FACULTY OF SCIENCE
UNIVERSITY OF COPENHAGEN

Bachelor Thesis in Mathematics

Casper Guldberg

Steenrod operations

- construction and application

Advisor: Jesper Grodal

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0.1 Abstract

English

This project is mainly concerned with the construction of the classical Steenrod operations, i.e. stable cohomology operations for $\mathbb{F}_p$ cohomology of topological spaces. First we shall define cohomology operations in general, and then we state the properties of the Steenrod operations, so that we know what we are aiming at. Then the construction itself will follow. This will be done at space level, and not on cochain level as some references prefer. It is then verified that the construction we make actually gives rise to the Steenrod operations, with the exception of some properties in the case p odd - there will not be space for this in the project. A few immediate applications are given, before we turn to a short discussion of the construction we have made, and propose some further development from here. Finally there are some technical details, which are not included in the main text in order to keep the exposition as smooth as possible, but nonetheless are needed for our construction to work.

Dansk

Dette projekt omhandler hovedsageligt konstruktionen af de klassiske Steenrod operationer, i.e. stabile kohomologioperationer for $\mathbb{F}_p$ kohomologi af topologiske rum. Først definerer vi kohomologioperationer generelt, og efterfølgende angiver vi egenskaberne for Steenrod operationerne, så vi ved hvad vi sigter efter. Derefter følger selve konstruktionen. Den bliver foretaget på rumniveau, og ikke på kokædeniveau som nogle referencer ellers foretrækker. Vi verificerer så at konstruktionen vi har lavet faktisk giver anledning til Steenrod operationerne, med undtagelse af nogle af egenskaberne i tilfældet p ulige - der vil ikke være plads til dette i projektet. Et par umiddelbare anvendelser bliver givet før vi fortsætter med en kort diskussion af den konstruktion vi har lavet, og foreslår nogle videreudviklinger herfra. Endeligt er der medtaget nogle tekniske detaljer som er undladt i hovedteksten for at holde fremstillingen så glidende som mulig, men som ikke desto mindre er nødvendige for at vores konstruktion fungerer.

Contents

0.2 Introduction

As a general motivation algebraic topology seeks to provide algebraic tools to distinguish topological spaces. The cohomology ring $H^*(X;R)$ for a space X being one example of this, turns out to be particularly interesting. Here we get the additional ring structure, not immediately provided by homology groups, and we can compute these ring with greater ease than the homotopy groups, which otherwise holds lots of information of the underlying space. Moreover, it turns out that we get even more structure, if we choose the coefficient ring R to be one of the finite fields of the form $\mathbb{F}_p$, for a prime p. In this case N. E. Steenrod introduced the stable cohomology operations[1], i.e. natural transformations $\theta : H^n(-,\mathbb{F}_p) \Rightarrow H^m(-,\mathbb{F}_p)$ with certain properties, which turn out to interact such that they generate an algebra.

The Steenrod algebra, as it is named, turns out to be interesting in its own right and may be studied without ever considering topological spaces, but the fact that cohomology operations with these properties can be explicitly constructed for any topological space, then impose further constraints on the existence and behavior of such spaces, and has given lots of previously inaccessible information about these.

I have chosen to follow the approach of [Hat02], to the construction of the operations, where most notation is also adopted from. This is because the construction works equally well for $p = 2$ and p odd, and can be done with fairly basic tools.

The reader is assumed to be somewhat familiar with the material covered in [Hat02], including some of the "optional" sections, and to have a basic understanding of functors. The most advanced machinery in play will be the natural bijection $\langle X, K(G,n)\rangle \simeq H^n(X;G)$, relating homotopy classes of maps into Eilenberg-MacLane spaces and cohomology, and the map $H^*(X;F) \otimes H^*(X;F) \to H^*(X \times X;F)$ given by the cohomological cross product, which is an isomorphism when certain conditions are meet. Also for a single argument we shall need Serre's spectral sequence for homology,[2] but this is postponed to the appendix.

[1] The classical reference is [Ste62], which I include for good measure

[2] Introduced in [Hat].

Chapter 1

The Steenrod operations

1.1 Properties of the Steenrod operations

Here we shall briefly introduce the operations, and list the defining properties of them.

