

# STRING TOPOLOGY OF FINITE GROUPS OF LIE TYPE

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ABSTRACT. We show that the mod  $\ell$  cohomology of any finite group of Lie type in characteristic  $p \neq \ell$  admits the structure of a module over the mod  $\ell$  cohomology of the free loop space of the classifying space  $BG$  of the corresponding compact Lie group  $G$ , via ring and module structures constructed from string topology, à la Chas–Sullivan. If a certain fundamental class in the homology of the finite group of Lie type is non-trivial, then this module structure becomes free of rank one, and provides, among other things, an isomorphism between the two cohomology rings equipped with the cup product, up to a filtration.

We verify the non-triviality of the fundamental class in a range of cases, including all split groups over  $\mathbb{F}_q$  corresponding to simply connected classical groups, as long as  $q$  is congruent to 1 mod  $\ell$ , and in fact without any congruence condition, if one replaces  $BG$  by a certain  $\ell$ -compact subgroup depending on the order of  $q$  mod  $\ell$ . With this modification, we know of no examples where the fundamental class is trivial, raising the possibility of a general structural answer to an open question of Tezuka, who speculated about the isomorphism of the two cohomology rings with cup product.

## 1. INTRODUCTION

The mod  $\ell$  cohomology ring of a finite group of Lie type  $\mathbf{G}(\mathbb{F}_q)$  over a finite field  $\mathbb{F}_q$  of characteristic  $p \neq \ell$  has been known since Quillen [Qui71a, §2] for large primes  $\ell$ , where it is a polynomial algebra tensor an exterior algebra. When furthermore  $q \equiv 1 \pmod{\ell}$ , the polynomial generators sit in degrees  $2d_i$  and the exterior generators in degrees  $2d_i - 1$ , with  $d_i$ 's the fundamental degrees of the root system corresponding to the reductive group scheme  $\mathbf{G}$  over  $\mathbb{Z}$  [Hum90, §3.7, Table 1]. For those  $\ell$ , the ring can, perhaps surprisingly, be observed to agree with that of  $LB\mathbf{G}(\mathbb{C}) = \text{map}(S^1, B\mathbf{G}(\mathbb{C}))$ , the free loop space on the classifying space of the complex points  $\mathbf{G}(\mathbb{C})$  of  $\mathbf{G}$ , with its standard topology, since the canonical spectral sequences in each case collapse at  $E_2$  and are seen to be abstractly isomorphic.

When  $\ell$  is small, more specifically when  $\ell$  is a torsion prime for  $\mathbf{G}$ , both  $H^*(B\mathbf{G}(\mathbb{F}_q); \mathbb{F}_\ell)$  and  $H^*(LB\mathbf{G}(\mathbb{C}); \mathbb{F}_\ell)$  become very hard to compute, and are in general unknown, but also extremely interesting. Isolated calculations of both rings have revealed that they sometimes agree, even in the presence of  $\ell$ -torsion in  $\mathbf{G}$ , see [Qui72], [Qui71b], [FP78], [Kle82], [MT91], [KK93]. Indeed Tezuka asked in 1998, in an unpublished note [Tez98], if they *always* agree, as long as  $q \equiv 1 \pmod{\ell}$  (or  $1 \pmod{4}$  in the case  $\ell = 2$ ). Further calculations supporting this “Tezuka conjecture” have been worked out by in [KMT00], [KK10], [KMN06], [KTY12], [Kam15], yet without any structural explanation why it might be true. The underlying spaces are certainly not homotopy equivalent, even at the prime  $\ell$ , as the most basic cases show: For  $\mathbf{T}$  a one-dimensional torus,  $B\mathbf{T}(\mathbb{F}_q) \simeq B\mathbb{Z}/(q-1)$  is a rationally trivial space depending heavily on  $q$ , whereas  $LB\mathbf{T}(\mathbb{C}) \simeq S^1 \times \mathbb{C}P^\infty$  is rationally non-trivial and independent of  $q$ .

The goal of this paper is to use string topology to produce a general structural relationship between  $H^*(LB\mathbf{G}(\mathbb{C}); \mathbb{F}_\ell)$  and  $H^*(B\mathbf{G}(\mathbb{F}_q); \mathbb{F}_\ell)$ : we show that when the former cohomology groups are equipped with a Chas–Sullivan-type string product, the latter cohomology groups become a module over the former. The non-vanishing of a single fundamental class, defined in an elementary fashion, implies that the module structure is free of rank 1, which in turn suffices to guarantee

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that the two cohomology rings (with respect to the cup product) are isomorphic up to a filtration. We prove the non-vanishing of the fundamental class in many cases. As long as  $q$  satisfies the aforementioned congruence condition (or alternatively allowing a modification of  $\mathbf{G}$  described below), we know of no case where the fundamental class does vanish, raising the question whether it is always non-zero.

Our structural theorems are most naturally formulated in the generality of connected  $\ell$ -compact groups, which we briefly recall (see e.g., [Gro10] for a survey). A connected  $\ell$ -compact group is a pointed simply connected space  $BG$  which is local with respect to homology with coefficients in  $\mathbb{F}_\ell$  [Bou75], and such that its based loop space  $G = \Omega BG$  has finite mod  $\ell$  cohomology; since any space can be  $\mathbb{F}_\ell$ -localized and this does not change the mod  $\ell$  homology, the locality condition is harmless for many purposes, when working at a prime  $\ell$ . We write  $d$  for the degree of the top non-trivial cohomology group of  $H^*(G; \mathbb{F}_\ell)$  and call it the dimension of the  $\ell$ -compact group. Connected  $\ell$ -compact groups and their automorphisms have been classified: they are in 1-1-correspondence with root data  $\mathbb{D}$  over the  $\ell$ -adic integers  $\mathbb{Z}_\ell$ , and  $\text{Out}(BG) \cong \text{Out}(\mathbb{D}_G)$  [AG09, Thm. 1.2]; in other words, the classification is wholly analogous to that of compact connected Lie groups, but with  $\mathbb{Z}$ -root data replaced with  $\mathbb{Z}_\ell$ -root data. Any compact connected Lie group  $K$  has an associated  $\ell$ -compact group, obtained as the  $\mathbb{F}_\ell$ -homology localization  $BK\hat{\ell}$ , a process which on the level of root data corresponds to tensoring with the  $\ell$ -adic integers  $\mathbb{Z}_\ell$ . (On a technical note, we remark that  $\mathbb{F}_\ell$ -homology localization here agrees with Bousfield–Kan  $\ell$ -completion; see e.g., [DW94, §11].) For  $\mathbf{G}$  a connected reductive algebraic group, and  $K$  a maximal compact subgroup of the complex algebraic group  $\mathbf{G}(\mathbb{C})$ , we have a homotopy equivalence  $BK \xrightarrow{\simeq} B\mathbf{G}(\mathbb{C})$  (see e.g., [AGMV08, §8.1]), so  $B\mathbf{G}(\mathbb{C})\hat{\ell}$  is a connected  $\ell$ -compact group as well.

Fundamental to our construction of the module structure on  $H^*(B\mathbf{G}(\mathbb{F}_q); \mathbb{F}_\ell)$  is that up to homotopy equivalence, the space  $B\mathbf{G}(\mathbb{F}_q)\hat{\ell}$  may be realized as a space of paths in the  $\ell$ -compact group  $BG = B\mathbf{G}(\mathbb{C})\hat{\ell}$ , as we will now explain. First, let us recall the definition of a general finite group of Lie type: Let  $\mathbf{G}$  be a connected reductive algebraic group scheme over  $\mathbb{Z}$  with  $\bar{\mathbb{F}}_p$ -rational points  $\mathbf{G}(\bar{\mathbb{F}}_p)$ , and let  $\sigma$  be a Steinberg endomorphism, i.e., an endomorphism of  $\mathbf{G}(\bar{\mathbb{F}}_p)$ , as algebraic group over  $\bar{\mathbb{F}}_p$ , which raised to some power becomes a standard Frobenius map  $\psi^q: \mathbf{G}(\bar{\mathbb{F}}_p) \rightarrow \mathbf{G}(\bar{\mathbb{F}}_p)$  induced by the  $q$ -th power map on  $\bar{\mathbb{F}}_p$ . A (necessarily finite) group which arises as the fixed-points  $\mathbf{G}(\bar{\mathbb{F}}_p)^\sigma$  is called a finite group of Lie type. A major example is of course the “untwisted case” where  $\sigma = \psi^q$  and  $\mathbf{G}(\bar{\mathbb{F}}_p)^{\psi^q} = \mathbf{G}(\mathbb{F}_q)$ . Classical groups  $\text{GL}_n(\mathbb{F}_q)$ ,  $\text{Sp}_n(\mathbb{F}_q)$ , etc. arise this way; see e.g., [MT11, Sec. 22.1] for more information.

Now by a theorem of Friedlander–Mislin [FM84, Thm. 1.4] (generalizing work of Quillen [Qui72]), there is a homotopy equivalence

$$BG = B\mathbf{G}(\mathbb{C})\hat{\ell} \leftarrow \simeq (B\mathbf{G}(\bar{\mathbb{F}}_p))\hat{\ell}. \quad (1.1)$$

for  $\ell \neq p$  relating characteristic  $p$  to characteristic 0, as long as we apply  $\mathbb{F}_\ell$ -localization. Furthermore, by another theorem of Quillen and Friedlander, we have homotopy equivalences

$$(B\mathbf{G}(\bar{\mathbb{F}}_p)^\sigma)\hat{\ell} \xrightarrow{\simeq} (B\mathbf{G}(\bar{\mathbb{F}}_p)\hat{\ell})^{h\sigma} \xrightarrow{\simeq} BG^{h\sigma} \quad (1.2)$$

for  $\sigma$  a Steinberg endomorphism, relating actual fixed-points to homotopy fixed-points, where we have continued to write  $\sigma$  for the self-equivalences of  $B\mathbf{G}(\bar{\mathbb{F}}_p)\hat{\ell}$  and  $BG$  induced by  $\sigma$ ; see [Fri76, Thm. 2.9], [Fri82, Thm. 12.2], and also [BMO12, Thm. 3.1]. In particular, in this picture the Frobenius map  $\psi^q$  of  $\mathbf{G}(\bar{\mathbb{F}}_p)$  gives rise to the “unstable Adams operation” self-equivalence  $\psi^q$  of  $BG$  which on the root datum corresponds to multiplication by  $q \in \mathbb{Z}_\ell^\times$ , a central element of  $\text{Out}(\mathbb{D}_G)$ . Here, and throughout, by the homotopy fixed points  $X^{h\sigma}$  of a self-map  $\sigma: X \rightarrow X$  we mean the space  $X^{h\sigma} = \{\alpha: I \rightarrow X \mid \sigma\alpha(1) = \alpha(0)\}$ , a subspace of the mapping space  $X^I$ . It also identifies with the homotopy pullback of the diagram  $X \xrightarrow{\Delta} X \times X \xleftarrow{(\sigma, 1)} X$  as well as the homotopy fixed-points  $X^{h\mathbb{N}_0}$  e.g., defined by the universal property of a left adjoint, letting

the monoid  $\mathbb{N}_0$  act on  $X$  via  $\sigma$ ; it also agrees with  $X^{h\mathbb{Z}}$  when  $\sigma$  is a homotopy equivalence. In particular the homotopy type of  $X^{h\sigma}$  only depends on the free homotopy class of  $\sigma$  (in fact often less, as we return to). See e.g., [BMO12, §4] and [BK72, Ch. XI§8] for details.

With the above dictionary in place, the rest of the paper will be formulated in the context of homotopy fixed-points on  $\ell$ -compact groups, which imply results about finite groups of Lie type in any characteristic  $p \neq \ell$  via (1.1) and (1.2). For general  $BG$  and  $\sigma$ , not coming from algebraic groups and Steinberg endomorphisms, the space  $BG^{h\sigma}$  is also quite interesting, and is often known to be an “exotic”  $\ell$ -local finite group, in the sense of [BLO03], though this so far builds on case-by-case considerations; see [LO02, Thm. 4.5] [BM07, Thm. A]. We emphasise that the group  $\text{Out}(BG) \cong \text{Out}(\mathbb{D}_G)$  can be described explicitly, just as in the Lie group case, with the main structural difference being that  $\mathbb{Z}_\ell$  has a lot more units than  $\mathbb{Z}$ : the calculation of  $\text{Out}(\mathbb{D})$  is reduced to  $\mathbb{D}$  simple simply connected in [AG09, §8.4] which is then tabulated in [AGMV08, Thm. 13.1] (see also [JMO92, JMO95] for earlier work).

Returning to string topology, Chas–Sullivan [CS99, CS04], and later authors, see e.g., [CG04, Tam10], observed that, for  $X$  a compact oriented manifold, both  $H^*(LX)$  and  $H_*(LX)$  carry additional “string” products and coproducts. These are constructed by reversing the direction of one of the two maps in the following diagram

$$LX \times LX = \text{map}(S^1 \amalg S^1, X) \leftarrow \text{map}(S^1 \vee S^1, X) \rightarrow \text{map}(S^1, X) = LX \quad (1.3)$$

via a transfer construction; here the first map is induced by joining the circles at the basepoint and the second map is induced by the pinch map. Furthermore, versions of these constructions, letting  $X$  be an orbifold, Borel construction, or classifying space have been considered by a number of authors [LUX08, BGNX07, BGNX12, GS08, GW08, CM12, HL15]. In particular, Chataur–Menichi showed in [CM12], that the shifted cohomology  $\mathbb{H}^*(LBG) = H^{*+d}(LBG; \mathbb{F}_\ell)$  can be endowed with a string topological product, which should be thought of as mixing the cup product on  $H^*(BG)$  with a dual of the Pontryagin product on  $H_*(G)$ , by choosing a transfer for the right-hand map in (1.3).

The group theoretic significance of these structures has however not been clear so far. What we prove in this paper is that the ring structure on  $\mathbb{H}^*(LBG)$  can be used to endow the cohomology  $H^*(BG^{h\sigma}; \mathbb{F}_\ell)$  with a module structure over this ring, which via (1.1) and (1.2) provides the cohomology groups of finite groups of Lie type with a large amount of new structure, allowing for new computations. As a part of our work, we give a new construction of an inherently unital and associative string product on  $\mathbb{H}^*(LBG)$  needed for the construction of the product structures on the level of spectral sequences in Theorem A below; see Definition 2.2 and Remark 2.3. It should agree with the product constructed in [CM12] (in the sign-corrected form of [KM16, Sec. 7]), although our work is logically independent of [CM12] and we will not verify the relationship here. We can now state our main result on the module structure precisely:

**Theorem A.** *Let  $\ell$  be a prime, and suppose  $BG$  is a connected  $\ell$ -compact group of dimension  $d$  and  $\sigma: BG \rightarrow BG$  a self-map. Set  $H^*(-) = H^*(-; \mathbb{F}_\ell)$ ,  $\mathbb{H}^* = H^{*+d}$ , and  $\mathbb{E}_r^{*,*} = E_r^{*,*+d}$ .*

- (i) *The cohomology groups  $H^*(BG^{h\sigma})$  admit a module structure over  $\mathbb{H}^*(LBG)$  equipped with the string product, via Definition 2.2.*
- (ii) *The Serre spectral sequence of the evaluation fibration  $LBG \rightarrow BG$ ,  $\omega \mapsto \omega(1)$ , induces a strongly convergent spectral sequence of algebras*

$$\mathbb{E}_2^{*,*}(LBG) \cong H^*(BG) \otimes \mathbb{H}^*(G) \implies \mathbb{H}^*(LBG).$$

*Here  $H^*(BG)$  is equipped with the cup product and the product on  $\mathbb{H}^*(G)$  is a dual of the Pontryagin product on  $H_*(G)$  (see Theorem 2.5).*

*Consider analogously the Serre spectral sequence*

$$\mathbb{E}_2^{*,*}(BG^{h\sigma}) \cong H^*(BG) \otimes \mathbb{H}^*(G) \implies \mathbb{H}^*(BG^{h\sigma})$$

of the fibration  $BG^{h\sigma} \rightarrow BG$ ,  $\alpha \mapsto \alpha(1)$ , with fibre homotopy equivalent to  $G$ .

(iii) The spectral sequence  $\{\mathbb{E}_r^{*,*}(BG^{h\sigma})\}$  is a module spectral sequence over  $\{\mathbb{E}_r^{*,*}(LBG)\}$ , and converges to  $\mathbb{H}^*(BG^{h\sigma})$  as a module over  $\mathbb{H}^*(LBG)$ . On the  $E_2$ -page, the module structure is free of rank 1 on a generator of  $\mathbb{E}_2^{0,0} \cong E_2^{0,d} \cong H^d(G) \cong \mathbb{F}_\ell$ .

The construction of the module structure is given in Section 2, and in overview form in Remark 2.3; the properties of the spectral sequence are proven in Section 3; and everything is put together to give Theorem A in the beginning of Section 4. More general coefficient rings than  $\mathbb{F}_\ell$  are possible; see Remark 4.19.

In the case where  $H^*(BG)$  is a polynomial ring, and  $\sigma$  acts as the identity on  $H^*(BG)$ , the spectral sequences collapse at the  $E_2$ -pages, providing a structured isomorphism between the  $E_\infty$ -pages, and hence a structured isomorphism between  $H^*(LBG)$  and  $H^*(BG^{h\sigma})$ , see Theorem D below and Theorem 4.3. In general, the spectral sequences of Theorem A(ii),(iii) are unknown, and have differentials—e.g., our product on  $\mathbb{H}^*(LBG)$  is expected to always be commutative, see Remark 2.4, whereas the Pontryagin product on  $H_*(G)$  is usually non-commutative in the presence of torsion, see Remark 2.6, forcing nontrivial differentials to appear in (ii). Nevertheless, we show that the question whether the  $E_\infty$ -pages are isomorphic is equivalent to a single class being nonzero. To state this precisely, consider the homotopy fibre sequence

$$G \xrightarrow{i} BG^{h\sigma} \longrightarrow BG \quad (1.4)$$

associated to the evaluation fibration  $BG^{h\sigma} \rightarrow BG$ ,  $\alpha \mapsto \alpha(1)$ .

**Definition 1.1.** For  $\sigma: BG \rightarrow BG$  a self-map, we say that  $BG^{h\sigma}$  has a  $[G]$ -fundamental class if the map  $i_*: H_d(G) \rightarrow H_d(BG^{h\sigma})$  is non-trivial, where  $d$  is the top non-trivial degree of  $H_*(G)$ .

In other words,  $BG^{h\sigma}$  is said to have a  $[G]$ -fundamental class, if a generator  $[G] \in H_d(G) \cong \mathbb{F}_\ell$  survives under  $i_*$ , or, formulated dually, if  $i^*: H^d(BG^{h\sigma}) \rightarrow H^d(G)$  is non-trivial.

**Theorem B.** *With setup as in Theorem A, the module  $H^*(BG^{h\sigma})$  is free of rank 1 over  $\mathbb{H}^*(LBG)$  if and only if  $BG^{h\sigma}$  has a  $[G]$ -fundamental class. An element  $x \in H^*(BG^{h\sigma})$  generates  $H^*(BG^{h\sigma})$  as a free rank 1 module over  $\mathbb{H}^*(LBG)$  if and only if  $x$  has degree  $d$  and  $i^*(x) \neq 0 \in H^d(G)$ .*

The last condition can alternatively be phrased as saying that  $x \in H^d(BG^{h\sigma})$  is a generator for the  $\mathbb{H}^*(LBG)$ -module structure if it evaluates non-trivially against  $i_*[G] \in H_d(BG^{h\sigma})$  for a generator  $[G] \in H_d(G) \cong \mathbb{F}_\ell$ . Note also that a  $[G]$ -fundamental class depends on the map  $BG^{h\sigma} \rightarrow BG$ , not just the space  $BG^{h\sigma}$ . We will return to this flexibility later on.

The existence of a fundamental class implies a strong link between the ring and  $H^*(BG)$ -module structures on  $H^*(LBG)$  and  $H^*(BG^{h\sigma})$ .

**Theorem C.** *With the setup of Theorem A, suppose that  $BG^{h\sigma}$  has a  $[G]$ -fundamental class (Definition 1.1), so that by Theorem B we may find an  $x \in H^d(BG^{h\sigma})$  such that the map*

$$- \circ x: H^*(LBG) \xrightarrow{\cong} H^*(BG^{h\sigma}) \quad (1.5)$$

given by module multiplication with  $x$  is an isomorphism of graded vector spaces. Then the following holds.

- (i) The isomorphism (1.5) is an isomorphism of  $H^*(BG)$ -modules, where the source and target are given the  $H^*(BG)$ -module structures induced by the fibrations  $LBG \rightarrow BG$  and  $BG^{h\sigma} \rightarrow BG$ , respectively. In particular, the induced map  $H^*(BG) \rightarrow H^*(BG^{h\sigma})$  is injective.
- (ii) If  $1 \circ x = 1$  (as may be arranged by replacing  $x$  by a multiple if necessary), the isomorphism (1.5) induces an algebra isomorphism

$$\text{gr } H^*(LBG) \xrightarrow{\cong} \text{gr } H^*(BG^{h\sigma})$$

between the associated graded algebras of  $H^*(LBG)$  and  $H^*(BG^{h\sigma})$  corresponding to the Serre spectral sequences of the fibrations  $LBG \rightarrow BG$  and  $BG^{h\sigma} \rightarrow BG$ , respectively.

Notice in particular that (i) implies that  $\sigma$  necessarily has to act as the identity on  $H^*(BG)$  for a  $[G]$ -fundamental class to exist, and thus  $\sigma$  will have to be a self-equivalence. Also note that without further assumptions, (ii) cannot be improved to an algebra isomorphism of the underlying algebras, again by the example of a torus: For  $BG = (BS^1)_2$  and  $\sigma = \cdot 3$ , the map  $i: G \rightarrow BG^{h\sigma}$  identifies with the map  $(S^1)_2 \rightarrow B\mathbb{Z}/2$  arising from the fibration  $B\mathbb{Z}/2 \rightarrow (BS^1)_2 \xrightarrow{2} (BS^1)_2$ , so  $BG^{h\sigma}$  has an  $[(S^1)_2]$ -fundamental class. However, the cohomology rings of  $L(BS^1)_2 \simeq (S^1 \times BS^1)_2$  and  $((BS^1)_2)^{h\sigma} \simeq B\mathbb{Z}/2$  are not isomorphic. It may be that this problem is limited to the case where  $\ell = 2$  and  $q \equiv 3 \pmod{4}$ , as suggested by Tezuka's question, though we do not know of a general result to that effect.

We now proceed to give examples of cases where a fundamental class exist, and for this we introduce some notation. Consider  $\sigma \in \text{Out}(BG) \cong \text{Out}(\mathbb{D})$  an " $\ell$ -compact Steinberg endomorphism", i.e.,  $\sigma = \tau\psi^q$  where  $\tau \in \text{Out}(\mathbb{D})$  is of finite order, and  $q \in \mathbb{Z}_\ell^\times$ . We write

$$B^\tau G(q) = BG^{h\tau\psi^q} \quad (1.6)$$

for short, further abbreviated to  $BG(q)$  when  $\tau = 1$ . The notation agrees with standard Lie theoretic notation by (1.2) when the latter makes sense, and is general enough to cover all classical Steinberg endomorphisms (also "very twisted" ones); again see [AGMV08, Thm. 13.1] for a tabulation of all possible  $\ell$ -compact group twistings. Let  $e$  be the order of  $q \pmod{\ell}$  and write  $\mu_e \leq \mathbb{Z}_\ell^\times$  for the group of  $e$ -th roots of unity. Inside  $\mathbb{Z}_\ell^\times$  we can hence write  $q = \zeta_e q'$  where  $\zeta_e$  is a primitive  $e$ -th root of unity, and  $q' \equiv 1 \pmod{\ell}$ . Then, at least when  $G$  is simply connected or  $BG$  has polynomial cohomology,

$$BG(q) \simeq (BG^{h\mu_e})(q'), \quad (1.7)$$

and the homotopy fixed points  $BG^{h\mu_e}$ , under the finite group  $\mu_e$  of order prime to  $\ell$ , will again be a connected  $\ell$ -compact group; see [BM07, Thm. E]. We say that  $BG(q)$  has a  $[G^{h\mu_e}]$ -fundamental class if  $BG(q) \simeq (BG^{h\mu_e})(q') \rightarrow BG^{h\mu_e}$  has a fundamental class in the sense of Definition 1.1. In particular,  $H^*BG(q)$  is then free of rank 1 as a module over  $\mathbb{H}^*LBG^{h\mu_e}$ . With the above terminology, we can state our main existence result.

**Theorem D.** *Let  $BG$  be a connected  $\ell$ -compact group with polynomial mod  $\ell$  cohomology or  $BG = B\text{Spin}(n)_2$ ,  $n \geq 10$ . Then  $BG(q)$  has a  $[G^{h\mu_e}]$ -fundamental class for every  $q \in \mathbb{Z}_\ell^\times$ . In particular, when  $G$  is simply connected,  $BG(q)$  has a  $[G^{h\mu_e}]$ -fundamental class except possibly in the following cases:  $\ell = 5$  and  $G$  contains an  $E_8$ -summand;  $\ell = 3$  and  $G$  contains an  $F_4$ ,  $E_6$ ,  $E_7$ , or  $E_8$ -summand; and  $\ell = 2$  and  $G$  contains an  $E_6$ ,  $E_7$ , or  $E_8$ -summand. Hence Theorems B and C provide structured isomorphisms between  $H^*(BG(q))$  and  $H^*(LBG^{h\mu_e})$  when  $G$  is simply connected away from those eight cases.*

Elaborating on the simply connected case stated above, it is known when  $BG$  is a polynomial ring, except for one loose end when  $\ell = 2$  and  $G$  not simple or simply connected:

- If  $\ell$  is odd,  $H^*(BG)$  is a polynomial ring iff  $\pi_1(G)$  is  $\ell$ -torsion free, and the universal cover of  $G$  does not contain a summand isomorphic to the  $\ell$ -completion of  $F_4$ ,  $E_6$ ,  $E_7$ , or  $E_8$  if  $\ell = 3$ , and  $E_8$  if  $\ell = 5$ . (See [AGMV08, Thms. 12.1, 12.2].)
- If  $\ell = 2$  and  $\pi_1(G)$  is 2-torsion free, then  $H^*(BG)$  is a polynomial ring iff the universal cover of  $G$  does not include 2-completions of  $\text{Spin}(n)$  for  $n \geq 10$ ,  $E_6$ ,  $E_7$ , or  $E_8$ . (See [AG09, Thm. 1.1 and 1.4].)
- If  $\ell = 2$  and  $G$  is simple, then  $H^*(BG)$  is a polynomial ring iff  $G$  is either  $\text{DI}(4)$  or the 2-completion of  $\text{SU}(n)$ ,  $\text{Sp}(n)$ ,  $\text{Spin}(n)$  ( $7 \leq n \leq 9$ ),  $G_2$ ,  $F_4$ ,  $\text{SO}(n)$  or  $\text{PSp}(2n+1)$ , the last two being non-simply connected. (See [AG09, Rem. 7.1].)

- If  $\ell = 2$  and  $G$  is not simple and  $\pi_1(G)$  contains 2-torsion, then there can also be new mixed examples with polynomial cohomology, e.g.,  $B(\mathrm{Sp}(1) \times \cdots \times \mathrm{Sp}(1))/\Delta$  has polynomial cohomology, where  $\Delta$  is a diagonally embedded copy of  $\mathbb{Z}/2$ , but a complete classification of such cases does not seem to be available (again see [AG09, Rem. 7.1] for more references).

Theorem D can be seen as generalizing a number of previous results in the literature where both  $H^*(BG(q))$  and  $H^*(LBG)$  were calculated to various degrees of precision and amount of structure, and subsequently observed to coincide. See e.g., [Kle82, Mil98, KMT00, Kam08, Grb06, KK10, KMN06, KTY12] for previous work on polynomial cases, and [Kam15] for the spin groups, where an isomorphism of graded abelian groups was established—it was noticing this last paper in an arXiv listing which originally alerted us to Tezuka’s question.

Even in the  $\ell$ -torsion cases for  $\ell = 2, 3, 5$  excluded in Theorem D, we still prove the following structural result about the set of classes  $[\sigma] \in \mathrm{Out}(BG)$  that admit a  $[G]$ -fundamental class.

**Theorem E.** *The subset of  $[\sigma] \in \mathrm{Out}(BG)$  for which  $BG^{h\sigma}$  has a  $[G]$ -fundamental class is an uncountable subgroup of  $\langle \sigma \in \mathrm{Out}(BG) \mid \sigma^* = 1 \in \mathrm{Aut}(H^*BG) \rangle$ . Moreover, this subgroup is closed in the topology induced on  $\mathrm{Out}(BG)$  from  $\mathrm{Out}(\mathbb{D}_G)$  when  $BG$  is semisimple.*

See also Corollary 4.6 for a more explicit statement about the elements  $\psi^q$ . We do not know examples where the subgroup in Theorem E is proper, and again this is true for the cases covered by Theorem D. We would hence like to end this introduction with some general questions.

**Question F** (Existence of a fundamental class). *Let  $BG$  be a connected  $\ell$ -compact group.*

- (1) *Does a  $[G]$ -fundamental class always exist for  $BG^{h\sigma}$  if  $\sigma^*$  acts as the identity on  $H^*(BG)$ ? (Notice that the condition on  $\sigma^*$  is necessary by Theorem E.)*
- (2) *Does a  $[G^{h\mu_e}]$ -fundamental class always exist for  $BG(q)$ , where  $e$  is the order of  $q \bmod \ell$ ? (A positive answer implies that  $\psi^q$  always induces the identity on  $H^*(BG)$  when  $q \equiv 1 \bmod \ell$ , and that every elementary abelian  $\ell$ -subgroup of  $\mathbf{G}(\mathbb{F}_q)$  is conjugate to a subgroup in  $\mathbf{G}(\mathbb{F}_q)$ , when  $BG = B\mathbf{G}(\mathbb{F}_q)\hat{\ell}$ , for  $\mathbf{G}$  a connected reductive algebraic group and  $q \equiv 1 \bmod \ell$ ; checking this in general would perhaps be the first step.)*

If some of these items should fail in full generality, it would still be interesting to identify the class of  $\ell$ -compact groups for which they hold—since  $\ell$ -compact groups and their automorphisms are classified, one should be able to produce a concrete list. The calculational literature often work with the explicit or implicit assumption that  $G$  is simply connected or simple, though it is not clear if this is necessary. (Indeed, Tezuka in his original question [Tez98] seems implicitly to be in the simply connected case.)

The second question which we would like to raise is to which extent the isomorphism (1.5) can be extended to respect the multiplicative and Steenrod algebra structure.

**Question G** (Multiplicative and Steenrod algebra structure).

- (1) *Can the class  $x$  in Theorem C(ii) always be chosen so as to produce an isomorphism of rings, or  $H^*(BG)$ -algebras, when  $\ell$  is odd? Is the same true for  $\ell = 2$ , if  $\sigma = \psi^q$  with  $q \equiv 1 \bmod 4$  (or if  $\sigma^*$  induces the identity on  $H^*(BG; \mathbb{Z}/4)$ )? And is there a unique and/or canonical such class, a “dual fundamental class”?*
- (2) *Can the class  $x$  always be chosen so as to produce an isomorphism over the subalgebra  $\mathcal{A}'$  of the Steenrod algebra of operations of even degrees (i.e., leaving out the Bockstein for  $\ell$  odd, and  $\mathrm{Sq}^{2n+1}$ ,  $n \geq 0$  for  $\ell = 2$ ). Over the whole Steenrod algebra for  $\sigma = \psi^q$  with  $q \equiv 1 \bmod \ell^2$  (or if  $\sigma^*$  induces the identity on  $H^*(BG; \mathbb{Z}/\ell^2)$ )?*
- (3) *Combining the previous points, can the class  $x$  be chosen so as to yield an isomorphisms of  $H^*(BG)$ -algebras, over  $\mathcal{A}'$ ? (See [Hen96] for more on this structure.)*

Note that for  $\ell$  big enough, a candidate for a good choice for the class  $x$  exists, namely the product of the exterior classes in cohomology; see also [Sol63, §3] [KK10, Prop. 7 and Thm. 8].

