mathcal{C}}$ satisfying $f(H') \le H$ (resp. $f(H') < H$). Hence we get corresponding identifications:

$$\begin{array}{ccc} H^*_G(|\mathcal{C}|; \mathcal{F}) & \cong & H^*_{NH}(|\mathcal{C}_{\le H}|, |\mathcal{C}_{<H}|; F(H)) \\ \downarrow & & \downarrow \\ H^*_G(|E\mathbf{A}_{\mathcal{C}}|; F) & \cong & H^*_{NH}(|E\mathbf{A}_{\mathcal{C}\le H}|, |E\mathbf{A}_{\mathcal{C}<H}|; F(H)). \end{array}$$

To show 1 we note that NH/CH acts freely on the pair $(|E\mathbf{A}_{\mathcal{C}\le H}|, |E\mathbf{A}_{\mathcal{C}<H}|)$ and argue as before. Likewise 2 follows dually. □

The same proof as Corollary 3.5 now reveals:

COROLLARY 6.4. *Let* $F : \mathbf{A}_{\mathcal{C}} \to \mathbf{Z}_{(p)}\text{-}\mathrm{mod}$ *be a functor and assume that* $|\mathcal{C}_{<H}|^Q$ *is* $\mathbf{F}_p$*-acyclic for all* $H \in \mathcal{C}$ *and all nontrivial* p*-subgroups* Q *of* NH/CH. *Then* $H^*_G(|\mathcal{C}|; F) \xrightarrow{\cong} H^*_G(|E\mathbf{A}_{\mathcal{C}}|; F)$.

Proof of Theorem 6.1. We already know that $\lim^*_{\mathbf{A}_{\mathcal{C}}} F = H^*_G(|E\mathbf{A}_{\mathcal{C}}|; F)$ from Propositions 2.6 and 2.10, and so to prove the first part of the theorem, we just have to show that the conditions of Corollary 6.4 apply.

We start by assuming that $\mathcal{C}$ is closed under passage to all nontrivial subgroups, and will later show that nonelementary abelian p-subgroups can be removed without changing either higher limits or Bredon cohomology. Hence, for all $H \in \mathcal{C}$, $\mathcal{C}_{<P} = \mathcal{S}_p(G)_{<P}$. If Q is a nontrivial p-subgroup of NP/CP, then $|\mathcal{S}_p(G)^Q_{<P}|$ is contractible by the inequalities $H \geq H^Q \leq P^Q$. (The fixed-point subgroup H^Q is not trivial since Q and H are p-groups.) Now Corollary 6.4 and Theorem 6.3 prove Theorem 6.1, except the statement about $\mathcal{C}'$ and the last rewriting in the atomic case.

We address the last rewriting first. If P is not elementary abelian, then the inequalities $H \leq H\Phi(P) \geq \Phi(P)$ show that $|\mathcal{S}_p(G)_{<P}|$ is NP-equivariantly contractible (see also [73]). (Here $\Phi(P)$ denotes the Frattini subgroup of P; alternatively, we use the inequalities $H \geq H \cap [P,P] \leq [P,P]$ if P is non-abelian, and the inequalities $H \geq {}_pH \leq {}_pP$ if P is abelian, instead.) Hence $\tilde{C}_*(|\mathcal{S}_p(G)_{<P}|)$ is NP-contractible if P is not elementary abelian. If $P = V$ is elementary abelian then $|\mathcal{S}_p(G)_{<V}|$ is the space associated to the ordered simplicial complex of flags of nontrivial proper sub-vector spaces of V, which again is equal to the barycentric subdivision of the Tits building of $\mathrm{GL}(V)$, by the definition of the Tits building [77]. But $\tilde{C}_*(|\mathcal{S}_p(G)_{<V}|)$ is NV-chain homotopy equivalent to its homology, which is just the Steinberg representation of $\mathrm{GL}(V)$ in dimension $\operatorname{rk} V - 2$. (One way to see this is to use Webb's Theorem 5.1.) This shows the last rewriting in the atomic case.

To see that we can replace $\mathcal{C}$ by $\mathcal{C}'$ we use the same argument as in the proof of Theorem 1.1. We can suppose that F is an atomic functor concentrated on conjugates of a subgroup $P \in \mathcal{C}$. If $P \in \mathcal{C} \setminus \mathcal{C}'$ then we are done by the formula for an atomic functor just proved. If $P \in \mathcal{C}'$ then we are done if we can show that $\mathcal{C}_{<P}$ and $\mathcal{C}'_{<P}$ are NP-homotopy equivalent. This is done by showing that we can remove subgroups in $\mathcal{C}_{<P} \setminus \mathcal{C}'_{<P}$ from $\mathcal{C}$ in order of decreasing size, without changing the NP homotopy type, using the pushout technique of the proof of Theorem 1.1 together with the NP'-contractibility of $\mathcal{S}_p(G)_{<P'}$ when P' is not elementary abelian, as shown above. ∎

Remark 6.5. A natural question to consider is whether one can obtain a similar description of higher limits of functors over quotient categories $\mathbf{D}$ of $\mathcal{C}_G$ or $(\mathcal{C}^{\mathrm{op}})_G$ other than just $\mathbf{A}_{\mathcal{C}}$ and $\mathbf{O}^{\mathrm{op}}_{\mathcal{C}}$. The proofs of Theorems 3.1 and 6.3 show what has to be satisfied. For quotient categories of $\mathcal{C}_G$ one needs that $|\mathcal{C}_{<H}|^Q$ is $\mathbf{F}_p$ acyclic for all $H \in \mathcal{C}$ and all nontrivial p-subgroups $Q \leq \operatorname{Aut}_{\mathbf{D}}(H)$ (and analogously for $(\mathcal{C}^{\mathrm{op}})_G$). We see that this property is rather special for $\mathbf{A}_{\mathcal{C}}$ (and $\mathbf{O}^{\mathrm{op}}_{\mathcal{C}}$), and generally does not hold for, for example, their opposite categories. For their opposite categories we however get formulas like the ones found in this paper for higher colimits.

7. Higher limits over orbit simplex categories

Let $(\operatorname{sd}\mathcal{C})/G$ be the orbit simplex category, i.e., the poset with objects G-conjugacy classes $[\sigma]$ of proper chains of subgroups $\sigma = (H_0 < \cdots < H_n)$ and with a morphism $[\sigma] \to [\sigma']$ if we can find representatives σ, σ', such that σ is a refinement of σ'. This category occurs when we work with the *normalizer* homology decomposition (cf. [30], [32]). Higher limits of functors on $((\operatorname{sd}\mathcal{C})/G)^{\mathrm{op}}$ can likewise be expressed as equivariant cohomology of the subgroup complex $|\mathcal{C}|$. Indeed, this follows by a formal argument, and does not require any assumptions on the collection $\mathcal{C}$ or the ground ring.

Since $(\operatorname{sd}\mathcal{C})/G$ is already a finite poset and hence $|(\operatorname{sd}\mathcal{C})/G| = |\operatorname{sd}\mathcal{C}|/G$ is a finite space (with associated topological space homeomorphic to $|\mathcal{C}|/G$), the only improvement will be to remove the subdivision sd. This follows easily from an argument of Słomińska [67]. (Alternatively one can use Remark 2.7: The topological spaces associated to $|\operatorname{sd}\mathcal{C}|$ and $|\mathcal{C}|$ are G-homeomorphic, and under the canonical homeomorphism the sheaves on the topological spaces associated to $|\operatorname{sd}\mathcal{C}|/G$ and $|\mathcal{C}|/G$ coincide.) The following proposition is due to her.

Proposition 7.1. *Let $\mathcal{C}$ be a collection of subgroups of a finite group G, and let $F : ((\operatorname{sd}\mathcal{C})/G)^{\mathrm{op}} \to R\text{-}\mathrm{mod}$ be an arbitrary functor. Then*

$$\lim_{(\operatorname{sd}\mathcal{C})/G}{}^{*} F \cong H^*_G(|\mathcal{C}|; \mathcal{F}),$$

where $\mathcal{F}$ is the G-local coefficient system given via $(\bar{\mathrm{d}}\,\mathcal{C})_G \to ((\operatorname{sd}\mathcal{C})/G)^{\mathrm{op}}$.

