Serre's theorem and the $\mathcal{N}il_l$ filtration of Lionel Schwartz

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Abstract. We give three different cohomological characterizations of classifying spaces of p-compact toral groups amongst finite Postnikov systems satisfying mild conditions. This leads to a unifying generalization of previous versions of Serre's theorem on the homotopy groups of a finite complex.

1. Introduction

In the heart of all generalizations of Serre's theorem on the homotopy groups of a finite complex (cf. e.g. [17, 12, 10, 11, 4, 6]) there have implicitly or explicitly been statements about the 'non-finiteness' of the mod p cohomology of a finite Postnikov system. The negations of these statements say that a space with 'finite' mod p cohomology has infinitely many nontrivial mod p homotopy groups (that is, infinitely many homotopy groups are not uniquely p-divisible). Such statements can then in some cases (but not all cf. Remark 4.5) afterwards be improved to showing that the space has infinitely many homotopy groups containing p-torsion, by employing a method of McGibbon and Neisendorfer [12]. We put these generalizations into a common framework by offering the following cohomological characterizations of classifying spaces of p-compact toral groups amongst finite Postnikov systems.

Theorem 1.1. *Let X be a connected finite Postnikov system with $\pi_1 X$ a finite p-group and $H^*(X;\mathbf{F}_p)$ of finite type. The following conditions are equivalent.*

1. $QH^*(X;\mathbf{F}_p) \in \overline{\mathcal{N}il}_2$.
2. $H^*(X;\mathbf{F}_p)$ *is of finite transcendence degree.*
3. $H^*(X;\mathbf{F}_p)$ *is noetherian.*
4. X *is $\mathbf{F}_p$-equivalent to the classifying space of a p-compact toral group.*

Here Q denotes the indecomposables, and $\overline{\mathcal{N}il}_2$ refers to the $\mathcal{N}il$ filtration of the category $\mathcal{U}$ of unstable modules over the Steenrod algebra of Schwartz [15]. (Note that locally finite modules over the Steenrod algebra lie in $\overline{\mathcal{N}il}_2$.) Recall that two spaces are said to be $\mathbf{F}_p$-equivalent if they become homotopy equivalent after

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Bousfield $H_*(-;\mathbf{F}_p)$ localization [1]. Furthermore, recall that a classifying space of a p-compact toral group is a space which fits as the total space in a fibration sequence over $K(P,1)$ with fiber $K(\hat{\mathbf{Z}}_p^n, 2)$ for some $n<\infty$, where $\hat{\mathbf{Z}}_p$ denotes the p-adic integers and P is a finite p-group (cf. [5]).

The theorem extends results in [6], and also the generalization of Serre's theorem of [4] can be easily recovered and generalized (Theorem 4.4).

To prove the results we study the $\mathcal{N}il_l$ filtration and its relation to the Eilenberg-Moore spectral sequence, using the methods of Schwartz [15], and use this to reduce the theorem to results of [6].

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2. Notation and preliminaries

By a space we mean, for simplicity, an object in the homotopy category of finite pointed CW-complexes. A connected finite Postnikov system is a connected space such that $\pi_i X = 0$ for i large. Throughout we abbreviate $H^*(X;\mathbf{F}_p)$ to H^*X where p is a fixed but arbitrary prime.

We now state some basic facts about the Eilenberg-Moore spectral sequence and the $\mathcal{N}il_l$ filtration.

2.1. The Eilenberg-Moore spectral sequence

The cohomological Eilenberg-Moore spectral sequence of a fibration $D\to E\to B$ over a connected space B is a second quadrant spectral sequence with E_2-term given by $E_2^{s,*} = \mathrm{Tor}^s_{H^*B}(\mathbf{F}_p, H^*E)$ as an unstable module over the Steenrod algebra $\mathcal{A}$ [14]. The columns $E_r^{s,*}$, $s\le 0, r\ge 2$ are likewise unstable $\mathcal{A}$-modules, and the differentials $d_r : E_r^{s,*} \to E_r^{s+r,*}$ are $\mathcal{A}$-linear of degree $-(r-1)$. The spectral sequence converges strongly to H^*D when H^*E and H^*B are of finite type and $\pi_1 B$ acts nilpotently on H^*D [3]. Specifically, there is a cocomplete filtration of $\mathcal{A}$-modules of H^*D:

$$H^*D \supset \cdots F_s \supset F_{s+1} \supset \cdots \supset F_0 \supset F_1 = 0$$

such that $\Sigma^{-s}F_s/F_{s+1} \simeq E_\infty^{s,*}$.