Definition 1.1.1. *We define cohomology operations as natural transformations of functors* $\Theta_{m,n,R,S} : H^m(-;R) \Rightarrow H^n(-;S)$*, where the cohomology functors are considered as functors from the category of topological spaces into the category of sets. I.e. we have a commutative diagram*

$$\begin{array}{ccc} H^m(Y,R) & \xrightarrow{\Theta_{m,n,R,S}} & H^n(Y,S) \\ \downarrow{f^*} & & \downarrow{f^*} \\ H^m(X,R) & \xrightarrow{\Theta_{m,n,R,S}} & H^n(X,S) \end{array}$$

for all spaces X, Y*, and maps* $f : X \to Y$.

Prior to introducing the Steenrod operations, we know the Bockstein homomorphisms as examples of cohomology operations. We recall that these are defined as connecting homomorphisms for the long exact sequence in cohomology associated to a short exact sequence of coefficient groups.[1] In particular from the coefficient sequence

$$0 \longrightarrow \mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_{p^2} \longrightarrow \mathbb{Z}_p \longrightarrow 0$$

we obtain $\beta_p : H^n(X,\mathbb{Z}_p) \to H^{n+1}(X,\mathbb{Z}_p)$ for any space X. Naturality follows form the fact that this is a connecting homomorphism in a long exact sequence for cohomology.

[1] [Hat02] p. 303.

Definition 1.1.2. *The Steenrod operations are cohomology operations*

$$Sq^i : H^n(-,\mathbb{F}_p) \Rightarrow H^{n+i}(-,\mathbb{F}_p), \quad \textit{for all } n \in \mathbb{N}$$

for $p = 2$*, and*

$$\begin{matrix} P^i : H^n(-,\mathbb{F}_p) \Rightarrow H^{n+2(p-1)i}(-,\mathbb{F}_p) \\ \beta_p : H^n(-,\mathbb{F}_p) \Rightarrow H^{n+1}(-,\mathbb{F}_p) \end{matrix} \quad \textit{both for all } n \in \mathbb{N}$$

for p *odd. For a fixed space* X*, we get components of the transformation, i.e. maps (also denoted)*[2] $Sq^i : H^n(X,\mathbb{F}_p) \to H^{n+i}(X,\mathbb{F}_p)$ *which are subject to the following:*

$$Sq^0(x) = x \tag{1.1}$$
$$Sq^i(x) = 0, \quad i < 0 \tag{1.2}$$
$$Sq^i(x+y) = Sq^i(x) + Sq^i(y) \tag{1.3}$$
$$Sq^i(xy) = \sum_j Sq^j(x)Sq^{i-j}(y) \tag{1.4}$$
$$Sq^i(\sigma(x)) = \sigma(Sq^i(x)) \tag{1.5}$$
$$Sq^i(x) = 0, \quad i > n \tag{1.6}$$
$$Sq^n(x) = x^2 \tag{1.7}$$
$$Sq^iSq^j(x) = \sum_k \binom{j-k-1}{i-2k} Sq^{i+j-k}Sq^k(x), \quad \textit{for } i < 2j \tag{1.8}$$
$$Sq^1 = \beta_2 \tag{1.9}$$

where σ *is the suspension map. The* P^i*'s are subject to the same conditions* (1.1) - (1.5)*, and the somewhat similar*

$$P^i(x) = 0, \quad 2i > n \tag{1.10}$$
$$P^i(x) = x^p, \quad 2i = n \tag{1.11}$$
$$P^iP^j(x) = \sum_k (-1)^{i+k} \binom{(p-1)(j-k)-1}{i-pk} P^{i+j-k}P^k(x), \quad i < pj \tag{1.12}$$
$$P^i\beta_pP^j(x) = \sum_k (-1)^{i+k} \binom{(p-1)(j-k)}{i-pk} \beta_pP^{i+j-k}P^k(x) - \binom{(p-1)(j-k)-1}{i-pk-1} P^{i+j-k}\beta_pP^k(x), \quad i \le pj \tag{1.13}$$

We shall now construct cohomology operations that satisfy the properties above.

[2]The underlying space X will be implicit.