Proof. Note that $|\mathcal{C}|$ is in fact an ordered simplicial complex, so that the subcomplex $|\sigma|$ generated by an n-simplex σ is contractible. Hence the functor $\sigma \mapsto C_*(|\sigma|; R)$, i.e., the functor which to σ associates the normalized chain complex of the n-simplex σ, gives a more economical projective resolution of R in $R(\operatorname{sd}\mathcal{C})/G\text{-}\mathrm{mod}$, than the one of Proposition 2.6. In this case we see that $\operatorname{Hom}_{R(\operatorname{sd}\mathcal{C})/G}(C_n(|-|; R), F) = \prod_{[\sigma]} F(\sigma)$, where $[\sigma]$ runs over the G-conjugacy classes of nondegenerate n-simplices in $|\mathcal{C}|$. Hence $\lim^*_{(\operatorname{sd}\mathcal{C})/G} F = H^*(|\mathcal{C}|/G; F) = H^*_G(|\mathcal{C}|; \mathcal{F})$ as wanted. $\square$

In the topological applications occurring when one uses the normalizer decomposition, F will most often be induced by a generic coefficient system $\mathfrak{F} : \mathbf{O}(G)^{\mathrm{op}} \to R\text{-}\mathrm{mod}$; i.e., F is the functor which on objects is given by $[(P_0 \le \cdots \le P_n)] \mapsto \mathfrak{F}(NP_0 \cap \cdots \cap NP_n)$. In this case we have $C^*_G(|\mathcal{C}|; \mathcal{F}) = C^*_G(|\mathcal{C}|; \mathfrak{F})$. (Note however that $\mathcal{F}$ is not exactly the G-local coefficient system induced by $\mathfrak{F}$.) Hence in this case we may replace $|\mathcal{C}|$ with any G-homotopy equivalent space, for the purpose of calculating $H^*_G(|\mathcal{C}|; \mathcal{F})$. More precisely we get:

Corollary 7.2. *Let $\mathcal{C}$ be a collection of subgroups of a finite group G and let $F : (\mathrm{sd}\,\mathcal{C})/G \to R\text{-}\mathrm{mod}$ be a functor induced by a generic coefficient system $\mathfrak{F}$. Then*

$$\lim_{(\mathrm{sd}\,\mathcal{C})/G}{}^{*} F \cong H^*_G(|\mathcal{C}'|; \mathfrak{F})$$

for any collection $\mathcal{C}'$ such that $|\mathcal{C}'|$ is G-homotopy equivalent to $|\mathcal{C}|$.

We end this section by examining the special case where $\mathfrak{F}$ is the generic coefficient system $\mathfrak{F}(-) = H^n(-; \mathbf{Z}_{(p)})$, $n \geq 0$, which illustrates that for a particular $\mathfrak{F}$ one can do better than Corollary 7.2.

Theorem 7.3. *Let $\mathcal{C}$ be a collection of p-subgroups of a finite group G, closed under passage to p-overgroups. Let $\mathfrak{F}(-) = H^n(-; \mathbf{Z}_{(p)})$, $n \geq 0$, and let $F : ((\mathrm{sd}\,\mathcal{C})/G)^{\mathrm{op}} \to \mathbf{Z}_{(p)}\text{-}\mathrm{mod}$ be the associated functor on objects given by $[(P_0 \leq \cdots \leq P_n)] \mapsto \mathfrak{F}(NP_0 \cap \cdots \cap NP_n)$. Then*

$$\lim_{(\mathrm{sd}\,\mathcal{C})/G}{}^{*} F \cong \lim_{(\mathrm{sd}\,\mathcal{C}')/G}{}^{*} F \cong H^*_G(|\mathcal{C}'|; \mathfrak{F}),$$

for $\mathcal{C}'$ any collection $\mathcal{C} \cap \mathcal{D}_p(G) \subseteq \mathcal{C}' \subseteq \mathcal{C}$.

Proof. As in the proof of Theorem 1.2 it is enough to prove the statement over $\mathbf{Z}_p$, and we therefore adopt the convention that all coefficients are over $\mathbf{Z}_p$. We want to show that we can successively remove conjugacy classes of subgroups in $\mathcal{C} \setminus \mathcal{C}'$ in order of increasing size, without changing the Bredon cohomology. So suppose that $\tilde{\mathcal{C}}$ is a subcollection obtained from $\mathcal{C}$ by removing G-conjugacy of subgroups in this way and let $\hat{\mathcal{C}}$ denote the collection obtained from $\tilde{\mathcal{C}}$ by removing the G-conjugacy class of a minimal subgroup $P \in \tilde{\mathcal{C}} \setminus \mathcal{C}'$. As in the proof of Theorem 1.1 we have a pushout square, which is a homotopy pushout of G-spaces

$$\begin{array}{ccc} G \times_{NP} |\,\mathbf{link}_{\tilde{\mathcal{C}}}\, P| & \longrightarrow & |\hat{\mathcal{C}}| \\ \big\downarrow & & \big\downarrow \\ G \times_{NP} |\,\mathbf{star}_{\tilde{\mathcal{C}}}\, P| & \longrightarrow & |\tilde{\mathcal{C}}|. \end{array}$$

We need to see that $\tilde{H}^*_{NP}(|\,\mathbf{link}_{\tilde{\mathcal{C}}}\, P|; \mathfrak{F}) = 0$. Writing out the definition, we must see that $H^n(NP; \tilde{C}_*(|\,\mathbf{link}_{\tilde{\mathcal{C}}}\, P|))$ is an acyclic chain complex. But as in the proof of Theorem 1.2, $\tilde{C}_*(|\,\mathbf{link}_{\tilde{\mathcal{C}}}\, P|)$ is an NP-chain homotopy equivalent to $\mathbb{St}_*(WP) \otimes \Sigma \tilde{C}_*(|\tilde{\mathcal{C}}_{<P'}|)$.

Consider the Lyndon-Hochschild-Serre spectral sequence:

$$\begin{aligned} E_2^{i,j} &= H^i(WP; H^j(P; \tilde{C}_r(|\tilde{\mathcal{C}}_{<P}|)) \otimes \mathbb{St}_m(WP)) \\ &\Rightarrow H^{i+j}(NP; \mathbb{St}_m(WP) \otimes \tilde{C}_r(|\tilde{\mathcal{C}}_{<P}|)). \end{aligned}$$

Since $H^j(P;\tilde{C}_r(|\tilde{\mathcal{C}}_{<P}|))\otimes \mathbb{S}\mathrm{t}_m(WP)$ is a projective WP-module, the spectral sequence collapses and yields the fact that the chain complex

$$H^n(NP;\tilde{C}_*(|\,\mathbf{link}_{\tilde{\mathcal{C}}}\,P|))$$

is chain homotopy equivalent to the complex

$$\mathrm{Hom}_{WP}(\mathbf{Z}_p, H^n(P;\tilde{C}_*(|\tilde{\mathcal{C}}_{<P}|))\otimes \mathbb{S}\mathrm{t}_*(WP)).$$

But, using Proposition 5.15, we obtain

$$\begin{aligned}\mathrm{Hom}_{WP}(\mathbf{Z}_p, H^n(P;\tilde{C}_r(|\tilde{\mathcal{C}}_{<P}|))\otimes \mathbb{S}\mathrm{t}_m(WP))&\\ \cong \mathrm{Hom}_{WP}(\mathbb{S}\mathrm{t}_m(WP), H^n(P;\tilde{C}_r(|\tilde{\mathcal{C}}_{<P}|)))&\end{aligned}$$

which is zero if P is non-p-centric by Corollary 5.4. It is also zero if P is p-centric, since in this case it equals

$$\mathrm{Hom}_{WP}(\mathbb{S}\mathrm{t}_m(NP/PC(P)), H^n(P;\tilde{C}_r(|\tilde{\mathcal{C}}_{<P}|))),$$

by Proposition 5.7, which is zero since $O_p(NP/PC(P))\neq e$. Hence in both cases our complex $H^n(NP;\tilde{C}_*(|\,\mathbf{link}_{\tilde{\mathcal{C}}}\,P|))$ is chain homotopy equivalent to the trivial complex, and therefore in particular acyclic, which finishes the proof of the theorem. □

Remark 7.4. The key feature in the above argument was being able to replace $\mathrm{St}_*(WP;\mathbf{Z}_p)$ with the chain homotopic complex $\mathbb{S}\mathrm{t}_*(WP)$. This will be possible more generally if just F extends from a functor $\mathbf{O}(G)^{\mathrm{op}}\to \mathbf{Z}_{(p)}$- mod to a $\mathbf{Z}_{(p)}$-linear functor $\mathbf{Z}_{(p)}G$- mod $\to \mathbf{Z}_{(p)}$- mod. (Having an extension to a $\mathbf{Z}_{(p)}$-linear functor $\mathbf{Z}_{(p)}G$-(permutation modules) $\to \mathbf{Z}_{(p)}$- mod is equivalent to F being a cohomological Mackey functor, by [84].) Note also that in concrete situations the method of the above proof can be used to determine if further subgroups can be removed.