2.2. The $\mathcal{N}il_l$ filtration

An unstable module M is said to be l-nilpotent if it is the colimit of unstable modules each having a finite filtration whose filtration quotients are l-suspensions. (For some equivalent definitions of l-nilpotency see for example [16], which is also a general reference for other facts concerning unstable modules over the Steenrod algebra.) Let $\mathcal{N}il_l$ denote the full subcategory of $\mathcal{U}$ of l-nilpotent modules. We hence get a decreasing filtration

$$\cdots \subset \mathcal{N}il_2 \subset \mathcal{N}il_1 = \mathcal{N}il \subset \mathcal{N}il_0 = \mathcal{U}$$

of $\mathcal{U}$ with the property that $\mathcal{N}il_1$ equals the usual subcategory of nilpotent modules $\mathcal{N}il$ and $\cap_l \mathcal{N}il_l = 0$. It is often more convenient to work with the slightly larger subcategory $\overline{\mathcal{N}il}_l$ which is the smallest Serre class in $\mathcal{U}$ that contains $\mathcal{N}il_l$ and $\mathcal{B}$, the category of locally finite modules, and is closed under colimits. A more concrete description of $\overline{\mathcal{N}il}_l$ is given by observing that $M \in \overline{\mathcal{N}il}_l$ if and only if $M^{\geq l} \in \mathcal{N}il_l$.

3. Relations between QH^*X and $H^*\Omega X$

In this section, we study the relation between the $\mathcal{N}il_l$ filtration and the Eilenberg-Moore spectral sequence using the methods of Schwartz [15]. Our Theorem 3.1 can fairly easily be derived directly from [15, 16], but since we need the results under slightly weaker assumptions than the ones used in [15, 16], and think that insight is gained from a direct proof, we give one.

Define the nilpotency degree of an unstable module M as the largest l, such that $M \in \overline{\mathcal{N}il}_l$. If M is locally finite we say that the nilpotency degree of M is infinite. We start by giving a proof of the key result about the nilpotency degree.

Theorem 3.1. *Let X be a connected space with $\pi_1 X$ a finite p-group and H^*X of finite type. For any $l \geq 0$, $QH^*X \in \overline{\mathcal{N}il}_{l+1}$ if and only if $H^*\Omega X \in \overline{\mathcal{N}il}_l$.*

Proof. Since we assume that $\pi_1 X$ is a finite p-group, it acts nilpotently on the $\mathbf{F}_p$-vector space $H^*\Omega X$, since the group-ring $\mathbf{F}_p \pi_1 X$ has nilpotent augmentation ideal. Hence the Eilenberg-Moore spectral sequence for the path-loop fibration over X converges strongly to $H^*\Omega X$.

For an unstable algebra K we have by [16, Lemma 8.7.6] that

$$QK \in \overline{\mathcal{N}il}_n \quad \text{implies that} \quad \operatorname{Tor}^s_K(\mathbf{F}_p, \mathbf{F}_p) \in \overline{\mathcal{N}il}_{n-s-1}. \tag{3.1}$$

Assume that $QH^*X \in \overline{\mathcal{N}il}_{l+1}$. This implies that $\operatorname{Tor}^s_{H^*X}(\mathbf{F}_p, \mathbf{F}_p) \in \overline{\mathcal{N}il}_{l-s}$, so $\Sigma^{-s} F_s / F_{s+1} \in \overline{\mathcal{N}il}_{l-s}$. Therefore $F_s/F_{s+1} \in \overline{\mathcal{N}il}_l$ for all s which gives $H^*\Omega X \in \overline{\mathcal{N}il}_l$.

