Algebraic models for finite G-spaces

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(joint work with Jeffrey H. Smith)

In my talk, based on [2], I explained how to assign certain finite algebraic models to G-spaces with finite mod p homology, for G a finite group. In general these models are too naïve to capture the mod p equivariant homotopy type, where by equivariant homotopy type we mean that two G-spaces are considered equivalent (called hG-equivalent) if they can be connected by a zig-zag of G-maps which are non-equivariant homotopy equivalences. We show, however, that the model does in fact capture the mod p equivariant homotopy type in the fundamental case when X has the mod p homology of a sphere—integral results can easily be obtained from the p-local results via the arithmetic square. Even when the model does not capture the homotopy type it still encodes important information about the group action, including all of classical Smith theory. We also show that for spheres the algebraic models are themselves determined by simple numerical information (Theorem 2), and that all models are realizable (Theorem 3). Our work also seems to point to that the models have independent algebraic interest. The models are given via the following theorem.

Theorem 1. *Let G be a finite group and consider the functor Φ : G-spaces $\to$ $\mathrm{Ch}(\mathrm{Vect}^{\mathbf{O}_p(G)^{\mathrm{op}}})$ which to a G-space X associates the functor on the opposite p-orbit category $\mathbf{O}_p(G)^{\mathrm{op}}$ given by*

$$G/P \mapsto \lim_n C_*(\mathrm{map}_G(EG \times G/P, P_n(\mathbb{F}_p)_n X); \mathbb{F}_p)$$

Then Φ sends a G-space with finite $\mathbb{F}_p$-homology to a perfect complex in $\mathrm{Vect}^{\mathbf{O}_p(G)^{\mathrm{op}}}$.

A perfect complex is a complex quasi-isomorphic to a finite chain complex of finitely generated projectives. Here $\mathrm{Ch}(\mathrm{Vect}^{\mathbf{O}_p(G)^{\mathrm{op}}})$ denotes chain complexes of functors from the opposite p-orbit category $\mathbf{O}_p(G)^{\mathrm{op}}$ to $\mathbb{F}_p$-vector spaces, C_* denotes singular chains reduced by setting $C_{-1} = \mathbb{F}_p$, map_G is the G-equivariant mapping space, P_n denotes the nth Postnikov truncation, and $(\mathbb{F}_p)_n X$ denotes the nth stage in the Bousfield-Kan tower of X converging to the Bousfield-Kan $\mathbb{F}_p$-completion of X. Under mild restrictions on X, the formula for $\Phi(X)(G/P)$ can be simplified to $C_*((X^\wedge_p)^{hP}; \mathbb{F}_p)$, and if X is a genuine finite complex even to $C_*(X^P; \mathbb{F}_p)$, using the generalized Sullivan Conjecture (now a theorem by Miller and Lannes). The technical description in general is forced upon us by properties of Lannes' T-functor.

Note that, up to quasi-isomorphism, $\Phi(X)$ only depends on X up to hG-equivalence. Also note that by the existence of minimal resolutions, every perfect complex has a minimal model, a finite chain complexes of projectives, well-defined up to *isomorphism.* To get a feeling for Theorem 1, one may note that in the case where X is a point, the result implies that the constant functor in $\mathrm{Vect}^{\mathbf{O}_p(G)^{\mathrm{op}}}$ has finite projective dimension; a celebrated result first proved by Jackowski-McClure-Oliver [3].

The abelian category $\text{Vect}^{\mathbf{O}_p(G)^{\text{op}}}$ can be viewed as modules over the category algebra $\mathbb{F}_p\mathbf{O}_p(G)^{\text{op}}$ of $\mathbf{O}_p(G)^{\text{op}}$, i.e., the algebra with $\mathbb{F}_p$-basis consisting of the morphisms in $\mathbf{O}_p(G)^{\text{op}}$ and multiplication given by composition. This is a finite dimensional algebra whose simple modules are described by the simple modules for $N_G(P)/P$, where P runs over the conjugacy classes of p-subgroups of G, and the projectives are given by left Kan extension. For instance for $G = C_p$ the orbit category looks like $G/G \to G/e$ so the simples are $k \to 0$ and $0 \to k$ with corresponding projectives $k \to k$ and $0 \to kC_p$, where we from now on let $k = \mathbb{F}_p$ for short. Examining the perfect complexes which can be built from them easily leads to the statements of classical Smith theory. For $G = \Sigma_3$ and $k = \mathbb{F}_2$ we get simples $k \Rightarrow 0$, $0 \Rightarrow k$, and $k \Rightarrow \text{St}$ (where $\Rightarrow$ denotes 3 arrows, and St denotes the 2-dimensional simple module for Σ_3) with corresponding projectives $k \Rightarrow k[G/C_2]$, $0 \Rightarrow k[G/C_3]$, and $0 \Rightarrow \text{St}$. Notice that here the constant functor is not projective, but has a projective resolution with $k \Rightarrow k[G/C_2]$ in degree zero and $0 \Rightarrow \text{St}$ in degree one.

