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The Steenrod problem of realizing polynomial cohomology rings

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Abstract

In this paper, we completely classify which graded polynomial R-algebras in finitely many even degree variables can occur as the singular cohomology of a space with coefficients in R, a 1960 question of N. E. Steenrod, for a commutative ring R satisfying mild conditions. In the fundamental case $R = \mathbb{Z}$, our result states that the only polynomial cohomology rings over $\mathbb{Z}$ that can occur are tensor products of copies of $H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[x_2]$, $H^*(B\,\mathrm{SU}(n);\mathbb{Z}) \cong \mathbb{Z}[x_4, x_6, \ldots, x_{2n}]$, and $H^*(B\,\mathrm{Sp}(n);\mathbb{Z}) \cong \mathbb{Z}[x_4, x_8, \ldots, x_{4n}]$, confirming an old conjecture. Our classification extends Notbohm's solution for $R = \mathbb{F}_p$, p odd. Odd degree generators, excluded above, only occur if R is an $\mathbb{F}_2$-algebra and in that case the recent classification of 2-compact groups by the authors can be used instead of the present paper. Our proofs are short and rely on the general theory of p-compact groups, but not on classification results for these.

1. *Introduction*

In 1960, N. E. Steenrod [**39**] asked which graded polynomial rings occur as the cohomology ring of a space. We answer the question in this paper.

Theorem 1.1. *If $H^*(X;\mathbb{Z})$ is a finitely generated polynomial algebra over $\mathbb{Z}$, for some space X, then $H^*(X;\mathbb{Z})$ is isomorphic, as a graded algebra, to a tensor product of copies of $H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[x_2]$, $H^*(B\,\mathrm{SU}(n);\mathbb{Z}) \cong \mathbb{Z}[x_4, x_6, \ldots, x_{2n}]$, and $H^*(B\,\mathrm{Sp}(n);\mathbb{Z}) \cong \mathbb{Z}[x_4, x_8, \ldots, x_{4n}]$.*

This has been a standard conjecture since the 1970s. In fact, we solve the question over any ground ring R, satisfying mild assumptions. As usual, $\mathbb{C}P^\infty$ denotes infinite complex projective space, and $B\,\mathrm{SU}(n)$ and $B\,\mathrm{Sp}(n)$ are the classifying spaces of the special unitary group and the symplectic group, respectively. We first describe the background of the problem and previously known results.

Steenrod proved in his 1960 paper [**39**] that if $H^*(X;\mathbb{Z}) \cong \mathbb{Z}[x]$ then $|x| = 2$ or 4, as a consequence of his newly introduced Steenrod operations, settling the one-variable case over $\mathbb{Z}$. In contrast, when taking coefficients in a field R of characteristic zero, every finitely generated graded polynomial ring on even degree generators can occur, since $H^*(K(\mathbb{Z}, 2n); R) \cong R[x_{2n}]$, as proved by Serre in his thesis [**37**, Chapitre VI, § 3, Proposition 4]. Note that by anti-commutativity, all generators have to be in even degrees, whenever $2 \neq 0$ in R.

Restrictions on realizable polynomial rings over $\mathbb{F}_p$ were studied in the 1960s and 1970s, mainly by ingenious use of Steenrod operations (see, for example, [**40**, **42**, **44**, **45**]). Key progress was made with the discovery of a connection with finite p-adic reflection groups by Sullivan [**43**], Clark and Ewing [**17**], and others in the early 1970s. Further significant progress

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was made with the connection to the category of unstable modules over the Steenrod algebra, starting with Wilkerson [50] and Adams and Wilkerson [1] in the late 1970s and early 1980s. Aguadé in his 1981 paper [2] used Adams and Wilkerson's work [1] to obtain a partial version of Theorem 1.1, under the strong additional hypothesis that all generators are in different degrees. (Theorem 1.1 is referred to in [2] as 'the standing conjecture'; see also the survey paper [3] for a historical account and references as of 1981.) Following additional partial results, Notbohm [33] in 1999 used the more powerful techniques of p-compact groups, as developed by Dwyer and Wilkerson [21, 23] and others since the early 1990s, to obtain a full answer in the case where the ground ring is $\mathbb{F}_p$ for p odd. Note, however, that 2-primary information is needed for Theorem 1.1 as, for example, $H^*(B\,\mathrm{Spin}(2n);\mathbb{Z}[\frac{1}{2}]) \cong \mathbb{Z}[\frac{1}{2}][x_4, x_8, \ldots, x_{4(n-1)}, x'_{2n}]$, for $n \geqslant 2$.

In this paper, we obtain a solution for any commutative Noetherian ring R of finite Krull dimension, under the assumption that all generators are in even degrees. The even degrees assumption is automatic unless R is an $\mathbb{F}_2$-algebra; in that case a solution however follows as a consequence of our recent classification of 2-compact groups; see [6, Theorem 1.4]. We use standard results about p-compact groups in the present paper, but remark that classification results for these, as well as Notbohm's results in the case $R = \mathbb{F}_p$, p odd, are not used. We also note that Theorem 1.1 can alternatively be deduced from our aforementioned result [6, Theorem 1.4], but the argument we present here is significantly more direct, as explained below. By the *type* of a graded polynomial ring, we mean the multiset (that is, unordered tuple) of degrees of its generators. The general result (implying Theorem 1.1) that we prove here is as follows.

THEOREM 1.2. *Let R be a commutative Noetherian ring of finite Krull dimension, let $\mathcal{P}$ be the set of prime numbers p that are not units in R, and let A be a graded polynomial R-algebra in finitely many variables, all in positive even degrees. Then there exists a space Y such that $A \cong H^*(Y;R)$ as graded R-algebras if, and only if, for each prime $p \in \mathcal{P}$, the type of A is a union of multisets of degrees in Tables 1–3 which occur at the prime p.*

TABLE 1. *Lie group cases.*

Lie group	Degrees	Occur for
S^1	2	all p
$\mathrm{SU}(n)$	$4, 6, \ldots, 2n$	all p
$\mathrm{Sp}(n)$	$4, 8, \ldots, 4n$	all p
$\mathrm{Spin}(2n)$	$4, 8, \ldots, 4(n-1), 2n$	$p \geqslant 3$
G_2	4, 12	$p \geqslant 3$
F_4	4, 12, 16, 24	$p \geqslant 5$
E_6	4, 10, 12, 16, 18, 24	$p \geqslant 5$
E_7	4, 12, 16, 20, 24, 28, 36	$p \geqslant 5$
E_8	4, 16, 24, 28, 36, 40, 48, 60	$p \geqslant 7$

