Uncompleting classifying spaces

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(joint work with Bob Oliver)

The information encoded in a finite group G is equivalent to the information encoded in its classifying space BG. In homotopy theory, one can complete a space at a prime, which produces the p-completed classifying space $BG^\wedge_p$. When G is abelian this procedure simply gives the classifying space BS of the Sylow p-subgroup of G, and all information prime to p is lost. When G is non-abelian the resulting space $BG^\wedge_p$ is a much more complicated object, but still an invariant of the "p-local structure" of the group, suitably defined.

In this talk we showed that often, when G is "non-abelian enough", this p-completion process can in fact be reversed! We explained theorems saying that for many "sufficiently complicated" groups G, BG can be recovered from from the p-completed space $BG^\wedge_p$ for just a single prime p. In other words the p-local structure in G in fact completely determines its global structure. The approach goes via a certain category $\mathcal{L}_p(G)$ called the p-local finite group of the group (see e.g., [1] or [2] for definitions). We propose the fundamental group $\pi_1(\mathcal{L}_p(G))$ as an interesting invariant of the p-local structure of the group.

For several groups we get "local-to-global" theorems:

Theorem 1 (Grodal-Oliver [3]). *Suppose that G is either*

(1) *A p-solvable group with $O_{p'}(G) = 1$.*
(2) *A finite group of Lie type of rank ≥ 3 with $O_{p'}(G) = 1$.*
(3) *Σ_{p^n} with $n \geq 3$ and $p = 2$ (probably also OK for p odd).*
(4) *Several of the larger sporadic groups for $p = 2$ e.g., the Monster, M_{24}, Co_3,....*

Then $\pi_1(\mathcal{L}_p(G)) = G$, and in particular G can be recovered from its p-local structure $\mathcal{L}_p(G)$.

For other groups G, such as linear groups over $\mathbb{F}_q$ for q a prime power different from p, easy examples show that the p-local structure cannot determine the group G uniquely. However, our work indicate that even in those cases $\pi_1(\mathcal{L}_p(G))$ can be a sort of "best global approximation" to the p-local finite group $\mathcal{L}_p(G)$, which is an interesting group, though not necessarily finite:

Theorem 2 (Grodal-Oliver [3]). *Suppose that $G = \mathrm{SO}_n(\mathbb{F}_q)$ for $q \equiv 3, 5(8)$, $n \leq 8$, then $\pi_1(\mathrm{SO}_n(\mathbb{F}_q)) \cong \mathrm{SO}_n(\mathbb{Z}[\frac{1}{2}])$*

References

[1] C. Broto, R. Levi, and B. Oliver, *Homotopy equivalences of p-completed classifying spaces of finite groups*, Invent. math. **151** (2003), 611–664
[2] C. Broto, R. Levi, and B. Oliver, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. **16** (2003), 779–856
[3] J. Grodal and B. Oliver, *Fundamental groups of p-local finite groups*, In preparation.