*Outline of the paper.* The string product and the module structure are constructed in section 2. In section 3 we set up the Serre spectral sequences and show how they interact with the product structures. Finally in section 4 we assemble the pieces and derive the theorems stated in the introduction. Our construction of the string structures requires a certain amount of parametrized homotopy theory which we present in Appendix A.

*Notation and conventions.* Throughout the paper,  $\ell$  will be a prime and  $BG$  will denote a fixed connected  $\ell$ -compact group. (The notion of an  $\ell$ -compact group was recalled earlier in the introduction; see also [Gro10].) We write  $G$  for the based loop space  $G = \Omega BG$ . Unless indicated otherwise, homology and cohomology is with  $\mathbb{F}_\ell$ -coefficients.

## 2. CONSTRUCTION OF THE PRODUCTS

Our aim in this section is to construct the string product on  $\mathbb{H}^*(LBG)$  and string module structure on  $\mathbb{H}^*(BG^{h\sigma})$ . Indeed, we will construct and study a more general pairing (Theorem 2.1 below) which will yield both of these structures as special cases. Subsection 2.1 will introduce this pairing along with many of its properties, and the remaining subsections are devoted to the actual construction of the pairing along with the verification of the asserted properties. For a quick indication of the basic idea behind the construction, the reader should see Remark 2.3.

**2.1. The pairing and its properties.** For a space  $B$  and continuous maps  $f, g: B \rightarrow BG$ , we write  $P(f, g) \rightarrow B$  for the pullback of the evaluation fibration

$$(\text{ev}_0, \text{ev}_1): BG^I \rightarrow BG \times BG, \quad \gamma \mapsto (\gamma(0), \gamma(1))$$

along the map  $(f, g): B \rightarrow BG$ . We call  $P(f, g)$  the *space of paths in  $BG$  from  $f$  to  $g$* . Explicitly,

$$P(f, g) = \{(b, \gamma) \in B \times BG^I \mid \gamma(0) = f(b), \gamma(1) = g(b)\},$$

with the map  $P(f, g) \rightarrow B$  given by projection onto the first coordinate. We may thus picture a point in  $P(f, g)$  as follows:

$$\left( b, \begin{array}{c} f(b) \quad \gamma \quad g(b) \\ \bullet \quad \curvearrowright \quad \bullet \end{array} \right).$$

We will obtain the string product and the module structure as special cases of the following result.

**Theorem 2.1.** *Let  $B$  be a space. For maps  $f, g, h: B \rightarrow BG$ , there is a pairing*

$$\circ: \mathbb{H}^*P(g, h) \otimes \mathbb{H}^*P(f, g) \longrightarrow \mathbb{H}^*P(f, h). \quad (2.1)$$

*This pairing satisfies the following properties:*

(i) *(Associativity): The equation*

$$(x \circ y) \circ z = x \circ (y \circ z)$$

*holds for all  $x, y$ , and  $z$  for which the pairings involved are defined.*

(ii) *(Existence of units): For every  $f: B \rightarrow BG$ , there exists an element  $\mathbb{1} = \mathbb{1}_f \in \mathbb{H}^*P(f, f)$  such that*

$$\mathbb{1}_f \circ x = x \quad \text{and} \quad y \circ \mathbb{1}_f = y$$

*for all  $x$  and  $y$  for which the pairings are defined.*

In particular, the pairing of Theorem 2.1 makes each  $\mathbb{H}^*P(f, f)$  into a graded  $\mathbb{F}_\ell$ -algebra. Several special cases of the path space construction are of particular interest:

- (1) For  $B = BG$  and  $f = g = \text{id}_{BG}$ , the projection  $(b, \gamma) \mapsto \gamma$  provides an identification  $P(\text{id}_{BG}, \text{id}_{BG}) = LBG$ , with the projection  $P(\text{id}_{BG}, \text{id}_{BG}) \rightarrow BG$  corresponding under this identification to the evaluation map  $LBG \rightarrow BG$ ,  $\gamma \mapsto \gamma(0) = \gamma(1)$ .

- (2) For  $B = BG$  and  $f = \sigma$ ,  $g = \text{id}_{BG}$ , the projection  $(b, \gamma) \mapsto \gamma$  provides an identification  $P(\sigma, \text{id}_{BG}) = BG^{h\sigma}$ , with the projection  $P(\sigma, \text{id}_{BG}) \rightarrow BG$  corresponding under this identification to the evaluation fibration  $\text{ev}_1: BG^{h\sigma} \rightarrow BG$ ,  $\gamma \mapsto \gamma(1)$ .
- (3) For  $B = \text{pt}$  and  $f = g: \text{pt} \rightarrow BG$  the inclusion of the basepoint, the projection  $(b, \gamma) \mapsto \gamma$  provides an identification  $P(f, f) = \Omega BG$ .

In the remainder of the paper, we will use the aforementioned identifications without further comment.

**Definition 2.2.** The *string product* on  $\mathbb{H}^*LBG$  is the multiplication on  $\mathbb{H}^*LBG$  obtained by taking  $f = g = h = \text{id}_{BG}$  in Theorem 2.1. The *string module structure* of  $\mathbb{H}^*(BG^{h\sigma})$  over  $\mathbb{H}^*LBG$  is the module structure obtained by taking  $f = \sigma$  and  $g = h = \text{id}_{BG}$ .

**Remark 2.3.** The reader may appreciate a quick summary of the origins the pairing (2.1) unnumbered by the technicalities of the actual constructions. Let  $B$ ,  $f$ ,  $g$ , and  $h$  be as in Theorem 2.1, and write  $P(f, g, h) \rightarrow B$  for the pullback of  $P(g, h) \times P(f, g) \rightarrow B \times B$  along the diagonal map  $\Delta: B \rightarrow B \times B$ . Explicitly,

$$P(f, g, h) = \{(b, \gamma_2, \gamma_1) \in B \times BG^I \times BG^I \mid \gamma_1(0) = f(b), \gamma_1(1) = \gamma_2(0) = g(b), \gamma_2(1) = h(b)\},$$

and we may picture a point in  $P(f, g, h)$  as follows:

$$\left( b, \begin{array}{c} f(b) \quad \gamma_1 \quad g(b) \quad \gamma_2 \quad h(b) \\ \curvearrowright \end{array} \right).$$

We have the commutative diagram

$$\begin{array}{ccc} P(g, h) \times P(f, g) & \xleftarrow{\text{split}} & P(f, g, h) & \xrightarrow{\text{concat}} & P(f, h) \\ \downarrow & & \searrow & & \swarrow \\ B \times B & \xleftarrow{\Delta} & & & B \end{array} \quad (2.2)$$

where the trapezoid on the left is a pullback square, with the map labeled ‘split’ given by

$$\text{split}: (b, \gamma_2, \gamma_1) \mapsto ((b, \gamma_2), (b, \gamma_1)),$$

and where the map labeled ‘concat’ is given by concatenation of paths:

$$\text{concat}: (b, \gamma_2, \gamma_1) \mapsto (b, \gamma_2 * \gamma_1).$$

The pairing (2.1) is given by a push–pull construction in the top of row of diagram (2.2): it is a degree shift of the composite

$$H^*P(g, h) \otimes H^*P(f, g) \xrightarrow{\times} H^*(P(g, h) \times P(f, g)) \xrightarrow{\text{split}^*} H^*P(f, g, h) \xrightarrow{\text{concat}_!} H^{*-d}P(f, h) \quad (2.3)$$

where  $\text{concat}_!$  is an ‘umkehr map’ whose existence is ensured by the fact that the fibres of  $P(f, g, h) \rightarrow B$  and  $P(f, h) \rightarrow B$  are dualizable and self-dual (up to a twist) in the category of  $H\mathbb{F}_\ell$ -local spectra. See Remark 2.43.

**Remark 2.4.** Our product on  $\mathbb{H}^*(LBG)$  should agree with the product constructed by Chataur and Menichi [CM12, Corollary 18] (with sign corrections by Kuribayashi and Menichi [KM16]), although we will not prove the comparison here. Indeed, Chataur and Menichi’s product also arises as a composite of an induced map and an umkehr map as in (2.3), but with the umkehr map  $\text{concat}_!$  constructed in a very different way. In addition to avoiding thorny sign issues, our approach to the product on  $\mathbb{H}^*(LBG)$  has the advantage of producing a multiplication which is manifestly unital and which, importantly for us, descends to the Serre spectral sequence of the evaluation fibration  $LBG \rightarrow BG$ . A drawback of our approach is that it becomes difficult to prove that the product on  $\mathbb{H}^*(LBG)$  is commutative, which we nevertheless expect to be the case in view of the commutativity of the Chataur–Menichi product. As pointed out in Remark 2.6 below, the algebras  $\mathbb{H}^*P(f, f)$  fail to be commutative in general, however.



As we remarked earlier, in the case where  $B = \text{pt}$  and  $f$  is the inclusion of the basepoint into  $BG$ , the path space  $P(f, f)$  recovers the based loop space  $\Omega BG$ . The resulting product on  $\mathbb{H}^*(\Omega BG)$  can be described in more familiar terms.

**Theorem 2.5.** *There exists an isomorphism  $\mathbb{H}^*(\Omega BG) \cong H_{-*}(\Omega BG)$  under which the product on  $\mathbb{H}^*(\Omega BG)$  corresponds to the Pontryagin product*

$$\text{concat}_* : H_*(\Omega BG) \otimes H_*(\Omega BG) \longrightarrow H_*(\Omega BG)$$

on  $H_*(\Omega BG)$  where  $\text{concat} : \Omega BG \times \Omega BG \rightarrow \Omega BG$  is the concatenation map.

**Remark 2.6.** The Pontryagin product on  $H_*(\Omega BG)$  is frequently noncommutative: for  $\ell$  odd, [Kan76, Theorem 1.1] implies that for  $BG = BK\hat{\imath}$  for a compact connected Lie group  $K$ , commutativity fails precisely when  $H^*(K; \mathbb{Z})$  has  $\ell$ -torsion; and for  $\ell = 2$ , an example where commutativity fails is given by  $BG = B\text{Spin}(10)\hat{\imath}$  [Bor54, Théorème 16.4]. Thus Theorem 2.5 implies that the graded rings  $\mathbb{H}^*P(f, f)$  fail to be commutative in general.

The multiplication on  $\mathbb{H}^*P(f, f)$  in fact makes it into an augmented  $H^*B$ -algebra:

**Theorem 2.7.** *Let  $B$  be a space, and let  $f : B \rightarrow BG$  be a map. Equip  $H^*B$  with the usual cup product and  $\mathbb{H}^*P(f, f)$  with the product of Theorem 2.1. Then there exist graded  $\mathbb{F}_\ell$ -algebra homomorphisms*

$$\iota = \iota_f : H^*B \longrightarrow \mathbb{H}^*P(f, f) \quad \text{and} \quad \rho = \rho_f : \mathbb{H}^*P(f, f) \longrightarrow H^*B$$

such that  $\rho_f \iota_f = \text{id}_{H^*(B)}$ .

**Remark 2.8.** Via the homomorphisms  $\iota_g$  and  $\iota_f$  and the  $\mathbb{H}^*P(g, g)$  and  $\mathbb{H}^*P(f, f)$ -actions on  $\mathbb{H}^*P(f, g)$  given by Theorem 2.1, the homology  $\mathbb{H}^*P(f, g)$  acquires the structure of an  $H^*B$ -bimodule. It follows formally from Theorem 2.1 that the pairing (2.1) in fact gives a pairing

$$\circ : \mathbb{H}^*P(g, h) \otimes_{H^*B} \mathbb{H}^*P(f, g) \longrightarrow \mathbb{H}^*P(f, h)$$

of  $H^*B$ -bimodules.

**Remark 2.9.** The pairings of Theorem 2.1 and the maps  $\iota$  and  $\rho$  of Theorem 2.7 depend on the choice of a piece of orientation data, which we will fix (arbitrarily) once and for all. See Remark 2.42 below.

In addition to the bimodule structure of Remark 2.8,  $\mathbb{H}^*P(f, g)$  has an additional  $H^*(B)$ -module structure induced by the projection  $P(f, g) \rightarrow B$  and the cup product on  $H^*P(f, g)$ .

**Definition 2.10.** Let  $\sigma^{-d} : H^*P(f, g) \rightarrow \mathbb{H}^{*-d}P(f, g)$  be the degree  $-d$  graded map which sends an element  $x \in H^*P(f, g)$  to itself, but now considered as an element of  $\mathbb{H}^*P(f, g)$ . Let  $\pi : P(f, g) \rightarrow B$  be the projection. We define an  $H^*(B)$ -module structure on  $\mathbb{H}^*P(f, g)$  by setting

$$a\sigma^{-d}(x) = (-1)^{|a||x|} \sigma^{-d}(\pi^*(a) \smile x) \in \mathbb{H}^*P(f, g). \quad (2.4)$$

for all  $a \in H^*(B)$  and  $\sigma^{-d}(x) \in \mathbb{H}^*P(f, g)$ .

**Theorem 2.11.** *Let  $B$  be a space, and let  $f, g, h : B \rightarrow BG$  be maps. The pairing*

$$\circ : \mathbb{H}^*P(g, h) \otimes \mathbb{H}^*P(f, g) \longrightarrow \mathbb{H}^*P(f, h)$$

is  $H^*(B)$ -bilinear with respect to the module structures of Definition 2.10: for all  $a \in H^{|a|}(B)$ ,  $b \in H^{|b|}(B)$  and  $x \in \mathbb{H}^{|x|}P(g, h)$ ,  $y \in \mathbb{H}^{|y|}P(f, g)$  we have

$$(ax) \circ (by) = (-1)^{|b||x|} (ab)(x \circ y).$$

The pairings for varying  $B$  are compatible in a manner we will now explain. Let  $f, g: B \rightarrow BG$ . For a map of spaces  $\varphi: A \rightarrow B$ , there is a pullback square

$$\begin{array}{ccc} P(f\varphi, g\varphi) & \xrightarrow{\bar{\varphi}} & P(f, g) \\ \downarrow & & \downarrow \\ A & \xrightarrow{\varphi} & B \end{array} \quad (2.5)$$

where  $\bar{\varphi}$  is the map  $\bar{\varphi}(a, \gamma) = (\varphi(a), \gamma)$ . Let us write

$$F_\varphi = \bar{\varphi}^*: \mathbb{H}^*P(f, g) \longrightarrow \mathbb{H}^*P(f\varphi, g\varphi) \quad (2.6)$$

for the resulting map on cohomology groups. We now have the following result.

**Theorem 2.12.** *Given a map of spaces  $\varphi: A \rightarrow B$ , the maps  $F_\varphi: \mathbb{H}^*P(f, g) \longrightarrow \mathbb{H}^*P(f\varphi, g\varphi)$  for  $f, g: B \rightarrow BG$  have the following properties:*

- (i)  $F_\varphi(\mathbb{1}_f) = \mathbb{1}_{f\varphi}$
- (ii)  $F_\varphi(x \circ y) = F_\varphi(x) \circ F_\varphi(y)$  for all  $x$  and  $y$  for which the pairing is defined.

Theorem 2.12 implies in particular the following homotopy invariance property for the pairing (2.1).

**Corollary 2.13.** *Let  $f_i, g_i, h_i: B \rightarrow BG$ ,  $i = 0, 1$  be maps, and let  $H_f: f_0 \simeq f_1$ ,  $H_g: g_0 \simeq g_1$ , and  $H_h: h_0 \simeq h_1$  be homotopies. Then there exist induced isomorphisms*

$$\begin{aligned} \Xi(H_f, H_g) &: \mathbb{H}^*P(f_0, g_0) \xrightarrow{\cong} \mathbb{H}^*P(f_1, g_1), \\ \Xi(H_g, H_h) &: \mathbb{H}^*P(g_0, h_0) \xrightarrow{\cong} \mathbb{H}^*P(g_1, h_1), \quad \text{and} \\ \Xi(H_f, H_h) &: \mathbb{H}^*P(f_0, h_0) \xrightarrow{\cong} \mathbb{H}^*P(f_1, h_1) \end{aligned}$$

making the following square commutative:

$$\begin{array}{ccc} \mathbb{H}^*P(g_0, h_0) \otimes \mathbb{H}^*P(f_0, g_0) & \xrightarrow{\circ} & \mathbb{H}^*P(f_0, h_0) \\ \Xi(H_g, H_h) \otimes \Xi(H_f, H_g) \downarrow \cong & & \cong \downarrow \Xi(H_f, H_h) \\ \mathbb{H}^*P(g_1, h_1) \otimes \mathbb{H}^*P(f_1, g_1) & \xrightarrow{\circ} & \mathbb{H}^*P(f_1, h_1) \end{array} \quad (2.7)$$

*Proof.* For  $i = 0, 1$ , write  $j_i: B \rightarrow B \times I$  for the inclusion  $b \mapsto (b, i)$ . Then  $P(f_i, g_i) = P(H_f j_i, H_g j_i)$ . The inclusions  $P(f_i, g_i) \hookrightarrow P(H_f, H_g)$  are homotopy equivalences, so the induced maps

$$F_{j_i}: \mathbb{H}^*P(H_f, H_g) \longrightarrow \mathbb{H}^*P(f_i, g_i)$$

are isomorphisms. Define

$$\Xi(H_f, H_g) = F_{j_1} \circ F_{j_0}^{-1}$$

and similarly for  $\Xi(H_g, H_h)$  and  $\Xi(H_f, H_h)$ . The commutativity of the square (2.7) now follows from Theorem 2.12.  $\square$

The maps  $F_\varphi$  satisfy the expected compatibility conditions with the maps  $\iota$  and  $\rho$ . In particular, the following result implies that the map (2.6) is  $\varphi^*$ -bilinear with respect to the bimodule structures of Remark 2.8.

**Theorem 2.14.** *Given a map of spaces  $\varphi: A \rightarrow B$  and a map  $f: B \rightarrow BG$ , the following diagram commutes:*

$$\begin{array}{ccc}
 H^*B & \xrightarrow{\varphi^*} & H^*A \\
 \downarrow \iota_f & & \downarrow \iota_{f\varphi} \\
 \mathbb{H}^*P(f, f) & \xrightarrow{F_\varphi} & \mathbb{H}^*P(f\varphi, f\varphi) \\
 \downarrow \rho_f & & \downarrow \rho_{f\varphi} \\
 H^*B & \xrightarrow{\varphi^*} & H^*A
 \end{array} \tag{2.8}$$

We note that it is immediate from the definition that the map (2.6) is  $\varphi^*$ -linear with respect to the module structures of Definition 2.10.

The rest of the section is structured as follows. In subsection 2.2, we will describe our strategy for proving Theorems 2.1 and 2.12. This strategy is then carried out in subsections 2.3 through 2.7. In subsection 2.8, we will prove Theorems 2.5, 2.7, and 2.14, and finally subsection 2.9 is devoted to the proof of Theorem 2.11.

**2.2. The strategy for proving Theorems 2.1 and 2.12.** Theorems 2.1 and 2.12 should remind the reader of the definition of a category and a functor, respectively. We will use the language of enriched category theory to organize their proofs.

**Definition 2.15.** A *category  $\mathcal{C}$  enriched in a monoidal category  $\mathcal{V}$*  consists of the following data: a collection of objects  $\text{Ob}\mathcal{C}$ , a hom-object  $\mathcal{C}(A, B) \in \mathcal{V}$  for every pair of objects  $A, B \in \text{Ob}\mathcal{C}$ , a composition law  $\mathcal{C}(B, C) \otimes \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, C)$  for every triple of objects  $A, B, C \in \text{Ob}\mathcal{C}$ , and an identity element  $I \rightarrow \mathcal{C}(A, A)$  for every object  $A \in \text{Ob}\mathcal{C}$ , where  $I$  denotes the monoidal unit in  $\mathcal{V}$ . These data are supposed to satisfy the evident analogues of the axioms of an ordinary category [Kel05, section 1.2].

**Definition 2.16.** An *enriched functor  $F: \mathcal{C} \rightarrow \mathcal{D}$*  between categories enriched in  $\mathcal{V}$  consists of a map  $F: \text{Ob}\mathcal{C} \rightarrow \text{Ob}\mathcal{D}$  and a map  $F = F_{A,B}: \mathcal{C}(A, B) \rightarrow \mathcal{C}(FA, FB)$  for every pair of objects  $A, B \in \text{Ob}\mathcal{C}$ , these data being subject to the evident analogues of the axioms for an ordinary functor [Kel05, section 1.2].

**Remark 2.17.** In the language introduced in Definitions 2.15 and 2.16, Theorem 2.1 can be reformulated as the assertion that given a space  $B$ , there is a category enriched in graded  $\mathbb{F}_\ell$ -vector spaces where objects are maps  $B \rightarrow BG$  and the hom-object from  $f: B \rightarrow BG$  to  $g: B \rightarrow BG$  is given by  $\mathbb{H}^*P(f, g)$ ; and Theorem 2.12 simply asserts that the maps  $F_\varphi$  give an enriched functor from the category associated to  $B$  to the one associated to  $A$ .

**Terminology 2.18** ((Symmetric) monoidal functors). By a (symmetric) monoidal functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  between (symmetric) monoidal categories, we mean a *strong* (symmetric) monoidal functor in the sense of Mac Lane [ML98, section XI.2], meaning that the monoidality and identity constraints

$$F_\otimes: F(X) \otimes F(Y) \longrightarrow F(X \otimes Y) \quad \text{and} \quad F_I: I_{\mathcal{D}} \longrightarrow F(I_{\mathcal{C}})$$

are assumed to be isomorphisms. Here  $I_{\mathcal{C}}$  and  $I_{\mathcal{D}}$  denote the unit objects of  $\mathcal{C}$  and  $\mathcal{D}$ , respectively. In a *lax* (symmetric) monoidal functor the requirement that the maps are isomorphisms is dropped, and in an *oplax* (symmetric) monoidal functor the direction of the maps is in addition reversed.

The following construction gives a basic way of obtaining new enriched categories and functors from existing ones.

**Construction 2.19.** Let  $M: \mathcal{V} \rightarrow \mathcal{W}$  be a lax monoidal functor. Then from a  $\mathcal{V}$ -enriched category  $\mathcal{C}$  we obtain a  $\mathcal{W}$ -enriched category  $M_*\mathcal{C}$  with  $\text{Ob } M_*\mathcal{C} = \text{Ob } \mathcal{C}$  and hom-objects

$$(M_*\mathcal{C})(A, B) = M\mathcal{C}(A, B)$$

by taking as the composition law the composite

$$MC(B, C) \otimes MC(A, B) \xrightarrow{M_\otimes} M(\mathcal{C}(B, C) \otimes \mathcal{C}(A, B)) \xrightarrow{M(\mu_{A,B,C})} MC(A, C)$$

and as the identity element for an object  $A$  the composite

$$I_{\mathcal{W}} \xrightarrow{M_I} M(I_{\mathcal{V}}) \xrightarrow{M(\iota_A)} MC(A, A),$$

where  $\mu_{A,B,C}$  and  $\iota_A$  refer to the composition law and the identity element in  $\mathcal{C}$ , respectively, and  $M_\otimes$  and  $M_I$  are the monoidality and identity constraints of  $M$ . Moreover, a  $\mathcal{V}$ -enriched functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  induces a  $\mathcal{W}$ -enriched functor  $M_*(F): M_*\mathcal{C} \rightarrow M_*\mathcal{D}$  by letting  $M_*(F) = F$  on objects, and by defining

$$M_*(F)_{A,B} = M(F_{A,B}): (M_*\mathcal{C})(A, B) \longrightarrow (M_*\mathcal{D})(FA, FB)$$

on morphisms.

In view of Remark 2.17 and Construction 2.19, one might try to prove Theorem 2.1 by first constructing an enriched category  $\mathcal{P}_B$  where the objects are maps  $f: B \rightarrow BG$ , where the hom-object from  $f$  to  $g$  is given by the fibration  $P(f, g) \rightarrow B$ , and where the composition law is given by diagram (2.2), and then applying Construction 2.19 to  $\mathcal{P}_B$  with a suitable monoidal functor  $M$  to obtain the category of Theorem 2.1. Moreover, Theorem 2.12 would follow if the pullback squares (2.5) assembled into an enriched functor  $K_\varphi: \mathcal{P}_B \rightarrow \mathcal{P}_A$  which, upon application of  $M$ , yielded  $F_\varphi$ . Modulo a few difficulties which we will point out along the way, this strategy is indeed the one we will follow. The functor  $M$  will be the composite

$$\begin{aligned} (\mathbf{Fib}^{\text{fop}})^{\text{op}} &\xrightarrow{D_{\text{fw}}^{\text{op}}} (\mathbf{hpSpectra}^\ell)^{\text{op}} \\ &\xrightarrow{U_{\text{fw}}^{\text{op}}} \mathbf{hpSpectra}^{\text{op}} \\ &\xrightarrow{r_!^{\text{op}}} \text{Ho}(\mathbf{Spectra})^{\text{op}} \\ &\xrightarrow{H^*} \mathbf{grMod}^{\mathbb{F}_\ell} \end{aligned} \tag{2.9}$$

Here  $H^*$  denotes cohomology with  $\mathbb{F}_\ell$  coefficients,  $\mathbf{grMod}^{\mathbb{F}_\ell}$  denotes the category of graded  $\mathbb{F}_\ell$ -modules, and  $\text{Ho}(\mathbf{Spectra})$  denotes the homotopy category of spectra. The categories  $\mathbf{hpSpectra}$  and  $\mathbf{hpSpectra}^\ell$  are the categories of parametrized spectra and parametrized  $H\mathbb{F}_\ell$ -local spectra obtained by applying the construction of appendix A.5 to the  $\infty$ -categories  $\mathbf{Spectra}$  and  $\mathbf{Spectra}^\ell$  of spectra and  $H\mathbb{F}_\ell$ -local spectra, respectively, and the functor  $U_{\text{fw}}$  is the functor obtained by applying the construction of appendix A.5 to the forgetful functor  $U: \mathbf{Spectra}^\ell \rightarrow \mathbf{Spectra}$ . The category  $\mathbf{Fib}^{\text{fop}}$  will be constructed in subsection 2.3; the functor  $D_{\text{fw}}$  in subsection 2.5; and the functor  $r_!$  in subsection 2.6, which also contains further discussion of the functor  $U_{\text{fw}}$ . Once all the categories and functors in (2.9) have been constructed, the proof of Theorems 2.1 and 2.12 will be completed by identifying  $H^*(r_!U_{\text{fw}}D_{\text{fw}}P(f, g))$  as  $\mathbb{H}^*(P(f, g))$ . This will be done in subsection 2.7.

**2.3. The category  $\mathbf{Fib}^{\text{fop}}$ .** Our goal in this subsection is to construct the category  $\mathbf{Fib}^{\text{fop}}$  appearing in (2.9). Write  $\mathcal{T}$  for the category of topological spaces and continuous maps. We start by constructing the category  $\mathbf{Fib}$  over  $\mathcal{T}$  of which  $\mathbf{Fib}^{\text{fop}}$  is the ‘‘fibrewise opposite category.’’

**Definition 2.20.** Call a space  $X$   $H\mathbb{F}_\ell$ -locally dualizable if  $L\Sigma_+^\infty X$  is a dualizable object in the sense of appendix B in the homotopy category  $\text{Ho}(\mathbf{Spectra}^\ell)$  of  $H\mathbb{F}_\ell$ -local spectra. Here  $L: \mathbf{Spectra} \rightarrow \mathbf{Spectra}^\ell$  denotes the localization functor.

**Example 2.21.** By [Bau04, Corollary 8], a sufficient condition for a space  $X$  to be  $H\mathbb{F}_\ell$ -locally dualizable is that the homology of  $X$  with  $\mathbb{F}_\ell$  coefficients is finitely generated in each degree

and vanishes outside a finite number of degrees. In particular, the space  $\Omega BG$  is an  $H\mathbb{F}_\ell$ -locally dualizable space.

**Definition 2.22** (The category  $\mathbf{Fib}$ ). For a space  $B$ , write  $\mathbf{Fib}_B$  for the full subcategory of the overcategory  $\mathcal{T}/B$  spanned by fibrations whose fibres are  $H\mathbb{F}_\ell$ -locally dualizable. For example, it follows from Example 2.21 that the fibrations  $P(f, g) \rightarrow B$  are objects of  $\mathbf{Fib}_B$  for all  $f, g: B \rightarrow BG$ , as their fibres are homotopy equivalent to  $\Omega BG$ . We equip  $\mathbf{Fib}_B$  with the symmetric monoidal structure given by the fibrewise direct product  $\times_B$ . For all spaces  $A$  and  $B$ , continuous maps  $f: A \rightarrow B$ , and objects  $\pi: E \rightarrow B$  of  $\mathbf{Fib}_B$ , choose once and for all a pullback square

$$\begin{array}{ccc} f^*E & \xrightarrow{\bar{f}\pi} & E \\ \pi_f \downarrow & & \downarrow \pi \\ A & \xrightarrow{f} & B \end{array} \quad (2.10)$$

Now for every  $f: A \rightarrow B$ , the universal property of pullbacks implies that there exists for every morphism  $\varphi: E \rightarrow E'$  in  $\mathbf{Fib}_B$  a unique morphism  $f^*(\varphi): f^*E \rightarrow f^*E'$  in  $\mathbf{Fib}_A$  with the property that the diagram

$$\begin{array}{ccc} f^*E & \xrightarrow{\bar{f}\pi} & E \\ f^*(\varphi) \downarrow & & \downarrow \varphi \\ f^*E' & \xrightarrow{\bar{f}\pi'} & E' \end{array}$$

commutes. In this way, we obtain a functor  $f^*: \mathbf{Fib}_B \rightarrow \mathbf{Fib}_A$  which is easily seen to be symmetric monoidal. The assignments  $B \mapsto \mathbf{Fib}_B$  and  $f \mapsto f^*$  along with the natural homeomorphisms  $(g \circ f)^*E \cong f^*g^*E$  now define a pseudofunctor (see Definition A.3) from  $\mathcal{T}^{\text{op}}$  to the 2-category  $\mathbf{smCat}$  of symmetric monoidal categories, symmetric monoidal functors, and symmetric monoidal transformations. We define  $\mathbf{Fib}$  to be the symmetric monoidal category obtained from this pseudofunctor by the Grothendieck construction of appendix A.4. Notice that the symmetric monoidal product on  $\mathbf{Fib}$  agrees with the cartesian product

$$(E \xrightarrow{\pi} B) \times (E' \xrightarrow{\pi'} B') = (E \times E' \xrightarrow{\pi \times \pi'} B \times B').$$

**Remark 2.23.** Alternatively, the category  $\mathbf{Fib}$  can be described as follows. The objects of  $\mathbf{Fib}$  are fibrations  $\pi: E \rightarrow B$  with  $H\mathbb{F}_\ell$ -locally dualizable fibres, and a map from  $\pi: E \rightarrow B$  to  $\pi': E' \rightarrow B'$  is a pair  $(\bar{f}, f)$  of continuous maps making the square

$$\begin{array}{ccc} E & \xrightarrow{\bar{f}} & E' \\ \pi \downarrow & & \downarrow \pi' \\ B & \xrightarrow{f} & B' \end{array}$$

commutative. The unique factorization of  $\bar{f}$  as a composite

$$E \xrightarrow{\varphi} f^*E \xrightarrow{\bar{f}\pi'} E'$$

where  $\varphi$  is a map over  $B$  provides the link between the two descriptions. The advantage of the admittedly more complicated description in Definition 2.22 is that it makes it simple to define the category  $\mathbf{Fib}^{\text{fop}}$ .