TABLE 2. *Exotic cases, first part (families 2a, 2b, and 3).*

W	Degrees	Conditions	Occur for
$G(m,r,n)$	$2m, 4m, \ldots, 2(n-1)m,\ 2mn/r$	$n \geqslant 2, m \geqslant 3,\ r \mid m$	$p \equiv 1 \pmod{m}$
D_{2m}	$4,\ 2m$	$m \geqslant 5,\ m \neq 6$	$p \equiv \pm 1 \pmod{m}$
C_m	$2m$	$m \geqslant 3$	$p \equiv 1 \pmod{m}$

TABLE 3. *Exotic cases, second part.*

W	Degrees	Occur for	
G_8	16, 24	$p \equiv 1$	(mod 4)
G_9	16, 48	$p \equiv 1$	(mod 8)
G_{12}	12, 16	$p \equiv 1, 3$	(mod 8)
G_{14}	12, 48	$p \equiv 1,\ 19$	(mod 24)
G_{16}	40, 60	$p \equiv 1$	(mod 5)
G_{17}	40, 120	$p \equiv 1$	(mod 20)
G_{20}	24, 60	$p \equiv 1,\ 4$	(mod 15)
G_{21}	24, 120	$p \equiv 1,\ 49$	(mod 60)
G_{22}	24, 40	$p \equiv 1,\ 9$	(mod 20)
G_{23}	4, 12, 20	$p \equiv 1,\ 4$	(mod 5)
G_{24}	8, 12, 28	$p \equiv 1,\ 2,\ 4$	(mod 7), $p \neq 2$
G_{29}	8, 16, 24, 40	$p \equiv 1$	(mod 4)
G_{30}	4, 24, 40, 60	$p \equiv 1,\ 4$	(mod 5)
G_{31}	16, 24, 40, 48	$p \equiv 1$	(mod 4)
G_{32}	24, 36, 48, 60	$p \equiv 1$	(mod 3)
G_{33}	8, 12, 20, 24, 36	$p \equiv 1$	(mod 3)
G_{34}	12, 24, 36, 48, 60, 84	$p \equiv 1$	(mod 3)

The space Y is rarely unique; see Remark 3.3. The assumption that R is Noetherian with finite Krull dimension can be replaced by the assumption that Y has finite type; see Proposition 2.5.

We briefly explain the origin of Tables 1–3. Table 1 lists the simple simply connected compact Lie groups and S^1, and the primes for which the homology of the group is p-torsion free, together with the degrees of the polynomial generators of the $\mathbb{F}_p$-cohomology of their classifying spaces, but with $\mathrm{Spin}(2n+1)$, p odd, left out for simplicity, since its $\mathbb{F}_p$-cohomology agrees with that of $\mathrm{Sp}(n)$. This information goes back to Borel [**13**] (see Proposition 4.3).

To explain Tables 2 and 3, we recall some facts about reflection groups. The Shephard–Todd–Chevalley theorem [**11**, Theorem 7.2.1] says that, for a field K of characteristic zero and a finite dimensional K-vector space V, a finite group $W \leqslant \mathrm{GL}(V)$ is a reflection group if and only if the ring $K[V]^W$ of W-invariant polynomial functions on V is a graded polynomial ring, where we grade $K[V]$ by giving the elements of V^* degree 2. The *degrees* of $W \leqslant \mathrm{GL}(V)$ is defined to be the type of this invariant ring. (We warn the reader that degrees are sometimes defined as half of what is the convention in this paper.) Shephard and Todd [**38**] classified the finite irreducible complex reflection groups as falling into three infinite families (labeled 1–3) and 34 sporadic cases (labeled G_i, $4 \leqslant i \leqslant 37$). Clark and Ewing [**17**] used this to give a classification of finite $\mathbb{Q}_p$-reflection groups (with additional clarification by Dwyer, Miller and Wilkerson [**21**, Proposition 5.5, Proof of Theorem 1.5]). This classification says that for a given prime p, a finite $\mathbb{C}$-reflection group gives rise to a unique $\mathbb{Q}_p$-reflection group if and only if its character field embeds in $\mathbb{Q}_p$, and all finite $\mathbb{Q}_p$-reflection groups arise this way. Tables 2 and 3 essentially list the finite irreducible $\mathbb{Q}_p$-reflection groups that are *exotic*, that is, those whose character field is not $\mathbb{Q}$, *except* the $\mathbb{Q}_2$-reflection group G_{24}; but, for simplicity, we have also removed the groups G_i for $i = 4,\ 5,\ 6,\ 7,\ 10,\ 11,\ 13,\ 15,\ 18,\ 19,\ 25,\ 26,$ and 27, whose degrees is readily obtainable as a multiset union of the degrees of the remaining groups, at any prime for which they exist as $\mathbb{Q}_p$-reflection groups.

To illustrate Theorem 1.2, if $\mathcal{P} = \emptyset$ (that is, if R is a $\mathbb{Q}$-algebra) there are no restrictions, recovering Serre's result mentioned earlier. If $2 \in \mathcal{P}$, the possible types of A are unions of the multisets $\{2\}$, $\{4, 6, \dots, 2n\}$, and $\{4, 8, \dots, 4n\}$, in particular recovering Theorem 1.1. If $\mathcal{P} = \{3\}$ the type of A is a union of $\{2\}$, $\{4, 6, \dots, 2n\}$, $\{4, 8, \dots, 4n\}$, $\{4, 8, \dots, 4(n-1), 2n\}$, $\{4, 12\}$, and $\{12, 16\}$, and similar lists can easily be compiled for $\mathcal{P} = \{p\}$ for any individual prime p. For arbitrary $\mathcal{P}$, it is a simple combinatorial problem to check whether any given multiset of degrees is realizable, as is described in the following corollary.

Corollary 1.3. *For any multiset of degrees* $\{2d_1, \ldots, 2d_r\}$, *there exist integers* $a_1, \ldots, a_m$ *and* N *such that the following conditions are equivalent for any commutative Noetherian ring* R *of finite Krull dimension.*

(1) *The graded polynomial* R*-algebra* $R[x_1, \ldots, x_r]$, *with* $|x_i| = 2d_i$, *is isomorphic to* $H^*(Y; R)$ *for some space* Y.

(2) *Every prime number* p, *which is not a unit in* R, *satisfies* $p \equiv a_i \pmod N$ *for some* i.

We give an algorithm for finding N and $a_1, \ldots, a_m$ in the course of the proof. Finding explicit generators for the commutative monoid of realizable multisets of degrees for a given collection of primes $\mathcal{P}$ is a harder combinatorial problem in general.