1.2 Construction of Steenrod operations

The goal of this chapter is to construct cohomology operations, by building the components at X, for any space X. To do this, we will consider a class in $H^n(K(\mathbb{Z}_p, n), \mathbb{Z}_p)$ as a map $K(\mathbb{Z}_p, n) \to K(\mathbb{Z}_p, n)$. From this map, we construct a new one with certain properties, which by some further work will translate back to an appropriate class in cohomology. Then we extend this construction by naturality to all cohomology classes, and finally we may define operations from whatever emerges on the other side.

1.2.1 Preliminary functors

We begin the actual construction here by first defining some functors. We shall only define them on objects here, in order to get to the point of the construction slightly faster, and refer to the appendix A.1 for more details.

Let (X, x_0) and (Y, y_0) be pointed spaces. Recall that the smash of two pointed spaces is given by $X \wedge Y := (X \times Y)/(X \vee Y)$, a quotient of the product space. Now

$$(X, x_0) \mapsto (X^{\wedge n}, [\underbrace{(x_0, \ldots, x_0)}_{n}])$$

defines a functor F_n on the category of pointed spaces.

Example 1.2.1 The functor F_n is illustrated below for $n = 3$.

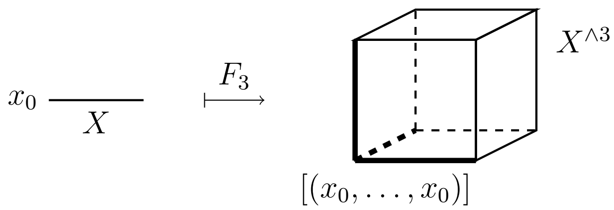


Figure 1.1: The fat lines indicate the points being identified to the new basepoint $[(x_0, \ldots, x_0)]$.

Now let G be a finite (discrete) group, acting freely on Y, and acting on X fixing x_0. Define a functor $Y \times_G -$, by mapping X to $Y \times_G X := (Y \times X)/G$, the orbit space under the diagonal action of G. We will eventually specialize this to the Borel construction,[3] but for now consider the following example.

[3][Bau06] chapter 7.

Example 1.2.2 For any space X, there is a $\mathbb{Z}_n$ action on F_nX given by permutation of the n identical factors. A map $f : X \to X'$ will then give us a $\mathbb{Z}_n$-equivariant map $F_nf : F_nX \to F_nX'$. We also have a free $\mathbb{Z}_n$ action on S^1 given by rotating S^1 by $2\pi/n$. So the composition of functors $S^1 \times_{\mathbb{Z}_n} F_nX$ makes sense, and for $n = 2$ it can be illustrated as follows.

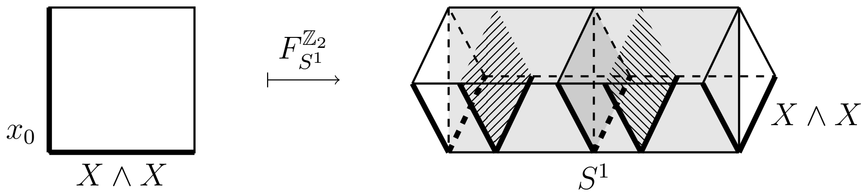


Figure 1.2: The fat lines still indicate identifications to the new basepoint now denoted by x_0. Light gray areas on left and right half of the space indicate identifications such that each slice is identified with the flipped slice translated by π along S^1. The darker gray is just a graphical indication of the overlap when seen from this point.

This example illustrates an important point in the construction which we will see later. One of the key properties that we want (1.7), is that the squaring operation in appropriate degree, should coincide with the cup product square of an element. This connects to the smash product, which can be obtained as fibre for the projection onto S^1. Among other things, this motivates that we consider the product fibre bundle $Y \times X \xrightarrow{\pi} Y$ given by the projection onto first factor. For this we have

Lemma 1.2.3. *Let X, Y be a pair of pointed G-spaces as above, and further let Y be Hausdorff. The product fibre bundle $Y \times X \xrightarrow{\pi} Y$ induces a fibre bundle $Y \times_G X \xrightarrow{\overline{\pi}} Y/G$, with fibre X.*

The proof of this lemma is rather tedious, and the content is not interesting related to this construction. Hence it can be found in the appendix A.1 in order to move things along here.