**Definition 2.24** (The category  $\mathbf{Fib}^{\text{fop}}$ ). Writing  $F$  for the pseudofunctor  $\mathcal{T}^{\text{op}} \rightarrow \mathbf{smCat}$  featuring in the construction of  $\mathbf{Fib}$ , we define the category  $\mathbf{Fib}^{\text{fop}}$  to be the category obtained by the Grothendieck construction of appendix A.4 from the pseudofunctor

$$\mathcal{T}^{\text{op}} \xrightarrow{F} \mathbf{smCat} \xrightarrow{\text{op}} \mathbf{smCat}$$

where  $\text{op}$  is the 2-functor sending a symmetric monoidal category to its opposite.

**2.4. The category  $\mathcal{P}_B$  enriched in  $(\mathbf{Fib}^{\text{fop}})^{\text{op}}$ .** Our goal now is, roughly speaking, to construct the category  $\mathcal{P}_B$  enriched in  $(\mathbf{Fib}^{\text{fop}})^{\text{op}}$  discussed at the end of subsection 2.2. We start by offering an alternative description of the category  $\mathbf{Fib}^{\text{fop}}$ .

**Observation 2.25.** Paralleling Remark 2.23, the category  $\mathbf{Fib}^{\text{fop}}$  can alternatively be described as follows: The objects of  $\mathbf{Fib}^{\text{fop}}$  are fibrations  $\pi: E \rightarrow B$  whose fibres are  $H\mathbb{F}_\ell$ -locally dualizable. For such fibrations  $\pi: E \rightarrow B$  and  $\pi': E' \rightarrow B'$ , write  $(\bar{f}, f): \pi \rightarrow \pi'$  for a commutative square

$$\begin{array}{ccc} E & \xrightarrow{\bar{f}} & E' \\ \pi \downarrow & & \downarrow \pi' \\ B & \xrightarrow{f} & B' \end{array}$$

and in keeping with fibred category theory, call such an arrow  $(\bar{f}, f)$  *cartesian* if the square is a pullback square. A morphism from  $\pi: E \rightarrow B$  to  $\pi': E' \rightarrow B'$  in  $\mathbf{Fib}^{\text{fop}}$  is then an equivalence class of zigzags

$$\pi \xleftarrow{(\alpha, \text{id}_B)} \tau \xrightarrow{(\bar{f}, f)} \pi'$$

where  $(\bar{f}, f)$  is cartesian. Two such zigzags

$$\pi \xleftarrow{(\alpha, \text{id}_B)} \tau \xrightarrow{(\bar{f}, f)} \pi' \quad \text{and} \quad \pi \xleftarrow{(\alpha', \text{id}_B)} \tau' \xrightarrow{(\bar{f}', f')} \pi'$$

are equivalent if  $f' = f$  and there exists an arrow  $(\theta, \text{id}_B): \tau \rightarrow \tau'$  such that  $\bar{f}' = \bar{f}\theta$  and  $\alpha = \alpha'\theta$ ; such a  $\theta$  is necessarily unique and a homeomorphism. In particular, employing the notation of diagram (2.10), for a map  $\varphi: f^*E' \rightarrow E$  in  $\mathbf{Fib}_B$ , the zigzag

$$\pi \xleftarrow{(\varphi, \text{id}_B)} \pi'_f \xrightarrow{(\bar{f}_{\pi'}, f)} \pi'$$

defines a morphism in  $\mathbf{Fib}^{\text{fop}}$  from  $\pi$  to  $\pi'$ , and any zigzag is equivalent to a unique zigzag of this form. This shows the equivalence of the old and new descriptions of morphisms in  $\mathbf{Fib}^{\text{fop}}$ . In the new description, the composite of

$$[\pi \xleftarrow{(\alpha, \text{id})} \tau \xrightarrow{(\bar{f}, f)} \pi'] \quad \text{and} \quad [\pi' \xleftarrow{(\alpha', \text{id})} \tau' \xrightarrow{(\bar{f}', f')} \pi'']$$

is represented by the zigzag given by the composites along the two sides of the diagram

$$\begin{array}{ccccc} & & \sigma & & \\ & & \swarrow & \searrow & \\ & (\tilde{\alpha}', \text{id}) & & (\bar{f}, f) & \\ & \swarrow & & \searrow & \\ (\alpha, \text{id}) & \tau & & \tau' & (\bar{f}', f') \\ & \swarrow & \searrow & \swarrow & \searrow \\ \pi & & \pi' & & \pi'' \end{array}$$

where  $\sigma$ ,  $\tilde{\alpha}'$  and  $\tilde{f}$  are determined by the requirement that  $(\tilde{f}, f)$  is cartesian and  $\tilde{\alpha}'$  is the unique morphism making the diamond in the middle commutative. The symmetric monoidal structure on  $\mathbf{Fib}^{\text{fop}}$  is given by the direct product

$$(E \xrightarrow{\pi} B) \times (E' \xrightarrow{\pi'} B') = (E \times E' \xrightarrow{\pi \times \pi'} B \times B').$$

Using the description of Observation 2.25 for  $\mathbf{Fib}^{\text{fop}}$ , we see that diagram (2.2) defines a morphism

$$(P(f, h) \rightarrow B) \longrightarrow (P(g, h) \rightarrow B) \times (P(f, g) \rightarrow B) \quad (2.11)$$

in  $\mathbf{Fib}^{\text{fop}}$ . For  $f: B \rightarrow BG$ , we also have a morphism (in  $\mathbf{Fib}^{\text{fop}}$ )

$$\left( P(f, f) \rightarrow B \right) \longrightarrow \left( \text{pt} \xrightarrow{\text{id}} \text{pt} \right) \quad (2.12)$$

into the monoidal unit given by the diagram

$$\begin{array}{ccccc} P(f, f) & \xleftarrow{s} & B & \xrightarrow{r} & \text{pt} \\ & \searrow & \downarrow \text{id} & & \downarrow \text{id} \\ & & B & \xrightarrow{r} & \text{pt} \end{array}$$

Here the map  $s$  is given by  $s(b) = (b, c_{f(b)})$  where  $c_{f(b)}$  denotes the constant path onto  $f(b) \in BG$ . Finally, the pullback square (2.5) gives a map (in  $\mathbf{Fib}^{\text{fop}}$ )

$$\left( P(f\varphi, g\varphi) \rightarrow A \right) \longrightarrow \left( P(f, g) \rightarrow B \right). \quad (2.13)$$

The reader might now expect us to make the following (incorrect) definition.

**“Definition” 2.26.** The category  $\mathcal{P}_B$  is the category enriched in  $(\mathbf{Fib}^{\text{fop}})^{\text{op}}$  whose objects are continuous maps  $B \rightarrow BG$ , where the hom-object of maps from  $f: B \rightarrow BG$  to  $g: B \rightarrow BG$  is  $P(f, g) \rightarrow B$ , and where the composition law and identity elements are given by the maps (2.11) and (2.12), respectively. The functor  $K_\varphi: \mathcal{P}_B \rightarrow \mathcal{P}_A$  is the enriched functor given by the maps (2.13).

The reason why the above definition is not quite correct is simple: the specified data for  $\mathcal{P}_B$  fail to satisfy the axioms of an enriched category. It is readily verified that the axioms are *almost* satisfied, however, the only problem being that composition of paths is associative and unital only up to homotopy, rather than strictly. The problem could be remedied for example by replacing  $\mathbf{Fib}$  by another category where morphisms are fibrewise homotopy classes of maps, or by working with Moore paths instead of ordinary paths. Instead of following either approach, we content ourselves with noting that the object  $\mathcal{P}_B$  given by Definition 2.26 is close enough to being an enriched category for the problem to disappear once we apply the functor

$$D_{\text{fw}}^{\text{op}}: (\mathbf{Fib}^{\text{fop}})^{\text{op}} \longrightarrow (\mathbf{hpSpectra}^\ell)^{\text{op}}$$

constructed in the next subsection: following Construction 2.19, we obtain a new object  $(D_{\text{fw}}^{\text{op}})_* \mathcal{P}_B$  which does satisfy the axioms for a category enriched in  $(\mathbf{hpSpectra}^\ell)^{\text{op}}$  as well as enriched functors

$$(D_{\text{fw}}^{\text{op}})_*(K_\varphi): (D_{\text{fw}}^{\text{op}})_* \mathcal{P}_B \longrightarrow (D_{\text{fw}}^{\text{op}})_* \mathcal{P}_A.$$

See Remark 2.28 below.

**2.5. The functor  $D_{\text{fw}}$ .** Our aim in this subsection is to construct the functor

$$D_{\text{fw}}: \mathbf{Fib}^{\text{fop}} \longrightarrow \mathbf{hpSpectra}^\ell$$

featuring in equation (2.9). Intuitively, this functor is the fibrewise analogue of the functor sending a space to its  $H\mathbb{F}_\ell$ -local dual. Despite the notation, we will not formally obtain the functor  $D_{\text{fw}}$  as a special case of Definition A.10. Instead, we will construct it by applying the Grothendieck construction to the pseudo natural transformation from the pseudofunctor  $B \mapsto \mathbf{Fib}_B^{\text{op}}$  to the

pseudofunctor  $B \mapsto \mathrm{Ho}(\mathbf{Spectra}_{/B}^\ell)$  given for a space  $B$  by the following composite of functors:

$$\begin{aligned}
\mathbf{Fib}_B^{\mathrm{op}} &\xrightarrow{T_B^{\mathrm{op}}} \mathrm{Ho}(\mathbf{Spaces}_{/B})^{\mathrm{op}} \\
&\xrightarrow{(\Sigma_{+B}^\infty)^{\mathrm{op}}} \mathrm{Ho}(\mathbf{Spectra}_{/B})^{\mathrm{op}} \\
&\xrightarrow{L_B^{\mathrm{op}}} \mathrm{Ho}(\mathbf{Spectra}_{/B}^\ell)^{\mathrm{op}} \\
&\xrightarrow{D_B} \mathrm{Ho}(\mathbf{Spectra}_{/B}^\ell)
\end{aligned} \tag{2.14}$$

Here the categories  $\mathrm{Ho}(\mathbf{Spaces}_{/B})$ ,  $\mathrm{Ho}(\mathbf{Spectra}_{/B})$  and  $\mathrm{Ho}(\mathbf{Spectra}_{/B}^\ell)$  are as constructed in appendix A.1; the functors  $\Sigma_{+B}^\infty$  and  $L_B$  are as constructed in appendix A.3; the functor  $D_B$  is the fibrewise dual functor

$$D_B = F_B(-, S_B): \mathrm{Ho}(\mathbf{Spectra}_{/B}^\ell)^{\mathrm{op}} \longrightarrow \mathrm{Ho}(\mathbf{Spectra}_{/B}^\ell);$$

and the functor  $T_B$  is the composite

$$\mathbf{Fib}_B \xrightarrow{i} \mathcal{T}/B \xrightarrow{\gamma} \mathrm{Ho}(\mathcal{T}/B) \xrightarrow{\simeq} \mathrm{Ho}(\mathbf{Spaces}_{/B}) \tag{2.15}$$

where  $i$  is the inclusion,  $\gamma$  is the projection onto the homotopy category, and the last arrow is the equivalence of categories alluded to in Remark A.2. The construction of  $D_{\mathrm{fw}}$  is now completed by the verification of the following lemma.

**Lemma 2.27.** *The composites of the functors (2.14) define a pseudo natural transformation between the pseudofunctors  $\mathcal{T}^{\mathrm{op}} \rightarrow \mathbf{smCat}$  given by  $B \mapsto \mathbf{Fib}_B^{\mathrm{op}}$  and  $B \mapsto \mathrm{Ho}(\mathbf{Spectra}_{/B}^\ell)$ .*

*Proof.* The main issues are verifying that the composite of the functors (2.14) is symmetric monoidal and that the composites commute (up to natural isomorphism) with the pullback functors

$$f^* = (f^*)^{\mathrm{op}}: \mathbf{Fib}_B^{\mathrm{op}} \longrightarrow \mathbf{Fib}_A^{\mathrm{op}}$$

and

$$f^*: \mathrm{Ho}(\mathbf{Spectra}_{/B}^\ell) \longrightarrow \mathrm{Ho}(\mathbf{Spectra}_{/A}^\ell)$$

for maps of spaces  $f: A \rightarrow B$ . As noted in appendix A.3, the functors  $\Sigma_{+B}^\infty$  and  $L_B$  are symmetric monoidal and compatible with the pullback functors. Let us consider the functor  $T_B$ . The symmetric monoidal structure on  $\mathrm{Ho}(\mathbf{Spectra}_{/B})$  is the cartesian one. Since the objects of  $\mathbf{Fib}_B$  are fibrant in a model structure on  $\mathcal{T}/B$  giving rise to  $\mathrm{Ho}(\mathcal{T}/B)$ , we see that  $\gamma \circ i$  and hence  $T_B$  preserve products. Thus  $T_B$  is symmetric monoidal. Moreover, the pullback functors

$$f^*: \mathrm{Ho}(\mathcal{T}/B) \longrightarrow \mathrm{Ho}(\mathcal{T}/A)$$

as constructed in [MS06] are the total right derived functors of the pointset-level pullback functors

$$f^*: \mathcal{T}/B \longrightarrow \mathcal{T}/A.$$

The compatibility of  $\gamma \circ i$  and hence that of the functors  $T_B$  with pullback functors therefore also follows from the fact that the objects of  $\mathbf{Fib}_B$  are fibrant in  $\mathcal{T}/B$ .

Let us now consider the functor  $D_B$ . For  $X$  and  $Y$  objects of  $\mathrm{Ho}(\mathbf{Spectra}_{/B}^\ell)$ , from (B.2) we obtain a natural morphism

$$D_B(X) \wedge_B^\ell D_B(Y) \longrightarrow D_B(X \wedge_B^\ell Y) \tag{2.16}$$

making  $D_B$  into a lax monoidal functor (see Notation A.1 for the definition of  $\wedge_B^\ell$ ). By the criterion of Corollary B.3, all objects in the image of the composite  $L_B \Sigma_{+B}^\infty T_B$  are dualizable. It follows that the morphism (2.16) is an equivalence for such  $X$  and  $Y$ . Thus the composite of all the functors in (2.14) is symmetric monoidal, as required.  $\square$



**Remark 2.28.** It is now straightforward to check that the object  $(D_{\text{fw}}^{\text{op}})_*\mathcal{P}_B$  obtained by following Construction 2.19 is a category enriched in  $(\mathbf{hpSpectra}^\ell)^{\text{op}}$ , as discussed at the end of subsection 2.4. The crucial point is to observe that the functor

$$\gamma: \mathcal{T}/B \longrightarrow \text{Ho}(\mathcal{T}/B)$$

in (2.15) sends fibrewise homotopic maps in  $\mathcal{T}/B$  to the same map in  $\text{Ho}(\mathcal{T}/B)$ . Moreover, for a map  $\varphi: A \rightarrow B$ , we obtain an enriched functor

$$(D_{\text{fw}}^{\text{op}})_*(K_\varphi): (D_{\text{fw}}^{\text{op}})_*\mathcal{P}_B \longrightarrow (D_{\text{fw}}^{\text{op}})_*\mathcal{P}_A.$$

**2.6. The functors  $U_{\text{fw}}$  and  $r_!$ .** In this subsection, we will discuss the functors  $U_{\text{fw}}$  and  $r_!$  featuring in (2.9) and show that the composite functor  $H^*r_!^{\text{op}}U_{\text{fw}}^{\text{op}}$  is lax symmetric monoidal. We start by promoting the functor  $U_{\text{fw}}$  (defined in appendix A.5) into a lax symmetric monoidal functor.

**Lemma 2.29.** *The functor*

$$U_{\text{fw}}: \mathbf{hpSpectra}^\ell \longrightarrow \mathbf{hpSpectra}$$

*admits the structure of a lax symmetric monoidal functor.*

*Proof.* The functor  $U_{\text{fw}}$  is part of the adjunction (A.12). It follows formally that  $U_{\text{fw}}$ , as a right adjoint of a symmetric monoidal functor, is a lax symmetric monoidal functor. Explicitly, the monoidality constraint for  $U_{\text{fw}}$  is given by the composite

$$\begin{aligned} U_{\text{fw}}X \bar{\wedge} U_{\text{fw}}Y &\longrightarrow U_{\text{fw}}L_{\text{fw}}(U_{\text{fw}}X \bar{\wedge} U_{\text{fw}}Y) \\ &\xrightarrow{\cong} U_{\text{fw}}(L_{\text{fw}}U_{\text{fw}}X \bar{\wedge}^\ell L_{\text{fw}}U_{\text{fw}}Y) \\ &\xrightarrow{\cong} U_{\text{fw}}(X \bar{\wedge}^\ell Y) \end{aligned} \tag{2.17}$$

where the first map is the counit of the  $(L_{\text{fw}}, U_{\text{fw}})$  adjunction; the second map is given by (the inverse of) the monoidality constraint for the symmetric monoidal functor  $L_{\text{fw}}$ ; and the last map is given by the counit of the  $(L_{\text{fw}}, U_{\text{fw}})$  adjunction. Similarly, the identity constraint for  $U_{\text{fw}}$  is given by the composite

$$S_{\text{pt}} \longrightarrow U_{\text{fw}}L_{\text{fw}}S_{\text{pt}} \xrightarrow{\cong} U_{\text{fw}}S_{\text{pt},\ell} \tag{2.18}$$

where we have written  $S_{\text{pt}}$  and  $S_{\text{pt},\ell}$  for the identity objects of  $\mathbf{hpSpectra}$  and  $\mathbf{hpSpectra}^\ell$ , respectively, and where the first map is again given by the counit of the  $(L_{\text{fw}}, U_{\text{fw}})$  adjunction and the second map is induced by the identity constraint for the symmetric monoidal functor  $L_{\text{fw}}$ .  $\square$

**Notation 2.30.** For  $B$  a space, we write  $r_B$ ,  $r^B$ , and sometimes just  $r$  for the unique map from  $B$  to the one-point space.

**Definition 2.31.** For a presentable symmetric monoidal  $\infty$ -category  $\mathcal{C}$ , we define an oplax symmetric monoidal functor (see Terminology 2.18)

$$r_!: \mathbf{hp}\mathcal{C} \longrightarrow \text{Ho}(\mathcal{C})$$

from the category  $\mathbf{hp}\mathcal{C}$  defined in appendix A.5 as follows. The functor  $r_!$  sends an object  $(B, X) \in \mathbf{hp}\mathcal{C}$  to the object  $r_!^B X$  and a morphism  $(f, \varphi): (B, X) \rightarrow (C, Y)$  to the composite

$$r_!^B X \xrightarrow{r_!^B(\varphi)} r_!^B f^* Y \xrightarrow{\cong} r_!^C f_! f^* Y \longrightarrow r_!^C Y$$

where the equivalence in the middle is induced by the natural equivalence  $f^*(r^C)^* \simeq (r^B)^*$  between right adjoints and the last map is induced by the counit of the  $(f_!, f^*)$  adjunction. (See

appendix A.2 for the definition of the base change functors  $g_!$  and  $g^*$  associated to a map  $g$  of spaces.) The monoidality constraint of  $r_!$  is given by the composite

$$\begin{aligned}
r_!^{B \times C}(X \bar{\otimes} Y) &\xrightarrow{=} r_!^{B \times C}((\pi_B^{B \times C})^* X \otimes_{B \times C} (\pi_C^{B \times C})^* Y) \\
&\longrightarrow r_!^{B \times C}((\pi_B^{B \times C})^* r_B^* r_!^B X \otimes_{B \times C} (\pi_C^{B \times C})^* r_C^* r_!^C Y) \\
&\xrightarrow{\simeq} r_!^{B \times C}(r_{B \times C}^* r_!^B X \otimes_{B \times C} r_{B \times C}^* r_!^C Y) \\
&\xrightarrow{\simeq} r_!^{B \times C} r_{B \times C}^*(r_!^B X \otimes r_!^C Y) \\
&\longrightarrow r_!^B X \otimes r_!^C Y
\end{aligned} \tag{2.19}$$

Here  $X$  and  $Y$  are objects of  $\mathbf{hpC}$  over the spaces  $B$  and  $C$ , respectively;  $\pi_B^{B \times C}: B \times C \rightarrow B$  and  $\pi_C^{B \times C}: B \times C \rightarrow C$  are the projections; the first map simply substitutes the definition of  $X \bar{\otimes} Y$  (see equation (A.11)); the second map is given by the units of the  $(r_!^B, r_B^*)$  and  $(r_!^C, r_C^*)$  adjunctions; the third map is given by the natural equivalences  $r_{B \times C}^* \simeq (\pi_B^{B \times C})^* r_B^*$  and  $r_{B \times C}^* \simeq (\pi_C^{B \times C})^* r_C^*$ ; the fourth map is given by the monoidality constraint for  $r_{B \times C}^*$ ; and the last map is given by the counit of the  $(r_!^{B \times C}, r_{B \times C}^*)$  adjunction. Finally, the identity constraint of  $r_!$  is given by the composite

$$r_!^{\text{pt}} S_{\text{pt}} \xrightarrow{\simeq} r_!^{\text{pt}} r_{\text{pt}}^* S \xrightarrow{\simeq} S$$

where  $S$  and  $S_{\text{pt}}$  are the monoidal units for  $\text{Ho}(\mathcal{C})$  and  $\text{Ho}(\mathcal{C}_{/\text{pt}})$ , respectively; where the first equivalence is given by the identity constraint for  $r_{\text{pt}}^*$ ; and where the second equivalence is the counit of the  $(r_!^{\text{pt}}, r_{\text{pt}}^*)$  adjunction.

**Proposition 2.32.** *The oplax symmetric monoidal functor*

$$r_!: \mathbf{hpC} \longrightarrow \text{Ho}(\mathcal{C})$$

*is in fact a symmetric monoidal functor.*

*Proof.* Since the identity constraint of  $r_!$  is an equivalence, it is enough to show that the monoidality constraint (2.19) is. A tedious diagram chase shows that the composite (2.19) agrees with the composite equivalence

$$\begin{aligned}
r_!^{B \times C}(X \bar{\otimes} Y) &\xrightarrow{=} r_!^{B \times C}((\pi_B^{B \times C})^* X \otimes_{B \times C} (\pi_C^{B \times C})^* Y) \\
&\xrightarrow{\simeq} r_!^B (\pi_B^{B \times C})_! ((\pi_B^{B \times C})^* X \otimes_{B \times C} (\pi_C^{B \times C})^* Y) \\
&\xrightarrow{\simeq} r_!^B (X \otimes_B (\pi_B^{B \times C})_! (\pi_C^{B \times C})^* Y) \\
&\xrightarrow{\simeq} r_!^B (X \otimes_B r_B^* r_!^C Y) \\
&\xrightarrow{\simeq} r_!^B X \otimes_B r_!^C Y
\end{aligned}$$

where the first map substitutes the definition of  $X \bar{\otimes} Y$ , where the second map is given by the equivalence  $r_!^{B \times C} \simeq r_!^B (\pi_B^{B \times C})_!$ , where the third and last equivalences follow from the projection formula (A.5), and where the second-last map is an instance of the commutation relation (A.7). In performing the diagram chase, one needs to know that the equivalence in the projection formula (A.5) is given by the composite

$$f_!(f^* Y \otimes_A X) \longrightarrow f_!(f^* Y \otimes_A f^* f_! X) \xrightarrow{\simeq} f_! f^*(Y \otimes_B f_! X) \longrightarrow Y \otimes_B f_! X$$

where the first and last maps are induced by the unit and the counit of the  $(f_!, f^*)$  adjunction, respectively, and where the middle map is induced by the monoidality constraint for  $f^*$ ; and that

the equivalence in the commutation relation (A.7) is the composite

$$\bar{f}_! \bar{g}^* \longrightarrow \bar{f}_! \bar{g}^* f^* f_! \xrightarrow{\cong} \bar{f}_! \bar{f}^* g^* f_! \longrightarrow g^* f_!$$

where the first map is induced by the unit of the  $(f_!, f^*)$  adjunction, where the middle map is induced the equivalence  $\bar{g}^* f^* \simeq \bar{f}^* g^*$ , and where the last map is induced by the counit of the  $(\bar{f}_!, \bar{f}^*)$  adjunction.  $\square$

Our goal in the remainder of the subsection is to show the following result.

**Proposition 2.33.** *The composite functor*

$$(\mathbf{hpSpectra}^\ell)^{\text{op}} \xrightarrow{U_{\text{fw}}^{\text{op}}} \mathbf{hpSpectra}^{\text{op}} \xrightarrow{r_!^{\text{op}}} \text{Ho}(\mathbf{Spectra})^{\text{op}} \xrightarrow{H^*} \mathbf{grMod}^{\mathbb{F}_\ell} \quad (2.20)$$

appearing in (2.9) is lax symmetric monoidal.

Notice that the result is not quite obvious, since while  $r_!^{\text{op}}$  and  $H^*$  are lax symmetric monoidal functors, the functor  $U_{\text{fw}}^{\text{op}}$  is oplax rather than lax symmetric monoidal. Write

$$(H\mathbb{F}_\ell \wedge -)_{\text{fw}} : \mathbf{hpSpectra} \longrightarrow \mathbf{hpMod}^{H\mathbb{F}_\ell}$$

for the symmetric monoidal functor induced by the symmetric monoidal functor

$$H\mathbb{F}_\ell \wedge - : \mathbf{Spectra} \longrightarrow \mathbf{Mod}^{H\mathbb{F}_\ell}$$

given by smashing with  $H\mathbb{F}_\ell$  (see appendix A.5). Also write  $F^{H\mathbb{F}_\ell}(-, -)$  for the internal hom in  $\text{Ho}(\mathbf{Mod}^{H\mathbb{F}_\ell})$ .

**Lemma 2.34.** *The diagram*

$$\begin{array}{ccccc} (\mathbf{hpSpectra}^\ell)^{\text{op}} & \xrightarrow{U_{\text{fw}}^{\text{op}}} & \mathbf{hpSpectra}^{\text{op}} & \xrightarrow{r_!^{\text{op}}} & \text{Ho}(\mathbf{Spectra})^{\text{op}} & \xrightarrow{H^*} & \mathbf{grMod}^{\mathbb{F}_\ell} \\ & & \downarrow (H\mathbb{F}_\ell \wedge -)_{\text{fw}}^{\text{op}} & & \downarrow (H\mathbb{F}_\ell \wedge -)^{\text{op}} & \searrow & \\ & & (\mathbf{hpMod}^{H\mathbb{F}_\ell})^{\text{op}} & \xrightarrow{r_!^{\text{op}}} & \text{Ho}(\mathbf{Mod}^{H\mathbb{F}_\ell})^{\text{op}} & \xrightarrow{\pi_{-*} F^{H\mathbb{F}_\ell}(-, H\mathbb{F}_\ell)} & \mathbf{grMod}^{\mathbb{F}_\ell} \end{array} \quad (2.21)$$

commutes up to natural isomorphism.

*Proof.* The commutativity up to natural isomorphism of the triangle on the right follows by the computation

$$\pi_{-*} F^{H\mathbb{F}_\ell}(H\mathbb{F}_\ell \wedge X, H\mathbb{F}_\ell) \cong \pi_{-*} F(X, H\mathbb{F}_\ell) = H^*(X).$$

The commutativity up to natural isomorphism of the square in the middle follows from the commutativity up to natural equivalence of the squares

$$\begin{array}{ccc} \text{Ho}(\mathbf{Spectra}_{/B}) & \xrightarrow{r_!} & \text{Ho}(\mathbf{Spectra})^{\text{op}} \\ (H\mathbb{F}_\ell \wedge -)_B \downarrow & & \downarrow H\mathbb{F}_\ell \wedge - \\ \text{Ho}(\mathbf{Mod}_{/B}^{H\mathbb{F}_\ell}) & \xrightarrow{r_!} & \text{Ho}(\mathbf{Mod}^{H\mathbb{F}_\ell}) \end{array}$$

for all spaces  $B$ , which in turn follows from the commutativity up to natural equivalence of the corresponding square of right adjoints

$$\begin{array}{ccc} \text{Ho}(\mathbf{Spectra}_{/B}) & \xleftarrow{r^*} & \text{Ho}(\mathbf{Spectra})^{\text{op}} \\ \text{forget}_B \uparrow & & \uparrow \text{Ho}(\text{forget}) \\ \text{Ho}(\mathbf{Mod}_{/B}^{H\mathbb{F}_\ell}) & \xleftarrow{r^*} & \text{Ho}(\mathbf{Mod}^{H\mathbb{F}_\ell}) \end{array}$$

where ‘forget’ stands for the forgetful functor  $\mathbf{Mod}^{H\mathbb{F}_\ell} \rightarrow \mathbf{Spectra}$ .  $\square$

**Lemma 2.35.** *The functor*

$$(H\mathbb{F}_\ell \wedge -)_{\text{fw}}: \mathbf{hpSpectra} \longrightarrow \mathbf{hpMod}^{H\mathbb{F}_\ell}$$

sends the monoidality (2.17) and identity (2.18) constraints of the lax symmetric monoidal functor

$$U_{\text{fw}}: \mathbf{hpSpectra}^\ell \longrightarrow \mathbf{hpSpectra}$$

to equivalences.

*Proof.* As equivalences in  $\mathbf{hpSpectra}$  are detected on fibres, it is enough to show that the identity and monoidality constraints are equivalences on fibres. But on fibres, up to equivalence both maps amount to unit maps of the adjunction

$$L: \mathbf{Ho}(\mathbf{Spectra}) \rightleftarrows \mathbf{Ho}(\mathbf{Spectra}^\ell): U.$$

The claim follows since the unit of the  $(L, U)$  adjunction is an  $H\mathbb{F}_\ell$ -equivalence.  $\square$

*Proof of Proposition 2.33.* Since the functors  $r_!^{\text{op}}$  and  $H^*$  in (2.20) are lax symmetric monoidal, it is enough to show the composite  $H^*r_!^{\text{op}}$  sends the identity and monoidality constraints of  $U_{\text{fw}}^{\text{op}}$  to isomorphisms. The monoidality constraint of  $H^*r_!^{\text{op}}U_{\text{fw}}^{\text{op}}$  is then given by the composite

$$H^*r_!U_{\text{fw}}(X) \otimes H^*r_!U_{\text{fw}}(Y) \longrightarrow H^*r_!(U_{\text{fw}}(X) \bar{\wedge} U_{\text{fw}}(Y)) \xrightarrow{\cong} H^*r_!U_{\text{fw}}(X \bar{\wedge}^\ell Y)$$

where the first map is the monoidality constraint for  $H^*r_!^{\text{op}}$  and the second map is the inverse of map obtained by applying  $H^*r_!$  to the monoidality constraint of  $U_{\text{fw}}$ , and the identity constraint is obtained similarly. In view of Lemma 2.34, the claim now follows from Lemma 2.35.  $\square$

**2.7. Identifying the result.** We have now constructed the categories  $(H^*r_!^{\text{op}}U_{\text{fw}}^{\text{op}}D_{\text{fw}}^{\text{op}})_*\mathcal{P}_B$  enriched in graded  $\mathbb{F}_\ell$ -modules we set out to construct at the end of subsection 2.2, along with the enriched functors

$$(H^*r_!^{\text{op}}U_{\text{fw}}^{\text{op}}D_{\text{fw}}^{\text{op}})_*(K_\varphi): (H^*r_!^{\text{op}}U_{\text{fw}}^{\text{op}}D_{\text{fw}}^{\text{op}})_*\mathcal{P}_B \longrightarrow (H^*r_!^{\text{op}}U_{\text{fw}}^{\text{op}}D_{\text{fw}}^{\text{op}})_*\mathcal{P}_A$$

between them. To complete the proof of Theorems 2.1 and 2.12, it now suffices to prove the following result.

**Theorem 2.36.** *For all  $f, g: B \rightarrow BG$ , there is an isomorphism*

$$H^*(r_!U_{\text{fw}}D_{\text{fw}}P(f, g)) \cong \mathbb{H}^*(P(f, g))$$

natural with respect to the homomorphisms induced by diagram (2.5).