The proof of the main theorem spans three short sections. The first two sections reduce the problem to p-compact groups: In Section 2, we show that if $H^*(Y; R)$ is a polynomial algebra, then the same holds for $H^*(Y; \mathbb{F}_p)$ for all prime numbers p that are not units in R, and hence that the $\mathbb{F}_p$-completion $Y_p^{\hat{}}$ is the classifying space of a p-compact group. The proof has a small ring-theoretic twist due to finite-type problems coming from the fact that we are working with cohomology. Conversely, Section 3 constructs a space Y with polynomial R-cohomology from a collection of p-compact groups with compatible cohomology rings, for p any non-unit in R. Finally, in Section 4, we determine the graded polynomial $\mathbb{F}_p$-algebras on even degree generators, which occur as the $\mathbb{F}_p$-cohomology of the classifying space of a p-compact group. The key Proposition 4.1 reduces this question to separate questions for classifying spaces of compact Lie groups and classifying spaces of exotic p-compact groups (without using classification results on p-compact groups). These are then easily solved, the first one essentially by Borel, and the second using earlier existence results for exotic p-compact groups. Combining the earlier steps now establishes the main result, Theorem 1.2, and its corollaries. We remark that Notbohm's approach to the case $R = \mathbb{F}_p$, p odd, in [**33**] differs from the one given here in Section 4. Notbohm instead uses an argument relying on various earlier, fairly elaborate, case-by-case calculations of invariant rings, which in particular does not extend to the case of $\mathbb{F}_2$.

1.1. *Notation and recollections*

We stress that by a graded polynomial algebra we mean a graded algebra generated by a set of homogeneous elements, which are algebraically independent (that is, we do not allow odd degree exterior generators). For the theorems of the paper, a space can be taken to mean any topological space, though for the purposes of the proofs we may, without loss of generality, restrict ourselves to CW-complexes (or even simplicial sets), as we will often tacitly do. We constantly use the theory of p-compact groups, and here recall some pertinent facts, referring to, for example, [**6**, **7**, **20**, **23**] for more information. A p-compact group consists of a triple $(X, BX, e\colon X \xrightarrow{\simeq} \Omega BX)$ such that BX is an $\mathbb{F}_p$-complete space and X has finite $\mathbb{F}_p$-cohomology. Proposition 2.2 recalls that spaces with polynomial $\mathbb{F}_p$-cohomology ring always come from p-compact groups. A standard result, central to this paper, states that for a connected p-compact group X, $H^*(BX; \mathbb{F}_p)$ is a polynomial algebra, concentrated in even degrees if and only if $H^*(X; \mathbb{Z}_p)$ is torsion free if and only if $H^*(BX; \mathbb{Z}_p) \xrightarrow{\cong} H^*(BT; \mathbb{Z}_p)^{W_X}$, where T and W_X are respectively the maximal torus and the Weyl group of X. (Parts of this result are due to Borel and parts to Dwyer, Miller and Wilkerson; see [**7**, Theorem 12.1].) In this case the type of $H^*(BX; \mathbb{F}_p)$ equals the degrees of W_X, which act as a reflection group on $\pi_2(BT) \otimes \mathbb{Q}$ with invariant ring $\mathbb{Q}_p[\pi_2(BT) \otimes \mathbb{Q}]^{W_X} \cong H^*(BX; \mathbb{Z}_p) \otimes \mathbb{Q}$ [**23**, Theorem 9.7]—this provides the link between polynomial cohomology rings concentrated in even degrees, torsion free p-compact groups, and finite $\mathbb{Z}_p$- and $\mathbb{Q}_p$-reflection groups. A couple of times in the proofs, we use root data for which we refer to [**6**, Section 8], although we do not use anything that is not already in [**24**] in a slightly different language.

2. *From R to $\mathbb{F}_p$*

In this section, we show how a polynomial cohomology ring over R produces polynomial cohomology rings over $\mathbb{F}_p$ for the prime numbers p that are not units in R.

PROPOSITION 2.1. *Suppose that R is a commutative Noetherian ring of finite Krull dimension, and Y is a space such that $H^*(Y;R)$ is a polynomial R-algebra in finitely many variables, each in positive degree. If a prime p is not a unit in R then $H^*(Y;\mathbb{F}_p)$ is a polynomial $\mathbb{F}_p$-algebra with the same type and $Y\hat{_p}$ is the classifying space of a p-compact group.*

The proof will occupy the rest of this section. The proof is straightforward if, for example, $R \subseteq \mathbb{Q}$ (using the exact sequence $0 \to R \xrightarrow{p} R \to \mathbb{F}_p \to 0$), but in general it is a bit more subtle, with complications arising even for $R = \mathbb{Z}/4$. First we explain the well-known fact that spaces with polynomial $\mathbb{F}_p$-cohomology ring give rise to p-compact groups.

PROPOSITION 2.2. *Let Y be a space. If $H^*(Y;\mathbb{F}_p)$ is a finitely generated polynomial algebra over $\mathbb{F}_p$, then $Y\hat{_p} \simeq BX$ for a p-compact group X. If, furthermore, $H^1(Y;\mathbb{F}_p) = 0$, then X is connected.*

Proof. By considering the fiber of the canonical map $Y \to K(H_1(Y;\mathbb{F}_p),1)$, one easily reduces to the case where $H_1(Y;\mathbb{F}_p) = 0$ (cf. [**6**, Proof of Theorem 1.4]), and, in fact, this is also the only case relevant for our main theorem. By [**14**, Proposition VII.3.2], Y is $\mathbb{F}_p$-good, and $Y\hat{_p}$ is $\mathbb{F}_p$-complete and simply connected. In particular, the Eilenberg–Moore spectral sequence of the path-loop fibration of $Y\hat{_p}$ converges [**19**] and shows that $H^*(\Omega Y\hat{_p};\mathbb{F}_p)$ is finite-dimensional over $\mathbb{F}_p$. Hence $Y\hat{_p}$ is the classifying space of a p-compact group, which is connected since $Y\hat{_p}$ is simply connected. □

Next we deal with the finiteness restrictions usually associated with universal coefficient theorems in cohomology. Recall that the *finitistic flat dimension* of a commutative ring R is defined to be the supremum over the flat dimensions of all R-modules with finite flat dimension.

LEMMA 2.3. *Let R be a commutative ring, and let C_* be a chain complex of R-modules (with differential of degree -1) concentrated in non-positive degrees such that C_n and $H_n(C_*)$ are flat for all n. If the finitistic flat dimension of R is finite, then $H_n(C_*) \otimes_R M \xrightarrow{\cong} H_n(C_* \otimes_R M)$ for any R-module M.*

