Spectral Sequence Abstract/Supplement

In this talk I plan to introduce spectral sequences by explaining how they arise from a filtration of an abelian group and then performing a couple of instructive applications of the Leray-Serre spectral sequence. I will also try to briefly touch the topics of local coefficients and convergence. Conspicuously absent will be any discussion of algebra structures and convergence. This supplement contains more precise definitions and statements of the theorems including the relevant multiplicative information.

I plan to follow Serre by constructing a "1-dimensional" form of a spectral sequence from the filtration of an abelian group which when I add the structure of a grading will create the standard "2-dimensional" form of a spectral sequence. And then for applications I would like to compute the cohomology module for ΩS^n and construct the Gysin exact sequence.

Definitions

Definition 1. A differential bigraded module over a ring R is a collection of R-Modules $\{E^{p,q}\}$ with $p, q \in \mathbb{Z}$ together with an R-linear map $d : E^{*,*} \to E^{*,*}$ called the differential of bidegree $(-s, s-1)$ or $(s, 1-s)$.

Definition 2. A spectral sequence (SS) is a collection of differential bigraded R-modules $\{E_r^{*,*}, d_r\}$with $r = 0, 1, 2, ...$ where all of the d_r have bidegree $(-r, r-1)$ (for a spectral sequence of homological type) or they all have bidegree $(r, 1-r)$ (for a SS of cohomological type) such that $E_{r+1}^{p,q} = H^{p,q}(E_r^{*,*})$.

Definition 3. A bigraded algebra $E^{*,*}$ is a bigraded algebra with multiplication μ if

$$\mu : E^{*,*} \otimes E^{*,*} \to E^{*,*}$$

is a map of bigraded modules (where the grading of $E^{*,*} \otimes E^{*,*}$ is determined by $(E^{*,*} \otimes E^{*,*})^{p,q} = \bigoplus_{m+n=p,\, r+s=q} E^{m,r} \otimes E^{n,s}$) that is associative and unital.

Definition 4. A differential bigraded algebra is a bigraded algebra with total degree one mapping

$$d : \bigoplus_{p+q=n} E^{p,q} \to \bigoplus_{r+s=n+1} E^{r,s}$$

that satisfies the Leibniz rule $d(e \cdot e') = d(e) \cdot e' + (-1)^{p+q} e \cdot d(e')$ where $e \in E^{p,q}$ and $e' \in E^{r,s}$.

Definition 5. A SS of algebras is a SS $\{E_r^{*,*}, d_r\}$ with algebra multiplications given by $\mu_r : E_r \otimes E_r \to E_r$ such that μ_{r+1} is induced by taking the homology of E_r in the sense that μ_{r+1} can be written

$$\mu_{r+1} : E_{r+1} \otimes E_{r+1} \xrightarrow{\cong} H(E_r) \otimes H(E_r) \longrightarrow H(E_r \otimes E_r) \xrightarrow{H(\mu_{r+1})} H(E_r) \xrightarrow{\cong} E_{r+1}$$

where the middle map is given by $[u] \otimes [v] \mapsto [u \otimes v]$.

Definition 6. A SS collapses at the rth term if $E_r = E_{r+1} = ... = E_\infty$.

Definition 7. A SS converges at (p, q) if there exists a finite r such that $E_r^{p,q} = E_{r+1}^{p,q} = ... = E_\infty^{p,q}$ in which case we say it converges to $E_\infty^{p,q}$. A SS converges if it converges for all p, q.

Definition 8. A first quadrant SS is a SS such that $E_*^{p,q} = 0$ if $p < 0$ or $q < 0$.

Definition 9. A morphism $f : \{E_r^{*,*}, d_r\} \to \left\{\overline{E}_r^{*,*}, \overline{d}_r\right\}$ is a sequence of maps $f_r : E_r^{*,*} \to \overline{E}_r^{*,*}$ of bigraded modules such that $f_r d_r = \overline{d}_r f_r$ and $f_{r+1} = H(f_r)$ that is f_{r+1} is induced by taking the homology with respect to d_r in $E_r^{*,*}$.

Definition 10. A filtered differential module is an R-module A equipped with a filtration F with an R-linear map $d : M \to M$ such that $d^2 = 0$ and with $dF^pA \subset F^pA$.

Definition 11. A filtered differential graded module is a filtered differential R-module A equipped with a grading such that d is a map of degree ± 1.

Theorems

Theorem 12. *Suppose (A, d, F, μ) is a filtered differential graded algebra such that $\mu(F^pA \otimes F^qA) \subset F^{p+q}A$ then the SS associated to (A, d, F) is a SS of algebras. And if the filtration is locally finite (That is for every graded piece A^n there exists integers$M(n)$ and $m(n)$ such that $A = F^{M(n)}A^n \supset ... \supset F^{m(n)}A^n = 0$) then the SS converges as an algebra.*

Theorem 13. *(Leray-Serre Spectral Sequence) Suppose $F \to E \to B$ is a fibration with F connected and B path-connected. Then there is a first quadrant SS $\{E_r^{*,*}, d_r\}$ converging to $H^*(E, R)$ as an algebra, with*

$$E_2^{p,q} \cong H^p(B, H^q(F, R))$$

where $H^q(F)$ is a system of local coefficients in the cohomology of the fiber F. This spectral sequence is natural with respect to fiber-preserving maps of fibrations and moreover we have for $u \in E_2^{p,q}$ and $v \in E_2^{p',q'}$ $u \cdot_2 v = (-1)^{p'q} u \cup v$. (There is an analogous result for homology)

Theorem 14. *(Gysin Sequence) Suppose $F \to E \to B$ is a fibration with B path connected and the system of local coefficients on Binduced by the fiber is simple. Suppose further that F is a homology n-sphere for $n \geq 1$. Then there is an exact sequence*

$$\longrightarrow H^k(B; R) \xrightarrow{\gamma} H^{n+k+1}(B; R) \xrightarrow{\pi^*} H^{n+k}(E; R) \xrightarrow{Q} H^{k+1}(B; R) \longrightarrow$$

where $\gamma(u) = z \cup u$ for some $z \in H^{n+1}(B; R)$ and, if n is even and $2 \neq 0$ in R then $2z = 0$.