To prove the other direction assume that $H^*\Omega X \in \overline{\mathcal{N}il}_l$, so $\Sigma \bar{H}^*\Omega X \in \overline{\mathcal{N}il}_{l+1}$. If $QH^*X \in \mathcal{B}$ we are done; else let n be the nilpotency degree of QH^*X. Consider the canonical map

$$QH^*X = E_2^{-1,*} \to E_\infty^{-1,*} \hookrightarrow \Sigma \bar{H}^*\Omega X$$

We claim that this map has kernel in $\overline{\mathcal{N}il}_{n+1}$. To see this, first note that since $QH^*X \in \overline{\mathcal{N}il}_n$ we have that $E_r^{-r-1} \in \overline{\mathcal{N}il}_{n+r}$ by (3.1). Therefore the image of $d_r : E_r^{-r-1,*} \to E_r^{-1,*}$ is in $\overline{\mathcal{N}il}_{n+1}$ by the fact that d_r is $\mathcal{A}$-linear of degree $-(r-1)$, and this establishes the claim. Since $\overline{\mathcal{N}il}_l$ is a Serre class we get that $n \geq \min\{n+1, l+1\}$, so $n \geq l+1$ as wanted. □

Corollary 3.2. *Let X be a connected space with $\pi_1 X$ a finite p-group and H^*X of finite type. We have that QH^*X is locally finite if and only if $H^*\Omega X$ is locally finite.* □

Remark 3.3. Note that one of the main results in [5] can be formulated as saying that if $H^*\Omega X$ is finite over $\mathbf{F}_p$ then QH^*X is finite over $\mathbf{F}_p$—here the converse is however far from being true.

Theorem 3.1 makes it desirable to understand the relationship between the nilpotency degree of K and QK for an unstable algebra K. In general the nilpotency degree can differ radically if K is not in $\overline{\mathcal{N}il}_1$, for example for any finite group G whose order is divisible by p, $QH^*BG \in \mathcal{B}$ but $H^*BG \notin \overline{\mathcal{N}il}_1$. However the next proposition shows that this is the only thing that can go wrong.

Proposition 3.4. *Let K be a connected unstable algebra and assume that $K \in \overline{\mathcal{N}il}_1$. Then $K \in \overline{\mathcal{N}il}_l$ if and only if $QK \in \overline{\mathcal{N}il}_l$.*

Proof. It is clear that $QK \in \overline{\mathcal{N}il}_l$ if $K \in \overline{\mathcal{N}il}_l$. To see the converse suppose that $QK \in \overline{\mathcal{N}il}_l$. The statement is trivially true for $l = 0, 1$, so suppose that $l \geq 2$ and assume by induction that the statement is true for $l - 1$.

It is straight forward to see that $\bar{K}^0 = 0$ and $\bar{K} \in \mathcal{N}il_{l-1}$ ensures that $\bar{K} \otimes \bar{K} \in \overline{\mathcal{N}il}_l$ (cf. [15, Cor. 1.9]). (Here $\bar{K}$ denotes the augmentation ideal of K.) Hence both QK and $\bar{K} \otimes \bar{K}$ lie in $\overline{\mathcal{N}il}_l$, so $\bar{K} \in \overline{\mathcal{N}il}_l$ as well, by the defining sequence of QK and the fact that $\overline{\mathcal{N}il}_l$ is a Serre class. This shows that $K \in \overline{\mathcal{N}il}_l$ as wanted. □

4. Generalizations of Serre's theorem

In this section we use the results of the preceding section to prove the promised generalizations of Serre's theorem on the homotopy groups of a finite complex.

Before doing this, however, we prove a general result which says that the transcendence degree of the cohomology ring decreases when passing to covers. To get the result in its best form we use the Sullivan p-adic completion (see [13]), which coincides with the Bousfield-Kan $\mathbf{F}_p$-completion on spaces with mod p cohomology of finite type [13, 3.4]. (We will actually only need the result for spaces with mod p cohomology of finite type, where references to [13] can be replaced by references to [9].)