Proof. Let $\mathrm{fd}(-)$ denote the flat dimension of an R-module. By the Künneth formula [**47**, Theorem 3.6.1] it suffices to show that $\mathrm{im}(C_n \xrightarrow{d_n} C_{n-1})$ is flat for all n. To see this, first note that by the short exact sequence

$$0 \to \mathrm{im}(d_{n+1}) \to \ker(d_n) \to H_n(C_*) \to 0,$$

we have $\mathrm{fd}(\mathrm{im}(d_{n+1})) = \mathrm{fd}(\ker(d_n))$, since $H_n(C_*)$ is flat, cf. [**47**, Exercise 4.1.2(3)]. Now, since C_n is flat, the short exact sequence

$$0 \to \ker(d_n) \to C_n \to \mathrm{im}(d_n) \to 0$$

shows that either $\mathrm{im}(d_n)$ and $\mathrm{im}(d_{n+1})$ are flat or $\mathrm{fd}(\mathrm{im}(d_{n+1})) = \mathrm{fd}(\ker(d_n)) = \mathrm{fd}(\mathrm{im}(d_n)) - 1$. Since $\mathrm{im}(d_n) = 0$ is flat for n positive, it follows that either $\mathrm{im}(d_n)$ is flat for all n or there exists an n_0 such that $\mathrm{im}(d_n)$ is flat for $n \geqslant n_0$ and $\mathrm{fd}(\mathrm{im}(d_n)) = n_0 - n$ for $n \leqslant n_0$. Since R has finite finitistic flat dimension the second possibility cannot occur, and we are done. □

Remark 2.4. A way to view the assumptions in the lemma is that they ensure convergence of the Künneth spectral sequence, derived from the spectral sequence of a double complex (see [**47**, § 5.6]). In complete generality the Künneth spectral sequence need not converge, the standard counterexample being $R = \mathbb{Z}/4$, $C_* = \cdots \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \cdots$ and $M = \mathbb{Z}/2$.

The preceding lemma, together with a result in commutative ring theory, gives the following result about the cohomology of spaces.

Proposition 2.5. *Let R be a commutative ring and let Y be a space such that $H^*(Y;R)$ is finite free over R in each degree. If either R is Noetherian with finite Krull dimension or Y has finite type, then*

$$H^*(Y;R)\otimes_R R/p \xrightarrow{\cong} H^*(Y;R/p) \xleftarrow{\cong} H^*(Y;\mathbb{F}_p)\otimes_{\mathbb{F}_p} R/p$$

as R/p-algebras.

Proof. Assume first that Y is arbitrary and that R is Noetherian with finite Krull dimension. Then the singular cochain complex $C^*(Y;R)$ is a complex of flat modules, since over a commutative Noetherian ring arbitrary products of flat modules are again flat (that is, commutative Noetherian rings are coherent; see [**16**, Theorem 2.1] or [**15**, Exercise VI.4, p. 122]). By a result of Auslander and Buchsbaum [**8**, Theorem 2.4], the finitistic flat dimension of R is bounded above by its Krull dimension (in fact they differ by at most 1 by a result of Bass [**10**, Corollary 5.3]), and in particular the assumptions imply that it is finite. Hence, since obviously $C^*(Y;R)\otimes_R R/p \cong C^*(Y;R/p)$, Lemma 2.3 implies that $H^*(Y;R)\otimes_R R/p \xrightarrow{\cong} H^*(Y;R/p)$. Now

$$H^*(Y;R/p) = H(C^*(Y;R/p)) = H(\mathrm{Hom}_{\mathbb{F}_p}(C_*(Y;\mathbb{F}_p),R/p)) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{F}_p}(H_*(Y;\mathbb{F}_p),R/p)$$

as R/p-modules, and in particular $H_*(Y;\mathbb{F}_p)$ is finite in each degree. Hence $H^*(Y;\mathbb{F}_p)\otimes_{\mathbb{F}_p} R/p \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{F}_p}(H_*(Y;\mathbb{F}_p),R/p)$, which combined with the previous isomorphisms gives the result under the ring-theoretic assumption on R.

Now assume that Y has finite type, and that R is an arbitrary commutative ring. Let $C^*(Y;-)$ denote the cellular cochain complex, and note that, by the finite-type assumption, $C^*(Y;\mathbb{Z})\otimes_{\mathbb{Z}} R \xrightarrow{\cong} C^*(Y;R)$. Now $H^*(Y;R)\otimes_R R/p \xrightarrow{\cong} H^*(Y;R/p)$ by a result of Dold [**18**, Satz 5.2], applied to the $\mathbb{Z}$-chain complex $C_* = C^{-*}(X;\mathbb{Z})$, $\Lambda = R$ and $M = R/p$. (Note that the proof of Dold's theorem is more involved than the proof of the ordinary Künneth theorem.) The second isomorphism in the proposition now follows as above. □

Lemma 2.6. *Let A be a graded connected $\mathbb{F}_p$-algebra that is finite-dimensional in each degree, and let B be an $\mathbb{F}_p$-algebra (viewed as a graded algebra concentrated in degree 0). If $A\otimes_{\mathbb{F}_p} B$ is a graded polynomial B-algebra, then A is a graded polynomial $\mathbb{F}_p$-algebra with the same type.*

Proof. By reducing B modulo a maximal ideal, we can without restriction assume that B is a field. Since B is a flat $\mathbb{F}_p$–module, $Q(A)\otimes_{\mathbb{F}_p} B \xrightarrow{\cong} Q(A\otimes_{\mathbb{F}_p} B)$, where $Q(\cdot)$ denotes the module of indecomposable elements. Picking a basis for the $\mathbb{F}_p$-vector space $Q(A)$ and lifting this to A produces a set of generators for A as an $\mathbb{F}_p$-algebra, which maps to a set of generators for $A\otimes_{\mathbb{F}_p} B$ as a B-algebra. By construction, they form a B-basis for $Q(A\otimes_{\mathbb{F}_p} B)$. Hence they are algebraically independent, since $A\otimes_{\mathbb{F}_p} B$ is assumed to be a graded polynomial B-algebra (to see this, note that mapping polynomial generators of $A\otimes_{\mathbb{F}_p} B$ to the constructed generators will produce an epimorphism $A\otimes_{\mathbb{F}_p} B \to A\otimes_{\mathbb{F}_p} B$, which also has to be a monomorphism since $A\otimes_{\mathbb{F}_p} B$ is finite-dimensional over B in each degree). Since B is a faithfully flat $\mathbb{F}_p$-module, the generators are algebraically independent in A as well, and hence A is a graded polynomial $\mathbb{F}_p$-algebra. ☐

Proof of Proposition 2.1. By Proposition 2.5 and Lemma 2.6, $H^*(X;\mathbb{F}_p)$ is a polynomial algebra with the same type as $H^*(X;R)$. It hence follows from Proposition 2.2 that X is $\mathbb{F}_p$-good and $X_p^\wedge$ is the classifying space of a p-compact group. ☐

3. *From* $\mathbb{F}_p$ *to* R

In this section, we show how spaces with polynomial $\mathbb{F}_p$-cohomology ring at different primes can be glued together, provided that they have the same type.