The proof of Theorem 2.36 is based on Proposition 2.38 below, which is essentially a fibrewise reformulation of Bauer's result on self-duality of  $\ell$ -compact groups [Bau04, Proposition 22]. See also [Rog08, Proposition 3.2.3]. Rather than deriving the proposition from Bauer's (or Rognes's) result, however, we will prove it from scratch, taking our cue from Rognes's treatment [Rog08]. In what follows, we consider the path space  $BG^I$  as a space over  $BG \times BG$  via the evaluation fibration

$$(\text{ev}_0, \text{ev}_1): BG^I \longrightarrow BG \times BG, \quad \alpha \longmapsto (\alpha(0), \alpha(1)) \tag{2.22}$$

and to simplify notation, we sometimes continue to write  $BG^I$  for the corresponding object

$$BG^I = L_{\text{fw}}\Sigma_{+BG^2}^\infty BG^I$$

in  $\mathbf{Ho}(\mathbf{Spectra}_{BG^2}^\ell)$  when this is unlikely to cause confusion. For  $1 \leq i < j \leq 3$ , let us write

$$\pi_{ij}: BG^3 \longrightarrow BG^2$$

for the projection onto the  $i$ -th and  $j$ -th coordinates. The following result plays the role of the "shear equivalence" of [Bau04, (4.6)] or [Rog08, Lemma 3.1.3].

**Lemma 2.37.** *There is an equivalence*

$$\pi_{12}^* D_{\text{fw}} BG^I \wedge_{BG^3}^\ell \pi_{23}^* BG^I \simeq \pi_{13}^* D_{\text{fw}} BG^I \wedge_{BG^3}^\ell \pi_{23}^* BG^I \quad (2.23)$$

in  $\text{Ho}(\mathbf{Spectra}_{/BG^3}^\ell)$ .

*Proof.* Let

$$\delta: \pi_{23}^* BG^I \longrightarrow \pi_{23}^* BG^I \wedge_{BG^3}^\ell \pi_{23}^* BG^I$$

be the map induced by the fibrewise diagonal map

$$\pi_{23}^* BG^I \longrightarrow \pi_{23}^* BG^I \times_{BG^3} \pi_{23}^* BG^I, \quad x \longmapsto (x, x)$$

of spaces over  $BG^3$ ; let

$$\alpha: \pi_{12}^* D_{\text{fw}} BG^I \wedge_{BG^3}^\ell \pi_{23}^* BG^I \longrightarrow \pi_{13}^* D_{\text{fw}} BG^I$$

be the adjoint to the composite

$$\pi_{12}^* D_{\text{fw}} BG^I \wedge_{BG^3}^\ell \pi_{23}^* BG^I \wedge_{BG^3}^\ell \pi_{13}^* BG^I \xrightarrow{1 \wedge^\ell c} \pi_{12}^* D_{\text{fw}} BG^I \wedge_{BG^3}^\ell \pi_{12}^* BG^I \xrightarrow{\text{ev}} S_{BG^3}$$

where  $c$  is induced by the map

$$\pi_{23}^* BG^I \times_{BG^3} \pi_{13}^* BG^I \longrightarrow \pi_{12}^* BG^I, \quad (x, y) \longmapsto x^{-1} * y$$

of spaces over  $BG^3$  and  $\text{ev}$  is adjoint to the identity map of  $\pi_{12}^* D_{\text{fw}} BG^I$ ; and let

$$\beta: \pi_{13}^* D_{\text{fw}} BG^I \wedge_{BG^3}^\ell \pi_{23}^* BG^I \longrightarrow \pi_{12}^* D_{\text{fw}} BG^I$$

be the adjoint to the composite

$$\pi_{13}^* D_{\text{fw}} BG^I \wedge_{BG^3}^\ell \pi_{23}^* BG^I \wedge_{BG^3}^\ell \pi_{12}^* BG^I \xrightarrow{1 \wedge^\ell c} \pi_{13}^* D_{\text{fw}} BG^I \wedge_{BG^3}^\ell \pi_{13}^* BG^I \xrightarrow{\text{ev}} S_{BG^3}$$

where  $c$  is induced by the map

$$\pi_{23}^* BG^I \times_{BG^3} \pi_{12}^* BG^I \longrightarrow \pi_{13}^* BG^I, \quad (x, y) \longmapsto x * y$$

of spaces over  $BG^3$  and  $\text{ev}$  is adjoint to the identity map of  $\pi_{13}^* D_{\text{fw}} BG^I$ . Now a diagram chase shows that the composite maps

$$\begin{aligned} \Phi: \pi_{12}^* D_{\text{fw}} BG^I \wedge_{BG^3}^\ell \pi_{23}^* BG^I &\xrightarrow{1 \wedge^\ell \delta} \pi_{12}^* D_{\text{fw}} BG^I \wedge_{BG^3}^\ell \pi_{23}^* BG^I \wedge_{BG^3}^\ell \pi_{23}^* BG^I \\ &\xrightarrow{\alpha \wedge^\ell 1} \pi_{13}^* D_{\text{fw}} BG^I \wedge_{BG^3}^\ell \pi_{23}^* BG^I \end{aligned}$$

and

$$\begin{aligned} \Psi: \pi_{13}^* D_{\text{fw}} BG^I \wedge_{BG^3}^\ell \pi_{23}^* BG^I &\xrightarrow{1 \wedge^\ell \delta} \pi_{13}^* D_{\text{fw}} BG^I \wedge_{BG^3}^\ell \pi_{23}^* BG^I \wedge_{BG^3}^\ell \pi_{23}^* BG^I \\ &\xrightarrow{\beta \wedge^\ell 1} \pi_{12}^* D_{\text{fw}} BG^I \wedge_{BG^3}^\ell \pi_{23}^* BG^I \end{aligned}$$

are inverses of each other. In symbols,

$$\Phi: \xi \wedge^\ell x \longmapsto (y \mapsto \xi(x^{-1} * y)) \wedge^\ell x \quad \text{and} \quad \Psi: \xi \wedge^\ell x \longmapsto (y \mapsto \xi(x * y)) \wedge^\ell x. \quad \square$$

Let

$$\pi_1, \pi_2: BG^2 \longrightarrow BG$$

be the projections onto the first and second coordinates, respectively, and let

$$Q = (\pi_2)_! D_{\text{fw}} BG^I \in \text{Ho}(\mathbf{Spectra}_{/BG}^\ell).$$

**Proposition 2.38.** *There is an equivalence*

$$D_{\text{fw}} BG^I \simeq BG^I \wedge_{BG^2}^\ell \pi_2^* Q \quad (2.24)$$

in  $\text{Ho}(\mathbf{Spectra}_{/BG^2}^\ell)$ .

*Proof.* By Example 2.21 and the criterion of Corollary B.3,  $BG^I$  is dualizable as an object of  $\mathrm{Ho}(\mathbf{Spectra}_{/BG^2}^\ell)$ . Applying  $(\pi_{13})!D_{\mathrm{fw}}$  to the equivalence (2.23), we therefore obtain an equivalence

$$(\pi_{13})!(\pi_{12}^*BG^I \wedge_{BG^3}^\ell \pi_{23}^*D_{\mathrm{fw}}BG^I) \simeq (\pi_{13})!(\pi_{13}^*BG^I \wedge_{BG^3}^\ell \pi_{23}^*D_{\mathrm{fw}}BG^I) \quad (2.25)$$

in  $\mathrm{Ho}(\mathbf{Spectra}_{/BG^2}^\ell)$ . Here we used that the double dual of a dualizable object is the object itself, and that the dual of a smash product of dualizable objects is the smash product of duals. The left hand side of the equivalence is

$$\begin{aligned} & (\pi_{13})!(\pi_{12}^*BG^I \wedge_{BG^3}^\ell \pi_{23}^*D_{\mathrm{fw}}BG^I) \\ & \simeq (\pi_{13})!(\pi_{12}^*\Delta_!S_{BG} \wedge_{BG^3}^\ell \pi_{23}^*D_{\mathrm{fw}}BG^I) \\ & \simeq (\pi_{13})!(\Delta \times 1)_!S_{BG^2} \wedge_{BG^3}^\ell \pi_{23}^*D_{\mathrm{fw}}BG^I) \\ & \simeq (\pi_{13})!(\Delta \times 1)_!((\Delta \times 1)^*\pi_{23}^*D_{\mathrm{fw}}BG^I) \\ & \simeq D_{\mathrm{fw}}BG^I \end{aligned}$$

Here  $\Delta$  denotes the diagonal map of  $BG$ . The first equivalence follows from the equivalence  $BG^I \simeq \Delta_!S_{BG}$  arising from the equivalence of (2.22) and  $\Delta: BG \rightarrow BG \times BG$  as spaces over  $BG \times BG$ . The second equivalence follows by the commutation relation (A.7) arising from the square

$$\begin{array}{ccc} BG^2 & \xrightarrow{\Delta \times 1} & BG^3 \\ \pi_1 \downarrow & & \downarrow \pi_{12} \\ BG & \xrightarrow{\Delta} & BG^2 \end{array}$$

The third equivalence follows from the projection formula (A.5), and the last equivalence follows from the observation that  $\pi_{13}(\Delta \times 1)$  and  $\pi_{23}(\Delta \times 1)$  are both the identity map of  $BG^2$ . On the other hand, the right hand side of (2.25) is

$$\begin{aligned} & (\pi_{13})!(\pi_{13}^*BG^I \wedge_{BG^3}^\ell \pi_{23}^*D_{\mathrm{fw}}BG^I) \\ & \simeq BG^I \wedge_{BG^2}^\ell (\pi_{13})!\pi_{23}^*D_{\mathrm{fw}}BG^I \\ & \simeq BG^I \wedge_{BG^2}^\ell \pi_2^*(\pi_2)_!D_{\mathrm{fw}}BG^I \\ & \simeq BG^I \wedge_{BG^2}^\ell \pi_2^*Q \end{aligned}$$

Here the first equivalence follows by the projection formula (A.5) and the second equivalence follows by the commutation relation (A.7) arising from the square

$$\begin{array}{ccc} BG^3 & \xrightarrow{\pi_{23}} & BG^2 \\ \pi_{13} \downarrow & & \downarrow \pi_2 \\ BG^2 & \xrightarrow{\pi_2} & BG \end{array}$$

The last equivalence holds by the definition of  $Q$ . The claim follows.  $\square$

The following result corresponds to [Bau04, Corollary 23].

**Proposition 2.39.** *The fibres of  $Q$  are  $(-d)$ -dimensional  $H\mathbb{F}_\ell$ -local spheres.*

*Proof.* Passing to fibres over a point  $(b, b) \in BG^2$  in (2.24), we obtain an equivalence

$$F(L\Sigma_+^\infty \Omega BG, S) \simeq L\Sigma_+^\infty \Omega BG \wedge^\ell Q_b$$

in  $\mathrm{Ho}(\mathbf{Spectra}^\ell)$ . It follows that

$$H^{-*}(\Omega BG) \cong H_*(\Omega BG) \otimes H_*(Q_b).$$

We deduce that

$$H_n(Q_b) \cong \begin{cases} \mathbb{F}_\ell & \text{if } n = -d \\ 0 & \text{otherwise} \end{cases}$$

Pick a map  $f_0: Q_b \rightarrow \Sigma^{-d}H\mathbb{F}_\ell$  representing the dual of a generator of  $H_{-d}(Q_b)$ . The long exact cohomology sequences associated with the short exact sequences

$$0 \longrightarrow \mathbb{Z}/\ell \longrightarrow \mathbb{Z}/\ell^{k+1} \longrightarrow \mathbb{Z}/\ell^k \longrightarrow 0 \quad (2.26)$$

of coefficients show that the map  $H^{-d}(Q_b; \mathbb{Z}/\ell^{k+1}) \rightarrow H^{-d}(Q_b; \mathbb{Z}/\ell^k)$  is an epimorphism for all  $k \geq 1$ . Thus the map  $f_0$  can be lifted along the tower

$$H\mathbb{Z}_\ell \simeq \text{holim}_k H\mathbb{Z}/\ell^k \longrightarrow \cdots \longrightarrow H\mathbb{Z}/\ell^3 \longrightarrow H\mathbb{Z}/\ell^2 \longrightarrow H\mathbb{Z}/\ell$$

to a map

$$f_1: Q_b \longrightarrow \Sigma^{-d}H\mathbb{Z}_\ell = \Sigma^{-d}H\pi_{-d}(LS^{-d}).$$

Working up the Postnikov tower of  $LS^{-d}$ , we can moreover find a lift

$$f_2: Q_b \longrightarrow LS^{-d}$$

of  $f_1$ : an induction on  $k$  using the long exact sequences associated to the short exact sequences (2.26) proves that  $H^n(Q_b; \mathbb{Z}/\ell^k) = 0$  for all  $n \neq -d$  and  $k \geq 1$ , showing that all the obstructions for finding a lift vanish. The map  $f_2$  induces a nontrivial map on mod  $\ell$  homology since the map  $f_0$  does. Thus the map  $f_2$  is an equivalence.  $\square$

We are now ready to prove the following result. For a space  $B$ , write

$$H\mathbb{F}_\ell \wedge_B: \text{Ho}(\mathbf{Spectra}/_B) \longrightarrow \text{Ho}(\mathbf{Mod}^{H\mathbb{F}_\ell}/_B)$$

for the functor induced by the functor  $H\mathbb{F}_\ell \wedge: \mathbf{Spectra} \rightarrow \mathbf{Mod}^{H\mathbb{F}_\ell}$  by the construction of appendix A.3.

**Proposition 2.40.** *For all  $f, g: B \rightarrow BG$ , there exists an equivalence*

$$H\mathbb{F}_\ell \wedge_B U_{\text{fw}} D_{\text{fw}} P(f, g) \simeq H\mathbb{F}_\ell \wedge_B \Sigma_B^{-d} \Sigma_{+B}^\infty P(f, g)$$

in  $\text{Ho}(\mathbf{hpMod}^{H\mathbb{F}_\ell})$  natural with respect to maps induced by diagram (2.5).

*Proof.* It suffices to construct the desired equivalence in the universal case where

$$f = \pi_1, g = \pi_2: BG \times BG \longrightarrow BG$$

in which case the space  $P(\pi_1, \pi_2) \rightarrow BG \times BG$  is simply  $(\text{ev}_0, \text{ev}_1): BG^I \rightarrow BG \times BG$ . Applying the composite functor

$$\mathbf{hpSpectra}^\ell \xrightarrow{U_{\text{fw}}} \mathbf{hpSpectra} \xrightarrow{(H\mathbb{F}_\ell \wedge -)_{\text{fw}}} \mathbf{hpMod}^{H\mathbb{F}_\ell} \quad (2.27)$$

to the equivalence (2.24), we obtain an equivalence

$$H\mathbb{F}_\ell \wedge_{BG^2} U_{\text{fw}} D_{\text{fw}} BG^I \simeq (H\mathbb{F}_\ell \wedge_{BG^2} U_{\text{fw}} BG^I) \wedge_{BG^2}^{H\mathbb{F}_\ell} (H\mathbb{F}_\ell \wedge_{BG^2} U_{\text{fw}} \pi_2^* Q)$$

in  $\text{Ho}(\mathbf{Mod}^{H\mathbb{F}_\ell}/_{BG^2})$ . To identify the right hand side, we have used the fact that the composite (2.27) is symmetric monoidal, as follows from Lemma 2.35. The first factor on the right hand side is equivalent to  $H\mathbb{F}_\ell \wedge_{BG^2} \Sigma_{+BG^2}^\infty BG^I$ , so the claim follows from the following lemma.  $\square$

**Lemma 2.41.** *There exists an equivalence*

$$H\mathbb{F}_\ell \wedge_{BG} U_{\text{fw}} Q \simeq H\mathbb{F}_\ell \wedge_{BG} r^* S^{-d}$$

in  $\text{Ho}(\mathbf{Mod}^{H\mathbb{F}_\ell}/_{BG})$ .

*Proof.* Since  $BG$  is simply connected, Theorem 3.4 below implies that there is a strongly convergent spectral sequence

$$E_2^{s,t} = H^s(BG) \otimes H^t(Q_b) \implies H^{s+t}(r_!^{BG} U_{\text{fw}} Q);$$

where  $b \in BG$  is a basepoint. By Proposition 2.39, the spectral sequence is concentrated on the  $t = -d$  line, so the spectral sequence collapses on the  $E_2$  page. Let

$$u \in H^{-d}(r_!^{BG} U_{\text{fw}} Q)$$

be the class corresponding to the class  $1 \otimes x \in H^0(BG) \otimes H^{-d}(Q_b)$  where  $x$  is a generator of  $H^{-d}(Q_b) \cong \mathbb{F}_\ell$ . By naturality of the spectral sequence, the class  $u$  now has the property that for every  $b \in BG$ , the restriction of  $u$  to  $H^{-d}(r_!^b U_{\text{fw}} Q_b) \cong H^{-d}(Q_b) \cong \mathbb{F}_\ell$  is a generator. The class  $u$  is equivalent to the data of a map

$$r_! U_{\text{fw}} Q \longrightarrow H\mathbb{F}_\ell \wedge S^{-d}$$

in  $\text{Ho}(\mathbf{Spectra})$ , which in turn via the adjunctions

$$r_! : \text{Ho}(\mathbf{Spectra}) \rightleftarrows \text{Ho}(\mathbf{Spectra}_{/BG}) : r^*$$

and

$$HF_\ell \wedge_{BG} : \text{Ho}(\mathbf{Spectra}_{/BG}) \rightleftarrows \text{Ho}(\mathbf{Mod}_{/BG}^{H\mathbb{F}_\ell}) : \text{forget}_{BG}$$

is equivalent to the data of a map

$$\tilde{u} : HF_\ell \wedge_{BG} U_{\text{fw}} Q \longrightarrow HF_\ell \wedge_{BG} r^* S^{-d}$$

in  $\text{Ho}(\mathbf{Mod}_{/BG}^{H\mathbb{F}_\ell})$ . Working through the adjunctions, it is easy to see that the property that  $u$  restricts to a generator for each fibre translates precisely to the property that  $\tilde{u}$  is an equivalence on all fibres. Since equivalences in  $\text{Ho}(\mathbf{Mod}_{/BG}^{H\mathbb{F}_\ell})$  are detected on fibres, the claim follows.  $\square$

**Remark 2.42.** The equivalence in Lemma 2.41 is not unique: the set of all such equivalences form an  $\mathbb{F}_\ell^\times$ -torsor, as follows by reversing the argument in the proof of Lemma 2.41. The choice of such an equivalence should be thought of as an  $H\mathbb{F}_\ell$ -orientation for the “sphere bundle”  $Q$ . For the purposes of constructing the pairings of Theorem 2.1 and the maps  $\iota$  and  $\rho$  of Theorem 2.7, we (arbitrarily) fix one such an orientation. The pairings and maps associated to different choices of orientation only differ from each other by multiplication by an element of  $\mathbb{F}_\ell^\times$ .

*Proof of Theorem 2.36.* The claim follows by applying the functor

$$(\mathbf{hpMod}^{H\mathbb{F}_\ell})^{\text{op}} \xrightarrow{r_!^{\text{op}}} \text{Ho}(\mathbf{Mod}^{H\mathbb{F}_\ell})^{\text{op}} \xrightarrow{\pi_{-*} F^{H\mathbb{F}_\ell}(-, H\mathbb{F}_\ell)} \mathbf{grMod}^{\mathbb{F}_\ell} \quad (2.28)$$

to the equivalence of Proposition 2.40 and using Lemma 2.34 to recognize the two sides of the resulting isomorphism as the two sides of the isomorphism of Theorem 2.36.  $\square$

**Remark 2.43.** Let us now reconcile our construction of the pairing of Theorem 2.1 with the description of this pairing given in Remark 2.3. Notice that the map of  $\mathbf{Fib}^{\text{fop}}$  defined by diagram (2.2) splits as the composite of the maps of  $\mathbf{Fib}^{\text{fop}}$  defined by the trapezoid and the triangle



in (2.2). Consider now the following diagram in  $\mathbf{hpMod}^{H\mathbb{F}_\ell}$ :

$$\begin{array}{ccc}
 & & (H\mathbb{F}_\ell \wedge_B \Sigma_B^{-d} \Sigma_{+B}^\infty P(g, h)) \bar{\wedge}^{H\mathbb{F}_\ell} (H\mathbb{F}_\ell \wedge_B \Sigma_B^{-d} \Sigma_{+B}^\infty P(f, g)) \\
 & & \nearrow \simeq \\
 & & (H\mathbb{F}_\ell \wedge_B U_{\text{fw}} D_{\text{fw}} P(g, h)) \bar{\wedge}^{H\mathbb{F}_\ell} (H\mathbb{F}_\ell \wedge_B U_{\text{fw}} D_{\text{fw}} P(f, g)) \\
 & & \uparrow H\mathbb{F}_\ell \wedge_{\text{fw}} \Sigma_B^{-2d} \Sigma_{+B}^\infty (\text{split}) \\
 H\mathbb{F}_\ell \wedge_{\text{fw}} U_{\text{fw}} D_{\text{fw}} (\text{split}) & & H\mathbb{F}_\ell \wedge_B \Sigma_B^{-2d} \Sigma_{+B}^\infty P(f, g, h) \\
 & \nearrow \simeq & \uparrow \text{concat}^\sharp \\
 H\mathbb{F}_\ell \wedge_B U_{\text{fw}} D_{\text{fw}} P(f, g, h) & & H\mathbb{F}_\ell \wedge_B \Sigma_B^{-d} \Sigma_{+B}^\infty P(f, h) \\
 & \nearrow \simeq & \uparrow \\
 H\mathbb{F}_\ell \wedge_{\text{fw}} U_{\text{fw}} D_{\text{fw}} (\text{concat}) & & H\mathbb{F}_\ell \wedge_B U_{\text{fw}} D_{\text{fw}} P(f, h) \\
 & \nearrow \simeq & \\
 H\mathbb{F}_\ell \wedge_B U_{\text{fw}} D_{\text{fw}} P(f, h) & & 
 \end{array}$$

Here the top slanted equivalence is obtained by smashing together the equivalences of Proposition 2.40 for  $P(g, h)$  and  $P(f, g)$ ; the bottom slanted equivalence is the equivalence of Proposition 2.40 for  $P(f, h)$ ; the middle slanted equivalence is obtained by pulling back the equivalence on top along the diagonal map  $\Delta: B \rightarrow B \times B$ ; and the map  $\text{concat}^\sharp$  is defined by the commutativity of the bottom parallelogram. The diagram then commutes by construction. Tracing through the definition of the pairing of Theorem 2.1 and making use of Lemma 2.34, one sees that the pairing can be obtained by applying the composite functor (2.28) to the above diagram and following in the resulting diagram the path down along the upper slanted equivalence, down along the left-hand vertical maps, and finally up along the bottom slanted equivalence. The description of the pairing of Theorem 2.1 given in Remark 2.3 now follows by the commutativity of the diagram: upon applying the composite functor (2.28), the top right-hand vertical map yields the map  $\text{split}^*$ , while the bottom right-hand vertical map gives the “umkehr map”  $\text{concat}_!$  in (2.3).

**2.8. Proof of Theorems 2.5, 2.7, and 2.14.** Having concluded the proof of Theorems 2.1 and 2.12 in the previous subsection, we now turn to the proof of Theorems 2.5, 2.7, and 2.14.

*Proof of Theorem 2.5.* The product on  $\mathbb{H}^* \Omega BG$  agrees under the isomorphism

$$\mathbb{H}^* \Omega BG \cong H^* D \Omega BG$$

provided by Theorem 2.36 with the composite

$$\begin{aligned}
 H^*(D \Omega BG) \otimes H^*(D \Omega BG) &\xrightarrow{\times} H^*(D \Omega BG \wedge^\ell D \Omega BG) \\
 &\xrightarrow{\cong} H^* D(\Omega BG \times \Omega BG) \\
 &\xrightarrow{(D \text{concat})^*} H^*(D \Omega BG)
 \end{aligned}$$

Here  $DX$  for a space  $X$  denotes the dual of  $L\Sigma_+^\infty X$  in  $\text{Ho}(\mathbf{Spectra}^\ell)$ , and the middle isomorphism is induced by the equivalence  $D(\Omega BG \times \Omega BG) \simeq D \Omega BG \wedge^\ell D \Omega BG$ . The claim now follows using the natural isomorphism  $H^* DX \cong H_{-*} X$  valid for  $H\mathbb{F}_\ell$ -locally dualizable spaces  $X$ .  $\square$

**Definition 2.44** (The maps  $\iota_f$  and  $\rho_f$  of Theorem 2.7). For a map  $f: B \rightarrow BG$ , consider the commutative triangles

$$\begin{array}{ccc} B & \xrightarrow{s} & P(f, f) \\ & \searrow \text{id}_B & \swarrow \pi \\ & & B \end{array} \quad \text{and} \quad \begin{array}{ccc} P(f, f) & \xrightarrow{\pi} & B \\ & \searrow \pi & \swarrow \text{id}_B \\ & & B \end{array} \quad (2.29)$$

where  $\pi$  is the projection and  $s$  is the section sending a point  $b \in B$  to the pair consisting of  $b$  and the constant path at  $f(b)$ . The triangles define morphisms

$$(B \xrightarrow{\text{id}} B) \longrightarrow (P(f, f) \rightarrow B) \quad \text{and} \quad (P(f, f) \rightarrow B) \longrightarrow (B \xrightarrow{\text{id}} B) \quad (2.30)$$

in the category  $(\mathbf{Fib}^{\text{fop}})^{\text{op}}$ . We define the maps

$$\iota = \iota_f: H^*B \longrightarrow \mathbb{H}^*P(f, f) \quad \text{and} \quad \rho = \rho_f: \mathbb{H}^*P(f, f) \longrightarrow H^*B$$

to be the morphisms obtained by first applying the composite functor (2.9) to the morphisms (2.30) and then using the isomorphism

$$H^*(r_!U_{\text{fw}}D_{\text{fw}}P(f, f)) \cong \mathbb{H}^*(P(f, f))$$

of Theorem 2.36 and the computation

$$\begin{aligned} H^*(r_!U_{\text{fw}}D_{\text{fw}}B) &\cong H^*(r_!U_{\text{fw}}S_{B,\ell}) \cong H^*(r_!^B U_B L_B S_B) \cong H^*(ULr_!^B U_B L_B S_B) \\ &\cong H^*(Ur_!^B L_B U_B L_B S_B) \cong H^*(Ur_!^B L_B S_B) \cong H^*(ULr_!^B S_B) \cong H^*(r_!^B S_B) \cong H^*(B). \end{aligned}$$

to recognize the source and the target. Here we have written  $S_B$  and  $S_{B,\ell}$  for the unit objects in  $\text{Ho}(\mathbf{Spectra}_{/B})$  and  $\text{Ho}(\mathbf{Spectra}_{/B}^\ell)$ , respectively. Here the first isomorphism in the top row follows from the observation that  $D_{\text{fw}}B \simeq S_{B,\ell}$  (where  $B$  on the left hand side is short for the object  $L_B \Sigma_{+B}^\infty B$  over  $B$  in  $\mathbf{hpSpectra}^\ell$ ); the second isomorphism in the top row substitutes the definitions of  $r_!$  and  $U_{\text{fw}}$ , and uses that  $L_B$ , as a symmetric monoidal functor, preserves unit objects; the last isomorphism in the top row and the second last one in the bottom row are induced by the unit of the  $(L, U)$  adjunction, an  $H\mathbb{F}_\ell$ -equivalence; the first and third isomorphisms on the bottom row are induced by the commutation relation  $Lr_!^B \simeq r_!^B L_B$  implied by the commutation relation  $r_B^* U_B \simeq Ur_B^*$  between right adjoints; and the second isomorphism in the bottom row is induced by the natural equivalence  $L_B U_B \simeq \text{id}$  (deriving from the natural equivalence  $LU \simeq \text{id}$ ).

*Proof of Theorem 2.7.* The diagrams

$$\begin{array}{ccc} B & \xrightarrow{\Delta} & B \times B \\ \text{id} \downarrow & & \downarrow \text{id} \times \text{id} \\ B & \xrightarrow{\Delta} & B \times B \end{array} \quad \text{and} \quad \begin{array}{ccc} B & \xrightarrow{r} & \text{pt} \\ \text{id} \downarrow & & \downarrow \text{id} \\ B & \xrightarrow{r} & \text{pt} \end{array}$$

define morphisms in  $\mathbf{Fib}^{\text{fop}}$  making the identity map  $(B \xrightarrow{\text{id}} B)$  into a monoid object in the category  $(\mathbf{Fib}^{\text{fop}})^{\text{op}}$ . Ignoring the problem that some of the required identities only hold up to fibrewise homotopy (cf. the discussion following ‘‘Definition’’ 2.26), the morphisms (2.12) and (2.11) (with  $g = h = f$ ) make  $P(f, f) \rightarrow B$  a monoid object in  $(\mathbf{Fib}^{\text{fop}})^{\text{op}}$ . It is straightforward to verify that the morphisms (2.30) in  $(\mathbf{Fib}^{\text{fop}})^{\text{op}}$  are monoid object homomorphisms whose composite is the identity. The claim now follows by applying the composite functor (2.9); by recognizing the image of the monoid object  $(B \xrightarrow{\text{id}} B)$  under this functor as  $H^*(B)$  equipped with the cup product; and by using Theorem 2.36 to recognize the image of  $(P(f, f) \rightarrow B)$  as the graded ring  $\mathbb{H}^*P(f, f)$ .  $\square$

*Proof of Theorem 2.14.* Consider the diagram in  $(\mathbf{Fib}^{\text{fop}})^{\text{op}}$

$$\begin{array}{ccc}
 (B \xrightarrow{\text{id}} B) & \longrightarrow & (A \xrightarrow{\text{id}} A) \\
 \downarrow & & \downarrow \\
 (P(f, f) \rightarrow B) & \longrightarrow & (P(f\varphi, f\varphi) \rightarrow A) \\
 \downarrow & & \downarrow \\
 (B \xrightarrow{\text{id}} B) & \longrightarrow & (A \xrightarrow{\text{id}} A)
 \end{array}$$

where the vertical arrows are induced by the diagrams (2.29) and the analogous diagrams for  $f\varphi$ , where the top and bottom horizontal arrows are induced by the pullback square

$$\begin{array}{ccc}
 A & \xrightarrow{\varphi} & B \\
 \text{id} \downarrow & & \downarrow \text{id} \\
 A & \xrightarrow{\varphi} & B
 \end{array}$$

and where the middle horizontal arrow is induced by square (2.5). It is straightforward to verify that the diagram commutes. The claim now follows by applying the composite functor (2.9) to the diagram and recognizing the result as diagram (2.8).  $\square$

**2.9. Proof of Theorem 2.11.** We finish the section with the somewhat lengthy proof of Theorem 2.11. By construction, Lemma 2.34, and Proposition 2.40, the pairing in Theorem 2.11 arises by applying the composite functor

$$\mathbf{hpMod}^{H\mathbb{F}_\ell} \xrightarrow{r_!} \mathbf{Ho}(\mathbf{Mod}^{H\mathbb{F}_\ell}) \xrightarrow{\pi_{-*}F^{H\mathbb{F}_\ell}(-, H\mathbb{F}_\ell)} \mathbf{grMod}^{\mathbb{F}_\ell}$$

to a certain map

$$H\mathbb{F}_\ell \wedge_B \Sigma_B^{-d} \Sigma_{+B}^\infty P(f, h) \longrightarrow (H\mathbb{F}_\ell \wedge_B \Sigma_B^{-d} \Sigma_{+B}^\infty P(g, h)) \bar{\wedge}^{H\mathbb{F}_\ell} (H\mathbb{F}_\ell \wedge_B \Sigma_B^{-d} \Sigma_{+B}^\infty P(f, g)) \quad (2.31)$$

in  $\mathbf{hpMod}^{H\mathbb{F}_\ell}$  covering the diagonal map  $\Delta: B \rightarrow B \times B$  in  $\mathcal{T}$ . The basic idea of our proof of Theorem 2.11 is as follows. For a general presentable symmetric monoidal  $\infty$ -category  $\mathcal{C}$  and space  $B$ , we construct a natural coalgebra structure on the object  $r_!^B S_B \in \mathbf{Ho}(\mathcal{C})$  along with, for any object  $X \in \mathbf{Ho}(\mathcal{C}/_B)$ , a natural  $r_!^B S_B$ -comodule structure on  $r_!^B X$ . When  $\mathcal{C} = \mathbf{Mod}^{H\mathbb{F}_\ell}$  and  $X = H\mathbb{F}_\ell \wedge_B \Sigma_B^{-d} \Sigma_{+B}^\infty P(f, g)$ , upon application of the functor  $\pi_{-*}F^{H\mathbb{F}_\ell}(-, H\mathbb{F}_\ell)$ , the coalgebra structure recovers the cup product on  $H^*B$  and the comodule structure recovers the module structure of Definition 2.10. This reduces the proof of Theorem 2.11 to proving sufficient compatibility and naturality properties for the coalgebra and comodule structures in  $\mathbf{Ho}(\mathcal{C})$ . Establishing these properties, along with the results needed for the comparison with the module structure of Definition 2.10, is done in Lemmas 2.46 through 2.50 below.