Example 7.5. To give an example where one cannot get away with just considering $\mathcal{D}_p(G)$, consider the generic coefficient system $F(H)=R(H)\otimes\mathbf{Z}_{(2)}$, where R denotes the complex representation ring. (This is a non-cohomological Mackey functor.) If $G=\Sigma_3\times\mathbf{Z}/2$ then $H^*_G(|\mathcal{S}_2(G)|;F)=R(G)\otimes\mathbf{Z}_{(2)}$ and $H^*_G(|\mathcal{D}_2(G)|;F)=R(\mathbf{Z}/2\times\mathbf{Z}/2)\otimes\mathbf{Z}_{(2)}$. These are obviously different, the first having rank 6, the latter 4. (See [74] for interesting work using this coefficient system.)

8. Induction sequences

The purpose of this section is to examine our results in the special case where F is a functor for which all the higher limits in fact vanish. In this case our results produce exact sequences expressing $F(G)$ in terms of F evaluated on subgroups, the one corresponding to the normalizer decomposition being the one found by Webb [80]. We do not assume that our functors are Mackey functors, although a Mackey structure is often needed in order to show that the higher limits indeed vanish.

We expand on work of Dwyer [32] and Webb [80] (see also Dress [29]), the main new results being Examples 8.8 and 8.9 and Corollaries 8.13, 8.14, and 8.17.

Our general setup is the following. Suppose we have a G-space X, a functor $F : (\mathcal{S}(G)^{\mathrm{op}})_G \to \mathbf{Z}\text{-}\mathrm{mod}$, and a map $(\bar{\mathrm{d}}\, X)_G \to (\mathcal{S}(G)^{\mathrm{op}})_G$ making F a G-local coefficient system on X. If $H^n_G(X;F) = 0$ for $n > 0$ and $H^0_G(X;F) = F(G)$, then writing the normalized cochain complex gives us an exact sequence

$$0 \to F(G) \to \prod_{\sigma_0 \in X_0/G} F(\sigma_0)^{G_{\sigma_0}} \to \prod_{\sigma_1 \in X_1/G} F(\sigma_1)^{G_{\sigma_1}} \to \cdots,$$

by setting $C^{-1}(X;F) = F(G)$. Here X_i denotes the set of nondegenerate i-simplices of X. This sequence expresses $F(G)$ in terms of subgroups in a quite explicit way. We will describe some instances of the above situation, as well as some related methods.

8.1. *Webb's exact sequence.* We will first try to explain the work of Webb in the current context—we refer the reader to [80], [81] for much more information, as well as many beautiful applications.

We say that a functor $\mathfrak{F} : \mathbf{O}(G)^{\mathrm{op}} \to R\text{-}\mathrm{mod}$ is projective (as a generic coefficient system) relative to a collection $\mathcal{C}$ if the canonical map $\oplus_{H \in \mathcal{C}} \mathfrak{F}\downarrow^G_H\uparrow^G_H \to \mathfrak{F}$, given as the unit of an adjunction, is split surjective in the category of functors $\mathbf{O}(G)^{\mathrm{op}} \to R\text{-}\mathrm{mod}$. (For example if $\mathcal{C}$ is a cohomological Mackey functor to $\mathbf{Z}_{(p)}\text{-}\mathrm{mod}$ then a transfer argument shows that $\mathfrak{F}$ is projective relative to $\mathcal{C} = \mathcal{S}_p(G)$.)

THEOREM 8.2 (Webb [80]). *Let $\mathfrak{F} : \mathbf{O}(G)^{\mathrm{op}} \to R\text{-}\mathrm{mod}$ be any functor. Let X be a G-space. Consider two collections $\mathcal{C}' \subseteq \mathcal{C}$ of subgroups of G both closed under passage to subgroups. Assume that $\mathfrak{F}$ is projective relative to $\mathcal{C}$, that $\mathfrak{F}(H) = 0$ for all $H \in \mathcal{C}'$, and that X^H is contractible for all $H \in \mathcal{C} \setminus \mathcal{C}'$.*

Then $H^*_G(X;\mathfrak{F}) = \mathfrak{F}(G)$ and there is a split exact sequence

$$0 \to \mathfrak{F}(G) \to \prod_{\sigma_0 \in X_0/G} \mathfrak{F}(G_{\sigma_0}) \to \prod_{\sigma_1 \in X_1/G} \mathfrak{F}(G_{\sigma_1}) \to \cdots.$$

Proof. Since $\oplus_{H \in \mathcal{C}} \mathfrak{F}\downarrow^G_H\uparrow^G_H \to \mathfrak{F}$ is assumed to be split surjective it is enough to prove the statements for $\mathfrak{F}\downarrow^G_H\uparrow^G_H$. However, since for any H-generic coefficient system $\mathfrak{F}$ we have the adjunction $C^*_G(X;\mathfrak{F}\uparrow^G_H) = C^*_H(X;\mathfrak{F})$ we can without restriction assume that $\mathcal{C}$ is the collection of all subgroups. The general acyclicity result now follows from Lemma 8.3 below, in the special case when $Y = *$. That the sequence in the theorem is in fact split exact follows from the proof of Lemma 8.3, since we may replace X by the G-contractible subspace X'. $\square$

LEMMA 8.3. *Let $f : X \to Y$ be a map of G-spaces, $\mathcal{C}$ a collection of subgroups of G closed under passage to subgroups, and $\mathfrak{F} : \mathbf{O}(G)^{\mathrm{op}} \to R\text{-}\mathrm{mod}$ an arbitrary generic coefficient system. Assume that $F(H) = 0$ for all $H \in \mathcal{C}$ and that $f^H : X^H \to Y^H$ is a homotopy equivalence for all $H \notin \mathcal{C}$. Then $H^*_G(Y;\mathfrak{F}) \xrightarrow{\cong} H^*_G(X;\mathfrak{F})$.*

Proof. Define $X' = \cup_{K\notin\mathcal{C}} X^K$ and note that $C^*_G(X;\mathfrak{F}) = C^*_G(X';\mathfrak{F})$ by the assumption on $\mathfrak{F}$. We define Y' similarly. It hence suffices to show that the restriction of f to a map $f' : X' \to Y'$ is a G-equivalence. If $H \notin \mathcal{C}$ then $X'^H = X^H$ and $Y'^H = Y^H$ since $\mathcal{C}$ is closed under passage to subgroups, so f'^H is a homotopy equivalence by our assumption. If $H \in \mathcal{C}$ we need to see that $f'^H : X'^H = \cup_{H\le K, K\notin\mathcal{C}} X^K \to \cup_{H\le K, K\notin\mathcal{C}} Y^K = Y'^H$ is a homotopy equivalence. But this is true since on each of the pieces X^K and on their intersections which are of the same form, the map $f'^H_{|X^K} : X^K \to Y^K$ is a homotopy equivalence. (The most appealing argument for why this implies that f'^H is a homotopy equivalence is perhaps to interpret the union as a homotopy colimit, and refer to the homotopy invariance property of homotopy colimits.) Hence f' is a homotopy equivalence on all fixed points, and so we conclude that f' is a G-equivalence. $\square$

It is interesting to compare the method of the above lemma to the 'method of discarded orbits' in [32, §6.7].

8.4. *Exact sequences arising from an acyclic collection.* Let $\mathcal{C}$ be a collection of subgroups in G and let $\mathfrak{F} : \mathbf{O}(G)^{\mathrm{op}} \to \mathbf{Z}_{(p)}\text{-}\mathrm{mod}$ be any functor. To $\mathfrak{F}$ we can associate three other functors. Let $F^\beta : \mathbf{O}_\mathcal{C}^{\mathrm{op}} \to \mathbf{Z}_{(p)}\text{-}\mathrm{mod}$ be the functor given by $H \mapsto \mathfrak{F}(H)$. Let $F^\alpha : \mathbf{A}_\mathcal{C} \to \mathbf{Z}_{(p)}\text{-}\mathrm{mod}$ be the functor given by $H \mapsto \mathfrak{F}(CH)$. Finally let $F^\delta : ((\mathrm{sd}\,\mathcal{C})/G)^{\mathrm{op}} \to \mathbf{Z}_{(p)}\text{-}\mathrm{mod}$ be the functor given by $[H_0 < \cdots < H_k] \mapsto \mathfrak{F}(NH_0 \cap \cdots \cap NH_k)$. Note that we have canonical maps $\mathfrak{F}(G) \to \lim^0_{\mathbf{O}_\mathcal{C}} F^\beta$, $F(G) \to \lim^0_{\mathbf{A}_\mathcal{C}} F^\alpha$, and $\mathfrak{F}(G) \to \lim^0_{(\mathrm{sd}\,\mathcal{C})/G} F^\delta$ given by restriction.

Definition 8.5. We say that $\mathfrak{F}$ is *subgroup $\mathcal{C}$-acyclic* if $\lim^i_{\mathbf{O}_\mathcal{C}} F^\beta = 0$ for $i > 0$ and $\mathfrak{F}(G) \xrightarrow{\cong} \lim^0_{\mathbf{O}_\mathcal{C}} F^\beta$. Likewise we say that $\mathfrak{F}$ is *centralizer or normalizer $\mathcal{C}$-acyclic* if the analogous conditions hold for F^α or F^δ.