PROPOSITION 3.1. *Let $\mathcal{P}$ be a set of prime numbers, let J be the set of prime numbers not in $\mathcal{P}$, and let $\{2d_1,\dots,2d_r\}$ be a fixed multiset. Suppose that for each $p\in\mathcal{P}$ there exists a space B_p such that $H^*(B_p;\mathbb{F}_p)$ is a polynomial algebra with type $\{2d_1,\dots,2d_r\}$. Then there exists a simply connected space Y of finite type, such that $H^*(Y;\mathbb{Z}[J^{-1}])$ is a polynomial algebra over $\mathbb{Z}[J^{-1}]$ with generators in degrees $\{2d_1,\dots,2d_r\}$. More generally, if R is a commutative ring with $R=R[J^{-1}]$, then $H^*(Y;R)$ is also a polynomial R-algebra with generators in the same degrees.*

Before the proof we need the following lemma, which is similar to a lemma used by Baker and Richter [**9**, Proposition 2.4] in a different context.

LEMMA 3.2. *Let J be a set of prime numbers and set $R=\mathbb{Z}[J^{-1}]$. Suppose that A is a graded connected R-algebra, finite free over R in each degree. If $A\otimes_R\mathbb{Q}$ is a graded polynomial $\mathbb{Q}$-algebra and $A\otimes_R\mathbb{Z}_p$ is a graded polynomial $\mathbb{Z}_p$-algebra for all primes $p\notin J$, then A is a graded polynomial R-algebra as well (with the same type as both $A\otimes_R\mathbb{Z}_p$, $p\notin J$, and $A\otimes_R\mathbb{Q}$).*

Proof. The proof is similar to the proof of Lemma 2.6. Since $\mathbb{Z}_p$ is flat over R for $p\notin J$, we have $Q(A)\otimes_R\mathbb{Z}_p\xrightarrow{\cong}Q(A\otimes_R\mathbb{Z}_p)$, which is free over $\mathbb{Z}_p$ in each degree by assumption. Hence $Q(A)$ does not have p-torsion when $p\notin J$, and since $Q(A)$ is finite over R in each degree, we conclude by the structure theorem for finite $\mathbb{Z}[J^{-1}]$-modules that $Q(A)$ is finite free over R in each degree. Likewise, since $\mathbb{Q}$ is flat over R, $Q(A)\otimes_R\mathbb{Q}\xrightarrow{\cong}Q(A\otimes_R\mathbb{Q})$. Pick homogeneous elements in A that project to an R-basis of $Q(A)$. Clearly, these elements generate A as an R-algebra. They are also algebraically independent, since they, via the isomorphism $Q(A)\otimes_R\mathbb{Q}\xrightarrow{\cong}Q(A\otimes_R\mathbb{Q})$, give rise to generators of the graded polynomial ring $A\otimes_R\mathbb{Q}$. Hence A is a graded polynomial R-algebra, with the same type as $A\otimes_R\mathbb{Q}$ and $A\otimes_R\mathbb{Z}_p$, $p\notin J$. ☐

Proof of Proposition 3.1. The proof is reminiscent of [**5**, Proof of Lemma 1.2], to which we also refer. By Proposition 2.2, $(B_p)_p^\wedge$ is $\mathbb{F}_p$-complete with the same $\mathbb{F}_p$-cohomology as B_p, so we may assume that B_p is $\mathbb{F}_p$-complete. Set $K=K(\mathbb{Z}[\mathcal{P}^{-1}],2d_1)\times\ldots\times K(\mathbb{Z}[\mathcal{P}^{-1}],2d_r)$. We will define Y as a homotopy pull-back

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \prod_{p\in\mathcal{P}} B_p \arrow[d] \\ K \arrow[r, "f"] & (\prod_{p\in\mathcal{P}} B_p)_{\mathbb{Q}} \end{tikzcd} \tag{3.1}$$

where the map f has to be constructed.

Since B_p is the classifying space of a connected p-compact group by Proposition 2.2, $\pi_*(B_p)$ and $H^*(B_p;\mathbb{Z}_p)$ are finite over $\mathbb{Z}_p$ in each degree. It follows that $H^*(B_p;\mathbb{Z}_p)$ is also a polynomial algebra over $\mathbb{Z}_p$, with type $\{2d_1,\ldots,2d_r\}$. By an essentially classical result (see [**5**, Theorem 2.1]),

$$\pi_n(B_p)\cong\pi_{n-1}((S^{2d_1-1}\times\ldots\times S^{2d_r-1})\hat{_p})$$

for $p>\max\{d_1,\ldots,d_r\}$ and similarly for the $\mathbb{Q}$-localization, so

$$\pi_n\left(\left(\prod_{p\in\mathcal{P}}B_p\right)_{\mathbb{Q}}\right)\cong\pi_{n-1}(S^{2d_1-1}\times\ldots\times S^{2d_r-1})\otimes\left(\prod_{p\in\mathcal{P}}\mathbb{Z}_p\right)\otimes\mathbb{Q}.$$

Since these homotopy groups are concentrated in even degrees, $(\prod_{p\in\mathcal{P}}B_p)_{\mathbb{Q}}$ is a product of Eilenberg–Mac Lane spaces, as follows from standard obstruction theory [**48**, Chapter IX].

Hence we can define $f\colon K\to(\prod_{p\in\mathcal{P}}B_p)_{\mathbb{Q}}$ to be the canonical map induced by the unique ring map $\mathbb{Z}[\mathcal{P}^{-1}]\to\mathbb{Q}\to(\prod_{p\in\mathcal{P}}\mathbb{Z}_p)\otimes\mathbb{Q}$. By the Mayer–Vietoris sequence in homotopy applied to the pull-back square (3.1), we see that

$$\pi_n(Y)=\left(\bigoplus_{i,2d_i=n}\mathbb{Z}\right)\oplus\left(\bigoplus_{p\in\mathcal{P}}\mathrm{Tor}(\mathbb{Q}/\mathbb{Z},\pi_n(B_p))\right).$$

In particular, $\pi_n(Y)$ is finite over $\mathbb{Z}$, since for a fixed n, $\pi_n(B_p)$ is p-torsion free, for p large [**37**, Chapitre V, §5, Proposition 4]. Since Y is simply connected, this implies that Y is homotopy equivalent to a CW-complex with finitely many cells in each dimension; that is, Y can be chosen to have finite type (compare, for example, [**46**, Theorem A]).