We begin by constructing the coalgebra and comodule structures. Let  $\mathcal{C}$  be a presentable symmetric monoidal  $\infty$ -category with tensor product  $\otimes$ . For each  $B \in \mathcal{T}$ , the functor

$$r_!^B: \mathbf{Ho}(\mathcal{C}/_B) \longrightarrow \mathbf{Ho}(\mathcal{C})$$

becomes an oplax symmetric monoidal functor if we equip it with the monoidality and identity constraints given by

$$(r_!^B)_{\otimes}: r_!^B(X \otimes_B Y) \xrightarrow{r_! \bar{\Delta}} r_!^{B \times B}(X \bar{\otimes} Y) \xrightarrow[\simeq]{(r_!)_{\otimes}^{-1}} r_!^B(X) \otimes r_!^B(Y)$$

and

$$(r_!^B)_I: r_!^B S_B \xrightarrow{r_!(\bar{r}_B)} r_!^{\text{pt}} S_{\text{pt}} \xrightarrow[\simeq]{(r_!)_I^{-1}} S,$$

respectively, where  $\tilde{\Delta}$  is the map in  $\mathbf{hpC}$  over the diagonal map  $\Delta: B \rightarrow B \times B$  defined by the canonical equivalence  $X \otimes_B Y \simeq \Delta^*(X \bar{\otimes} Y)$  and  $\tilde{r}_B$  is the map in  $\mathbf{hpC}$  over  $r_B: B \rightarrow \text{pt}$  defined by the canonical equivalence  $S_B \simeq r_B^* S_{\text{pt}}$ . It follows that the functor  $(r_B)_!$  preserves coalgebras and comodules. The equivalence  $S_B \simeq S_B \otimes_B S_B$  given by the left (equivalently, the right) unit constraint in  $\text{Ho}(\mathcal{C}/_B)$  makes  $S_B$  into a coalgebra such that the left unit constraint  $X \simeq S_B \otimes_B X$  makes every  $X$  in  $\text{Ho}(\mathcal{C}/_B)$  into an  $S_B$ -coalgebra. The desired coalgebra and comodule structures on  $r_!^B S_B$  and  $r_!^B X$  follow. Explicitly, the comultiplication and counit on  $r_!^B S_B$  are given by the composite

$$r_!^B S_B \xrightarrow[r_!^B(\lambda^{-1})]{\simeq} r_!^B(S_B \otimes_B S_B) \xrightarrow{(r_!^B)_\otimes} r_!^B S_B \otimes r_!^B S_B$$

and the map

$$r_!^B S_B \xrightarrow{(r_!^B)_I} S,$$

respectively, and the comodule structure on  $r_!^B X$  is given by the composite

$$\nu_X: r_!^B X \xrightarrow[r_!^B(\lambda^{-1})]{\simeq} r_!^B(S_B \otimes_B X) \xrightarrow{(r_!^B)_\otimes} r_!^B S_B \otimes r_!^B X.$$

Here  $\lambda$  refers to the left unit constraint.

**Remark 2.45.** In the case  $\mathcal{C} = \mathbf{Spaces}$ , for a space  $B$  and a parametrized space  $\pi: X \rightarrow B$  over  $B$ , these constructions recover the usual coalgebra structure on  $B$  given by the diagonal map  $\Delta: B \rightarrow B \times B$  and the coalgebra structure on  $X$  given by the map  $(\pi, \text{id}): X \rightarrow B \times X$ .

The following two lemmas follow by tedious diagram chases.

**Lemma 2.46.** *Given a map  $f: B \rightarrow C$  of spaces, the evident map  $\bar{f}: S_B \rightarrow S_C$  in  $\mathbf{hpC}$  covering  $f$  induces a coalgebra homomorphism*

$$r_!(\bar{f}): r_!^B S_B \longrightarrow r_!^C S_C.$$

Moreover, given a map  $\tilde{f}: X \rightarrow Y$  in  $\mathbf{hpC}$  covering  $f$ , the diagram

$$\begin{array}{ccc} r_!^B X & \xrightarrow{r_!(\tilde{f})} & r_!^C Y \\ \nu_X \downarrow & & \downarrow \nu_Y \\ r_!^B S_B \otimes r_!^B X & \xrightarrow{r_!(\tilde{f}) \otimes r_!(\tilde{f})} & r_!^C S_C \otimes r_!^C Y \end{array}$$

commutes, showing that the induced map  $r_!(\tilde{f}): r_!^B X \rightarrow r_!^C Y$  is  $r_!(\bar{f})$ -linear.  $\square$

**Lemma 2.47.** *For  $X \in \text{Ho}(\mathcal{C}/_B)$ ,  $Y \in \text{Ho}(\mathcal{C}/_C)$ , the comodule structures  $\nu_X$ ,  $\nu_Y$  and  $\nu_{X \bar{\otimes} Y}$  are compatible in the sense that the following diagram commutes:*

$$\begin{array}{ccc} r_!^B X \otimes r_!^C Y & \xrightarrow{\nu_X \otimes \nu_Y} & r_!^B S_B \otimes r_!^B X \otimes r_!^C S_C \otimes r_!^C Y \\ \downarrow (r_!)_\otimes \simeq & & \simeq \downarrow 1 \otimes \chi \otimes 1 \\ & & r_!^B S_B \otimes r_!^C S_C \otimes r_!^B X \otimes r_!^C Y \\ & & \simeq \downarrow (r_!)_\otimes \otimes (r_!)_\otimes \\ & & r_!^{B \times C}(S_B \bar{\otimes} S_C) \otimes r_!^{B \times C}(X \bar{\otimes} Y) \\ & & \simeq \downarrow \\ r_!^{B \times C}(X \bar{\otimes} Y) & \xrightarrow{\nu_{X \bar{\otimes} Y}} & r_!^{B \times C} S_{B \times C} \otimes r_!^{B \times C}(X \bar{\otimes} Y) \end{array}$$

Here  $\chi$  denotes the symmetry constraint and the bottom right vertical map is induced by the evident equivalence  $S_B \bar{\otimes} S_C \simeq S_{B \times C}$  in  $\text{Ho}(\mathcal{C}/_{B \times C})$ .  $\square$

The following lemma follows, in essence, by specializing to the case  $B = \text{pt}$ ,  $C = B$ ,  $X = T$ , and  $Y = X$  in Lemma 2.47.

**Lemma 2.48.** *Given  $T \in \text{Ho}(\mathcal{C})$  and  $X \in \text{Ho}(\mathcal{C}/_B)$ , the diagram*

$$\begin{array}{ccc}
 T \otimes r_1^B X & \xrightarrow{1 \otimes \nu_X} & T \otimes r_1^B S_B \otimes r_1^B X \\
 \downarrow \theta_X \simeq & & \downarrow \simeq \chi \otimes 1 \\
 & & r_1^B S_B \otimes T \otimes r_1^B X \\
 & & \downarrow \simeq 1 \otimes \theta_X \\
 r_1^B(r_B^* T \otimes_B X) & \xrightarrow{\nu_{r_B^* T \otimes_B X}} & r_1^B S_B \otimes r_1^B(r_B^* T \otimes_B X)
 \end{array}$$

commutes, where  $\theta_X$  is the composite

$$T \otimes r_1^B X \xrightarrow{\simeq} r_1^{\text{pt}} T \otimes r_1^B X \xrightarrow{(r_1)_{\otimes}} r_1^{\text{pt} \times B}(T \bar{\otimes} X) \xrightarrow{\simeq} r_1^B(r_B^* T \otimes_B X)$$

with the first and last arrows given by the evident equivalences.  $\square$

Suppose now  $\mathcal{D}$  is another presentable symmetric monoidal  $\infty$ -category and  $F: \mathcal{C} \rightarrow \mathcal{D}$  is a symmetric monoidal  $\infty$ -functor with a right adjoint  $G$ . For every space  $B$ , the commutation equivalence  $r_B^* G \simeq G_B r_B^*$  between right adjoints then induces an equivalence

$$c: r_1^B F_B \xrightarrow{\simeq} F r_1^B$$

between left adjoints.

**Lemma 2.49.** *The equivalence  $c$  is oplax symmetric monoidal.*

*Proof.* In terms of the calculus of mates [KS74], the commutation equivalence

$$\begin{array}{ccc}
 \text{Ho}(\mathcal{C}/_B) & \xrightarrow{F_B} & \text{Ho}(\mathcal{D}/_B) \\
 r_1^B \downarrow & \swarrow \simeq c & \downarrow r_1^B \\
 \text{Ho}(\mathcal{C}) & \xrightarrow{F} & \text{Ho}(\mathcal{D})
 \end{array} \tag{2.32}$$

can be obtained from the commutation equivalence

$$\begin{array}{ccc}
 \text{Ho}(\mathcal{C}/_B) & \xleftarrow{G_B} & \text{Ho}(\mathcal{D}/_B) \\
 r_B^* \uparrow & \searrow \simeq & \uparrow r_B^* \\
 \text{Ho}(\mathcal{C}) & \xleftarrow{G} & \text{Ho}(\mathcal{D})
 \end{array} \tag{2.33}$$

in two stages by first passing to the mate of (2.33) with respect to the  $(F_B, G_B)$  and  $(F, G)$  adjunctions to obtain the transformation

$$\begin{array}{ccc}
 \text{Ho}(\mathcal{C}/_B) & \xrightarrow{F_B} & \text{Ho}(\mathcal{D}/_B) \\
 r_B^* \uparrow & \searrow & \uparrow r_B^* \\
 \text{Ho}(\mathcal{C}) & \xrightarrow{F} & \text{Ho}(\mathcal{D})
 \end{array} \tag{2.34}$$

and then passing to the mate of (2.34) with respect to the  $(r_1^B, r_B^*)$  adjunctions to obtain (2.32). The transformation (2.34) is simply the usual commutation equivalence  $F_B r_B^* \simeq r_B^* F$ , which is

symmetric monoidal. Moreover, the  $(r_!^B, r_B^*)$  adjunctions are in fact adjunctions of oplax symmetric monoidal functors, as it is easily verified that the monoidality and unit constraints of  $r_!^B$  agree with the mates of the monoidality and unit constraints of  $r_B^*$ , as required [Kel74, section 1.3]. Thus the commutation equivalence (2.32) is oplax symmetric monoidal, as claimed.  $\square$

From Lemma 2.49, one easily deduces the following result.

**Lemma 2.50.** *Equip  $Fr_!^B S_B^C$  with the coalgebra structure given by the composite*

$$Fr_!^B S_B^C \xrightarrow{\simeq} F(r_!^B S_B^C \otimes r_!^B S_B^C) \xrightarrow[\simeq]{F_\otimes^{-1}} Fr_!^B S_B^C \otimes Fr_!^B S_B^C$$

where the first arrow is induced by the coalgebra structure on  $r_!^B S_B^C$ . Then the composite map

$$r_!^B S_B^D \xrightarrow[\simeq]{r_!^B(F_B)_I} r_!^B F_B S_B^C \xrightarrow[\simeq]{c} Fr_!^B S_B^C$$

is an equivalence of coalgebra objects in  $\text{Ho}(\mathcal{D}/_B)$ . Moreover, for  $X \in \text{Ho}(\mathcal{C}/_B)$ , the diagram

$$\begin{array}{ccc} r_!^B F_B X & \xrightarrow{\nu_{F_B X}} & r_!^B S_B^D \otimes r_!^B F_B X \\ \downarrow c \simeq & & \simeq \downarrow r_!^B(F_B)_I \otimes 1 \\ & & r_!^B F_B S_B^C \otimes r_!^B F_B X \\ & & \simeq \downarrow c \otimes c \\ & & Fr_!^B S_B^C \otimes Fr_!^B X \\ & & \simeq \downarrow F_\otimes \\ Fr_!^B X & \xrightarrow{F(\nu_X)} & F(r_!^B S_B^C \otimes r_!^B X) \end{array}$$

commutes.  $\square$

We now turn to the proof of Theorem 2.11. Write  $(H\mathbb{F}_\ell)_B$  for the identity object in  $\text{Ho}(\mathbf{Mod}_{/B}^{H\mathbb{F}_\ell})$ . Upon application of the functor  $\pi_{-*} F^{H\mathbb{F}_\ell}(-, H\mathbb{F}_\ell)$ , the coalgebra structure on  $r_!^B (H\mathbb{F}_\ell)_B$  induces an algebra structure on  $H^*(B)$ . In view of Remark 2.45, applying Lemma 2.50 with  $F$  the functor

$$\mathbf{Spaces} \xrightarrow{\Sigma_+^\infty} \mathbf{Spectra} \xrightarrow{H\mathbb{F}_\ell \wedge -} \mathbf{Mod}^{H\mathbb{F}_\ell},$$

we see that this algebra structure agrees with the usual cup product on  $H^*(B)$ . Moreover, upon application of the functor  $\pi_{-*} F^{H\mathbb{F}_\ell}(-, H\mathbb{F}_\ell)$ , the comodule structure of

$$r_!^B (H\mathbb{F}_\ell \wedge_B \Sigma_B^{-d} \Sigma_{+B}^\infty P(f, g))$$

over  $r_!^B (H\mathbb{F}_\ell)_B$  induces on  $\mathbb{H}^* P(f, g)$  a module structure over  $H^*(B)$ . In view of Remark 2.45, Lemma 2.50 applied with  $F$  the functor  $\Sigma_+^\infty: \mathbf{Spaces} \rightarrow \mathbf{Spectra}$ , Lemma 2.48 applied with  $T = S^{-d}$ , and Lemma 2.50 applied with  $F$  the functor  $H\mathbb{F}_\ell \wedge (-): \mathbf{Spectra} \rightarrow \mathbf{Mod}^{H\mathbb{F}_\ell}$  imply that this module structure agrees with that of Definition 2.10. Theorem 2.11 now follows by applying Lemma 2.46 to the map (2.31) and using Lemma 2.47.

### 3. SPECTRAL SEQUENCES

For a fibration  $\pi: X \rightarrow B$ , let us write  $\mathbb{E}(X)$  for the strongly convergent spectral sequence

$$\mathbb{E}_r^{s,t}(X) \implies \mathbb{H}^{s+t}(X)$$

obtained from the Serre spectral sequence  $E(X)$  of  $\pi$  by setting  $\mathbb{E}_r^{s,t}(X) = E_r^{s,t+d}(X)$ . Our aim is to prove the following analogues of Theorems 2.1 and 2.12.

**Theorem 3.1.** *Let  $B$  be a space. For maps  $f, g, h: B \rightarrow BG$ , there is a pairing*

$$\circ: \mathbb{E}_r^{s_1, t_1}(P(g, h)) \otimes \mathbb{E}_r^{s_2, t_2}(P(f, g)) \longrightarrow \mathbb{E}_r^{s_1+s_2, t_1+t_2}(P(f, h))$$

*of spectral sequences compatible with the pairing*

$$\circ: \mathbb{H}^*P(g, h) \otimes \mathbb{H}^*P(f, g) \longrightarrow \mathbb{H}^*P(f, h).$$

*of Theorem 2.1 on targets. This pairing is associative, and it is unital in the following sense: for every map  $f: B \rightarrow BG$ , the spectral sequence*

$$\mathbb{E}_r^{s, t}(P(f, f)) \Longrightarrow \mathbb{H}^{s+t}(P(f, f))$$

*has a permanent cycle of degree  $(0, 0)$  which is a unit for the pairing on each page of the spectral sequence and which corresponds to the unit  $\mathbb{1}_f \in \mathbb{H}^0(P(f, f))$  on the target.*

**Theorem 3.2.** *Given a map of spaces  $\varphi: A \rightarrow B$ , the maps of spectral sequences*

$$\mathbb{E}(P(f, g)) \longrightarrow \mathbb{E}(P(f\varphi, g\varphi)) \quad (3.1)$$

*induced by (2.5) preserve the pairings and unit elements of Theorem 3.1. Moreover, these maps are compatible with the maps*

$$F_\varphi: \mathbb{H}^*(P(f, g)) \longrightarrow \mathbb{H}^*(P(f\varphi, g\varphi))$$

*of (2.6) on targets.*

Our strategy for proving Theorems 3.1 and 3.2 is similar to the one we employed to prove Theorems 2.1 and 2.12. Starting with the category  $(D_{\text{fw}}^{\text{op}})_* \mathcal{P}_B$  enriched in  $(\mathbf{hpSpectra}^\ell)^{\text{op}}$  and the enriched functor  $(D_{\text{fw}}^{\text{op}})_*(K_\varphi)$  constructed in Remark 2.28, we will apply Construction 2.19 with a lax monoidal functor

$$(\mathbf{hpSpectra}^\ell)^{\text{op}} \xrightarrow{U_{\text{fw}}^{\text{op}}} \mathbf{hpSpectra}^{\text{op}} \xrightarrow{E} \mathbf{SS} \quad (3.2)$$

to obtain a category  $(EU_{\text{fw}}^{\text{op}} D_{\text{fw}}^{\text{op}})_* \mathcal{P}_B$  enriched in the category  $\mathbf{SS}$  of spectral sequences along with  $\mathbf{SS}$ -enriched functors  $(EU_{\text{fw}}^{\text{op}} D_{\text{fw}}^{\text{op}})_*(K_\varphi)$ . The proof is then completed by showing that the hom-objects  $E(U_{\text{fw}} D_{\text{fw}} P(f, g))$  in  $(EU_{\text{fw}}^{\text{op}} D_{\text{fw}}^{\text{op}})_* \mathcal{P}_B$  are isomorphic to the shifted Serre spectral sequences  $\mathbb{E}(P(f, g))$ , and observing that under these isomorphisms, the functor  $(EU_{\text{fw}}^{\text{op}} D_{\text{fw}}^{\text{op}})_*(K_\varphi)$  is given by the map (3.1). We will start by making explicit the category  $\mathbf{SS}$ , after which we will turn to the construction of the functor  $E$ .

**Definition 3.3.** A *spectral sequence*  $E$  consists of the following data: a sequence  $E_1^{*,*}, E_2^{*,*}, \dots$  of bigraded  $\mathbb{F}_\ell$ -vector spaces; a differential  $d_r$  of bidegree  $(r, 1-r)$  on each  $E_r^{*,*}$ ; and an isomorphism

$$\varphi_r: H(E_r^{*,*}) \xrightarrow{\cong} E_{r+1}^{*,*}$$

for each  $r$ . A *morphism*  $f: E \rightarrow D$  of spectral sequences consists of a sequence

$$f_r: E_r^{*,*} \longrightarrow D_r^{*,*}, \quad r = 1, 2, \dots$$

of morphisms commuting with the differentials and having the property that  $f_{r+1}$  corresponds to  $H(f_r)$  under the isomorphisms  $\varphi_r$ . The *tensor product* of spectral sequences  $E$  and  $D$  is the spectral sequence  $E \otimes D$  with

$$(E \otimes D)_r^{s, t} = \bigoplus_{\substack{s_1+s_2=s \\ t_1+t_2=t}} E_r^{s_1, t_1} \otimes D_r^{s_2, t_2},$$

differential

$$d_r(x \otimes y) = d_r(x) \otimes y + (-1)^{|x|} x \otimes d_r(y)$$

where  $|x| = s + t$  for  $x \in E_r^{s,t}$ , and isomorphisms  $\varphi_r$  given by the Künneth theorem. There results a symmetric monoidal category of spectral sequences which we denote by **SS**. The symmetry constraint in **SS** is given by

$$E_r^{s_1,t_1} \otimes D_r^{s_2,t_2} \longrightarrow D_r^{s_2,t_2} \otimes E_r^{s_1,t_1}, \quad x \otimes y \longmapsto (-1)^{(s_1+t_1)(s_2+t_2)} y \otimes x,$$

while the monoidal unit is given by the spectral sequence which on each page is a single copy of  $\mathbb{F}_\ell$  concentrated in degree  $(0, 0)$ .

The functor  $E: \mathbf{hpSpectra} \rightarrow \mathbf{SS}$  is given by May and Sigurdsson's generalization of the Serre spectral sequence to parametrized spectra [MS06, Theorem 20.4.1], which we recall in the relevant special case in the following theorem. The case  $X = \Sigma_{+B}Y$  recovers the ordinary Serre spectral sequence of a fibration  $Y \rightarrow B$ .

**Theorem 3.4.** *Let  $X$  be a parametrized spectrum over a space  $B$ . Then there is a spectral sequence*

$$E_2^{s,t} = H^s(B; \mathcal{L}^t(X, H\mathbb{F}_\ell)) \implies H^{s+t}(r_!X; \mathbb{F}_\ell) \quad (3.3)$$

where  $\mathcal{L}^*(X, H\mathbb{F}_\ell)$  is the local coefficient system [MS06, Definition 20.3.4] with fibre over  $b \in B$  given by  $H^*(X_b; \mathbb{F}_\ell)$ . In the case where  $X$  is bounded from below in the sense that there exists a  $t_0 \in \mathbb{Z}$  such that  $H^t(X_b; \mathbb{F}_\ell) = 0$  for all  $t < t_0$  and  $b \in B$ , the spectral sequence converges strongly to the indicated target.

*Proof.* Take  $J = r^*H\mathbb{F}_\ell$  in [MS06, Theorem 20.4.1.(ii)]. To identify the target, use the equivalence  $F(r_!X, H\mathbb{F}_\ell) \simeq r_*F_B(X, r^*H\mathbb{F}_\ell)$ .  $\square$

**Remark 3.5.** Reinterpreting the construction in [MS06, Theorem 20.4.1.(ii)], we see that the spectral sequence (3.3) can be constructed as follows: Let  $\Gamma B$  be a functorial CW approximation to  $B$  (such as  $|\mathbf{Sing}_\bullet B|$ ), and write  $\Gamma_n B$  for the  $n$ -skeleton of  $\Gamma B$ . Let  $X_n$  be the restriction to  $\Gamma_n B$  of the pullback of  $X$  over  $\Gamma B$ . Then the spectral sequence (3.3) is the spectral sequence arising from the sequence

$$r_!^{\Gamma_0 B} X_0 \longrightarrow r_!^{\Gamma_1 B} X_1 \longrightarrow \dots \quad (3.4)$$

of spectra by applying  $\mathbb{F}_\ell$ -cohomology to obtain the unrolled exact couple

$$\begin{array}{ccccccc} H^*(r_!^{\Gamma_0 B} X_0) & \longleftarrow & H^*(r_!^{\Gamma_1 B} X_1) & \longleftarrow & H^*(r_!^{\Gamma_2 B} X_2) & \longleftarrow & \dots \\ & \searrow & \nearrow & \searrow & \nearrow & & \\ & H^*(r_!^{\Gamma_1 B} X_1, r_!^{\Gamma_0 B} X_0) & & H^*(r_!^{\Gamma_2 B} X_2, r_!^{\Gamma_1 B} X_1) & & \dots & \end{array}$$

It is easily verified that the spectral sequence of Theorem 3.4 is functorial.

**Proposition 3.6.** *Given a map  $(f, \varphi): (B, X) \rightarrow (C, Y)$  in  $\mathbf{hpSpectra}$ , there is an induced map*

$$E_r^{s,t}(Y) \longrightarrow E_r^{s,t}(X)$$

between the spectral sequences of Theorem 3.4 compatible with the map

$$H^*(r_!^C Y; \mathbb{F}_\ell) \longrightarrow H^*(r_!^B X; \mathbb{F}_\ell)$$

induced by  $(f, \varphi)$  on the targets.

*Proof.* Using the cellularity of the map  $\Gamma f: \Gamma B \rightarrow \Gamma C$ , one sees that the map  $(f, \varphi)$  induces a map from the sequence (3.4) to the corresponding sequence for  $Y$ .  $\square$

An argument analogous to [Whi78, section XIII.8] gives



**Theorem 3.7.** *Given parametrized spectra  $X$  over  $B$  and  $Y$  over  $C$ , there is an associative pairing of spectral sequences*

$$E_r^{s,t}(X) \otimes E_r^{s',t'}(Y) \longrightarrow E_r^{s+s',t+t'}(X \bar{\wedge} Y)$$

which on the  $E_2$ -page is given by the cross product

$$H^s(B; \mathcal{L}^t(X, H\mathbb{F}_\ell)) \otimes H^{s'}(C; \mathcal{L}^{t'}(Y, H\mathbb{F}_\ell)) \xrightarrow{\times} H^{s+s'}(B \times C; \mathcal{L}^{t+t'}(X \bar{\wedge} Y, H\mathbb{F}_\ell))$$

where the pairing on local coefficient systems is given by the products

$$H^t(X_b; \mathbb{F}_\ell) \otimes H^{t'}(Y_c; \mathbb{F}_\ell) \xrightarrow{\times} H^{t+t'}(X_b \wedge Y_c; \mathbb{F}_\ell)$$

for  $b \in B$  and  $c \in C$ . If  $X$  and  $Y$  (and hence  $X \bar{\wedge} Y$ ) are bounded from below in the sense of Theorem 3.4, ensuring strong convergence, the pairing on the  $E_\infty$ -page is the one induced by the cross product

$$H^*(r_!^B X; \mathbb{F}_\ell) \otimes \tilde{H}^*(r_!^C Y; \mathbb{F}_\ell) \xrightarrow{\times} H^*(r_!^B X \wedge r_!^C Y; \mathbb{F}_\ell). \quad \square$$

Summarizing and elaborating Theorem 3.4, Proposition 3.6 and Theorem 3.7, we obtain the desired lax symmetric monoidal functor

$$E: \mathbf{hpSpectra}^{\text{op}} \longrightarrow \mathbf{SS}. \quad (3.5)$$

Our next goal is to show that the composite functor (3.2) is lax symmetric monoidal; as before, this is not quite obvious, since the functor  $U_{\text{fw}}^{\text{op}}$  is oplax symmetric monoidal. We need the following lemma.

**Lemma 3.8.** *There exists a functor  $\tilde{E}: (\mathbf{hpMod}^{H\mathbb{F}_\ell})^{\text{op}} \rightarrow \mathbf{SS}$  such that the functor (3.5) factors as the composite*

$$\mathbf{hpSpectra}^{\text{op}} \xrightarrow{(H\mathbb{F}_\ell \wedge -)_{\text{fw}}^{\text{op}}} (\mathbf{hpMod}^{H\mathbb{F}_\ell})^{\text{op}} \xrightarrow{\tilde{E}} \mathbf{SS}.$$

*Proof.* Working as in Remark 3.5, given a parametrized  $H\mathbb{F}_\ell$ -module  $Y$  over  $B$ , we can associate to it a sequence

$$r_!^{\Gamma_0 B} Y_0 \longrightarrow r_!^{\Gamma_1 B} Y_1 \longrightarrow \dots \quad (3.6)$$

of  $H\mathbb{F}_\ell$ -modules. By applying the functor  $\pi_{-*} F^{H\mathbb{F}_\ell}(-, H\mathbb{F}_\ell)$ , we obtain an exact couple giving rise to the spectral sequence  $\tilde{E}(Y)$ . As in the proof of Lemma 2.34, notice that for each space  $C$ , the diagram of functors

$$\begin{array}{ccc} \text{Ho}(\mathbf{Spectra}_{/C}) & \xrightarrow{(H\mathbb{F}_\ell \wedge -)_C} & \text{Ho}(\mathbf{Mod}_{/C}^{H\mathbb{F}_\ell}) \\ r_! \downarrow & & \downarrow r_! \\ \text{Ho}(\mathbf{Spectra}) & \xrightarrow{H\mathbb{F}_\ell \wedge -} & \text{Ho}(\mathbf{Mod}^{H\mathbb{F}_\ell}) \end{array}$$

commutes up to natural equivalence since the corresponding diagram of right adjoints

$$\begin{array}{ccc} \text{Ho}(\mathbf{Spectra}_{/C}) & \xleftarrow{\text{forget}_C} & \text{Ho}(\mathbf{Mod}_{/C}^{H\mathbb{F}_\ell}) \\ r^* \uparrow & & \uparrow r^* \\ \text{Ho}(\mathbf{Spectra}) & \xleftarrow{\text{forget}} & \text{Ho}(\mathbf{Mod}^{H\mathbb{F}_\ell}) \end{array}$$

does. It follows that in the case  $Y = H\mathbb{F}_\ell \wedge_B X$  for a parametrized spectrum  $X$  over  $B$ , the sequence (3.6) is naturally equivalent to the sequence

$$H\mathbb{F}_\ell \wedge r_!^{\Gamma_0 B} X_0 \longrightarrow H\mathbb{F}_\ell \wedge r_!^{\Gamma_1 B} X_1 \longrightarrow \dots$$

obtained by applying  $H\mathbb{F}_\ell \wedge (-)$  to (3.4). The claim now follows from the natural isomorphism

$$\pi_{-*} F^{H\mathbb{F}_\ell}(H\mathbb{F}_\ell \wedge -, H\mathbb{F}_\ell) \cong H^*$$

of functors  $\mathrm{Ho}(\mathbf{Spectra}) \rightarrow \mathbf{grMod}$ . □

**Corollary 3.9.** *The composite of the functors*

$$(\mathbf{hpSpectra}^\ell)^{\mathrm{op}} \xrightarrow{U_{\mathrm{fw}}^{\mathrm{op}}} \mathbf{hpSpectra}^{\mathrm{op}} \xrightarrow{E} \mathbf{SS}$$

*is lax symmetric monoidal.*

*Proof.* As in the proof of Proposition 2.33, it suffices to show that the functor  $E$  takes the monoidality and identity constraints of  $U_{\mathrm{fw}}^{\mathrm{op}}$  to isomorphisms. That  $E$  has this property follows from Lemmas 2.35 and 3.8. □

We can now apply Construction 2.19 to the enriched categories  $(D_{\mathrm{fw}}^{\mathrm{op}})_* \mathcal{P}_B$  and enriched functors  $(D_{\mathrm{fw}}^{\mathrm{op}})_*(K_\varphi)$  of Remark 2.28 to obtain categories  $(EU_{\mathrm{fw}}^{\mathrm{op}} D_{\mathrm{fw}}^{\mathrm{op}})_* \mathcal{P}_B$  enriched in spectral sequences, along with enriched functors  $(EU_{\mathrm{fw}}^{\mathrm{op}} D_{\mathrm{fw}}^{\mathrm{op}})_*(K_\varphi)$  between them. Explicitly, the hom-object from  $f: B \rightarrow BG$  to  $g: B \rightarrow BG$  in  $(EU_{\mathrm{fw}}^{\mathrm{op}} D_{\mathrm{fw}}^{\mathrm{op}})_* \mathcal{P}_B$  is the spectral sequence

$$E_r^{s,t}(U_{\mathrm{fw}} D_{\mathrm{fw}} P(f, g)) \implies H^{s+t}(r_! U_{\mathrm{fw}} D_{\mathrm{fw}} P(f, g))$$

whose target is the hom-object from  $f$  to  $g$  in the category  $(H^* r_!^{\mathrm{op}} U_{\mathrm{fw}}^{\mathrm{op}} D_{\mathrm{fw}}^{\mathrm{op}})_* \mathcal{P}_B$  constructed in subsections 2.2—2.6. We note that the composition law in  $(EU_{\mathrm{fw}}^{\mathrm{op}} D_{\mathrm{fw}}^{\mathrm{op}})_* \mathcal{P}_B$  is compatible with that in  $(H^* r_!^{\mathrm{op}} U_{\mathrm{fw}}^{\mathrm{op}} D_{\mathrm{fw}}^{\mathrm{op}})_* \mathcal{P}_B$ , and that the functors  $(EU_{\mathrm{fw}}^{\mathrm{op}} D_{\mathrm{fw}}^{\mathrm{op}})_*(K_\varphi)$  and  $(H^* r_!^{\mathrm{op}} U_{\mathrm{fw}}^{\mathrm{op}} D_{\mathrm{fw}}^{\mathrm{op}})_*(K_\varphi)$  in the two settings are also compatible. Theorems 3.1 and 3.2 now follow from the following result identifying the hom-objects in  $(EU_{\mathrm{fw}}^{\mathrm{op}} D_{\mathrm{fw}}^{\mathrm{op}})_* \mathcal{P}_B$ .