By the rewritings of Propositions 2.6, 2.10, and 7.1, this definition can be stated as stating that $\mathfrak{F}$ is subgroup (resp. centralizer and normalizer) $\mathcal{C}$-acyclic if and only if the G-space $|E\mathbf{O}_\mathcal{C}|$ (resp. $|E\mathbf{A}_\mathcal{C}|$ and $|\mathcal{C}|$) is acyclic with respect to the generic coefficient system $\mathfrak{F}$.

We now give some important examples of acyclicity before giving our exact sequences.

Example 8.6. [The collection $\mathcal{S}_p^e(G)$] Consider the case $\mathcal{C} = \mathcal{S}_p^e(G)$, and let M be an arbitrary $\mathbf{Z}_{(p)}G$-module. By the analysis of Section 2, the spaces $|E\mathbf{O}_\mathcal{C}|$, $|E\mathbf{A}_\mathcal{C}|$ and $|\mathcal{C}|$ are P-contractible, where P is a Sylow p-subgroup of G. Since $H^n(-;M)$, $n \geq 0$, is a cohomological Mackey functor, a transfer argument shows that $H^n(-;M)$ is both subgroup, centralizer and normalizer $\mathcal{S}_p^e(G)$-acyclic.

Example 8.6 (The collection $\mathcal{S}_p(G)$). Let $\mathfrak{F}(-) = H^n(-;M) : \mathbf{O}(G)^{\mathrm{op}} \to \mathbf{Z}_{(p)}$- mod, where M is some $\mathbf{Z}_{(p)}G$-module. By our analysis in Sections 3 and 6, the spaces $|E\mathbf{O}_{\mathcal{S}_p(G)}|$, $|E\mathbf{A}_{\mathcal{S}_p(G)}|$ and $|\mathcal{S}_p(G)|$ are P-equivalent. Hence by an application of the transfer, we see that

$$H^*_G(|E\mathbf{O}_{\mathcal{S}_p(G)}|;\mathfrak{F}) \cong H^*_G(|\mathcal{S}_p(G)|;\mathfrak{F}) \cong H^*_G(|E\mathbf{A}_{\mathcal{S}_p(G)}|;\mathfrak{F}).$$

For $n > 0$ the formula in Theorem 1.1 shows that

$$H^*_G(|E\mathbf{O}_{\mathcal{S}_p(G)}|;\mathfrak{F}) = H^*_G(|E\mathbf{O}_{\mathcal{S}_p^e(G)}|;\mathfrak{F}),$$

since $H^n(e;M) = 0$ for $n > 0$. So, by the previous example, we see that $H^n(-;M)$, $n > 0$, is both subgroup, centralizer and normalizer $\mathcal{S}_p(G)$-acyclic. (Alternatively, a proof can be obtained by appealing to Lemma 8.3.) For $n = 0$, by definition, $H^*_G(|\mathcal{S}_p(G)|;M) = M^G$ if and only if $\mathrm{Hom}_G(\mathrm{St}_*(G), M)$ is an acyclic cochain complex. Hence $H^n(-;M)$ is subgroup $\mathcal{S}_p(G)$-acyclic if and only if it is centralizer $\mathcal{S}_p(G)$-acyclic if and only if it is normalizer $\mathcal{S}_p(G)$-acyclic, iff $\mathrm{Hom}_G(\mathrm{St}_*(G), M)$ is an acyclic cochain complex.

By removing subgroups we can now establish sharpness of smaller collections.

Example 8.8 (The collection $\mathcal{D}_p(G)$). Example 8.6 combined with Theorem 1.2 and 7.3 and Corollary 7.2 shows that $\mathfrak{F} = H^n(-;\mathbf{Z}_{(p)})$ is subgroup and normalizer $\mathcal{D}_p(G)$-acyclic, $n \geq 0$.

Example 8.9 (The collections $\mathcal{D}^e_{p,M}(G)$ and $\mathcal{D}_{p,M}(G)$). The previous example can be generalized to deal with arbitrary coefficient modules. Let M be a $\mathbf{Z}_{(p)}G$-module and let K be the kernel of the homomorphism $G \to \mathrm{Aut}\, M$. As in [30] a p-subgroup P is said to be M-centric if $Z(P)\cap K$ is a Sylow p-subgroup in $C(P)\cap K$. (If P is M-centric then we can write $C(P)\cap K = (Z(P)\cap K)\times L$ for some subgroup L where $p \nmid |L|$.) Let $\mathcal{D}^e_{p,M}(G)$ denote the collection of M-centric subgroups which satisfy $O_p(NP/(P(C(P)\cap K))) = e$, and let $\mathcal{D}_{p,M}(G) = \mathcal{D}^e_{p,M}(G) \cap \mathcal{S}_p(G)$. Note that $\mathcal{D}_p(G) \subseteq \mathcal{D}^e_{p,M}(G) \subseteq \mathcal{B}^e_p(G)$. In the extreme case where M is the trivial module and $p||G|$ then $\mathcal{D}_{p,M}(G) = \mathcal{D}^e_{p,M}(G) = \mathcal{D}_p(G)$, since e is noncentric. On the other hand if G acts faithfully on M, then $\mathcal{D}^e_{p,M}(G) = \mathcal{B}^e_p(G)$.

The proofs of Theorems 1.2 and 7.3 reveal that $H^n(-;M)$ is subgroup and normalizer $\mathcal{D}^e_{p,M}(G)$-acyclic for $n \geq 0$. Likewise $H^n(-;M)$ is subgroup and normalizer $\mathcal{D}_{p,M}(G)$-acyclic for $n > 0$, and for $n = 0$ it is $\mathcal{D}_{p,M}(G)$-acyclic if and only if it is $\mathcal{S}_p(G)$-acyclic.

Remark 8.10. By a result of Diethelm-Stammbach [28] (see also [40], [70]), for any finite p-group P and any $W \leq \mathrm{Out}(P)$, every simple $\mathbf{F}_pW$-module appears as a composition factor of $H^n(P;\mathbf{F}_p)$ for some n. Hence properties of $\mathrm{St}_*(W)$ will not in the most naive way allow one to remove more subgroups from the collection $\mathcal{D}_p(G)$ when examining acyclicity properties of the functors $H^n(-;\mathbf{Z}_{(p)})$, if one wants the collection to work for *all* $n \geq 0$. (But, see Example 10.11.) It would be interesting to get smaller collections than $\mathcal{D}^e_{p,M}(G)$ for the functors $H^n(-;M)$, using more elaborate information about M.

Remark 8.11. For the subgroup and normalizer $\mathcal{D}_{p,M}(G)$-acyclicity of $H^n(-;M)$, $n > 0$ we can furthermore remove those subgroups $P \in \mathcal{D}^e_{p,M}(G)$ such that $M \downarrow^G_P$ is projective. This is obvious for subgroup acyclicity, and for normalizer acyclicity it follows from Lemma 8.3.

Example 8.12. [The collection $\mathcal{A}_p(G)$] Theorem 6.1 and Corollary 7.2 show that $H^n(-;M)$ is centralizer and normalizer $\mathcal{A}_p(G)$-acyclic for $n > 0$ and has the same centralizer and normalizer $\mathcal{A}_p(G)$-acyclicity properties as $\mathcal{S}_p(G)$-acyclicity properties for $n = 0$.

Theorem 1.1, Theorem 6.1, and Proposition 7.1 immediately produce three exact sequences, where the last one is a version of Webb's result.

Corollary 8.13. *Let $\mathcal{C}$ be a collection of p-subgroups of a finite group G, closed under passage to p-radical overgroups. Let $\mathfrak{F} : \mathbf{O}(G)^{\mathrm{op}} \to \mathbf{Z}_{(p)}\text{-}\mathrm{mod}$ be any subgroup $\mathcal{C}$-acyclic functor. Then there is an exact sequence*

$$0 \to \mathfrak{F}(G) \to \oplus_{[P]\in\mathcal{C}'/G}\mathfrak{F}(P)^{NP} \to \oplus_{[P_0<P_1]\in|\mathcal{C}'|_1/G}\mathfrak{F}(P_0)^{NP_0\cap NP_1} \to \cdots \to 0$$

where $\mathcal{C}' = \mathcal{C} \cap \mathcal{B}^e_p(G)$. The first boundary map is given by restriction and the latter comes from the boundary map in the cochain complex $C^*_G(|\mathcal{C}'|;F^\beta)$.

If for all $P \in \mathcal{C}$, CP acts trivially on $\mathfrak{F}(P)$ then $\mathcal{C}'$ may be replaced by $\mathcal{C} \cap \mathcal{D}_p(G)$.