For $p\in\mathcal{P}$, the pull-back square above shows that the map $Y\to B_p$ induces an isomorphism on mod p homology so $H^*(Y;\mathbb{Z}_p)\xleftarrow{\cong}H^*(B_p;\mathbb{Z}_p)$. Hence $H^*(Y;\mathbb{Z}_p)$ is a polynomial ring over $\mathbb{Z}_p$ with the same type as $H^*(B_p;\mathbb{F}_p)$. Since $H_*(Y;\mathbb{Z}[J^{-1}])$ is finite over $\mathbb{Z}[J^{-1}]$ in each degree, the universal coefficient theorem [**47**, 3.6.5] (applied to the $\mathbb{Z}[J^{-1}]$-module $\mathbb{Z}_p$) shows that $H_*(Y;\mathbb{Z}[J^{-1}])$ is in fact finite free over $\mathbb{Z}[J^{-1}]$ in each degree, and hence so is $H^*(Y;\mathbb{Z}[J^{-1}])$. In particular,

$$H^*(Y;\mathbb{Z}[J^{-1}])\otimes_{\mathbb{Z}[J^{-1}]}\mathbb{Z}_p\cong H^*(Y;\mathbb{Z}_p)\cong H^*(B_p;\mathbb{Z}_p).$$

Lemma 3.2 now shows that $H^*(Y;\mathbb{Z}[J^{-1}])$ is a polynomial $\mathbb{Z}[J^{-1}]$-algebra as wanted. Finally, let R be a commutative ring with $R=R[J^{-1}]$. Since $H_*(Y;\mathbb{Z}[J^{-1}])$ is finite free over $\mathbb{Z}[J^{-1}]$ in each degree, the universal coefficient theorem again implies that $H^*(Y;R)\cong H^*(Y;\mathbb{Z}[J^{-1}])\otimes_{\mathbb{Z}[J^{-1}]}R$, proving the last claim. □

REMARK 3.3. The space Y in Proposition 3.1 is rarely unique, even up to $\mathbb{Z}[J^{-1}]$-localization, due to a multitude of factors: First, given a multiset of degrees and a prime p, there may be several choices of p-compact groups with the same degrees; this is a finite, essentially combinatorial, problem (see [**6**, Theorem 1.5] and [**33**, Corollary 1.8]).

Secondly, the classifying space BX of a connected p-compact group may lift to many $\mathbb{Z}_{(p)}$-homotopy types of spaces of finite type over $\mathbb{Z}_{(p)}$, with examples at least going back to Bousfield and Mislin [**34**, Appendix]. This uses a variant of Wilkerson's double coset formula [**49**, Proof of Theorem 3.8] resulting from the pull-back $K\to(BX)_{\mathbb{Q}}\leftarrow BX$, for K a rational space with the same rational degrees as BX. In fact, a more careful analysis along the same lines reveals that BX has a unique lifting as above if and only if X is a p-compact torus or X is a product of rank one p-compact groups and the rational degrees $\{2d_1,\ldots,2d_r\}$ have the property that none of the degrees can be written as a non-negative integral linear combination of the others. Moreover, if the lift is non-unique, there are uncountably many lifts.

Thirdly, given a set of $\mathbb{Z}_{(p)}$-homotopy types Y_p as in the second step, one for each prime in $\mathcal{P}$ (that is, $p\notin J$), with the same rational degrees, Zabrodsky mixing associated to the pull-back

$K \to (\prod_{p\in\mathcal{P}} Y_p)_{\mathbb{Q}} \leftarrow \prod_{p\in\mathcal{P}} Y_p$ produces, in general, infinitely many $\mathbb{Z}[J^{-1}]$-homotopy types. Again, a more careful analysis with the corresponding double coset formula reveals that if $2 \leqslant |\mathcal{P}| < \infty$, then there are countably infinitely many lifts, unless $(Y_p)\hat{_p}$ satisfies the conditions of the second step for each $p \in \mathcal{P}$, in which case there is only one lift. If $\mathcal{P}$ is infinite then there are uncountably many lifts unless the rational degrees are $\{2, 2, \ldots, 2\}$; that is, the torus case. (See also, for example, [**29**, **31**, **34**, **36**, **49**] for related information.)

4. *p-compact groups with polynomial $\mathbb{F}_p$-cohomology concentrated in even degrees*

In this section, we provide the necessary results on p-compact groups, and use this, together with the results of the previous sections, to prove Theorem 1.2 and its corollaries. The key step is the following result, which in fact also holds without the even degree assumption [**6**, Theorem 1.4], though we do not know a proof in that generality, which does not go via the classification of 2-compact groups [**6**, Theorem 1.2].

PROPOSITION 4.1. *If X is a p-compact group such that $H^*(BX; \mathbb{F}_p)$ is a finitely generated polynomial algebra over $\mathbb{F}_p$ concentrated in even degrees, then we can write*

$$H^*(BX; \mathbb{F}_p) \cong H^*(BG; \mathbb{F}_p) \otimes H^*(BY; \mathbb{F}_p)$$

as $\mathbb{F}_p$-algebras (and in fact as algebras over the Steenrod algebra), for G a compact connected Lie group and Y a product of exotic p-compact groups.

First, we need a lemma.

LEMMA 4.2. *Let X be a p-compact group such that $H^*(BX; \mathbb{F}_p)$ is a finitely generated polynomial algebra over $\mathbb{F}_p$ concentrated in even degrees. Then $\mathcal{C}_X(\nu)$ is connected for any elementary abelian p-subgroup $\nu\colon BE \to BX$ of X.*

Proof. Let $\nu\colon BE \to BX$ be an elementary abelian p-subgroup of X. By [**24**, Proof of Theorem 8.1] we have $H^*(B\mathcal{C}_X(\nu); \mathbb{F}_p) \cong T_{E,\nu^*}(H^*(BX; \mathbb{F}_p))$, where T_{E,ν^*} denotes the component of Lannes' T-functor corresponding to ν^* [**28**, 2.5.2]. Since T_{E,ν^*} preserves objects concentrated in even degrees [**28**, Proposition 2.1.3] as well as finitely generated graded polynomial algebras [**26**, Theorem 1.2], we conclude that $H^*(B\mathcal{C}_X(\nu); \mathbb{F}_p)$ is also a finitely generated polynomial algebra over $\mathbb{F}_p$ concentrated in even degrees and, in particular, $\mathcal{C}_X(\nu)$ is connected by Proposition 2.2. □

Proof of Proposition 4.1. By Proposition 2.2, X is connected. The classification of $\mathbb{Z}_p$-root data [**6**, Theorem 8.1] says that the $\mathbb{Z}_p$-root datum $\mathbf{D}_X$ of X can be written as a direct product $\mathbf{D}_X \cong (\mathbf{D}_G \otimes \mathbb{Z}_p) \times \mathbf{D}_1 \times \ldots \times \mathbf{D}_n$, where G is a compact connected Lie group and the $\mathbf{D}_i$ are exotic $\mathbb{Z}_p$-root data. By the product decomposition theorem [**25**, Theorem 1.4] we have an associated decomposition $BX \simeq BX' \times BX_1 \times \ldots \times BX_n$, where X' is a connected p-compact group with $\mathbf{D}_{X'} \cong \mathbf{D}_G \otimes \mathbb{Z}_p$ and the X_i are exotic p-compact groups with $\mathbf{D}_{X_i} \cong \mathbf{D}_i$.