**Theorem 3.10.** *For all  $f, g: B \rightarrow BG$ , there is an isomorphism*

$$E(U_{\mathrm{fw}} D_{\mathrm{fw}} P(f, g)) \cong \mathbb{E}(P(f, g))$$

*of spectral sequences natural with respect to the homomorphisms induced by diagram (2.5).*

*Proof.* In view of Proposition 2.40, this follows from Lemma 3.8. □

Tracing through definitions, we obtain the following description of the pairing between spectral sequences on the  $E_2$  page.

**Proposition 3.11.** *Let  $f, g, h: B \rightarrow BG$  be maps, and pick a basepoint  $b \in B$ . On the  $E_2$  page, the pairing*

$$\circ: \mathbb{E}_r^{s_1, t_1}(P(g, h)) \otimes \mathbb{E}_r^{s_2, t_2}(P(f, g)) \longrightarrow \mathbb{E}_r^{s_1+s_2, t_1+t_2}(P(f, h))$$

*of spectral sequences is given by the map*

$$H^*(B) \otimes \mathbb{H}^*(P(g, h)_b) \otimes H^*(B) \otimes \mathbb{H}^*(P(f, g)_b) \longrightarrow H^*(B) \otimes \mathbb{H}^*(P(f, h)_b)$$

*induced by the cup product on  $B$  and the pairing*

$$\circ: \mathbb{H}^*(P(g, h)_b) \otimes \mathbb{H}^*(P(f, g)_b) \longrightarrow \mathbb{H}^*(P(f, h)_b)$$

*arising from the identifications  $P(g, h)_b = P(gi, hi)$ ,  $P(f, g)_b = P(fi, gi)$ , and  $P(f, h)_b = P(fi, hi)$  for  $i$  is the inclusion of  $b$  into  $B$ .* □

#### 4. PROOF OF THEOREMS A—E

Our aim in this section is to prove Theorems A—E stated in the introduction. Theorem A follows easily by combining various results from sections 2 and 3.

*Proof of Theorem A.* Part (i) follows from Theorem 2.1; see Definition 2.2. Parts (ii) and (iii) follow from Theorem 3.1 and Proposition 3.11. In part (ii), the assertion about product on  $\mathbb{H}^*(G)$  follows from Theorem 2.5. In part (iii), the assertion about the  $E_2$ -page follows from Proposition 3.11 and Corollary 2.13 (which allows one to compare the  $\mathbb{H}^*(G)$ -module structure on the fibre of  $BG^{h\sigma} \rightarrow BG$  to that on  $\mathbb{H}^*(G)$  itself). □

The following result is an elaboration of Theorem B.

**Theorem 4.1.** *Let  $B$  be a path connected space, let  $f, g: B \rightarrow BG$  be maps, let  $F \simeq \Omega BG$  be a fibre of the fibration  $P(f, g) \rightarrow B$ , and let  $i: F \hookrightarrow P(f, g)$  be the inclusion. Then the following are equivalent conditions on an element  $x \in \mathbb{H}^0 P(f, g)$ :*

- (1)  $\mathbb{H}^* P(f, g)$  is free of rank 1 with basis  $\{x\}$  as a graded left module over  $\mathbb{H}^* P(g, g)$ .
- (2)  $\mathbb{H}^* P(f, g)$  is free of rank 1 with basis  $\{x\}$  as a graded right module over  $\mathbb{H}^* P(f, f)$ .
- (3)  $\mathbb{H}^* P(f, g)$  is generated by  $x$  as a graded left module over  $\mathbb{H}^* P(g, g)$ .
- (4)  $\mathbb{H}^* P(f, g)$  is generated by  $x$  as a graded right module over  $\mathbb{H}^* P(f, f)$ .
- (5)  $i^*(x) \neq 0 \in H^d F$ .
- (6) The map

$$\mathbb{E}(P(g, g)) \longrightarrow \mathbb{E}(P(f, g)), \quad z \longmapsto z \circ (1 \otimes i^*(x))$$

is an isomorphism from the Serre spectral sequence of  $P(g, g) \rightarrow B$  to that of  $P(f, g) \rightarrow B$ .

- (7) The map

$$\mathbb{E}(P(f, f)) \longrightarrow \mathbb{E}(P(f, g)), \quad z \longmapsto (1 \otimes i^*(x)) \circ z$$

is an isomorphism from the Serre spectral sequence of  $P(f, f) \rightarrow B$  to that of  $P(f, g) \rightarrow B$ .

Moreover, the following conditions are equivalent:

- (8) There exists an element  $x \in \mathbb{H}^0 P(f, g)$  satisfying conditions (1)–(7).
- (9) The map  $i_*: H_d F \rightarrow H_d P(f, g)$  is nontrivial.
- (10) The Serre spectral sequences of  $P(f, g) \rightarrow B$  and  $P(g, g) \rightarrow B$  are isomorphic.
- (11) The Serre spectral sequences of  $P(f, g) \rightarrow B$  and  $P(f, f) \rightarrow B$  are isomorphic.
- (12) The parametrized  $H\mathbb{F}_\ell$ -modules  $H\mathbb{F}_\ell \wedge_B \Sigma_{+B}^\infty P(f, g)$  and  $H\mathbb{F}_\ell \wedge_B \Sigma_{+B}^\infty P(g, g)$  are equivalent objects of  $\text{Ho}(\mathbf{Mod}_{/B}^{H\mathbb{F}_\ell})$ .
- (13) The parametrized  $H\mathbb{F}_\ell$ -modules  $H\mathbb{F}_\ell \wedge_B \Sigma_{+B}^\infty P(f, g)$  and  $H\mathbb{F}_\ell \wedge_B \Sigma_{+B}^\infty P(f, f)$  are equivalent objects of  $\text{Ho}(\mathbf{Mod}_{/B}^{H\mathbb{F}_\ell})$ .
- (14) The generator of  $E_2^{0,d}(P(f, g)) = H^0 B \otimes H^d F \cong \mathbb{F}_\ell$  is a permanent cycle in the Serre spectral sequence of  $P(f, g) \rightarrow B$ .

*Proof.* The implications (1)  $\Rightarrow$  (3) and (2)  $\Rightarrow$  (4) are obvious. To show (3)  $\Rightarrow$  (5), let  $j: \Omega BG \hookrightarrow P(g, g)$  be the inclusion of a fibre. By assumption, there exists some element  $y \in \mathbb{H}^{-d} P(g, g)$  such that  $y \circ x = 1 \in \mathbb{H}^{-d} P(f, g)$ . We now have

$$j^*(y) \circ i^*(x) = i^*(y \circ x) = i^*(1) = 1 \in \mathbb{H}^{-d}(F),$$

showing that  $i^*(x) \neq 0 \in \mathbb{H}^0 F$ . Here the first equality follows from Theorem 2.12. The implication (4)  $\Rightarrow$  (5) follows similarly.

Let us now show that (5)  $\Rightarrow$  (6). By naturality of the Serre spectral sequence, the element  $1 \otimes i^*(x) \in \mathbb{E}_2^{0,0}(P(f, g)) = H^0(B) \otimes \mathbb{H}^0(F)$  is the image of  $x$  under the composite

$$\mathbb{H}^0 P(f, g) \longrightarrow \mathbb{E}_\infty^{0,0}(P(f, g)) \longrightarrow \mathbb{E}_2^{0,0}(P(f, g))$$

of the quotient and inclusion maps. Thus  $1 \otimes i^*(x)$  is a permanent cycle, and multiplication by it does define a morphism of spectral sequences

$$m_{1 \otimes i^*(x)}: \mathbb{E}(P(g, g)) \longrightarrow \mathbb{E}(P(f, g)), \quad z \mapsto z \circ (1 \otimes i^*(x)).$$

By Proposition 3.11, on the  $E_2$  page, this map is given by the map

$$\text{id} \otimes m_{i^*(x)}: H^*(B) \otimes \mathbb{H}^*(\Omega BG) \longrightarrow H^*(B) \otimes \mathbb{H}^*(F)$$

where  $m_{i^*(x)}$  is the multiplication map  $z \mapsto z \circ i^*(x)$  (and we interpret  $i^*(x)$  as an element of  $\mathbb{H}^0(F)$ ). By Corollary 2.13, there is an isomorphism  $\mathbb{H}^*(F) \cong \mathbb{H}^*(\Omega BG)$  under which the map  $m_{i^*(x)}$  corresponds to the map

$$m_y: \mathbb{H}^*(\Omega BG) \longrightarrow \mathbb{H}^*(\Omega BG), \quad z \mapsto z \circ y$$

for some  $y \in \mathbb{H}^0(\Omega BG)$ . Since  $i^*(x)$  is nonzero, so is  $y$ . Thus  $y$  is a nonzero multiple of the unit element for the product on  $\mathbb{H}^*(\Omega BG)$ . It follows that the map  $m_y$  and hence the map  $m_{i^*(x)}$  are isomorphisms. Thus the map  $m_{1 \otimes i^*(x)}$  is an isomorphism on the  $E_2$  pages, and hence on all further pages as well, giving an isomorphism of spectral sequences. Again, the implication (5)  $\Rightarrow$  (7) follows similarly.

We now prove the implication (6)  $\Rightarrow$  (1). Since  $x \in \mathbb{H}^0 P(f, g)$  is a lift of the element  $1 \otimes i^*(x) \in \mathbb{E}_\infty^{0,0}(P(f, g))$ , the multiplication map

$$m_x: \mathbb{H}^* P(g, g) \longrightarrow \mathbb{H}^* P(f, g), \quad z \mapsto z \circ x$$

induces on the associated graded modules corresponding to the Serre spectral sequences of  $P(g, g)$  and  $P(f, g)$  the isomorphism

$$m_{1 \otimes i^*(x)}: \mathbb{E}_\infty^{*,*}(P(g, g)) \xrightarrow{\cong} \mathbb{E}_\infty^{*,*}(P(f, g)).$$

Therefore the map  $m_x$  itself must be an isomorphism. Thus  $\mathbb{H}^* P(f, g)$  is free of rank 1 over  $\mathbb{H}^* P(g, g)$  with basis  $\{x\}$ . The implication (7)  $\Rightarrow$  (2) follows similarly.

In view of condition (5), condition (8) is equivalent to the map  $i^*: H^d P(f, g) \rightarrow H^d F$  being nontrivial, which in turn is equivalent to condition (9). Thus (8)  $\Leftrightarrow$  (9). The implications (8)  $\Rightarrow$  (10) and (8)  $\Rightarrow$  (11) follow from conditions (6) and (7). To show (10)  $\Rightarrow$  (14), it suffices to show that the generator of  $E_2^{0,d}(P(g, g)) = H^0(B) \otimes H^d(\Omega BG) \cong \mathbb{F}_\ell$  is a permanent cycle. Let  $j: \Omega BG \hookrightarrow P(g, g)$  be the inclusion of a fibre. As before, the element  $1 \otimes j^*(\mathbb{1}) \in \mathbb{E}_2^{0,0}(P(g, g)) = H^0(B) \otimes \mathbb{H}^0(\Omega BG)$  is a permanent cycle. Since  $j^*(\mathbb{1}) = \mathbb{1} \neq 0 \in \mathbb{H}^0(\Omega BG)$  by Theorem 2.12, it is nonzero, and hence generates  $\mathbb{E}_2^{0,0}(P(g, g)) \cong \mathbb{F}_\ell$ . Thus the claim follows. Again, the implication (11)  $\Rightarrow$  (14) follows similarly. Finally, to show that (14)  $\Rightarrow$  (8), it suffices to observe that by naturality of the Serre spectral sequence, a lift of the nontrivial permanent cycle from  $E_\infty^{0,d}(P(f, g))$  to an element of  $H^d P(f, g)$  satisfies condition (5).

In view of Lemma 3.8, it is clear that (12)  $\Rightarrow$  (10) and (13)  $\Rightarrow$  (11). Let us show that (8)  $\Rightarrow$  (12). Suppose  $x \in \mathbb{H}^0 P(f, g)$  satisfies condition (1). As observed at the beginning of subsection 2.9, the product

$$\circ: \mathbb{H}^* P(g, g) \otimes \mathbb{H}^* P(f, g) \longrightarrow \mathbb{H}^* P(f, g)$$

arises by applying the composite functor

$$\mathbf{hpMod}^{H\mathbb{F}_\ell} \xrightarrow{r_!} \mathbf{Ho}(\mathbf{Mod}^{H\mathbb{F}_\ell}) \xrightarrow{\pi_{-*} F^{H\mathbb{F}_\ell}(-, H\mathbb{F}_\ell)} \mathbf{grMod}^{\mathbb{F}_\ell} \quad (4.1)$$

to a certain map

$$H\mathbb{F}_\ell \wedge_B \Sigma_B^{-d} \Sigma_{+B}^\infty P(f, g) \longrightarrow (H\mathbb{F}_\ell \wedge_B \Sigma_B^{-d} \Sigma_{+B}^\infty P(g, g)) \bar{\wedge}^{H\mathbb{F}_\ell} (H\mathbb{F}_\ell \wedge_B \Sigma_B^{-d} \Sigma_{+B}^\infty P(f, g)) \quad (4.2)$$

in  $\mathbf{hpMod}^{H\mathbb{F}_\ell}$  covering the diagonal map  $\Delta: B \rightarrow B \times B$  in  $\mathcal{T}$ . The element  $x \in \mathbb{H}^0 P(f, g)$  is represented by a map

$$r_!^B(H\mathbb{F}_\ell \wedge_B \Sigma_B^{-d} \Sigma_{+B}^\infty P(f, g)) \longrightarrow H\mathbb{F}_\ell$$

in  $\mathbf{Ho}(\mathbf{Mod}^{H\mathbb{F}_\ell})$  which by adjunction amounts to a map

$$\tilde{x}: H\mathbb{F}_\ell \wedge_B \Sigma_B^{-d} \Sigma_{+B}^\infty P(f, g) \longrightarrow H\mathbb{F}_\ell$$

in  $\mathbf{hpMod}^{H\mathbb{F}_\ell}$  covering the map  $r^B: B \rightarrow \text{pt}$  in  $\mathcal{T}$ . Composing (4.2) with the map  $1 \bar{\wedge}^{H\mathbb{F}_\ell} \tilde{x}$  yields a map

$$H\mathbb{F}_\ell \wedge_B \Sigma_B^{-d} \Sigma_{+B}^\infty P(f, g) \longrightarrow H\mathbb{F}_\ell \wedge_B \Sigma_B^{-d} \Sigma_{+B}^\infty P(g, g) \quad (4.3)$$

in  $\mathbf{hpMod}^{H\mathbb{F}_\ell}$  covering the identity map of  $B$ ; upon application of the composite (4.1), this map recovers the multiplication-by- $x$ -map  $\mathbb{H}^* P(g, g) \rightarrow \mathbb{H}^* P(f, g)$ . The restriction of (4.3) to fibres over a point  $b \in B$  is a map

$$H\mathbb{F}_\ell \wedge \Sigma^{-d} \Sigma^\infty P(f, g)_b \longrightarrow H\mathbb{F}_\ell \wedge \Sigma^{-d} \Sigma^\infty P(g, g)_b \quad (4.4)$$

in  $\text{Ho}(\mathbf{Mod}^{H\mathbb{F}_\ell})$  which, upon application of  $\pi_{-*}F^{H\mathbb{F}_\ell}(-, H\mathbb{F}_\ell)$ , recovers the map  $\mathbb{H}^*P(g, g)_b \rightarrow \mathbb{H}^*P(f, g)_b$  given by multiplication by the restriction  $x_b$  of  $x$  to  $\mathbb{H}^*P(f, g)_b$ . In view of the equivalence of conditions (1) and (5), the assumption on  $x$  ensures that this multiplication-by- $x_b$ -map is an isomorphism for each  $b \in B$ . It follows that the map (4.4) is an equivalence for all  $b$ , whence the map (4.3) is an equivalence, and the claim follows by applying  $\Sigma_B^d$  to (4.3). The implication (8)  $\Rightarrow$  (13) follows similarly.  $\square$

*Proof of Theorem B.* The claim follows from the equivalence of conditions (8) and (9) in Theorem 4.1 by taking  $g = \text{id}_{BG}$  and  $f = \sigma$ .  $\square$

We now turn to the proof of Theorem C. We will deduce part (ii) of the theorem from the following result.

**Theorem 4.2.** *Let  $B$  be a path connected space, let  $f, g: B \rightarrow BG$  be maps, let  $F \simeq \Omega BG$  be a fibre of the fibration  $P(f, g) \rightarrow B$ , and let  $i: F \hookrightarrow P(f, g)$  be the inclusion. Suppose  $x \in \mathbb{H}^0P(f, g)$  is a generator of  $\mathbb{H}^*P(f, g)$  as a free rank 1 module over  $\mathbb{H}^*P(g, g)$  satisfying  $1 \circ i^*(x) = 1 \in H^0F$ , where  $\circ$  refers to the pairing*

$$\circ: \mathbb{H}^*(\Omega BG) \otimes \mathbb{H}^*(F) \longrightarrow \mathbb{H}^*(F).$$

*Then the isomorphism*

$$E(P(g, g)) \xrightarrow{\cong} E(P(f, g)), \quad z \mapsto z \circ (1 \otimes i^*(x)) \quad (4.5)$$

*of Theorem 4.1.(6) from the Serre spectral sequence of  $P(g, g) \rightarrow B$  to that of  $P(f, g) \rightarrow B$  is an isomorphism of spectral sequences of algebras, where the spectral sequences are equipped with the usual algebra structures induced by cup product. In particular, the map*

$$H^*P(g, g) \longrightarrow H^*P(f, g), \quad z \longrightarrow z \circ x$$

*induces an algebra isomorphism*

$$\text{gr } H^*P(g, g) \xrightarrow{\cong} \text{gr } H^*P(f, g)$$

*on the associated graded algebras of  $H^*P(g, g)$  and  $H^*P(f, g)$  corresponding to the Serre spectral sequences.*

*Proof.* To show that the map (4.5) respects the algebra structures, it is enough to show that it does so on the  $E_2$  pages. By Proposition 3.11, on  $E_2$  pages the map (4.5) is given by

$$H^*(B) \otimes H^*(\Omega BG) \longrightarrow H^*(B) \otimes H^*(F), \quad \alpha \otimes \beta \mapsto \alpha \otimes (\beta \circ i^*(x)),$$

so it is enough to show that the map

$$H^*(\Omega BG) \longrightarrow H^*(F), \quad \beta \mapsto \beta \circ i^*(x) \quad (4.6)$$

is a ring homomorphism. Picking a path connecting  $f(b_0)$  and  $g(b_0)$  where  $b_0$  is the point in  $B$  over which  $F$  is a fibre of  $P(f, g) \rightarrow B$  and  $\Omega BG$  is a fibre of  $P(g, g) \rightarrow B$ , from Corollary 2.13 we obtain an isomorphism

$$\Xi: H^*(F) \xrightarrow{\cong} H^*(\Omega BG)$$

of  $\mathbb{H}^*(\Omega BG)$ -modules. An inspection of the proof of Corollary 2.13 shows that  $\Xi$  is induced by a zigzag of homotopy equivalences of spaces, so  $\Xi$  is also an algebra isomorphism with respect to cup products. Thus it is enough to show that the composite

$$H^*(\Omega BG) \longrightarrow H^*(\Omega BG), \quad \beta \mapsto \beta \circ \Xi(i^*(x)) \quad (4.7)$$

of the map (4.6) and  $\Xi$  is a ring homomorphism, which we will do by showing that this map is in fact the identity map. We have (in  $\mathbb{H}^*(\Omega BG)$ )

$$1 \circ \Xi(i^*(x)) = \Xi(1 \circ i^*(x)) = \Xi(1) = 1$$

where the first equality follows by the  $\mathbb{H}^*(\Omega BG)$ -linearity of  $\Xi$ , the second equality holds by the assumption on  $x$ , and the final equality holds since  $\Xi$  is an algebra homomorphism. Since by  $\mathbb{F}_\ell$ -linearity the group  $H^d(\Omega BG) \cong \mathbb{F}_\ell$  contains at most one element  $y$  with the property that  $1 \circ y = 1 \in H^0(\Omega BG)$ , and both  $\mathbb{1}$  and  $\Xi(i^*(x))$  have this property, we must have  $\Xi(i^*(x)) = \mathbb{1}$ . Thus the map (4.7) is the identity map as claimed.  $\square$

*Proof of Theorem C.* The first part of part (i) of the theorem is immediate from Theorem 2.11. The claimed injectivity of the induced map  $H^*(BG) \rightarrow H^*(BG^{h\sigma})$  then follows from the existence of a section for the fibration  $LBG \rightarrow BG$ , which implies that  $H^*(LBG)$  and hence  $H^*(BG^{h\sigma})$  are faithful as  $H^*(BG)$ -modules. Part (ii) follows from the special case  $f = \sigma$ ,  $g = \text{id}_{BG}$  of Theorem 4.2 by observing that the isomorphism

$$E_\infty^{*,*}(P(g, g)) \xrightarrow{\cong} E_\infty^{*,*}(P(f, g)), \quad z \mapsto z \circ (1 \otimes i^*(x))$$

agrees with the map

$$\text{gr } H^*P(g, g) \longrightarrow \text{gr } H^*P(f, g)$$

induced by the multiplication map

$$H^*P(g, g) \longrightarrow H^*P(f, g), \quad z \mapsto z \circ x. \quad \square$$

We now turn to the proof of Theorem D, focusing first on the polynomial case. The following theorem describes what happens in the polynomial case when  $\sigma$  acts as the identity on  $H^*(BG)$ .

**Theorem 4.3.** *Suppose that  $H^*BG$  is a polynomial ring and  $\sigma: BG \rightarrow BG$  a map inducing the identity on  $H^*BG$ . For the fibration sequence  $G \xrightarrow{i} BG^{h\sigma} \rightarrow BG$  of (1.4),  $H^*(BG^{h\sigma}) \xrightarrow{i^*} H^*(G)$  is surjective and the Serre spectral sequence satisfies  $E_2 = E_\infty$ . In particular  $BG^{h\sigma}$  has a  $[G]$ -fundamental class.*

*Proof.* Recall that for a pullback diagram of spaces

$$\begin{array}{ccc} X \times_B Y & \longrightarrow & Y \\ \downarrow & & \downarrow g \\ X & \xrightarrow{f} & B \end{array}$$

with at least one of  $f$  and  $g$  a fibration and the space  $B$  simply connected, the Eilenberg–Moore spectral sequence is a strongly convergent second-quadrant spectral sequence of algebras

$$E_2^{s,t} = \text{Tor}_{-s,t}^{H^*(B)}(H^*(X), H^*(Y)) \implies H^{s+t}(X \times_B Y)$$

where  $t$  is the internal degree and  $-s$  the homological degree. The entries on the  $E_\infty$ -page of the spectral sequence are filtration quotients

$$E_\infty^{s,t} = F^s H^{s+t}(X \times_B Y) / F^{s+1} H^{s+t}(X \times_B Y)$$

for a certain descending filtration

$$H^*(X \times_B Y) \supset \cdots \supset F^{-2} \supset F^{-1} \supset F^0 \supset F^1 = 0.$$

of  $H^*(X \times_B Y)$ . In particular, we may interpret the line  $E_\infty^{0,*}$  as a subring of  $H^*(X \times_B Y)$ . By naturality of the spectral sequence, it is easy to see that this subring is precisely the image of the map

$$H^*(X \times Y) \longrightarrow H^*(X \times_B Y)$$

induced by the inclusion of  $X \times_B Y$  into  $X \times Y$ .

Suppose now  $H^*(BG) = \mathbb{F}_\ell[x_1, \dots, x_n]$ , and let  $\sigma: BG \rightarrow BG$  be a map inducing the identity on cohomology. Consider the Eilenberg–Moore spectral sequence for the pullback square

$$\begin{array}{ccc} BG^{h\sigma} & \longrightarrow & BG^I \\ \text{ev}_1 \downarrow & & \downarrow (\text{ev}_0, \text{ev}_1) \\ BG & \xrightarrow{(\sigma, 1)} & BG \times BG \end{array}$$

By the assumption on  $\sigma$ , the  $E_2$ -page of the spectral sequence amounts to

$$\text{Tor}^{\mathbb{F}_\ell[x_1, \dots, x_n, x'_1, \dots, x'_n]}(\mathbb{F}_\ell[x_1, \dots, x_n], \mathbb{F}_\ell[x_1, \dots, x_n])$$

where  $\mathbb{F}_\ell[x_1, \dots, x_n, x'_1, \dots, x'_n]$  acts on the two copies of  $\mathbb{F}_\ell[x_1, \dots, x_n]$  via the map  $x_i \mapsto x_i$ ,  $x'_i \mapsto x_i$ . Interpreting

$$\mathbb{F}_\ell[x_1, \dots, x_n, x'_1, \dots, x'_n] = \mathbb{F}_\ell[x_1, \dots, x_n, y_1, \dots, y_n]$$

with  $y_i = x'_i - x_i$ , it is now easy to compute that the  $E_2$ -page is

$$\mathbb{F}_\ell[x_1, \dots, x_n] \otimes \Lambda_{\mathbb{F}_\ell}(z_1, \dots, z_n)$$

where the  $x_i$ 's occur on the line  $s = 0$  and the  $z_i$ 's on the line  $s = -1$ . There can be no differentials, so the spectral sequence collapses on the  $E_2$ -page. From the freeness of the  $E_2$ -page as an  $H^*(BG)$ -module we now deduce that the  $H^*(BG^{h\sigma})$  is free as an  $H^*(BG)$  module (with the module structure induced by  $\text{ev}_1: BG^{h\sigma} \rightarrow BG$ ). It follows that the Eilenberg–Moore spectral sequence of the pullback diagram

$$\begin{array}{ccc} G & \xrightarrow{i} & BG^{h\sigma} \\ \downarrow & & \downarrow \text{ev}_1 \\ \text{pt} & \longrightarrow & BG \end{array}$$

is concentrated on the line  $s = 0$ , which in turn implies that the map  $i^*: H^*(BG^{h\sigma}) \rightarrow H^*(G)$  is an epimorphism. In particular it is non-zero in degree  $d$ , so  $BG^{h\sigma}$  has a fundamental class by Definition 1.1 and the remark that follows it. Also, the Serre spectral sequence collapses at  $E_2$ : the surjectivity of  $i^*$  guarantees that there can be no differentials originating on the  $E_r^{0,*}$  line, whence there cannot be any nonzero differentials at all by multiplicativity.  $\square$

We now establish how  $\psi^q$  acts on  $H^*(BG)$  in the cases we consider.

**Proposition 4.4.** *Suppose that  $H^*(BG)$  is a polynomial ring. If  $\ell$  is odd then  $H^*(BG)$  is concentrated in even degrees and  $\psi^q$  acts as  $q^n$  on  $H^{2n}(BG)$  for any  $n$ ; in particular it acts as the identity if  $q \equiv 1 \pmod{\ell}$ . If  $\ell = 2$  then  $\psi^q$  acts as the identity on  $H^*(BG)$  for all  $q \in \mathbb{Z}_2^\times$ .*

*Proof.* If  $\ell$  is odd then  $H^*(BG; \mathbb{Z}_\ell) \xrightarrow{\sim} H^*(BT; \mathbb{Z}_\ell)^W$  (see [AGMV08, Thm. 12.1]), and the result follows since  $\psi^q$  induces multiplication by  $q$  on  $H^2(BT; \mathbb{Z}_\ell)$ , by definition.

Now, suppose that  $\ell = 2$ . Here  $H^*(BG) \rightarrow H^*(BT)$  need not be injective, unless  $G$  is 2–torsion free (see [AGMV08, Thm. 12.1]). However, by the theory of unstable modules over the Steenrod algebra (see e.g., [AG09, Prop. 7.2]), there exists an elementary abelian 2–subgroup  $V$ , which, up to conjugation, contains every other elementary abelian 2–subgroup. In particular, we can choose a representative which contains  ${}_2T$ , the elements of order 2 in the maximal torus. Let  $W_0$  denote the stabilizer of  ${}_2T$  in  $W$ , which by e.g., [AGMV08, Lem. 11.3] is an elementary abelian 2–subgroup of  $W$ . Now, recall Tits' model for  $N_G(T)$  from [Tit66], elaborated and extended to 2–compact groups in [DW05] and [AG08]:  $N_G(T)$  can be constructed from the root datum by first constructing the reflection extension  $1 \rightarrow \mathbb{Z}[\Sigma] \rightarrow \rho(W) \rightarrow W \rightarrow 1$ , where  $\Sigma$  is the set of reflections in  $W$ , and then constructing  $N_G(T)$  as a push-forward along a  $W$ -map  $f: \mathbb{Z}[\Sigma] \rightarrow T$  sending each reflection  $\sigma$  to a certain element of order two  $h_\sigma$  of  $T$ ; see [AG08, Sec. 2-3]. Let  $\rho_0$  denote the preimage of  $W_0$  in  $\rho(W)$ , and consider the subgroup  $A$  of  $N_G(T)$  generated by  ${}_2T$

and the image of  $\rho_0$  under  $\rho(W) \rightarrow N_G(T)$ , the map to the push-forward. By construction  $A$  is an abelian subgroup of  $N_G(T)$ . Likewise by construction it will contain  $V$ . We hence just have to see that  $\psi^q$  acts trivially on  $A$ . However, this is a consequence of [AG08, Thm. B+C], which explains exactly how  $\psi^q$  acts on  $N_G(T)$ , namely as a quotient of a map which multiplies by  $q$  on  $T$  and is the identity on  $\rho(W)$  (see Step 2 of the proof of Thm. B in [AG08] for the definition of the homomorphism  $s: \text{Out}(\mathbb{D}) \rightarrow \text{Out}(N_G(T))$ ).  $\square$

*Proof of Theorem D when  $H^*(BG)$  is a polynomial ring.* As in the introduction, let  $e$  denote the order of  $q \bmod \ell$ , and write  $q = \zeta_e q'$ , so that  $BG(q) \simeq (BG^{h\mu_e})(q')$  as in (1.7). Note that by [BM07, Thm B(4)],  $BG^{h\mu_e}$  again has polynomial mod  $\ell$  cohomology ring. The result now follows from Theorem 4.3, noting that the assumptions are satisfied by Proposition 4.4.  $\square$

We will postpone the proof of Theorem D in the remaining case where  $BG = B\text{Spin}(n)\hat{2}$  until the end of the section. Write  $D$  for the set

$$D = \{[\sigma] \in \text{Out}(BG) \mid BG^{h\sigma} \text{ has a } [G]\text{-fundamental class}\}$$

and equip  $\text{Out}(BG)$  with the  $\ell$ -adic topology induced by the action on  $H^*(BG; \mathbb{Z}/\ell^k)$ ,  $k \geq 1$ , as explained e.g. in [BMO12, p. 7]. Under the isomorphism  $\text{Out}(BG) \cong \text{Out}(\mathbb{D}_G)$ , this topology agrees with the natural topology on  $\text{Out}(\mathbb{D}_G)$ ; see Proposition 4.16 below. Our next goal is to establish the following theorem and corollary elaborating on Theorem E.