Corollary 8.14. *Let $\mathcal{C}$ be a collection of nontrivial p-subgroups of a finite group G, closed under passage to nontrivial elementary abelian subgroups. Let $\mathfrak{F} : \mathbf{O}(G)^{\mathrm{op}} \to \mathbf{Z}_{(p)}\text{-}\mathrm{mod}$ be any centralizer $\mathcal{C}$-acyclic functor. Then there* *is an exact sequence*

$$0 \to \mathfrak{F}(G) \to \oplus_{[V]\in \mathcal{C}'/G}\mathfrak{F}(CV)^{NV} \to \oplus_{[V_0<V_1]\in|\mathcal{C}'|_1/G}\mathfrak{F}(CV_1)^{NV_0\cap NV_1} \to \cdots \to 0,$$

where $\mathcal{C}' = \mathcal{C}\cap\mathcal{A}_p(G)$. The first boundary map is given by restriction and the latter comes from the boundary map in the cochain complex $C^*_G(|\mathcal{C}'|;F^\alpha)$.

Corollary 8.15 ([80]). *Let $\mathcal{C}$ be any collection of subgroups of a finite group G. Let $\mathfrak{F}:\mathbf{O}(G)^{\mathrm{op}} \to R\text{-}\mathrm{mod}$ be any normalizer $\mathcal{C}$-acyclic functor. Then there is an exact sequence*

$$0 \to \mathfrak{F}(G) \to \oplus_{[H]\in\mathcal{C}/G}\mathfrak{F}(NH) \to \oplus_{[H_0<H_1]\in|\mathcal{C}|_1/G}\mathfrak{F}(NH_0\cap NH_1) \to \cdots \to 0,$$

where the boundary map comes from the boundary map in the cochain complex $\tilde{C}^*_G(|\mathcal{C}|;\mathfrak{F})$, where $\mathfrak{F}$ is viewed as a generic coefficient system.

Example 8.16. Consider the sporadic group M_{11}. According to the Atlas [26] there are two conjugacy classes of elementary abelian 2-subgroups: one of rank one V_1 with centralizer $CV_1 = \mathrm{GL}_2(\mathbf{F}_3)$ and one of rank two V_2 which is self-centralizing $CV_2 = V_2$. Furthermore $NV_1 = CV_1$, $NV_2 = \Sigma_4$, and $NV_1 \cap NV_2 = D_8$. Example 8.12 and Corollary 8.14 hence produce an exact sequence for any $n>0$ and any $\mathbf{Z}_{(2)}M_{11}$-module N,

$$0 \to H^n(M_{11};N) \to H^n(\mathrm{GL}_2(\mathbf{F}_3);N)\oplus H^n(V_2;N)^{\Sigma_3} \to H^n(V_2;N)^{\mathbf{Z}/2} \to 0.$$

The chain complexes of Theorem 4.1 and Theorem 6.2 likewise give us exact sequences.

Corollary 8.17. *Let G be a finite group of Lie type of characteristic p, and let $\mathcal{C}$ be a collection of p-subgroups closed under passage to p-radical overgroups. Let $\mathfrak{F}:\mathbf{O}(G)^{\mathrm{op}} \to \mathbf{Z}_{(p)}\text{-}\mathrm{mod}$ be any subgroup $\mathcal{C}$-acyclic functor. Then there is an exact sequence*

$$0 \to \mathfrak{F}(G) \to \mathfrak{F}(U_\emptyset)^{P_\emptyset} \to \bigoplus_{J\subseteq S;|J|=1;U_J\in\mathcal{C}} \mathrm{Hom}_{L_J}(\mathrm{St}_{L_J},\mathfrak{F}(U_J)) \to \cdots \to 0.$$

The first boundary map is restriction to the Sylow p-subgroup $U_\emptyset$ in G and the following ones are described in Theorem 4.7.

Corollary 8.18. *Let $\mathcal{C}$ be a collection of nontrivial p-subgroups of a finite group G, closed under passage to nontrivial elementary abelian p-subgroups. Let $\mathfrak{F}:\mathbf{O}(G)^{\mathrm{op}} \to \mathbf{Z}_{(p)}\text{-}\mathrm{mod}$ be any centralizer $\mathcal{C}$-acyclic functor. Then we have an exact sequence*

$$\begin{aligned} 0 \;\to\; & \mathfrak{F}(G) \to \bigoplus_{[V]\in\mathcal{C}\cap\mathcal{A}_p(G)/G;\,\mathrm{rk}\,V=1} \mathfrak{F}(CV)^{NV} \\ \to & \bigoplus_{[V]\in\mathcal{C}\cap\mathcal{A}_p(G)/G;\,\mathrm{rk}\,V=2} \mathrm{Hom}_{NV/CV}(\mathrm{St}_{\mathrm{GL}(V)},\mathfrak{F}(CV)) \to \cdots \to 0. \end{aligned}$$

The first boundary map is given by restriction and the following ones are described in Theorem 6.2.

Theorem 1.3 can also be used to obtain information about $\mathfrak{F}(G)$. However, in general, if G is not a finite group of Lie type, then we do not get an exact sequence, but we can get two other expressions. One is obtained by looking at what happens at the $(0,0)$-spot in the spectral sequence, where we get a formula generalizing the Cartan-Eilenberg formula for stable elements in group cohomology [24] to be explained in Section 10. Another is obtained by trading in the spectral sequence for an expression in the Grothendieck group of modules, where we get a somewhat generalized version of a result of Webb, which we now explain.

Define the Lefschetz module L of a chain complex C_* to be the virtual module $L(C_*) = \sum_i (-1)^i C_i$, where the alternating sum is taken in the Grothendieck group of modules under direct sum. Recall that the Steinberg module $\mathrm{St}(G)$ of an arbitrary finite group G is defined to be the virtual $\mathbf{Z}_{(p)}G$-module

$$\mathrm{St}(G) = L(\mathrm{St}_*(G)).$$

(This definition of course coincides with the pre-existing definition for a finite group of Lie type up to a sign.) Note that this is a projective virtual $\mathbf{Z}_{(p)}G$-module, since its reduction modulo p is projective.

Corollary 8.19 (cf. Webb [79, Thm. 6.8]). *Let G be a finite group and let $\mathcal{C}$ be a collection of p-subgroups closed under passage to p-radical overgroups. Let $\mathfrak{F} : \mathbf{O}(G)^{\mathrm{op}} \to$ (finitely generated $\mathbf{Z}_{(p)}$-mod) be any subgroup $\mathcal{C}$-acyclic functor. Then*

$$\mathfrak{F}(G) \cong - \sum_{P \in (\mathcal{C} \cap \mathcal{B}_p^e(G))/G} \mathrm{Hom}_{WP}(\mathrm{St}(WP), \mathfrak{F}(P)).$$

If furthermore CP acts trivially on $\mathfrak{F}(P)$ for all $P \in \mathcal{C}$ then

$$\mathfrak{F}(G) \cong - \sum_{P \in (\mathcal{C} \cap \mathcal{D}_p(G))/G} \mathrm{Hom}_{WP}(\mathrm{St}(NP/PC(P)), \mathfrak{F}(P)).$$

Proof. Let $\mathcal{F}$ be the G-local coefficient system on $\mathcal{C}$ given by precomposing $\mathfrak{F}$ with the composite $\mathcal{C}_G \to \mathbf{O}_{\mathcal{C}} \hookrightarrow \mathbf{O}(G)$. By Theorem 1.1 and our assumption we have that $\mathfrak{F}(G) = H^0_G(|\mathcal{C}|; \mathcal{F})$, and the higher cohomology is trivial. Hence $\mathfrak{F}(G) = L(C^*_G(|\mathcal{C}|; \mathcal{F}))$. Let $\mathfrak{F}_P$ denote the atomic functor associated to $\mathfrak{F}$, which is $F(Q)$ on subgroups Q conjugate to P and zero otherwise, and let $\mathcal{F}_P$ denote the corresponding G-local coefficient system. By filtering $\mathcal{F}$ by the functors $\mathcal{F}_P$, as in the proof of Corollary 3.5 and using the fact that this filtration of $C^*_G(|\mathcal{C}|; \mathcal{F})$ splits as $\mathbf{Z}_{(p)}$-modules, we get that

$$L(C^*_G(|\mathcal{C}|; \mathcal{F})) \cong \sum_{[P] \in \mathcal{C}/G} L(C^*_G(|\mathcal{C}|; \mathcal{F}_P)).$$

But the proof of Theorem 1.1 shows that $C^*_G(|\mathcal{C}|;\mathcal{F}_P)$ is $\mathbf{Z}_{(p)}$-chain homotopy equivalent to $\mathrm{Hom}_{WP}(\Sigma\mathrm{St}_*(WQ),\mathfrak{F}(P))$ where Σ here denotes a suspension of chain complexes. Combining these statements and using the definition of $\mathrm{St}(G)$ we prove the corollary in the first case. The case where CP acts trivially on $\mathfrak{F}(P)$ follows from the first statement by Corollary 5.4 and Proposition 5.7. $\square$

Remark 8.20. The above techniques can also be used to obtain information even if acyclicity does not hold, although in that case the results often come as an equality between virtual $\mathbf{F}_p$-modules, rather than as exact sequences. We illustrate this with an example.