Theorem 4.1. *Let X be a Sullivan p-adically complete space. Then the transcendence degree of $H^*X\langle 1\rangle$ is less than or equal to the transcendence degree of H^*X.*

Proof. First note that we may assume that X is connected. Recall that, by the work of Morel [13], for any Sullivan p-adically complete space X and any elementary abelian p-group V $[BV, X] = \mathrm{Hom}_{\mathcal{K}}(H^*X, H^*BV)$, where $[-,-]$ denotes free homotopy classes of maps and $\mathrm{Hom}_{\mathcal{K}}$ denotes Hom in the category of unstable algebras over the Steenrod algebra. By [8] the transcendence degree of H^*X is equal to the transcendence degree of the functor $[B-, X] = \mathrm{Hom}_{\mathcal{K}}(H^*X, H^*B-)$ from elementary abelian p-groups to profinite sets. One definition of the transcendence degree of the functor $[B-, X]$ is the rank of the largest elementary abelian p-group V such that there exists $s \in [BV, X]$ which cannot be written as $s = s' \circ B\varphi$, where $\varphi \in \operatorname{End} V$ is singular (see [8],[6, Prop. 5.8]). Since $[BV, X]$ is obtained from $[BV, X]_{pt}$ by taking the quotient under the action of $\pi_1 X$ we especially see that $[B-, X]$ and $[B-, X]_{pt}$ have the same transcendence degree.

The principal fibration $\pi_1 X \to X\langle 1\rangle \to X$ shows that the map $[BV, X\langle 1\rangle]_{pt} \to [BV, X]_{pt}$ is injective, so the transcendence degree of $[B-, X\langle 1\rangle]$ is less than or equal to that of $[B-, X]$. Since $X\langle 1\rangle$ is again Sullivan p-adically complete (see [13, 1.3,1.4]) we conclude that the transcendence degree of $H^*X\langle 1\rangle$ is less than or equal to that of H^*X. □

Theorem 4.2. *Assume that X is a connected finite Postnikov system, with $\pi_1 X$ a finite p-group and H^*X of finite type. Then $QH^*X \in \overline{\mathcal{N}il}_2$ if and only if X is $\mathbf{F}_p$-equivalent to the classifying space of a p-compact toral group.*

Proof. If X is $\mathbf{F}_p$-equivalent to a the classifying space of a p-compact toral group, then ΩX is $\mathbf{F}_p$-equivalent to a disjoint union of circles, so especially $H^*\Omega X \in \overline{\mathcal{N}il}_1$, so $QH^*X \in \overline{\mathcal{N}il}_2$ by Theorem 3.1.

To see the converse, assume that $QH^*X \in \overline{\mathcal{N}il}_2$. Since $\pi_1 X$ is a finite p-group the Bousfield-Kan $\mathbf{F}_p$-completion of X is $\mathbf{F}_p$-complete and is again a finite Postnikov system by [2, II 5.1], so we may assume that X is $\mathbf{F}_p$-complete. Hence ΩX is also $\mathbf{F}_p$-complete [2] [5, 11.9]. By for example the Eilenberg-Moore spectral sequence, $H^*\Omega X$ is also of finite type so both X and ΩX are Sullivan p-adically complete. By Theorem 3.1 $H^*(\Omega X) \in \overline{\mathcal{N}il}_1$, so Theorem 4.1 implies that also $H^*((\Omega X)\langle 1\rangle) \in \overline{\mathcal{N}il}_1$. Likewise, as above, $H^*((\Omega X)\langle 1\rangle)$ is of finite type.