The proof of Theorem 1 requires first developing a criterion for recognizing perfect complexes in terms of certain $kN_G(P)/P$-modules, and then using Lannes theory together with the theory of support varieties to verify that these conditions are met.

We say that a perfect complex $\mathbb{X}$ of $k\mathbf{O}_p(G)^{\text{op}}$-modules is a $k\mathbf{O}_p(G)^{\text{op}}$-sphere if $H(\mathbb{X}(G/e))$ is one-dimensional. An oriented dimension function consists of a functor $n : \mathbf{O}_p(G)^{\text{op}} \to (\mathbb{Z}, \leq)$, together with an action of G on k. Every $k\mathbf{O}_p(G)^{\text{op}}$-sphere has an associated oriented dimension function given by letting $n(G/P)$ be the (by Smith theory) unique n such that $H_n(\mathbb{X}(G/P)) \neq 0$, and taking as orientation the action of G on $H(\mathbb{X}(G/e))$. The next theorem gives a classification of $k\mathbf{O}_p(G)^{\text{op}}$-spheres.

Theorem 2. *Let $\mathbb{X}$ and $\mathbb{Y}$ be $k\mathbf{O}_p(G)^{\text{op}}$-spheres. Then $\mathbb{X}$ and $\mathbb{Y}$ are quasi-isomorphic if and only if $\mathbb{X}$ and $\mathbb{Y}$ have the same oriented dimension functions.*

The proof involves setting up the right obstruction theory, and then showing that the existence and uniqueness obstructions to extending a map vanish, once we have chosen the map on $\mathbb{X}(G/S)$ for a Sylow p-subgroup S. The vanishing result takes as one input extending properties of Steinberg complexes in [1] to the perfect complexes we are considering.

We conjecture that a given oriented dimension function is realizable by a $k\mathbf{O}_p(G)^{\text{op}}$-sphere if and only if it satisfies the so-called Borel-Smith conditions (where only-if is easy). We have proved this for certain classes of groups including many groups of small order, and groups with normal Sylow p-subgroup (in particular p-groups).

The last theorem which we list here states that there is a 1-1-correspondence between p-complete G-spheres, up to homotopy, and $k\mathbf{O}_p(G)^{\text{op}}$-spheres subject to obvious low-dimensional restrictions.

Theorem 3. *Let X and Y be G-spaces which homotopy equivalent to $\mathbb{F}_p$-complete spheres. Then X and Y are hG-equivalent if and only if $\Phi(X)$ and $\Phi(Y)$ are quasi-isomorphic.*

Furthermore, we have a 1-1*-correspondence between* $\mathbb{F}_p$*-complete* G*-spheres up to* hG*-equivalence and* $k\mathbf{O}_p(G)^{\mathrm{op}}$*-spheres where* $H(\mathbb{X}(G/S))$ *has degree no less than* -1 *and the* NS/S*-action on* $H(\mathbb{X}(G/S))$ *is trivial if in degree* -1 *and factors through* ± 1 *if in degree* 0*, where* S *is a Sylow* p*-subgroup.*

In addition to the results and methods of Theorem 1 and 2, the proof uses that the G-action on $\pi_*(X)$ of has a filtration where the quotients are kG-modules of dimension less than p, which allows for passing between X and its abelianization $\mathbb{F}_pX$ via obstruction theory and tower arguments.

Note that Theorem 2 and 3 in particular shows that maps $[BG, B\,\mathrm{Aut}({S^n}^\wedge_p)]$ are determined by the restriction to the normalizer of a Sylow p-subgroup in G. This shows that homotopical group actions on spheres are better behaved than homotopical representations $[BG, BU(n)^\wedge_p]$, where there are many exotic maps due to non-vanishing of certain higher limit obstructions, as first discovered by Jackowski-McClure-Oliver [3].

References

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[3] S. Jackowski, J. McClure, and B. Oliver. Homotopy classification of self-maps of BG via G-actions. I+II. *Ann. of Math. (2)*, 135(2):183–226,227–270, 1992.