Suppose that G is a finite group and H is a normal subgroup. Then G acts on $\mathcal{S}_p(G/H)$ by conjugation. Consider the Bredon equivariant cohomology of $|\mathcal{S}_p(G/H)|$ with values in the generic coefficient system $\mathfrak{F}(-) = H^n(-;M)$, where M is any $\mathbf{F}_pG$-module. By a spectral sequence argument, as in the proof of Theorem 7.3, one sees that

$$\tilde{H}^*_G(|\mathcal{S}_p(G/H)|;H^n(-;M)) = \tilde{H}^*_G(|\mathcal{S}_p(G/H)|;H^0(-;H^n(H;M))),$$

which need not equal zero. However, writing out the defining chain complexes on the left- and the right-hand side and rearranging we obtain

$$\begin{aligned} H^n(G;M) \quad &\cong \quad \left(\sum_{[\sigma]\in|\mathcal{S}_p(G/H)|/G}(-1)^{|\sigma|}H^n(G_\sigma;M)\right) \\ &\qquad - \ \mathrm{Hom}_{G/H}(\mathrm{St}(G/H),H^n(H;M)). \end{aligned}$$

This result is due to Adem-Milgram [1, Thm. 2.2].

Remark 8.21. Note that all our exact sequences exist if just $\lim^*_{\mathbf{D}} F = 0$ for $* > 0$, without making assumptions on $\lim^0_{\mathbf{D}} F$, if one is willing to replace $\mathfrak{F}(G)$ with $\lim^0_{\mathbf{D}} F$. This weaker assumption is satisfied more frequently in nature. In fact, for *any* Mackey functor $\mathfrak{F} : \mathbf{O}(G)^{\mathrm{op}} \to \mathbf{Z}_{(p)}$- mod we have $\lim^*_{\mathbf{O}_{\mathcal{S}^e_p(G)}} F^\beta = 0$ for $* > 0$ (see [44], [47]). The condition that $\mathfrak{F}(G) \to \lim^0_{\mathbf{O}_{\mathcal{S}^e_p(G)}} F^\beta$ is an isomorphism is in practice more restrictive, and requires $\mathfrak{F}$ to be projective relative to p-subgroups [81]. (Or, almost equivalently, satisfy an 'induction theorem' relative to p-subgroups [81], [9, 5.6].) For a Mackey functor $\mathfrak{F}$ which does not satisfy this, one can instead consider the smallest collection $\mathcal{C}$ which is closed under passage to subgroups, contains a collection relative to which $\mathfrak{F}$ is projective, and which satisfies the condition of Remark 3.11. $\mathfrak{F}$ is now subgroup $\mathcal{C}$-acyclic and Theorem 1.1 still applies. We hope to examine some interesting examples of this situation in detail in a future paper.

9. Sharpness

In this short section we explain how to obtain sharpness results from the acyclicity results of the previous section. We will be brief and refer the reader to [32] for a much fuller treatment.

Let M be some $\mathbf{Z}_{(p)}G$-module. A collection $\mathcal{C}$ is called M-ample if $H^*(G;M) \xrightarrow{\cong} H^*_{hG}(|\mathcal{C}|;M)$. A collection $\mathcal{C}$ is called subgroup M-sharp if $H^n(-;M)$ is subgroup $\mathcal{C}$-acyclic for all $n \geq 0$. We define centralizer M-sharp and normalizer M-sharp similarly. (We abbreviate $\mathbf{Z}_{(p)}$-sharp to just sharp, and similarly for ample.)

Recall that for any G-space X we have an isotropy spectral sequence

$$E_2^{i,j} = H^i_G(X; H^j(-;M)) \Rightarrow H^{i+j}_{hG}(X;M).$$

The rewritings of Proposition 2.6, 2.10 and 7.1 identify the E_2-term of the isotropy spectral sequence for $|E\mathbf{O}_\mathcal{C}|$, $|E\mathbf{A}_\mathcal{C}|$, and $|\mathcal{C}|$ with the higher limits over $\mathbf{O}_\mathcal{C}^{\mathrm{op}}$, $\mathbf{A}_\mathcal{C}$, and $((\mathrm{sd}\,\mathcal{C})/G)^{\mathrm{op}}$ respectively. Therefore the definition of M-sharpness says that the E_2-term of the isotropy spectral sequence must be concentrated on the vertical axis, where we must have $E^{0,*} = H^*(G;M)$. In particular, if $\mathcal{C}$ is either subgroup, centralizer or normalizer M-sharp then $\mathcal{C}$ is M-ample, since $|E\mathbf{O}_\mathcal{C}|_{hG} \simeq |\mathcal{C}|_{hG} \simeq |E\mathbf{A}_\mathcal{C}|_{hG}$. (Compare the isotropy spectral sequence of $|E\mathbf{O}_\mathcal{C}|$, $|E\mathbf{A}_\mathcal{C}|$, or $|\mathcal{C}|$ to that of a point.)

We can now use the acyclicity results in the previous sections to obtain sharpness statements. We start by demonstrating that $\mathcal{S}_p(G)$ is normalizer M-sharp if and only if it is M-ample. We have seen that the isotropy spectral sequence for $|\mathcal{S}_p(G)|$ is concentrated on the two axes. On the horizontal axis we have $E_2^{*,0} = H^*_G(|\mathcal{S}_p(G)|;M)$, whereas the vertical axis is $E_2^{0,*} = H^*(G;M)$, for $* > 0$. Furthermore, this spectral sequence has to collapse at E_2. To see this note that the map $\mathcal{S}_p(G) \to *$ has a P-equivariant section given by $* \mapsto ZP$, and so there can be no differentials in the isotropy spectral sequence for the P-space $|\mathcal{S}_p(G)|$. But by a transfer argument, the E_2-term for the G-space $|\mathcal{S}_p(G)|$ is a retract of the E_2-term for $|\mathcal{S}_p(G)|$ viewed as a P-space; hence there can be no differential for the G-space either. Example 8.9 now gives us the following result.

Theorem 9.1. *The collection $\mathcal{D}^e_{p,M}(G)$ of Example* 8.9 *is subgroup M-sharp and normalizer M-sharp* (*and is in particular M-ample*). *The collection $\mathcal{D}_{p,M}(G)$ of Example* 8.9 *is subgroup M-sharp if and only if it is normalizer M-sharp if and only if it is M-ample if and only if* $\mathrm{Hom}_G(\mathrm{St}_*(G),M)$ *is an acyclic chain complex.*

Note that in the extreme case where $p \nmid |G|$ we get that $\mathcal{D}_{p,M}(G)$ is M-ample if and only if $M^G = 0$. The above result generalizes the result of Dwyer [30], [32] that the collection of p-centric and p-radical subgroups is subgroup sharp, and the result originally due to Webb [80] that the collection of p-radical subgroups is normalizer sharp. The consequence that the collection of p-radical and p-centric subgroups is normalizer sharp was suspected by Smith-Yoshiara [69]. We remark that the analysis shows that in the cases we have studied the collections lying in between our sharp collections and $\mathcal{S}_p^e(G)$ are also sharp.

10. Stable elements and Alperin's fusion theorem

In this section we explain how our work relates to Cartan-Eilenberg's formula for stable elements in group cohomology and Alperin's fusion theorem.

Let $\mathcal{C}$ be a collection of p-subgroups of a finite group G, closed under passage to p-radical overgroups, and let $F : \mathbf{O}_{\mathcal{C}}^{\mathrm{op}} \to \mathbf{Z}_{(p)}$- mod be any functor. We call $\lim^0_{\mathbf{O}_{\mathcal{C}}} F$ the stable elements of F with respect to $\mathcal{C}$. (These are the stable elements of Cartan-Eilenberg [24] if $\mathcal{C} = \mathcal{S}_p^e(G)$ and $F = H^n(-;M)$.)

More amenable formulas for the stable elements can be obtained from Alperin's fusion theorem. This is mentioned e.g. in Holt [41], and a related approach to the fusion theorem and stable elements is presented in the pioneering work of Puig [58, Ch. III §2].

Our point is that $\lim^0_{\mathbf{O}_{\mathcal{C}}} F$ in our model for $\lim^*_{\mathbf{O}_{\mathcal{C}}}\mathrm{F}$ gives the same kind of reduction in the formula for stable elements, and the method gives some additional insight as well. We start by explaining the results, and will afterward relate them to Alperin's fusion theorem.