We finish the proof by showing that $H^*(BX'; \mathbb{F}_p) \cong H^*(BG; \mathbb{F}_p)$ as $\mathbb{F}_p$-algebras. Lemma 4.2 shows that $\mathcal{C}_{X'}(\nu)$ is connected for every elementary abelian p-subgroup $\nu\colon BE \to BX'$ of X'. If ν is *toral*, that is, if ν factors through the maximal torus $BT \to BX'$ of X', a result of Dwyer and Wilkerson [**24**, Theorem 7.6] (see also [**6**, Proposition 8.4(3)]) shows that $\pi_0(\mathcal{C}_{X'}(\nu))$ can be computed from the root datum $\mathbf{D}_{X'}$. Since $\mathbf{D}_{X'} \cong \mathbf{D}_{G\hat{_p}}$, it follows that $\mathcal{C}_{G\hat{_p}}(\nu)$ is also connected for every toral elementary abelian p-subgroup ν of $G\hat{_p}$. If E is an elementary abelian subgroup of G and $i\colon E \to G$ is the inclusion, we have $C_G(E)\hat{_p} \cong \mathcal{C}_{G\hat{_p}}(Bi)$, so $C_G(E)$ is connected for every elementary abelian p-subgroup E of G that is contained in a maximal torus. By results of Steinberg [41, Theorems 2.27 and 2.28] and Borel [13, Theorem B], it follows that G does not have p-torsion, or equivalently that $H^*(BG;\mathbb{F}_p)$ is a polynomial algebra concentrated in even degrees. But, by [7, Theorem 12.1], this implies that we have isomorphisms

$$H^*(BX';\mathbb{Z}_p) \xrightarrow{\cong} H^*(BT;\mathbb{Z}_p)^W \xleftarrow{\cong} H^*(BG;\mathbb{Z}_p),$$

since the Weyl group W and its action on T is determined by $\mathbf{D}_{X'} \cong \mathbf{D}_{G\hat{_p}}$. In particular, $H^*(BX';\mathbb{F}_p) \cong H^*(BG;\mathbb{F}_p)$ as algebras over the Steenrod algebra, as required. □

The following proposition, which determines the compact Lie groups whose classifying space has polynomial cohomology, goes back to Borel [13, Theorem 2.5].

PROPOSITION 4.3. *A finitely generated graded polynomial $\mathbb{F}_p$-algebra A concentrated in even degrees is isomorphic to $H^*(BG;\mathbb{F}_p)$ for some compact connected Lie group G if and only if the type of A is a union of multisets of degrees occurring in Table 1.*

Proof. In [13, Theorem 2.5] Borel determined, for all simple simply connected compact Lie groups G, the prime numbers p such that $H^*(BG;\mathbb{F}_p)$ is a polynomial algebra concentrated in even degrees. This is the data in Table 1. ($G = \mathrm{Spin}(2n+1)$, $p \geqslant 3$, is omitted since $H^*(B\,\mathrm{Spin}(2n+1);\mathbb{F}_p) \cong H^*(B\,\mathrm{Sp}(n);\mathbb{F}_p)$ as $\mathbb{F}_p$-algebras.)

Assume now that G is an arbitrary compact connected Lie group such that $H^*(BG;\mathbb{F}_p)$ is a polynomial $\mathbb{F}_p$-algebra concentrated in even degrees. There exists a fibration $BG' \to BG \to BS$, where G' denotes the commutator subgroup of G and S is a torus. Since $\pi_2(BG) \to \pi_2(BS)$ is surjective by the long exact sequence in homotopy, $H^2(BS;\mathbb{F}_p) \to H^2(BG;\mathbb{F}_p)$ is injective. Because $H^*(BS;\mathbb{F}_p)$ is generated by elements in degree 2 and $H^*(BG;\mathbb{F}_p)$ is polynomial, the homomorphism $H^*(BS;\mathbb{F}_p) \to H^*(BG;\mathbb{F}_p)$ is injective and $H^*(BG;\mathbb{F}_p)$ is free over $H^*(BS;\mathbb{F}_p)$. Hence the Eilenberg–Moore spectral sequence for the fibration collapses and we get an isomorphism $H^*(BG;\mathbb{F}_p) \cong H^*(BS;\mathbb{F}_p) \otimes H^*(BG';\mathbb{F}_p)$ of $\mathbb{F}_p$-algebras. This shows that $H^*(BG';\mathbb{F}_p)$ is a polynomial algebra concentrated in even degrees as well, so we can assume that G has finite fundamental group. But, since $H^3(BG;\mathbb{F}_p) = 0$, $\pi_1(G)$ cannot contain p-torsion, so we can assume that G is simply connected. Hence G splits as a product of simple simply connected compact Lie groups, and in particular the type of $H^*(BG;\mathbb{F}_p)$ is a union of multisets of degrees from Table 1. □

The next proposition covers the exotic case.

PROPOSITION 4.4. *A finitely generated graded polynomial $\mathbb{F}_p$-algebra A concentrated in even degrees is isomorphic to $H^*(BY;\mathbb{F}_p)$ for Y a product of exotic p-compact groups if and only if the type of A is a union of multisets of degrees occurring in Tables 2 and 3.*

Proof. The degrees of every exotic $\mathbb{Q}_p$-reflection group, p odd, is realized as the type of a polynomial $\mathbb{F}_p$-cohomology ring. This is the culmination of the work of a number of people: Sullivan [43, p. 166–167], Clark and Ewing [17], Quillen [35, § 10], Zabrodsky [51, 4.3], Aguadé [4] and Notbohm and Oliver [32]. See also [33] and [7, § 7] for a unified treatment. In particular, the multisets of degrees in Tables 2 and 3 are all realized.

Conversely, suppose that X is an exotic p-compact group, with $H^*(BX;\mathbb{F}_p)$ a polynomial $\mathbb{F}_p$-algebra concentrated in even degrees. As used before, this implies that $H^*(BX;\mathbb{Z}_p)$ is also polynomial and $H^*(BX;\mathbb{Z}_p) \cong H^*(BT;\mathbb{Z}_p)^W$. In particular, the type of $H^*(BX;\mathbb{F}_p)$ agrees with the type of $H^*(BT;\mathbb{Z}_p)^W \otimes \mathbb{Q}$, that is, with the degrees of the $\mathbb{Q}_p$-reflection group W, which is exotic by assumption. For $p = 2$, the classification of finite $\mathbb{Q}_2$-reflection groups says that there is only one exotic $\mathbb{Q}_2$-reflection group, namely G_{24}. By the classification of $\mathbb{Z}_p$-root data [**6**, Theorem 8.1], there is a unique $\mathbb{Z}_2$-root datum associated to G_{24}, namely $\mathbf{D}_X = \mathbf{D}_{\mathrm{DI}(4)}$, the root datum of the exotic 2-compact group DI(4) constructed by Dwyer and Wilkerson [**22**]. Using the formula for the component group of a centralizer [**24**, Theorem 7.6] (see also [**6**, Proposition 8.4(3)]) one sees that $\mathcal{C}_X({}_2\breve{T})$ is not connected, where ${}_2\breve{T}$ denotes the maximal elementary abelian 2-subgroup of the discrete torus $\breve{T}$. This contradicts Lemma 4.2, so the classifying space of a 2-compact group with this root datum cannot have polynomial $\mathbb{F}_2$-cohomology ring concentrated in even degrees, and we must have p odd. (Alternatively, one can see directly that the polynomial algebra $\mathbb{F}_2[x_8, x_{12}, x_{28}]$ cannot be an unstable algebra over the mod 2 Steenrod algebra $\mathcal{A}_2$; for degree reasons the ideal generated by x_8 and x_{12} is closed under $\mathcal{A}_2$, so the quotient $\mathbb{F}_2[x_{28}]$ would have to be an unstable algebra over $\mathcal{A}_2$. However, this is not possible since 28 is not a power of 2.)