We therefore have that $(\Omega X)\langle 1\rangle$ is a one-connected finite Postnikov system with cohomology in $\overline{\mathcal{N}il}_1$ and of finite type, which by [6, Thm. 1.1] implies that $\bar{H}^*((\Omega X)\langle 1\rangle) = 0$. So X is homotopy equivalent to its second Postnikov stage P_2X, since both spaces are $\mathbf{F}_p$-complete. Since $H^*\Omega P_2 X \in \overline{\mathcal{N}il}_1$, the abelian group $\pi_2 X$ cannot have p-torsion, as this would imply the existence of an element of infinite height in the mod p reduced cohomology ring of $\Omega_0 P_2 X = K(\pi_2 X, 1)$ by standard group cohomology (or [9]). (Ω_0 denotes the zero component of the loop space.) Hence X is an $\mathbf{F}_p$-complete space with homotopy only in dimensions 1 and 2, and $\pi_2 X$ p-torsion free. But this means that $\pi_2 X$ is a torsion free Ext-p complete abelian group and hence isomorphic to $\hat{\mathbf{Z}}_p^n$ for some n (cf. [2, p. 181],[7]). Since H^*X is of finite type we have $n < \infty$. This completes the proof. □

Combining Theorem 4.2 with the results in [6] now enables us to give a proof of the main Theorem 1.1.

Proof of Theorem 1.1. By [5, Prop. 6.9, 12.1] 4. implies 3. Condition 3. obviously implies 1. and 2., and the implication 1. implies 4. follows from Theorem 4.2. The remaining 2. implies 4. follows easily from [6]. Namely, assume that the transcendence degree of H^*X is finite. We can assume that X is $\mathbf{F}_p$-complete. By Theorem 4.1, $H^*X\langle 1\rangle$ likewise has finite transcendence degree. Now, $X\langle 1\rangle$ is an $\mathbf{F}_p$-complete one connected finite Postnikov system with $H^*X\langle 1\rangle$ of finite type, which by [6, Thm. 1.2] means that $X\langle 1\rangle$ is homotopy equivalent to $K(\hat{\mathbf{Z}}_p^n, 2)$ for some $n < \infty$. Hence X is $\mathbf{F}_p$-equivalent to the classifying space of a p-compact toral group. □

Remark 4.3. Note how 1. and 2. are complementary, in the sense that 2. is a statement about the ring structure of H^*X, whereas 1. is a statement about the Steenrod algebra action on what is left when you kill all products.

Using the 'McGibbon-Neisendorfer trick', which basically says that an arbitrary space cannot have locally finite cohomology *and* have infinitely many nontrivial homotopy groups which are all p-torsion free, we can now easily strengthen the result of Dwyer and Wilkerson [4]. We show that the 2-connected assumption on their result was only necessary to exclude classifying spaces of p-compact toral groups.

Theorem 4.4. *Assume that X is a connected space, with $\pi_1 X$ a finite p-group and H^*X of finite type. If QH^*X is locally finite as a module over the Steenrod algebra then X is either $\mathbf{F}_p$-equivalent to the classifying space of a p-compact toral group, or it contains p-torsion in infinitely many of its homotopy groups.*

Proof. Assume that X is not $\mathbf{F}_p$-equivalent to the classifying space of a p-compact toral group. By Theorem 4.2, X cannot be $\mathbf{F}_p$-equivalent to a finite Postnikov system, in other words $\pi_i(X; \mathbf{Z}/p) \neq 0$ for infinitely many i. But $H^*\Omega X$ is also locally finite by Corollary 3.2 which by [12, p. 255] implies that $\pi_i X$ actually has p-torsion for infinitely many i. □

Remark 4.5. The above theorem is not a true generalization of Theorem 4.2 for a good reason. The assumptions in Theorem 4.4 cannot be weakened to $QH^*X \in \overline{\mathcal{N}il}_2$ as is demonstrated by setting $X = BSU$, the classifying space of the infinite special unitary group. Indeed, $H^*SU \in \overline{\mathcal{N}il}_1$, so $QH^*BSU \in \overline{\mathcal{N}il}_2$ by Theorem 3.1. But BSU is obviously neither the classifying space of a p-compact toral group nor does it contain any p-torsion in its homotopy groups by Bott periodicity. (See also [11].)

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