Let $\mathcal{H}_p(G)$ denote the collection of p-subgroups Q in G such that $\mathcal{S}_p(NQ/Q)$ is either empty or disconnected (see Remark 5.9). In particular $\mathcal{S}_p(NQ/Q)$ will be noncontractible so that $\mathcal{H}_p(G) \subseteq \mathcal{B}_p^e(G)$. If G is a finite group of Lie type of characteristic p then one easily identifies $\mathcal{H}_p(G)$ inside $\mathcal{B}_p^e(G)$ as exactly the unipotent radicals of parabolic subgroups of rank less than or equal to one.

Remark 10.1. The role of the subgroups in $\mathcal{H}_p(G)$ for the p-local structure of G has been examined especially by Goldschmidt [37] and Puig [58], and a link to subgroup complexes was noted by Quillen [59, §6]. (Puig calls the non-Sylow p-subgroups in $\mathcal{H}_p(G)$ 1-essential (or just essential) [58, p. 25] and non-Sylow subgroups in $\mathcal{H}_p(G) \cap \mathcal{D}_p(G)$ C-essential [58, p. 38]; see also [75] where the notation is modified.)

Remark 10.2. Easy examples show that $\mathcal{H}_p(G)$ is in general not an ample collection. For example take $G = \mathrm{GL}_4(\mathbf{F}_2)$. If $\mathcal{H}_2(G)$ was an ample collection it would be sharp and Corollary 8.17 should give an exact sequence e.g. for $F = H^1(-;\mathbf{F}_2)$. However this is not the case, as can perhaps most easily be seen by comparing it to the corresponding sequence for the collection $\mathcal{B}_p(G)$.

Intuitively, one should think of the G-poset $\mathcal{H}_p(G)$ as giving generators and relations for the cohomology of G, whereas the G-poset $\mathcal{B}_p(G)$ also encodes relations amongst the relations necessary to get exact (sequence) expressions for the cohomology of G.

Examination of what happens at the $(0,0)$-spot of the spectral sequence of Theorem 1.3 gives the following result which says that one can determine the stable elements 'locally', i.e., via information coming from p-subgroups and their normalizers; one does not have to consider conjugation between different subgroups as it appears from the definition.

Theorem 10.3. *Let $\mathcal{C}$ be a collection of p-subgroups in G closed under passage to p-overgroups, and let $F : \mathbf{O}_{\mathcal{C}}^{\mathrm{op}} \to \mathbf{Z}_{(p)}\text{-}\mathrm{mod}$ be any functor. Let P be a Sylow p-subgroup in G. Then*

$$\lim_{\mathbf{O}_{\mathcal{C}}}{}^0 F = \{x \in F(P)^{WP} |\, \mathrm{res}^P_Q(x) \in F(Q)^{WQ} \textit{ for all } [Q] \in \mathcal{C} \cap \mathcal{H}_p(G)/G\}$$

where Q is an arbitrary representative of $[Q]$ such that $Q \leq P$.

If CQ acts trivially on $F(Q)$ for all subgroups $Q \leq P$, then $\mathcal{C} \cap \mathcal{H}_p(G)$ may be replaced by $\mathcal{C} \cap \mathcal{H}_p(G) \cap \mathcal{D}_p(G)$.

Proof. Choose a height function on $\mathcal{C} \cap \mathcal{H}_p(G)$ such that $\mathrm{ht}(P) = 0$. Consider the spectral sequence of Theorem 1.3. We have that $E_1^{i,-i} \neq 0$ only if $i = 0$, since $\mathrm{St}_*(H)$ is (-1)-connected if $p||H|$. Hence, we can calculate $\lim^0_{\mathbf{O}_{\mathcal{C}}} F$ as the intersection of the kernels of the differentials originating at the $(0,0)$-spot. By Proposition 5.8 we have

$$\begin{aligned} E_1^{i,-(i-1)} &= \bigoplus_{[Q] \in (\mathcal{C} \cap \mathcal{B}_p^e(G))/G; \mathrm{ht}(Q) = -i} \mathrm{Hom}_{WQ}(\tilde{H}_0(|\mathcal{S}_p(WQ)|), F(Q)) \\ &= F(Q)^{\widetilde{WQ}}/F(Q)^{WQ}. \end{aligned}$$

In particular, only differentials going to a subgroup $Q \in \mathcal{C} \cap \mathcal{H}_p(G)$ can contribute anything.

Finally, one easily verifies from the construction of the spectral sequence (see the proof of Theorem 1.3) that the kernel of the differential $d_n : E_n^{0,0} \to E_n^{n,-(n-1)}$ is just equal to the kernel of $F(P)^{NP} \overset{\mathrm{res}^P_Q}{\to} F(Q) \to F(Q)/F(Q)^{WQ}$. This proves the first part of the theorem.

The last part follows since if CQ acts trivially on $F(Q)$ then Theorem 1.3 holds with $\mathcal{C} \cap \mathcal{B}_p^e(G)$ replaced by $\mathcal{C} \cap \mathcal{D}_p(G)$. □

Specializing to the case of group cohomology, we find that the above result takes the following form, in view of the results of Section 8.

COROLLARY 10.4. *Let M be any $\mathbf{Z}_{(p)}G$-module, and let P be a Sylow p-subgroup in G. Then the following formula for the group cohomology of G with coefficients in M holds*:

$$\begin{aligned} H^*(G;M) \quad &= \quad \{x \in H^*(P;M)^{WP} \,|\, \mathrm{res}^P_Q(x) \in H^*(Q;M)^{WQ} \\ &\qquad \textit{for all } [Q] \in (\mathcal{H}_p(G) \cap \mathcal{D}^e_{p,M}(G))/G\}, \end{aligned}$$

where Q is an arbitrary representative of $[Q]$ such that $Q \le P$.

Notice also that Theorem 6.2 yields a dual version of Theorem 10.3.

THEOREM 10.5. *Let $\mathcal{C}$ be any collection of p-subgroups of G closed under passage to nontrivial elementary abelian p-subgroups, and let $F : \mathbf{A}_{\mathcal{C}} \to \mathbf{Z}_{(p)}$-mod be any functor. Then*

$$\begin{aligned} \lim_{\mathbf{A}_{\mathcal{C}}}{}^0 F \quad = \quad &\Big\{ \prod x_V \in \prod_{[V]\in\mathcal{C}'/G,\,\mathrm{rk}\,V = 1} F(V)^{NV} \,|\, \sum_{[V],[V]\le[W]} \iota(x_V) \in F(W)^{NW} \\ &\qquad \textit{for all } [W] \in \mathcal{C}'/G, \mathrm{rk}\,W = 2 \Big\}. \end{aligned}$$

Here $\mathcal{C}' = \mathcal{C} \cap \mathcal{A}_p(G)$ and ι is the inclusion map $V \le W$, where representatives V such that $V \le W$ are chosen.

For group cohomology Theorem 10.5 gives an expression for $H^*(G;M)$ in terms of centralizers of elementary abelian p-subgroups of rank 1 and 2.

Example 10.6 (The extra-special group p^{1+2}_+). Consider a finite group G with Sylow p-subgroup the extra-special group $P = p^{1+2}_+ = \langle a,b,c | a^p = b^p = c^p = 1, [a,c] = [b,c] = 1, [a,b] = 1\rangle$. If $Q \le P$ satisfies $C_P Q = ZQ$ then $Q = P$ or $Q \cong \mathbf{Z}/p \times \mathbf{Z}/p$. Furthermore, if $Q \cong \mathbf{Z}/p \times \mathbf{Z}/p$ and $W = NQ/CQ \le \mathrm{Out}(Q) = \mathrm{GL}_2(\mathbf{F}_p)$ satisfies $O_p(W) = e$ then an easy computation reveals that $\mathrm{SL}_2(\mathbf{F}_p) \le W$, since $p||W|$. Therefore

$$\mathcal{D}_p(G)/G = \{[P], [V_1], \dots, [V_k]\}$$

where $V_1, \dots, V_k$ are G-conjugacy class representatives of the rank 2 elementary abelian subgroups of P with the property that $\mathrm{SL}_2(\mathbf{F}_p) \le NV_i/CV_i$.

Corollary 10.4, when applied to this example, gives a generalization of [72, Thm. 0.2], where a formula of this type was proved for certain sporadic groups. The fact that the collection $\mathcal{D}_p(G)$ is ample (Theorem 9.1) likewise generalizes [83, Thm. 1.8].

Example 10.7 (Swan groups). Suppose that for a p-group P the condition

$$C_P(Q) = ZQ \text{ and } N_P(Q)/Q \cap O_p(\mathrm{Out}(Q)) = e,$$

mentioned in Remark 3.14, is not satisfied for any proper subgroups Q of P.