**Theorem 4.5.** *The set  $D$  has the following properties:*

- (i)  $D$  is contained in the kernel of the homomorphism  $\text{Out}(BG) \rightarrow \text{Aut}(H^*BG)$ ,  $[\sigma] \mapsto \sigma^*$ , i.e., for any  $\sigma \in D$ ,  $\sigma^*$  acts as the identity on  $H^*(BG)$ .
- (ii)  $D$  is a subgroup of  $\text{Out}(BG)$ .
- (iii)  $D$  has the following closure property: Suppose  $x, y \in \text{Out}(BG)$  generate the same closed subgroup of  $\text{Out}(BG)$ . Then  $x \in D$  if and only if  $y \in D$ .
- (iv) When  $BG$  is semisimple,  $D$  is a closed subgroup of  $\text{Out}(BG)$ .
- (v)  $\psi^q \in D$  for some  $q \in \mathbb{Z}_\ell^\times$  of infinite order.

**Corollary 4.6.** *If  $\ell$  is odd, the set of  $q \in \mathbb{Z}_\ell^\times$  for which  $BG^{h\psi^q}$  has a  $[G]$ -fundamental class is a subgroup of  $\mathbb{Z}_\ell^\times$  splitting as an internal direct product  $\mu_e H_n$  for some  $e \mid (\ell - 1)$  and  $1 \leq n < \infty$  where  $\mu_e \leq \mathbb{Z}_\ell^\times$  is the subgroup of  $e$ -th roots of unity and*

$$H_n = \{q \in \mathbb{Z}_\ell^\times \mid q \equiv 1 \pmod{\ell} \text{ and } v_\ell(q - 1) \geq n\}.$$

If  $\ell = 2$ , it is a subgroup of  $\mathbb{Z}_2^\times$  of one of the following forms:

$$H'_n \text{ or } \pm H'_n \text{ for } 2 \leq n < \infty, \quad \text{or} \quad H'_n \cup (-1 + 2^{n-1})H'_n \text{ for } 3 \leq n < \infty,$$

where

$$H'_n = \{q \in \mathbb{Z}_2^\times \mid q \equiv 1 \pmod{4} \text{ and } v_2(q - 1) \geq n\}.$$

**Remark 4.7.** As remarked in the introduction, we do not know an example of an  $\ell$ -compact group  $BG$  for which  $D$  is a proper subgroup of the kernel of the homomorphism  $\text{Out}(BG) \rightarrow \text{Aut}(H^*(BG))$  in Theorem 4.5.(ii). If  $D$  is the whole kernel, it of course is a closed subgroup of  $\text{Out}(BG)$ .

We will prove Theorem 4.5 in a series of auxiliary results.

**Lemma 4.8.** *Suppose  $\sigma \in D$ . Then  $\sigma^*: H^*BG \rightarrow H^*BG$  is the identity map.*

*Proof.* The existence of a section for the evaluation fibration  $LGB \rightarrow BG$  implies that the Serre spectral sequence for  $LGB \rightarrow BG$  has no differentials hitting the bottom row. By Theorem 4.1, the same is therefore true for the Serre spectral sequence of the evaluation fibration  $\text{ev}_1: BG^{h\sigma} \rightarrow BG$ . It follows that the map

$$\text{ev}_1^*: H^*BG \longrightarrow H^*BG^{h\sigma}$$



is injective. Since the maps  $\text{ev}_0, \text{ev}_1: BG^{h\sigma} \rightarrow BG$  are homotopic, we have  $\text{ev}_0^* = \text{ev}_1^*$ . The claim now follows from the equation

$$\text{ev}_1^* \sigma^* = (\sigma \circ \text{ev}_1)^* = \text{ev}_0^* = \text{ev}_1^*. \quad \square$$

**Lemma 4.9.** *D is a subgroup of  $\text{Out}(BG)$ .*

*Proof.* The cohomology of  $BG^{h\text{id}_{BG}} = LBG$  is certainly free of rank 1 over itself, so  $[\text{id}_{BG}] \in D$  by Theorem B. Suppose now  $[\sigma] \in D$ , and let  $\sigma^{-1}$  be a homotopy inverse of  $\sigma$ . We then have the following commutative diagram:

$$\begin{array}{ccccc} BG^{h\sigma^{-1}} = P(\sigma^{-1}, \text{id}_{BG}) & \xrightarrow{\simeq} & P(\sigma^{-1}, \sigma\sigma^{-1}) & \xrightarrow{\simeq} & P(\text{id}_{BG}, \sigma) & \xrightarrow{\simeq} & P(\sigma, \text{id}_{BG}) = BG^{h\sigma} \\ & \searrow & & & \searrow & & \searrow \\ & & BG & \xrightarrow[\simeq]{\sigma^{-1}} & BG & & \end{array}$$

Here the top left hand map is a fibre homotopy equivalence induced by a homotopy between  $\text{id}_{BG}$  and  $\sigma\sigma^{-1}$ ; the trapezoid in the middle is a pullback square; and the top right hand map is the homeomorphism given by path reversal. The middle map in the top row is a homotopy equivalence because the map  $\sigma^{-1}$  is. It follows that  $[\sigma^{-1}] \in D$ . Finally, suppose  $[\sigma], [\sigma'] \in D$ . Our aim is to show that  $[\sigma\sigma'] \in D$ . In view of Theorem B and the equivalence between (8) and (14) in Theorem 4.1, it suffices to show that the generator of

$$\mathbb{E}_2^{0,0}(P(\sigma\sigma', \text{id}_{BG})) \cong \mathbb{F}_\ell$$

in the shifted Serre spectral sequence of  $P(\sigma\sigma', \text{id}_{BG}) \rightarrow BG$  is a permanent cycle. Under our pairings between spectral sequences, this generator factors as the product of the generators of  $\mathbb{E}_2^{0,0}(P(\sigma', \text{id}_{BG}))$  and  $\mathbb{E}_2^{0,0}(P(\sigma\sigma', \sigma'))$ , so it is enough to show that these two factors are permanent cycles. That the first factor is a permanent cycle follows from the assumption that  $[\sigma'] \in D$  by Theorem B and the equivalence between (8) and (14) in Theorem 4.1. That the second factor is also a permanent cycle follows similarly from the assumption that  $[\sigma] \in D$  by making use of the pullback diagram

$$\begin{array}{ccc} P(\sigma\sigma', \sigma') & \xrightarrow{\simeq} & P(\sigma, \text{id}_{BG}) \\ \downarrow & & \downarrow \\ BG & \xrightarrow[\simeq]{\sigma'} & BG \end{array}$$

where the top horizontal map is a homotopy equivalence because  $\sigma'$  is.  $\square$

**Proposition 4.10.** *Suppose  $[\sigma], [\sigma'] \in \text{Out}(BG)$  generate the same closed subgroup of  $\text{Out}(BG)$ . Then  $BG^{h\sigma}$  and  $BG^{h\sigma'}$  are homotopy equivalent over  $BG$ , i.e., we can choose a homotopy equivalence so that the following diagram commutes*

$$\begin{array}{ccc} BG^{h\sigma} & \xrightarrow{\simeq} & BG^{h\sigma'} \\ & \searrow & \swarrow \\ & & BG \end{array}$$

*In particular  $[\sigma] \in D$  if and only if  $[\sigma'] \in D$ .*

*Proof.* We have a homotopy equivalence  $BG^{h\sigma} \xrightarrow{\simeq} BG^{h\sigma'}$ , by [BMO12, Thm. 2.4], generalizing [BM07, Prop. 6.5], and an inspection of the proof shows that this homotopy equivalence can be chosen so that the indicated diagram commutes. The hypotheses of the theorem are satisfied in our situation:  $H^*(BG)$  is Noetherian by [DW94, Thm. 2.4], and  $\text{Out}(BG)$  is detected on  $\hat{H}^*(BG; \mathbb{Z}_\ell) = H^*(BG; \mathbb{Z}_\ell)$  since  $H^*(BG; \mathbb{Z}_\ell) \otimes \mathbb{Q} \cong (H^*(BT; \mathbb{Z}_\ell) \otimes \mathbb{Q})^{W_G}$  by [DW94, Thm. 9.7(iii)] and  $\text{Out}(BG)$  is detected on  $BT$ , as part of the classification of connected  $\ell$ -compact groups

[AG09, Thm. 1.2] (in fact  $\text{Out}(BG) \cong \text{Out}(\mathbf{D}_G)$ ). The statement about  $D$  is a consequence of the first claim.  $\square$

The following result can be read off from [AGMV08, Theorem 13.1].

**Lemma 4.11.** *Suppose  $\mathbb{D}$  is an irreducible  $\mathbb{Z}_\ell$ -root datum. Then the image of the homomorphism  $\mathbb{Z}_\ell^\times \rightarrow \text{Out}(\mathbb{D})$  sending  $q \in \mathbb{Z}_\ell^\times$  to the multiplication-by- $q$  map has finite index in  $\text{Out}(\mathbb{D})$ .  $\square$*

**Lemma 4.12.** *Any group homomorphism from a finitely generated free  $\mathbb{Z}_\ell$ -module into  $\text{Out}(BG)$  is continuous.*

*Proof.* The topology on  $\text{Out}(BG)$  is easily seen to be induced from the discrete topologies on the groups  $\text{Aut}(H^*(BG; \mathbb{Z}/\ell^k))$ ,  $k \geq 1$  via the evident maps  $\text{Out}(BG) \rightarrow \text{Aut}(H^*(BG; \mathbb{Z}/\ell^k))$ , so that a homomorphism into  $\text{Out}(BG)$  is continuous if and only if its composite with all these maps is. By [ACF<sup>+</sup>13, Theorem 4.2], the cohomology ring  $H^*(BG; \mathbb{Z}/\ell^k)$  is Noetherian and hence [Mat89, Theorem 13.1] a finitely generated  $\mathbb{Z}/\ell^k$ -algebra. It follows that  $\text{Aut}(H^*(BG; \mathbb{Z}/\ell^k))$  is finite for every  $k$ . The claim now follows if we can show that all group homomorphisms from finitely generated free  $\mathbb{Z}_\ell$ -modules into finite groups are continuous, or, what is the same, that every finite-index subgroup of a finitely generated free  $\mathbb{Z}_\ell$ -module is open. But the latter claim holds as a special case of a theorem of Serre, which asserts that finite-index subgroups of topologically finitely generated pro- $\ell$ -groups are open (see e.g. [Ser02, section 4.2] for an indication of the proof).  $\square$

**Lemma 4.13.** *When  $BG$  is semisimple,  $D$  is a closed subgroup of  $\text{Out}(BG)$ .*

*Proof.* By Lemma 4.9,  $D$  is a subgroup of  $\text{Out}(BG)$ . It remains to show that it is closed. In view of Lemma 4.11, the isomorphisms

$$\mathbb{Z}_\ell^\times \cong \mathbb{Z}_\ell \times (\mathbb{Z}/\ell)^\times \quad (\text{for } \ell \text{ odd}) \quad \text{and} \quad \mathbb{Z}_2^\times \cong \mathbb{Z}_2 \times (\mathbb{Z}/4)^\times \quad (\text{for } \ell = 2)$$

(see e.g. [Gou97, Corollary 4.5.10]), and the description of the group  $\text{Out}(BG) \cong \text{Out}(\mathbb{D}_G)$  given in [AG09, Propositions 8.14 and 8.15], the assumption that  $BG$  is semisimple implies that we can find a group homomorphism  $\varphi: E \rightarrow \text{Out}(BG)$  where  $E$  is a finitely generated free  $\mathbb{Z}_\ell$ -module and the image  $\varphi E$  has finite index in  $\text{Out}(BG)$ .

The preimage  $\varphi^{-1}(D)$  is, of course, a subgroup of  $E$ . We wish to show that it in fact is a  $\mathbb{Z}_\ell$ -submodule. Suppose  $x \in \varphi^{-1}(D)$ . By Lemma 4.12, the homomorphism  $\psi: \mathbb{Z}_\ell \rightarrow \text{Out}(BG)$ ,  $\psi(b) = \varphi(bx)$  is continuous. For  $a \in \mathbb{Z}_\ell$ , we therefore have  $\psi\langle a \rangle \subset \overline{\psi\langle a \rangle} \subset \overline{\psi\langle a \rangle}$  where the first inclusion follows from the continuity of  $\psi$  and the second one from the observation that  $\overline{\psi\langle a \rangle}$  is closed, being the image of a compact set in a continuous map to a Hausdorff space. Thus  $\overline{\langle \varphi(ax) \rangle} = \overline{\langle \psi(a) \rangle} = \overline{\psi\langle a \rangle} = \overline{\psi\langle a \rangle}$ , so that the closed subgroup of  $\text{Out}(BG)$  generated by  $\varphi(ax)$  only depends on the closed subgroup of  $\mathbb{Z}_\ell$  generated by  $a$ . For any  $b \in \mathbb{Z}_\ell$ , we can find an integer  $n$  generating the same closed subgroup of  $\mathbb{Z}_\ell$  as  $b$  does. Since  $nx \in \varphi^{-1}(D)$ , Proposition 4.10 implies that  $bx \in \varphi^{-1}(D)$ . Thus  $\varphi^{-1}(D)$  is a  $\mathbb{Z}_\ell$ -submodule of  $E$  as claimed.

Since  $\mathbb{Z}_\ell$  is a principal ideal domain, as a submodule of the finitely generated free  $\mathbb{Z}_\ell$ -module  $E$ , the preimage  $\varphi^{-1}(D)$  is also such. In particular,  $\varphi^{-1}(D)$  is compact. Since by Lemma 4.12 the homomorphism  $\varphi|: \varphi^{-1}(D) \rightarrow \text{Out}(BG)$  is continuous, it follows that the image  $\varphi\varphi^{-1}(D)$  is closed in  $\text{Out}(BG)$ . Moreover, the index of  $\varphi\varphi^{-1}(D)$  in  $D$  is finite because the index of  $\varphi E$  in  $\text{Out}(BG)$  is. Thus  $D$  is a finite union of closed sets and therefore closed, as claimed.  $\square$

**Lemma 4.14.**  *$[\psi^q] \in D$  for some  $q \in \mathbb{Z}_\ell^\times$  of infinite order.*

*Proof.* Suppose first that  $BG$  is the  $\ell$ -completion of the classifying space of a compact connected Lie group. In the proof of [Kam08, Theorem 1.5], it is shown that the Serre spectral sequences of the fibrations  $LBG \rightarrow BG$  and  $BG^{h\psi^q} \rightarrow BG$  are isomorphic when  $q$  is a suitable prime power congruent to 1 modulo  $\ell$ . The claim now follows from the equivalence of (10) and (8)

in Theorem 4.1. Now for the general case we know by the classification of  $\ell$ -compact groups (specifically [AG09, Thm. 1.1] [AGMV08, Thm. 1.2]) that  $BG$  splits a product  $BH \times BK$ , where  $BH$  is the  $\ell$ -completion of the classifying space of a compact connected Lie group, and  $BK$  is an exotic  $\ell$ -compact group with polynomial cohomology ring. Hence by the already-proven polynomial case of Theorem D, if  $\psi^q \in D$  for  $BH$ , we will also have  $\psi^q \in D$  for the product.  $\square$

*Proof of Theorem 4.5.* The various parts of the theorem were established in Proposition 4.10 and Lemmas 4.8, 4.9, 4.13 and 4.14 above.  $\square$

We now proceed to derive Corollary 4.6 from Theorem 4.5.

**Lemma 4.15.** *The homomorphism*

$$\theta: \mathbb{Z}_\ell^\times \longrightarrow \text{Out}(BG), \quad q \longmapsto [\psi^q]$$

*is continuous.*

*Proof.* The group  $\mathbb{Z}_\ell^\times$  has an open subgroup isomorphic to  $\mathbb{Z}_\ell$ . See e.g. [Gou97, Corollary 4.5.10]. By Lemma 4.12, the restriction of  $\theta$  to this subgroup is continuous, whence  $\theta$  also is.  $\square$

*Proof of Corollary 4.6.* Let

$$K = \{q \in \mathbb{Z}_\ell^\times \mid BG^{h\psi^q} \text{ has a fundamental class}\} \subset \mathbb{Z}_\ell^\times.$$

As an inverse image of the subgroup  $D \leq \text{Out}(BG)$  under the homomorphism  $\theta$  of Lemma 4.15,  $K$  is a subgroup of  $\mathbb{Z}_\ell^\times$ . Moreover, Theorem 4.5.(iii) and Lemma 4.15 imply that  $K$  has the following closure property: if  $q, q' \in \mathbb{Z}_\ell^\times$  generate the same closed subgroup of  $\mathbb{Z}_\ell^\times$ , then  $q \in K$  if and only if  $q' \in K$ .

Consider now the case  $\ell$  odd. Recall that in this case there exists a canonically split short exact sequence

$$0 \longrightarrow \mathbb{Z}_\ell \xrightarrow{\text{exp} \circ \ell} \mathbb{Z}_\ell^\times \longrightarrow (\mathbb{Z}/\ell)^\times \longrightarrow 0 \quad (4.8)$$

where the  $\ell$  in  $\text{exp} \circ \ell$  refers to multiplication by  $\ell$ , the map out of  $\mathbb{Z}_\ell^\times$  is given by reduction mod  $\ell$ , and the image of the splitting is precisely the set of roots of unity in  $\mathbb{Z}_\ell^\times$ . See e.g. [Gou97, Corollary 4.5.10]. Let  $K' \leq \mathbb{Z}_\ell$  be the inverse image of  $K \leq \mathbb{Z}_\ell^\times$ . By the aforementioned closure property of  $K$  and continuity of  $\text{exp} \circ \ell$ , the question of whether an element  $x \in \mathbb{Z}_\ell$  belongs to  $K'$  only depends on the closed subgroup generated by  $x$ . Since this subgroup consists of all elements of  $\mathbb{Z}_\ell$  with  $\ell$ -adic valuation at least equal to that of  $x$ , we see that

$$K' = \{x \in \mathbb{Z}_\ell \mid v_\ell(x) \geq k\}$$

where  $0 \leq k \leq \infty$  is the minimum of the  $\ell$ -adic valuations of elements of  $K'$ . Moreover, Theorem 4.5.(v) implies that  $k < \infty$ . It follows that the image  $H$  of  $K'$  under  $\text{exp} \circ \ell$  is

$$H = \{q \in \mathbb{Z}_2^\times \mid q \equiv 1 \pmod{\ell} \text{ and } v_\ell(q - 1) \geq k + 1\} = H_n$$

for  $n = k + 1$ . Now let  $e$  be the order of the image of  $K$  in  $(\mathbb{Z}/\ell)^\times \cong \mathbb{Z}/(\ell - 1)$ . To prove the claim, it suffices to show that  $K$  contains a primitive  $e$ -th root of unity. Let  $x$  be any element of  $K$  mapping to an element of order  $e$  in  $(\mathbb{Z}/\ell)^\times$ . By the split exact sequence (4.8), we can write  $x$  as  $x = uy$ , where  $u \in \mathbb{Z}_\ell^\times$  is a primitive  $e$ -th root of unity and  $y \equiv 1 \pmod{\ell}$ . Then  $y^e = x^e \in H$ . Since  $\ell \nmid e$ , we have  $v_\ell(y^e - 1) = v_\ell(y - 1)$ , and therefore also  $y \in H$ . Thus  $u = xy^{-1} \in K$ , as desired.

Consider now the case  $\ell = 2$ . In this case, there exists a (canonically split) short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{\text{exp} \circ 4} \mathbb{Z}_2^\times \longrightarrow (\mathbb{Z}/4)^\times \longrightarrow 0$$

where the map out of  $\mathbb{Z}_2^\times$  is given by reduction mod 4. Let  $K'$  be the preimage of  $K$  under the embedding  $\exp \circ 4$ . Again, we have

$$K' = \{x \in \mathbb{Z}_2 \mid v_2(x) \geq k\}$$

for some  $0 \leq k < \infty$ . The image  $H$  of  $K'$  under  $\exp \circ 4$  is then

$$H = \{q \in \mathbb{Z}_2^\times \mid q \equiv 1 \pmod{4} \text{ and } v_2(q-1) \geq k+2\} = H'_n$$

where  $n = k+2$ . The quotient  $K/H$  embeds into  $(\mathbb{Z}/4)^\times \cong \mathbb{Z}/2$ , so the index of  $H$  in  $K$  is 1 or 2. In the former case  $K = H$ . Assume that the latter case holds. Then  $K$  contains an element  $x$  such that  $x \equiv -1 \pmod{4}$ , and  $K = H \cup xH$ . If in fact  $-1 \in K$ , we thus have  $K = \pm H$ . Let us assume  $-1 \notin K$ . Then  $x = -1 + 2^e x'$  for some  $2 \leq e < \infty$  and  $x' \in \mathbb{Z}_2^\times$ , and we may assume that  $x$  has been chosen so that  $e$  is minimal. Since  $x^2 \in H$ , we must have  $e \geq n-1$ . On the other hand, if we had  $e > n$ , consideration of the element  $(1+2^n)x$  would lead to a contradiction with the minimality of  $e$ . Thus  $e = n$  or  $e = n-1$ . In the former case, we would have  $1+2^e x' x^{-1} \in H$ , and therefore

$$-1 = x(1+2^e x' x^{-1}) \in K,$$

contradicting the assumption that  $-1 \notin K$ . Thus  $e = n-1$ . Now  $1+2^e(1-x')x^{-1} \in H$ , so

$$xH = x(1+2^e(1-x')x^{-1})H = (-1+2^e)H.$$

Thus the claim follows.  $\square$

For a  $\mathbb{Z}_\ell$ -root datum  $\mathbb{D}$ , the outer automorphism group  $\text{Out}(\mathbb{D})$  is by definition a quotient of a closed subgroup of  $\text{GL}_{\mathbb{Z}_\ell}(L)$  by a finite group, where  $L$  denotes the underlying finitely generated free  $\mathbb{Z}_\ell$ -module of  $\mathbb{D}$ . See [AG09, p. 388]. Thus the natural topology on  $\text{GL}_{\mathbb{Z}_\ell}(L)$  induces a topology on  $\text{Out}(\mathbb{D})$  which makes  $\text{Out}(\mathbb{D})$  into a profinite group.

**Proposition 4.16.** *Under the isomorphism  $\text{Out}(BG) \cong \text{Out}(\mathbb{D}_G)$ , the topologies on  $\text{Out}(BG)$  and  $\text{Out}(\mathbb{D}_G)$  coincide.*

*Proof.* Since  $\text{Out}(\mathbb{D}_G)$  is compact and  $\text{Out}(BG)$  is Hausdorff, it suffices to show that the isomorphism  $\text{Out}(\mathbb{D}_G) \xrightarrow{\cong} \text{Out}(BG)$  is continuous. As in the proof of Lemma 4.12, the continuity of this homomorphism follows if we can show that finite-index subgroups of  $\text{Out}(BG)$  are open. It is a result of Nikolov and Segal [NS07] (generalizing the theorem of Serre used in the proof of Lemma 4.12) that finite-index subgroups of *all* topologically finitely generated profinite groups are open. Thus it is enough to show that the group  $\text{Out}(\mathbb{D})$  is topologically finitely generated for every  $\mathbb{Z}_\ell$ -root datum  $\mathbb{D}$ . When  $\mathbb{D}$  is irreducible,  $\text{Out}(\mathbb{D})$  is topologically finitely generated by Lemma 4.11. That  $\text{Out}(\mathbb{D})$  is topologically finitely generated in general now follows from the description of  $\text{Out}(\mathbb{D})$  given in [AG09, Theorem 8.13 and Propositions 8.14 and 8.15] and the observation that open subgroups of topologically finitely generated profinite groups are topologically finitely generated.  $\square$

**Remark 4.17.** The Nikolov–Segal theorem used in the above proof is a very deep result ultimately relying on the classification of finite simple groups. It would be preferable to find a more direct proof for Proposition 4.16.

*Proof of Theorem E.* The theorem is immediate from Theorem 4.5.(i), (ii), and (iv); Proposition 4.16; Corollary 4.6, which implies that the set of  $q \in \mathbb{Z}_\ell^\times$  for which  $BG^{h\psi^q}$  has a  $[G]$ -fundamental class is uncountable; and the fact, evident from the defining property of  $\psi^q$ , that the kernel of the homomorphism  $\mathbb{Z}_\ell^\times \rightarrow \text{Out}(BG)$ ,  $q \mapsto [\psi^q]$  is finite.  $\square$

We now return to Theorem D to prove the result in the spin case, building on the work of Kameko [Kam15].

**Theorem 4.18.** *For  $\ell = 2$  and  $n \geq 2$ ,  $(B\text{Spin}(n)\hat{2})^{h\psi^q}$  has a  $[\text{Spin}(n)\hat{2}]$ -fundamental class for all  $q \in \mathbb{Z}_2^\times$ .*

*Proof.* Since the mod 2 cohomology of  $B\text{Spin}(2)\hat{2}$  is a polynomial ring, for  $n = 2$ , the claim follows from Proposition 4.4 and Theorem 4.3. Let us assume that  $n \geq 3$ . For brevity, let us write  $B\text{SO}_n$  and  $B\text{Spin}_n$  for  $B\text{SO}(n)\hat{2}$  and  $B\text{Spin}(n)\hat{2}$ , respectively. In view of Corollary 4.6, it is enough to prove that  $(B\text{Spin}_n)^{h\psi^q}$  has a fundamental class when  $q = 3$  or  $q = 5$ . So let us assume that  $q$  is one of these numbers; in fact, all that matters for the argument that follows is that  $q$  is an odd prime power. By choosing suitable models for  $B\text{SO}_n$  and  $B\text{Spin}_n$  and for the self-maps  $\psi^q$  of  $B\text{SO}_n$  and  $B\text{Spin}_n$ , we may assume the following:

- (1) The maps  $\psi^q: B\text{SO}_n \rightarrow B\text{SO}_n$  and  $\psi^q: B\text{Spin}_n \rightarrow B\text{Spin}_n$  are basepoint-preserving.
- (2) The map  $p: B\text{Spin}_n \rightarrow B\text{SO}_n$  induced by the projection  $\text{Spin}(n) \rightarrow \text{SO}(n)$  preserves basepoints and commutes with the action of the maps  $\psi^q$ .

For example, by replacing  $\psi^q: B\text{SO}_n \rightarrow B\text{SO}_n$  by a homotopic map if necessary, we may assume that it preserves the basepoint, after which models for  $B\text{Spin}_n$  and  $\psi^q: B\text{Spin}_n \rightarrow B\text{Spin}_n$  with the desired properties can be obtained by passing to functorial 2-connected covers.

Let  $E_{n,q}$  and  $E_n$  be the spaces obtained as pullbacks

$$\begin{array}{ccc} E_{n,q} & \longrightarrow & B\text{SO}_n^{h\psi^q} \\ \downarrow & & \downarrow \text{ev}_1 \\ B\text{Spin}_n & \xrightarrow{p} & B\text{SO}_n \end{array} \quad \text{and} \quad \begin{array}{ccc} E_n & \longrightarrow & L\text{B}\text{SO}_n \\ \downarrow & & \downarrow \text{ev}_1 \\ B\text{Spin}_n & \xrightarrow{p} & B\text{SO}_n \end{array} \quad (4.9)$$

Let

$$\pi_{n,q}: B\text{Spin}_n^{h\psi^q} \longrightarrow E_{n,q} \quad \text{and} \quad \pi_n: L\text{B}\text{Spin}_n \longrightarrow E_n$$

be the maps over  $B\text{Spin}_n$  induced by the evaluation maps

$$\text{ev}_1: B\text{Spin}_n^{h\psi^q} \longrightarrow B\text{Spin}_n \quad \text{and} \quad \text{ev}_1: L\text{B}\text{Spin}_n \longrightarrow B\text{Spin}_n$$

and the maps

$$B\text{Spin}_n^{h\psi^q} \longrightarrow B\text{SO}_n^{h\psi^q} \quad \text{and} \quad L\text{B}\text{Spin}_n \longrightarrow L\text{B}\text{SO}_n$$

induced by  $p$ . Then, up to homotopy,  $\pi_{n,q}$  and  $\pi_n$  are two-fold covering spaces, and there are maps of fibre sequences

$$\begin{array}{ccc} \mathbb{Z}/2 & \longrightarrow & \Omega B\text{Spin}_n \xrightarrow{\Omega p} \Omega B\text{SO}_n \\ \parallel & & \downarrow i_{n,q} \quad \downarrow j_{n,q} \\ \mathbb{Z}/2 & \longrightarrow & B\text{Spin}_n^{h\psi^q} \xrightarrow{\pi_{n,q}} E_{n,q} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{Z}/2 & \longrightarrow & \Omega L\text{B}\text{Spin}_n \xrightarrow{\Omega p} \Omega L\text{B}\text{SO}_n \\ \parallel & & \downarrow i_n \quad \downarrow j_n \\ \mathbb{Z}/2 & \longrightarrow & L\text{B}\text{Spin}_n \xrightarrow{\pi_n} E_n \end{array}$$

where the vertical arrows are inclusions of fibres of the various spaces over  $B\text{Spin}_n$  over the basepoint. We obtain the following commutative diagram whose rows are long exact Gysin sequences:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^d(E_{n,q}) & \xrightarrow{\pi_{n,q}^*} & H^d(B\text{Spin}_n^{h\psi^q}) & \xrightarrow{(\pi_{n,q})!} & H^d(E_{n,q}) & \xrightarrow{\cup e_{n,q}} & H^{d+1}(E_{n,q}) & \longrightarrow & \cdots \\ & & \downarrow j_{n,q}^* & & \downarrow i_{n,q}^* & & \downarrow j_{n,q}^* & & \downarrow j_{n,q}^* & & \\ \cdots & \longrightarrow & H^d(\Omega B\text{SO}_n) & \xrightarrow{(\Omega p)^*} & H^d(\Omega B\text{Spin}_n) & \xrightarrow{(\Omega p)!} & H^d(\Omega B\text{SO}_n) & \xrightarrow{\cup e} & H^{d+1}(\Omega B\text{SO}_n) & \longrightarrow & \cdots \\ & & \uparrow j_n^* & & \uparrow i_n^* & & \uparrow j_n^* & & \uparrow j_n^* & & \\ \cdots & \longrightarrow & H^d(E_n) & \xrightarrow{\pi_n^*} & H^d(L\text{B}\text{Spin}_n) & \xrightarrow{(\pi_n)!} & H^d(E_n) & \xrightarrow{\cup e_n} & H^{d+1}(E_n) & \longrightarrow & \cdots \end{array}$$

Here  $d = \dim \text{SO}(n) = \dim \text{Spin}(n)$  and  $e$ ,  $e_n$  and  $e_{n,q}$  are the Euler classes for the respective double covers.

Our task is to show that the map  $i_{n,q}^*$  in the above diagram is nonzero. To do this, it suffices to show that the composite map

$$(\Omega p)_! \circ i_{n,q}^* = j_{n,q}^* \circ (\pi_{n,q})_!: H^d(B\text{Spin}_n^{h\psi^q}) \longrightarrow H^d(\Omega BSO_n)$$

is nonzero. Thus it is enough to show that there exists an element  $x \in H^d(E_{n,q})$  such that  $j_{n,q}^*(x) \neq 0$  and  $x \cup e_{n,q} = 0$ .