It only remains to be shown that for p odd, the degrees of any exotic $\mathbb{Q}_p$-reflection group that does not occur in Tables 2 and 3 can be written as a multiset union of degrees of those that do; the cases G_4, G_5, G_6, G_7, G_{10}, G_{11}, G_{13}, G_{15}, G_{18}, G_{19}, G_{25}, G_{26}, and G_{27} are left out of Table 3 since these are covered by the groups $G(6,3,2)$, $G(6,1,2)$, $C_4 \times C_{12}$, $C_{12} \times C_{12}$, $G(12,1,2)$, $C_{24} \times C_{24}$, G_8, $G(12,1,2)$, $G(30,1,2)$, $C_{60} \times C_{60}$, $G(6,2,3)$, $G(6,1,3)$, and $C_6 \times G_{20}$, respectively. □

Proof of Theorem 1.2. Suppose that we are in the setup of the theorem, with a fixed commutative Noetherian ring R of finite Krull dimension. If $H^*(Y;R)$ is a polynomial R-algebra, then Proposition 2.1 shows that for $p \in \mathcal{P}$, $H^*(Y;\mathbb{F}_p)$ is a polynomial $\mathbb{F}_p$-algebra with the same type and $Y\hat{_p}$ is the classifying space of a p-compact group. Hence by Propositions 4.1, 4.3, and 4.4, the type of $H^*(Y;R)$ satisfies the restrictions of Theorem 1.2, as required.

Conversely, suppose that we are given a multiset of degrees, which satisfies the degree restrictions for all $p \in \mathcal{P}$. Then for each $p \in \mathcal{P}$ there exists, by Propositions 4.3 and 4.4, a space B_p such that $H^*(B_p;\mathbb{F}_p)$ is a polynomial $\mathbb{F}_p$-algebra with the given type. By Proposition 3.1, we can construct a space Y such that $H^*(Y;R)$ is a polynomial R-algebra with the given type. □

Proof of Corollary 1.3. By inspection, a given multiset of degrees occurs at most finitely many times in Tables 1–3. It follows that a given finite multiset L can only be decomposed in finitely many ways as a union of multisets of degrees from Tables 1–3 (ignoring for now restrictions on primes p). We next have to determine the prime numbers p for which each of these decompositions can occur.

For short, say that a set of integers is *listable* if it can be written as a union of congruence classes modulo some number N. Finite intersections and finite unions of listable sets are again listable and there is an obvious algorithm for computing them as listable sets. Notice that each entry in the tables occur for a prime number p if and only if p belongs to some listable set. (This follows directly from the tables, together with the observation that for any integer k we have $p \geqslant k$ if and only if p belongs to a union of congruence classes modulo $N = \prod_{l<k,\, l \text{ prime}} l$; for example, $p \geqslant 5$ if and only if $p \equiv \pm 1 \bmod 6$.) A given decomposition of L (as a union of multisets of degrees corresponding to entries in the tables), can be used at a prime p if and only if each of the entries occur at p, and by the above this is equivalent to p belonging to an explicit listable set depending only on the decomposition. Since there are only finitely many decompositions of L as above, we conclude that the set of primes p for which there exists some decomposition of L at p is a listable set. The result now follows from Theorem 1.2. □

REMARK 4.5. A different line of argument for Theorem 1.2 in the case $R = \mathbb{F}_2$ is as follows. As before, using product splitting for p-compact groups, one can reduce to the case where

$H^*(Y;\mathbb{F}_2) \cong H^*(BX;\mathbb{F}_2)$ for X a simply connected simple 2-compact group, and the type of $H^*(BX;\mathbb{F}_2)$ agrees with the degrees of W_X. We hence have to rule out the degrees of $W(D_n)$ ($n \geqslant 4$), $W(E_6)$, $W(E_7)$, $W(E_8)$, $W(F_4)$, $W(G_2)$, and $W(\mathrm{DI}(4))$. All but the first family can be eliminated directly, even as algebras over the Steenrod algebra, by [**45**, Theorem 1.4], and the same can be seen to hold for $W(D_n)$, $n \geqslant 5$, n odd, using [**42**]. The case $W(D_n)$, $n \geqslant 4$, n even, can actually exist as an algebra over the Steenrod algebra (when n is a power of 2) but cannot occur as the cohomology ring of a space by the following argument. Consider $X' = \mathcal{C}_X({}_2\breve{T})$. By [**24**, Theorem 7.6], $W_{X'}$ is the subgroup of elements in W, which act trivially on L modulo 2, and hence $W_{X'}$ is a normal 2-subgroup of W (cf. [**7**, Lemma 11.3]). By assumption and Lemma 4.2, X' is connected, so $W_{X'}$ is generated by reflections. Since all reflections in W are conjugate and $-1 \in W$, we conclude $W_{X'} = W$. In particular, W is a 2-group, a contradiction.

Remark 4.6. The spaces in Theorems 1.1 and 1.2 can be realized as cohomology rings of discrete groups by a theorem of Kan and Thurston [**27**], but the group can rarely be taken to be finite: If $H^*(BG;R)$ is a finitely generated polynomial algebra, for G a finite group and R any commutative ring, then all polynomial generators are in degree 1. This follows from a result of Benson and Carlson [**12**, Corollary 6.6], but one can also argue as follows: If all generators are in even degrees, $H^*(BG;\mathbb{F}_p)$ is a polynomial ring with the same type as $H^*(BG;R)$ for the primes p that are not units in R. Hence $BG\hat{_p}$ is a connected p-compact group, and in particular every element of p-power order in G is divisible by p (see [**23**, Proposition 5.6] and [**30**, (4)]). Therefore the order of G is prime to p, and $H^*(BG;R) = R$. If there are generators in odd degrees, then R has to be an $\mathbb{F}_2$-algebra and a small modification of the argument shows that all generators have to be in degree one (see the proof of Proposition 2.2).

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