Then we conclude that for *any* finite group G having P as Sylow p-subgroup $\mathcal{D}_p(G)/G = [P]$ and hence $H^*(G;\mathbf{F}_p) \cong H^*(P;\mathbf{F}_p)^{WP}$, by Corollary 10.4. This was originally proved by Martino-Priddy [49, Thm. 2.3].

Remark 10.8. Using Goldschmidt's or Puig's version of Alperin's fusion theorem (see below) one sees that $\mathcal{D}_p(G)/G = [P]$ for the Sylow p-subgroup P of G if and only if NP controls (strong) p-fusion in G. Furthermore, Mislin's theorem [52] tells us that $H^*(G;\mathbf{F}_p) \xrightarrow{\cong} H^*(P;\mathbf{F}_p)^{WP}$ implies that NP controls (strong) p-fusion in G, so that the three conditions $\mathcal{D}_p(G)/G = [P]$, NP controls (strong) p-fusion in G, and $H^*(G;\mathbf{F}_p) \xrightarrow{\cong} H^*(P;\mathbf{F}_p)^{WP}$, and are in fact equivalent. In general, one can interpret the number $\dim|\mathcal{D}_p(G)|$ as a measure of how complicated the p-fusion is in G.

Remark 10.9. The proof of Theorem 10.3 in the case where CQ acts trivially on $F(Q)$ allows for the seemingly stronger statement that we may restrict attention to p-centric subgroups $Q \in \mathcal{C}$ for which $\mathcal{S}_p(NQ/QC(Q))$ is empty or disconnected. However, using some group theory one can show that this is equivalent to assuming $Q \in \mathcal{C} \cap \mathcal{H}_p(G) \cap \mathcal{D}_p(G)$ (see [58, Ch. III, Cor. 2]).

Remark 10.10. Note that the proof of Theorem 10.3 actually shows more than stated in the theorem. We already know that $\mathrm{res}^P_Q(x) \in F(Q)^{\widetilde{WQ}}$, so we only have to calculate invariants under a subgroup which together with $\widetilde{WQ}$ generate WQ. Also, the proof shows that each $Q \in \mathcal{C} \cap \mathcal{H}_p(G)$ can only contribute if $\mathrm{Hom}_{WQ}(\tilde{H}_0(\mathcal{S}_p(WQ)), F(Q)) \neq 0$. Remark 5.12 gives us a good handle on when this happens, and can allow us to rule out more subgroups if we know something about $F(Q)$. This can for instance be useful for low dimensional group cohomology calculations, as we shall see in the next example.

Example 10.11. Suppose that G is a finite group, whose Sylow p-subgroup P has nilpotency class at most c, i.e., any c-fold commutator between elements in P is trivial. If $Q \leq P$ and $g \in WQ$ of order p, then Miyamoto has proved [54, Lemma 2], using induction and the Lyndon-Hochschild-Serre spectral sequence, that $(1-g)^{(c-1)n+1}$ acts as zero on the WQ-composition factors of $H^n(Q;\mathbf{F}_p)$. Now, if $(c-1)n+1 \leq p-1$ then, by Example 5.13, $\mathrm{Hom}_{WQ}(\tilde{H}_0(\mathcal{S}_p(WQ)), H^n(Q;\mathbf{F}_p)) = 0$ for all $Q < P$. Hence,

$$\text{if } c \leq \frac{p-2}{n} + 1 \text{ then } H^n(G;\mathbf{F}_p) = H^n(P;\mathbf{F}_p)^{NP}.$$

This result, due to Miyamoto [54], generalizes both Swan's classical theorem, which is the case $c = 1$, as well as results for $n = 1$ and 2 of Glauberman and Holt [41] respectively.

Remark 10.12. The conditions on subgroups are stated in a number of different ways in the literature. For the convenience of the reader, we state some relations between these.

1. A subgroup Q of P is a *tame intersection* in P if there exists a Sylow p-subgroup P' of G such that $Q = P \cap P'$ and that $N_P(Q)$ and $N_{P'}(Q)$ are both Sylow p-subgroups in NQ. Puig observed [58, p. 42] (see also [59, §6]) that if $Q \in \mathcal{H}_p(G)$ and if $N_P(Q)$ is a Sylow p-subgroup in NQ, then such P' always exists. Hence, by Sylow's theorem, any $Q \leq P$ with $Q \in \mathcal{H}_p(G)$ is G-conjugate to a subgroup $Q' \leq P$ with Q' a tame intersection in P.

2. A subgroup $Q \leq P$ is p-centric in G if and only if for all subgroups $Q' \leq P$ with Q' G-conjugate to Q we have $C_P(Q') = ZQ'$. (Use Sylow's theorem.)

3. If Q is assumed to be a tame intersection in P and $C_P(Q) = ZQ$ then Q is p-centric. To see this just note that since $N_P(Q)$ is assumed to be a Sylow p-subgroup in NQ and $CQ \trianglelefteq NQ$, $C_P(Q) = CQ \cap N_P(Q)$ is also a Sylow p-subgroup in CQ.

4. Note that we always have $O_{p'}(NQ) = O_{p'}(CQ)$. Hence, if Q is p-centric then the condition that $O_p(NQ/QC(Q)) = e$ is equivalent assuming that Q is a Sylow p-subgroup of $O_{p',p}(NQ)$. If Q is assumed to be a tame intersection in P, it is again equivalent to assuming $O_{p',p}(NQ) \cap P = Q$.

By 2 we can view the p-centric condition as being a G-conjugacy invariant refinement of the condition $C_P(Q) = ZQ$. Also, 1 and 3 show that the tame intersection condition picks out distinguished representatives from a G-conjugacy class of p-centric subgroups. (The name p-centric subgroup was coined by Dwyer [30], inspired the p-centric diagrams of [34]. In parts of group theory p-centric subgroups are called self-centralizing [75, p. 324].)

Definition 10.13. [2] Let G be a finite group and P a fixed Sylow p-subgroup in G. A set $\mathbb{X}$ of pairs $\{(Q,T)\}$, where Q is a subgroup of P and T is a subset of NQ, is called a *conjugation family* provided the following condition is satisfied: For any subset A of P and any $g \in G$ such that $A^g \leq P$, there exist pairs $(Q_1,T_1),\dots,(Q_n,T_n)$ in $\mathbb{X}$ and elements $x_1,\dots,x_n,y$ of G such that $g = x_1 \cdots x_n y$, $x_i \in T_i$ for $i = 1,\dots,n$, $y \in NP$, and $A^{(x_1 \cdots x_i)} \leq Q_{i+1}$ for $i = 0,\dots,n-1$. ($A^g = g^{-1}Ag$.)

Note that for each conjugation family $\mathbb{X}$ we can form a new conjugation family replacing each (Q,T) with $(Q,\bar{T})$, where $\bar{T}$ is the subgroup generated by the union of the subsets $\{U | (Q,U) \in \mathbb{X}\}$, which will be just as good for practical purposes. Hence we can when convenient assume that (Q,T) is uniquely determined by Q, and that T is a group. We say that a conjugation family is *invariant* if it, in addition to the just stated requirements, satisfies the fact that if $(Q,T) \in \mathbb{X}$ and $Q^g \subseteq P$ then $(Q^g, T^g) \in \mathbb{X}$.

It is not obvious that conjugation families do indeed exist. However the following is immediate.

PROPOSITION 10.14. *Let* $\mathcal{C} = \mathcal{S}_p^e(G)$ *and let* $F : \mathbf{O}_{\mathcal{C}}^{\mathrm{op}} \to \mathbf{Z}_{(p)}\text{-mod}$ *be any functor. Assume that* $\mathbb{X}$ *is a conjugation family in* G*. Then*

$$\lim_{\mathbf{O}_{\mathcal{C}}}{}^0 F = \{\alpha \in F(P)^{WP} \,|\, \mathrm{res}_Q^P(\alpha) \in F(Q)^T \text{ for all } (Q,T) \in \mathbb{X}\}.$$

If $\mathbb{X}$ *is invariant then it is enough to check the restriction condition on one representative* (Q,T) *for each* G*-conjugacy class of pairs in* $\mathbb{X}$*.*

Proof. The first part of the statement follows by writing out the definition of $\lim^0$ (see e.g. [41]). The last part of the statement follows by a downward induction on the size of Q. (The statement is trivial for $Q = P$.) $\square$

Goldschmidt's version of Alperin's fusion theorem [2], [37] (stated in a stronger form due to Miyamoto [53, Cor. 1], and slightly reformulated) says that the set of pairs $\{(Q,T)\}$ where Q is a tame intersection subgroup in P such that $Q \in \mathcal{H}_p(G)$ and $T = NQ$ if $Q \in \mathcal{D}_p(G)$ and $T = CQ$ otherwise, is a conjugation family. Hence, Proposition 10.14 combined with the fusion theorem produces a statement very close to Theorem 10.3. (Compare also [58, Ch. III §2].)

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