To show the existence of such an element  $x$ , we will compare the upper part of the diagram with the lower part. As in [Kam15, Propositions 4.1 and 4.3], the Eilenberg–Moore spectral sequences for the pullback squares (4.9) yield ring isomorphisms

$$H^*(E_{n,q}) \cong H^*(B\text{Spin}_n) \otimes_{H^*(BSO_n)} H^*(BSO_n^{h\psi^q})$$

and

$$H^*(E_n) \cong H^*(B\text{Spin}_n) \otimes_{H^*(BSO_n)} H^*(LBSO_n).$$

Let

$$k_{n,q}: \Omega BSO_n \longrightarrow BSO_n^{h\psi^q} \quad \text{and} \quad k_n: \Omega BSO_n \longrightarrow LBSO_n$$

be inclusions of fibres of the evaluation fibrations  $BSO_n^{h\psi^q} \rightarrow BSO_n$  and  $LBSO_n \rightarrow BSO_n$ . Under the above isomorphisms, the maps  $j_{n,q}^*$  and  $j_n^*$  then correspond to the maps induced by

$$k_{n,q}^*: H^*(BSO_n^{h\psi^q}) \longrightarrow H^*(\Omega BSO_n) \quad \text{and} \quad k_n^*: H^*(LBSO_n) \longrightarrow H^*(\Omega BSO_n)$$

(along with the augmentation  $H^*(B\text{Spin}_n) \rightarrow \mathbb{F}_2$ ), respectively. By [Kam15, Theorem 1.7], there exists a  $H^*(BSO_n)$ -algebra isomorphism

$$H^*(BSO_n^{h\psi^q}) \cong H^*(LBSO_n).$$

Under this isomorphism, the maps  $k_{n,q}^*$  and  $k_n^*$  agree; to see this, observe that both halves of the diagram on p. 525 of Kameko’s paper [Kam15] are pullbacks, so  $k_{n,q}$  and  $k_n$  factor (up to homotopy) through the same map from  $\Omega BSO_n$  to Kameko’s space  $\tilde{B}A_{n-1}$ . We conclude that there exists a ring isomorphism

$$H^*(E_{n,q}) \cong H^*(E_n) \tag{4.10}$$

under which the maps  $j_{n,q}^*$  and  $j_n^*$  agree.

Using for example the Serre spectral sequence, it is easy to see that  $H^1 E_{n,q} \cong H^1 E_n \cong \mathbb{F}_2$ . The Euler classes  $e_{n,q} \in H^1 E_{n,q}$  and  $e_n \in H^1 E_n$  both pull back to the class  $e \in H^1 \Omega BSO_n$ , which in turn pulls back to the (non-trivial) Euler class of the double cover  $\text{Spin}(2) \rightarrow \text{SO}(2)$ . Thus the classes  $e_{n,q}$  and  $e_n$  must be the unique non-trivial degree 1 classes in their respective cohomology groups, and hence they must correspond under the isomorphism (4.10).

We have reduced the task of constructing the desired class  $x \in H^d E_{n,q}$  to the task of finding a class  $y \in H^d E_n$  such that  $j_n^*(y) \neq 0$  and  $y \cup e_n = 0$ . By Theorem 2.12, the map  $i_n^*$  sends the class  $\mathbb{1} \in H^d(LB\text{Spin}_n)$  to the nontrivial class  $\mathbb{1} \in H^d(\Omega B\text{Spin}_n)$ . Moreover, since  $H^{d+1}(\Omega BSO_n) = 0$ , the map  $(\Omega p)_!$  is an epimorphism and hence an isomorphism, as its source and target are both one-dimensional. Thus  $(\Omega p)_! i_n^*(\mathbb{1}) \neq 0$ . Now the class  $y = (\pi_n)_!(\mathbb{1}) \in H^d(E_n)$  is as desired.  $\square$

*Proof of Theorem D when  $BG = B\text{Spin}(n)\hat{2}$ ,  $n \geq 10$ .* The claim is immediate from Theorem 4.18.  $\square$

**Remark 4.19.** Most of our results generalize effortlessly from cohomology with  $\mathbb{F}_\ell$ -coefficients to cohomology with coefficients in an arbitrary commutative ring  $R$  such that all maps inducing an isomorphism on homology with  $\mathbb{F}_\ell$ -coefficients do so on homology with  $R$ -coefficients; examples of such rings  $R$  include  $R = \mathbb{Z}/\ell^k$  for all  $k \geq 1$ . All of the results in sections 2 and 3, and hence Theorem A, hold for cohomology with  $R$ -coefficients with little change required in the proofs. (In the description of the  $E_2$ -pages of the spectral sequences in Theorem A, the tensor product  $H^*(BG) \otimes \mathbb{H}^*(G)$  should, of course, be replaced with  $H^*(BG; \mathbb{H}^*(G; R))$ , and similarly elsewhere.) Moreover, the resulting products are natural with respect to the maps induced by

ring homomorphisms  $R \rightarrow R'$  as long as long as the orientations over  $R$  and  $R'$  (see Remark 2.42) are chosen in a compatible way.

Theorem 4.1 and its proof also generalize, with the following modifications: Condition (9) should be removed, in condition (5) replace  $i^*(x) \neq 0$  with the statement that  $i^*(x) \in H^d(F; R)$  is an  $R$ -module generator; and in conditions (6) and (7) replace  $1 \otimes i^*(x)$  with the element  $\bar{x} \in \mathbb{E}_2^{0,0} = H^0(B; \mathbb{H}^0(F; R))$  corresponding to  $i^*(x) \in \mathbb{H}^0(F; R)$  under the isomorphism  $H^0(B; \mathbb{H}^0(F; R)) \cong \mathbb{H}^0(F; R)$ . Theorem 4.2 also generalizes in a straightforward fashion, as does Theorem 4.5 (with the exception of part (v)) when we interpret  $D$  as

$$D = \{[\sigma] \in \text{Out}(BG) \mid H^*(BG^{h\sigma}; R) \text{ is free of rank 1 over } \mathbb{H}^*(LBG; R)\}.$$

Even though  $\mathbb{Z}_\ell$  does not satisfy the aforementioned condition on  $R$ , one also obtains a multiplication on  $\mathbb{H}^*(LBG; \mathbb{Z}_\ell) \cong \lim_k \mathbb{H}^*(LBG; \mathbb{Z}/\ell^k)$  and a module structure on  $H^*(BG^{h\sigma}; \mathbb{Z}_\ell) \cong \lim_k \mathbb{H}^*(BG^{h\sigma}; \mathbb{Z}/\ell^k)$  by passing to limits over  $\mathbb{Z}/\ell^k$ -coefficients. The resulting module structure is never free of rank 1, however, when  $BG^{h\sigma}$  recovers  $\ell$ -completion of the classifying space of a finite group of Lie type, as in this case  $H^*(BG^{h\sigma}; \mathbb{Z}_\ell)$  is torsion in positive degrees, unlike  $H^*(LBG; \mathbb{Z}_\ell)$ . This illustrates the difficulty for the module structure to be free of rank 1 over  $\mathbb{Z}/\ell^k$  for  $k$  large. We will not describe the behavior over  $\mathbb{Z}/\ell^k$  or  $\mathbb{Z}_\ell$ -coefficients further in this paper, but note that a key question seems to be understanding the (non-)divisibility and order of the “integral fundamental class”, namely the image of the fundamental class of  $G$  under the map  $H_d^{\mathbb{Z}_\ell}(G) \rightarrow H_d^{\mathbb{Z}_\ell}(BG^{h\sigma})$ , where we have written  $H_*^{\mathbb{Z}_\ell}(X)$  for  $\lim_k H_*(X; \mathbb{Z}/\ell^k)$ .

## APPENDIX A. PARAMETRIZED HOMOTOPY THEORY

In this appendix, we will give a brief summary of the parametrized homotopy theory needed in this paper, referring the reader to the monograph [MS06] and the  $\infty$ -categorical treatment [ABG<sup>+</sup>14, ABG11] for a fuller discussion. Informally, for  $\mathcal{C}$  a category, a parametrized  $\mathcal{C}$ -object  $X$  over a space  $B$  is a collection  $\{X_b\}_{b \in B}$  of objects of  $\mathcal{C}$  together with glue banding them together in a way compatible with the topology of  $B$ . The object  $X_b$  is called the *fibre* of  $X$  over  $b \in B$ . We will need to work with several different types of parametrized objects: parametrized spaces, parametrized spectra, parametrized  $H\mathbb{F}_\ell$ -local spectra, and parametrized  $H\mathbb{F}_\ell$ -modules.

**A.1. The categories  $\text{Ho}(\mathcal{C}/_B)$ .** A convenient way to construct the necessary categories of parametrized objects is the  $\infty$ -categorical framework of [ABG11]. Let  $\mathcal{C}$  be a presentable  $\infty$ -category such as one of the  $\infty$ -categories **Spaces**, **Spectra**, **Spectra** $^\ell$ , and **Mod** $^{H\mathbb{F}_\ell}$  of spaces, spectra,  $H\mathbb{F}_\ell$ -local spectra, and  $H\mathbb{F}_\ell$ -modules, respectively. A parametrized  $\mathcal{C}$ -object over a space  $B$  is then simply an  $\infty$ -functor

$$X: (\Pi_\infty B)^{\text{op}} \longrightarrow \mathcal{C}$$

where  $\Pi_\infty B$  denotes the  $\infty$ -groupoid of  $B$ . The fibre of  $X$  over  $b \in B$  is the value of  $X$  on the object  $b \in (\Pi_\infty B)^{\text{op}}$ . We write

$$\mathcal{C}/_B = \text{Fun}((\Pi_\infty B)^{\text{op}}, \mathcal{C})$$

for the  $\infty$ -category of such  $\infty$ -functors, and observe that we can identify  $\mathcal{C}/_{\text{pt}} = \mathcal{C}$ . As for the purposes of this paper it will suffice to work on the level of homotopy categories, we will focus our discussion on the homotopy categories  $\text{Ho}(\mathcal{C}/_B)$  (which are ordinary rather than  $\infty$ -categories).

**Notation A.1.** In the cases of interest of us, the  $\infty$ -category  $\mathcal{C}$  will be symmetric monoidal: for example, we equip the  $\infty$ -category **Spaces** with the cartesian monoidal structure, denoted by  $\times$ , and the  $\infty$ -categories **Spectra**, **Spectra** $^\ell$ , and **Mod** $^{H\mathbb{F}_\ell}$  with the appropriate smash product, denoted  $\wedge$ ,  $\wedge^\ell$ , and  $\wedge^{H\mathbb{F}_\ell}$ , respectively. Denoting the symmetric monoidal product on  $\mathcal{C}$  by  $\otimes$ , we then have on  $\text{Ho}(\mathcal{C}/_B)$  an induced closed symmetric monoidal structure with symmetric monoidal product  $\otimes_B$ .

**Remark A.2.** While the  $\infty$ -categorical approach outlined above is technically convenient, we will rely on a much more down-to-earth approach to the categories  $\mathrm{Ho}(\mathbf{Spaces}/_B)$  to populate them with examples. Recall that  $\mathcal{T}$  denotes the category of topological spaces and continuous maps. We write  $\mathcal{T}/B$  for the usual category of spaces over a space  $B$ : an object in  $\mathcal{T}/B$  is a continuous map  $X \rightarrow B$ , and a map in  $\mathcal{T}/B$  from  $X \rightarrow B$  to  $Y \rightarrow B$  is a continuous map  $\varphi: X \rightarrow Y$  making the triangle

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ & \searrow & \swarrow \\ & B & \end{array}$$

commute. The category  $\mathcal{T}/B$  admits a model structure where a morphism  $\varphi$  as above is a weak equivalence if it is a weak equivalence of the underlying spaces. We refer the reader to [MS06] for further details and development. The two approaches are compared and shown to be equivalent in [ABG11, Appendix B]. In particular, the approaches lead to the same homotopy category  $\mathrm{Ho}(\mathbf{Spaces}/_B)$ , so to give an object of  $\mathrm{Ho}(\mathbf{Spaces}/_B)$ , it suffices to give a continuous map  $X \rightarrow B$ .

**A.2. The base change functors  $f_!$ ,  $f^*$ , and  $f_*$ .** A key feature of the parametrized theory are the various base change functors relating the categories  $\mathrm{Ho}(\mathcal{C}/_B)$  for varying base spaces  $B$ . Given a map of spaces  $f: A \rightarrow B$ , precomposition with  $(\Pi_\infty f)^{\mathrm{op}}$  induces a functor

$$f^*: \mathrm{Ho}(\mathcal{C}/_A) \longrightarrow \mathrm{Ho}(\mathcal{C}/_B)$$

which has both a left adjoint  $f_!$  and a right adjoint  $f_*$ , and which is an equivalence of categories if  $f$  is a weak equivalence. In the case where  $\mathcal{C}$  is symmetric monoidal, the functor  $f^*$  is a closed symmetric monoidal functor, giving together with its adjoints rise to a ‘‘Wirthmüller context’’ [FHM03]. A number of natural equivalences then relate the base change functors and the closed symmetric monoidal structures on  $\mathrm{Ho}(\mathcal{C}/_A)$  and  $\mathrm{Ho}(\mathcal{C}/_B)$ : for  $X \in \mathrm{Ho}(\mathcal{C}/_A)$  and  $Y, Z \in \mathrm{Ho}(\mathcal{C}/_B)$ , we have

$$f^* S_B \simeq S_A \tag{A.1}$$

$$f^*(Y \otimes_B Z) \simeq f^* Y \otimes_A f^* Z \tag{A.2}$$

$$F_B(Y, f_* X) \simeq f_* F_A(f^* Y, X) \tag{A.3}$$

$$f^* F_B(Y, Z) \simeq F_A(f^* Y, f^* Z) \tag{A.4}$$

$$f_!(f^* Y \otimes_A X) \simeq Y \otimes_B f_! X \tag{A.5}$$

$$F_B(f_! X, Y) \simeq f_* F_A(X, f^* Y). \tag{A.6}$$

where we have written  $S_B$  for the unit object and  $F_B(-, -)$  for the internal hom in  $\mathrm{Ho}(\mathcal{C}/_B)$ . See [ABG11, Proposition 6.8]. The base change functors also satisfy certain commutation relations: given a homotopy cartesian square

$$\begin{array}{ccc} A & \xrightarrow{\bar{f}} & B \\ \bar{g} \downarrow & & \downarrow g \\ C & \xrightarrow{f} & D \end{array}$$

of spaces, there exist natural equivalences

$$\bar{f}_! \bar{g}^* \simeq g^* f_! \tag{A.7}$$

and

$$\bar{f}_* \bar{g}^* \simeq g^* f_*, \tag{A.8}$$



as follows from [ABG11, Proposition 5.12]. Equivalences between parametrized objects are detected on fibres: a map  $\varphi: X \rightarrow Y$  in  $\mathrm{Ho}(\mathcal{C}/_B)$  is an equivalence if and only if the map  $\varphi_b: X_b \rightarrow Y_b$  is for every  $b \in B$ , where we have written  $\varphi_b = i_b^*(\varphi)$ ,  $X_b = i_b^*X$  and  $Y_b = i_b^*Y$  for  $i_b: \{b\} \hookrightarrow B$  the inclusion.

**A.3. Fibrewise constructions: the functors  $T_B$ .** Constructions for unparametrized objects generalize to parametrized objects by performing them fibrewise. Formally, an  $\infty$ -functor  $T: \mathcal{C} \rightarrow \mathcal{D}$  induces, via postcomposition, functors

$$T_B: \mathrm{Ho}(\mathcal{C}/_B) \longrightarrow \mathrm{Ho}(\mathcal{D}/_B)$$

commuting with the pullback functors  $f^*$ . If  $T$  is symmetric monoidal, so are the functors  $T_B$ , and an adjoint pair of  $\infty$ -functors

$$T: \mathcal{C} \rightleftarrows \mathcal{D}: V$$

induces an adjoint pair of functors

$$T_B: \mathrm{Ho}(\mathcal{C}/_B) \rightleftarrows \mathrm{Ho}(\mathcal{D}/_B): V_B.$$

For example, from the adjunction

$$\Sigma_+^\infty: \mathbf{Spaces} \rightleftarrows \mathbf{Spectra}: \Omega^\infty$$

we obtain an adjunction

$$\Sigma_{+B}^\infty: \mathrm{Ho}(\mathbf{Spaces}/_B) \rightleftarrows \mathrm{Ho}(\mathbf{Spectra}/_B): \Omega_B^\infty$$

with  $\Sigma_{+B}^\infty$  symmetric monoidal. Similarly, from the adjunction

$$L: \mathbf{Spectra} \rightleftarrows \mathbf{Spectra}^\ell: U \tag{A.9}$$

where  $U$  is the inclusion and  $L$  is the localization functor, we obtain an adjunction

$$L_B: \mathrm{Ho}(\mathbf{Spectra}/_B) \rightleftarrows \mathrm{Ho}(\mathbf{Spectra}^\ell/_B): U_B \tag{A.10}$$

with  $L_B$  symmetric monoidal.

**A.4. The Grothendieck construction.** Given a presentable  $\infty$ -category  $\mathcal{C}$ , it will be convenient for us to assemble the categories  $\mathrm{Ho}(\mathcal{C}/_B)$  for varying base spaces  $B$  into a single category. This can be done using the *Grothendieck construction*, which we now review.

**Definition A.3.** Let  $\mathcal{B}$  be an ordinary category, and let  $\mathcal{S}$  be a 2-category, such as the 2-category  $\mathbf{Cat}$  of categories, functors and natural transformations, or the 2-category  $\mathbf{smCat}$  of symmetric monoidal categories, symmetric monoidal functors, and symmetric monoidal transformations. Recall that a *pseudofunctor*  $\mathcal{B}^{\mathrm{op}} \rightarrow \mathcal{S}$  is then a contravariant “functor” from  $\mathcal{B}$  to  $\mathcal{S}$  such that the identities  $(g \circ f)^* = f^* \circ g^*$  and  $\mathrm{id}_B^* = \mathrm{id}_{F(B)}$  for induced maps are only required to hold up to coherent 2-isomorphisms. See e.g. [Bor94a, section 7.5]. Also recall that a *pseudo natural transformation*  $\eta: F \rightarrow G$  between pseudofunctors  $F, G: \mathcal{B}^{\mathrm{op}} \rightarrow \mathcal{S}$  is a “natural transformation” from  $F$  to  $G$  such that the identity  $G(f) \circ \eta_C = \eta_B \circ F(f)$  for morphisms  $f: B \rightarrow C$  in  $\mathcal{B}$  is only required to hold up to a coherent 2-isomorphism. See e.g. [Bor94a, section 7.5]. Finally, recall that a *modification*  $\xi: \eta \rightarrow \theta$  between pseudo natural transformations  $\eta, \theta: F \rightarrow G$  consists of a 2-morphism  $\xi_A: \eta_A \rightarrow \theta_A$  for each object  $A$  of  $\mathcal{B}$  satisfying a compatibility condition with the coherence isomorphisms of  $\eta$  and  $\theta$ . See e.g. [SP14, Definition A.8]. We write  $[\mathcal{B}^{\mathrm{op}}, \mathcal{S}]$  for the 2-category of pseudofunctors, pseudo natural transformations and modifications from  $\mathcal{B}^{\mathrm{op}}$  to  $\mathcal{S}$ .

**Definition A.4.** For  $\mathcal{B}$  an ordinary category, let us write  $\mathbf{Cat}/\mathcal{B}$  for the 2-category with objects pairs  $(\mathcal{E}, \pi)$  where  $\mathcal{E}$  is a category and  $\pi: \mathcal{E} \rightarrow \mathcal{B}$  is a functor, with morphisms from  $(\mathcal{E}, \pi)$  to  $(\mathcal{E}', \pi')$  functors  $F: \mathcal{E} \rightarrow \mathcal{E}'$  such that the triangle

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{F} & \mathcal{E}' \\ & \searrow \pi & \swarrow \pi' \\ & \mathcal{B} & \end{array}$$

commutes, and with 2-morphisms between  $F, G: (\mathcal{E}, \pi) \rightarrow (\mathcal{E}', \pi')$  natural transformations  $\eta: F \rightarrow G$  such that  $\pi'(\eta_E) = \text{id}_{\pi(E)}$  for all objects  $E$  of  $\mathcal{E}$ .

**Definition A.5.** The *Grothendieck construction* (see e.g. [Bor94b, Theorem 8.3.1]) is a 2-functor

$$\int: [\mathcal{B}^{\text{op}}, \mathbf{Cat}] \longrightarrow \mathbf{Cat}/\mathcal{B}.$$

For a pseudofunctor  $F: \mathcal{B}^{\text{op}} \rightarrow \mathbf{Cat}$ , the category  $\int F$  over  $\mathcal{B}$  is constructed as follows:

- The objects of  $\int F$  are pairs  $(B, X)$  where  $B$  is an object of  $\mathcal{B}$  and  $X$  is an object of  $F(B)$ .
- A morphism from  $(B, X)$  to  $(C, Y)$  in  $\int F$  is a pair  $(f, \varphi)$  where  $f: B \rightarrow C$  is a morphism in  $\mathcal{B}$  and  $\varphi: X \rightarrow f^*(Y)$  is a morphism in  $F(B)$ .
- The composite of two such morphisms  $(f, \varphi): (B, X) \rightarrow (C, Y)$  and  $(g, \psi): (C, Y) \rightarrow (D, Z)$  is the pair  $(g \circ f, \psi * \varphi)$  where  $\psi * \varphi: X \rightarrow F(g \circ f)(Z)$  is the composite

$$X \xrightarrow{\varphi} f^*(Y) \xrightarrow{f^*(\psi)} f^*g^*(Z) \xrightarrow{\cong} (g \circ f)^*(Z).$$

Here the last arrow is the coherence isomorphism that is part of the data of a pseudofunctor.

- The functor  $\int F \rightarrow \mathcal{B}$  is projection onto first coordinate.

For a pseudo natural transformation  $\eta: F \rightarrow G$  between pseudofunctors  $\mathcal{B}^{\text{op}} \rightarrow \mathbf{Cat}$ , the functor

$$\int \eta: \int F \longrightarrow \int G$$

is defined as follows:

- On objects,  $(\int \eta)(B, X) = (B, \eta_B(X))$ .
- On morphisms,  $\int \eta$  sends a morphism  $(f, \varphi): (B, X) \rightarrow (C, Y)$  to the morphism  $(f, \zeta)$  where  $\zeta$  is the composite

$$\eta_B(X) \xrightarrow{\eta_B(\varphi)} \eta_B F(f)(Y) \xrightarrow{\cong} G(f)\eta_C(Y).$$

Here the last arrow is the coherence isomorphism that is part of the data of a pseudo natural transformation.

Finally, for a modification  $\xi: \eta \rightarrow \theta$  between pseudo natural transformations  $\eta, \theta: F \rightarrow G$ , the natural transformation

$$\int \xi: \int \eta \longrightarrow \int \theta$$

is defined by setting

$$(\int \xi)_{(B, X)} = (\text{id}_B, \xi_B): (B, \eta_B(X)) \longrightarrow (B, \theta_B(X)).$$

**Remark A.6.** Notice that the fibre of the functor  $\int F \rightarrow \mathcal{B}$  over an object  $B$  of  $\mathcal{B}$ , meaning the subcategory of  $\int F$  consisting of the objects mapping to  $B$  and the morphisms mapping to  $\text{id}_B$ , is canonically isomorphic to the category  $F(B)$ . In this way, we may think of the category  $\int F$  as obtained by assembling the categories  $F(B)$  for varying  $B$  together.

**Remark A.7.** The Grothendieck construction in fact provides an equivalence of 2-categories from  $[\mathcal{B}^{\text{op}}, \mathbf{Cat}]$  to the sub-2-category of  $\mathbf{Cat}/\mathcal{B}$  consisting of fibrations over  $\mathcal{B}$  and morphisms and transformations of such (see e.g. [Bor94b, Theorem 8.3.1]). We will not use the theory of fibrations in this paper, however.

**Remark A.8.** For a cartesian monoidal category  $\mathcal{B}$ , the Grothendieck construction of Definition A.4 lifts to the symmetric monoidal context, providing a 2-functor

$$\int: [\mathcal{B}^{\text{op}}, \mathbf{smCat}] \longrightarrow \mathbf{smCat}/\mathcal{B}$$

where the target is the 2-category of symmetric monoidal categories, symmetric monoidal functors, and symmetric monoidal transformations over  $\mathcal{B}$ . See [Shu08, Theorem 12.7]. Explicitly, the tensor product on  $\int F$  is given by the formula

$$(B, X) \otimes (C, Y) = (B \times C, \pi_B^* X \otimes_{B \times C} \pi_C^* Y) \quad (\text{A.11})$$

where  $\pi_B: B \times C \rightarrow B$  and  $\pi_C: B \times C \rightarrow C$  are the projections and  $\otimes_{B \times C}$  refers to the tensor product in  $F(B \times C)$ . The identity object for  $\otimes$  is the pair  $(\text{pt}, I_{\text{pt}})$  where  $\text{pt}$  is the terminal object of  $\mathcal{B}$  and  $I_{\text{pt}}$  is the identity object of  $F(\text{pt})$ . Notice that the symmetric monoidal structure on  $F(B)$  can be recovered from that on  $\int F$ : for  $X, Y \in F(B)$ , we have a natural isomorphism

$$X \otimes_B Y \cong \Delta^*(X \otimes Y)$$

where  $\Delta: B \rightarrow B \times B$  is the diagonal map.

#### A.5. The categories $\mathbf{hpC}$ and functors $F_{\text{fw}}$ .

**Definition A.9.** For a presentable  $\infty$ -category  $\mathcal{C}$ , we denote by  $\mathbf{hpC}$  the category obtained by applying the Grothendieck construction of appendix A.4 to the functor  $\mathcal{T}^{\text{op}} \rightarrow \mathbf{Cat}$  given by  $B \mapsto \text{Ho}(\mathcal{C}/_B)$ ,  $f \mapsto f^*$ . We think of  $\mathbf{hpC}$  as the category of parametrized  $\mathcal{C}$ -objects over varying base spaces, and will often write just  $X$  for an object  $(B, X)$  of  $\mathbf{hpC}$ , leaving the base space  $B$  as implicitly understood. The letters  $\mathbf{h}$  and  $\mathbf{p}$  stand for ‘homotopy’ and ‘parametrized,’ respectively.

**Definition A.10.** For an  $\infty$ -functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  between presentable  $\infty$ -categories, we write

$$F_{\text{fw}}: \mathbf{hpC} \longrightarrow \mathbf{hpD}$$

for the functor obtained by the Grothendieck construction from the pseudo natural transformation with components  $F_B: \text{Ho}(\mathcal{C}/_B) \rightarrow \text{Ho}(\mathcal{D}/_B)$ . The subscript  $\text{fw}$  stands for ‘fibrewise’: intuitively,  $F_{\text{fw}}$  is simply the functor obtained by applying  $F$  to a parametrized object fibrewise.

By Remark A.8, if  $\mathcal{C}$  is symmetric monoidal, so is the resulting category  $\mathbf{hpC}$ , and if  $F: \mathcal{C} \rightarrow \mathcal{D}$  is symmetric monoidal, so is the functor  $F_{\text{fw}}: \mathbf{hpC} \rightarrow \mathbf{hpD}$ . If  $\otimes$  denotes the symmetric monoidal product in  $\mathcal{C}$ , we write  $\bar{\otimes}$  for the symmetric monoidal product in  $\mathbf{hpC}$ . Finally, we note that an adjoint pair

$$T: \mathcal{C} \rightleftarrows \mathcal{D}: V$$

of functors gives rise to an adjoint pair of functors

$$T_{\text{fw}}: \mathbf{hpC} \rightleftarrows \mathbf{hpD}: V_{\text{fw}}.$$

Thus, for example, the adjunction (A.9) gives an adjunction

$$L_{\text{fw}}: \mathbf{hpSpectra} \rightleftarrows \mathbf{hpSpectra}^\ell: U_{\text{fw}} \quad (\text{A.12})$$

with  $L_{\text{fw}}$  symmetric monoidal.

## APPENDIX B. DUALIZABILITY IN A SYMMETRIC MONOIDAL CATEGORY

In this appendix, we briefly recall the notion of dualizability in the context of symmetric monoidal categories. All of the material in this appendix is well known.

Let  $\mathcal{C}$  be a symmetric monoidal category with tensor product  $\otimes$  and identity object  $I$ . An object  $X$  of  $\mathcal{C}$  is called *dualizable* if there exists an object  $Y$  of  $\mathcal{C}$  and maps  $\eta: I \rightarrow X \otimes Y$  and  $\varepsilon: Y \otimes X \rightarrow I$  such that the following composites are identity maps:

$$X \cong I \otimes X \xrightarrow{\eta \otimes 1} X \otimes Y \otimes X \xrightarrow{1 \otimes \varepsilon} X \otimes I \cong X$$

$$Y \cong Y \otimes I \xrightarrow{1 \otimes \eta} Y \otimes X \otimes Y \xrightarrow{\varepsilon \otimes 1} I \otimes Y \cong Y$$

Here  $I$  denotes the identity object of  $\mathcal{C}$ , and the displayed isomorphisms are given by the left and right unit constraints of  $\mathcal{C}$ , as appropriate. The object  $Y$  is called a *dual* of  $X$ , and it is unique up to a unique isomorphism compatible with  $\eta$  and  $\varepsilon$ . If  $X$  is dualizable with dual  $Y$ , replacing  $\eta$  and  $\varepsilon$  with  $\chi\eta$  and  $\varepsilon\chi$  (where  $\chi$  denotes the symmetry constraint) shows that  $Y$  is dualizable with dual  $X$ .

Suppose now  $\mathcal{C}$  is closed symmetric monoidal with internal hom  $F$ . In this case, if  $X$  is dualizable, its dual is isomorphic to the object  $DX = F(X, I)$  via an isomorphism under which the map  $\varepsilon$  corresponds to the usual evaluation map  $\varepsilon: DX \otimes X \rightarrow I$ . Observe that the adjoint of the composite

$$F(X, Y) \otimes F(Z, W) \otimes X \otimes Z \xrightarrow{1 \otimes \chi \otimes 1} F(X, Y) \otimes X \otimes F(Z, W) \otimes Z \xrightarrow{\varepsilon \otimes \varepsilon} Y \otimes W$$

provides a natural morphism

$$\otimes: F(X, Y) \otimes F(Z, W) \longrightarrow F(X \otimes Z, Y \otimes W). \quad (\text{B.1})$$

Under sufficient dualizability conditions, this map is an isomorphism; see [LMSM86, Proposition III.1.3].

**Lemma B.1.** *The morphism  $\otimes: F(X, Y) \otimes F(Z, W) \rightarrow F(X \otimes Z, Y \otimes W)$  of (B.1) is an isomorphism if  $X$  and  $Z$  are dualizable.  $\square$*

In particular, taking  $Y = W = I$  in (B.1), we obtain a natural morphism

$$DX \otimes DZ \longrightarrow D(X \otimes Z) \quad (\text{B.2})$$

which is an isomorphism if  $X$  and  $Z$  are dualizable.

The following characterization of dualizable objects in closed symmetric monoidal categories is well known, but lacking a good reference, we sketch the easy proof.

**Lemma B.2.** *Let  $\mathcal{C}$  be a closed symmetric monoidal category. Then an object  $X$  of  $\mathcal{C}$  is dualizable if and only if the map*

$$\mu: X \otimes DX \longrightarrow F(X, X)$$

*adjoint to the composite*

$$X \otimes DX \otimes X \xrightarrow{1 \otimes \varepsilon} X \otimes I \xrightarrow{\cong} X$$

*is an isomorphism.*

*Proof.* If  $X$  is dualizable, the map  $\mu$  is an isomorphism as the special case  $X = I, Y = X, Z = X, W = I$  of Lemma B.1. Conversely, if  $\mu$  is an isomorphism, the composite

$$\eta: I \longrightarrow F(X, X) \xrightarrow{\mu^{-1}} X \otimes DX$$

(where the first arrow is the adjoint of the identity map of  $X$ ) pairs with the evaluation map

$$\varepsilon: DX \otimes X \longrightarrow I$$

to show that  $X$  and  $DX$  are duals in  $\mathcal{C}$ .  $\square$

Since equivalences in  $\text{Ho}(\mathcal{C}/B)$  for  $\mathcal{C}$  a presentable symmetric monoidal  $\infty$ -category are detected fibrewise, Lemma B.2 implies the following criterion for detecting dualizable objects in  $\text{Ho}(\mathcal{C}/B)$ .

**Corollary B.3.** *Let  $\mathcal{C}$  be a presentable symmetric monoidal  $\infty$ -category, and let  $B$  be a space. Then an object of  $\text{Ho}(\mathcal{C}/B)$  is dualizable if and only if all its fibres are.  $\square$*

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