The action on X fixes the basepoint x_0, so the inclusion $Y \times \{x_0\} \hookrightarrow Y \times X$ induces an inclusion $Y/G \hookrightarrow Y \times_G X$, such that the composition $Y/G \hookrightarrow Y \times_G X \to Y/G$ is the identity. We thus have a section of the fibre bundle, and may consider the quotient $Y \times_G X/(Y/G)$ where this particular section is collapsed. The fibres X are still embedded in this quotient, as each fibre only has a single point collapsed, when forming the quotient by the section. We may thus define a functor $\Lambda_{Y,G,n}$ that sends a space X to $(Y \times_G X^{\wedge n})/(Y/G)$.

Further, the fibres $X^{\wedge n}$ from the induced fibre bundle in Lemma 1.2.3 are embedded in $\Lambda_{Y,G,n}X$. In particular p a fixed prime, we may consider

$$\boxed{\Gamma_Y := Y \times_{\mathbb{Z}_p} (-)^{\wedge p} \quad \text{and} \quad \Lambda_Y := \Lambda_{Y,\mathbb{Z}_p,p}}$$

Here the $\mathbb{Z}_p$ action on $X^{\wedge p}$ is generated by the map $T : X^{\wedge p} \to X^{\wedge p}$ permuting the factors once cyclically, and hence the action fixes the basepoint. Continuing on example 1.2.2 from before, Λ_{S^1} may be illustrated as follows for $p = 2$.

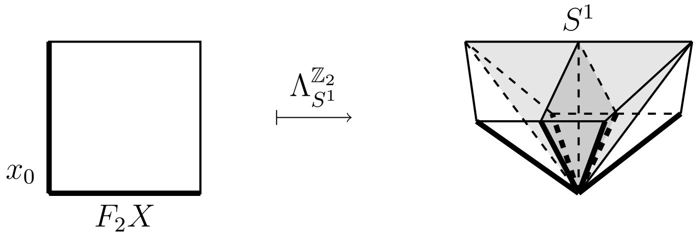


Figure 1.3: Legend as for figure 1.2, but the bottom S^1 is now collapsed.

1.2.2 A map with a certain property

Having defined the functors Γ_Y and Λ_Y, we will now continue by using the natural bijection $\mu : \langle X, K(G,n)\rangle \xrightarrow{\sim} H^n(X;G)$, where $\langle -,-\rangle$ denotes homotopy classes of pointed maps. This bijection only holds when X is a CW-complex,[4] but as we show in A.2, this is an assumption we can make on all the spaces involved. I will mostly suppress μ in the notation, and whenever we apply μ to a map, it is of course meant to be the homotopy class of the map.

Adopting the notation from [Hat02], we define for $n > 0$, K_n to be a CW-complex, which is an Eilenberg-MacLane space $K(\mathbb{Z}_p, n)$ with $(n-1)$-skeleton a single point, and n-skeleton S^n. Also in most of this section we will suppress the coefficient ring in the notation for cohomology, but it is assumed to be $\mathbb{F}_p$, for the appropriate p.

The goal of this section is then to construct a map $\Lambda_{S^\infty}K_n \to K_{np}$, by finding a homotopy on $K_n^{\wedge p}$, which defines a map on Γ_{S^1}. Then composing this with the quotient map to Λ_{S^1} and extending to all of Λ_{S^∞}. This map

[4] as seen in [Hat02] p. 393

will play a central role in the actual definition of the Steenrod operation in the next section.

Let $\iota_n \in H^n(K_n)$ be the fundamental class given by $\mu(id_{K_n})$. There is a reduced cross product $\tilde{H}^*(K_n) \otimes \tilde{H}^*(K_n) \xrightarrow{\times} \tilde{H}^*(K_n \wedge K_n)$, by which in our case is an isomorphism since we are using field coefficients and $H^k(K_n)$ is finitely generated for all k.[5] This is referred to as either the cross product isomorphism, or the Künneth formula. We shall only deal with cohomology in positive degree, so it will not always be specified whether we are using reduced or unreduced cohomology.

Due to the above isomorphism we identify $\iota_n^{\otimes p}$ with $\iota_n^{\times p} \in \tilde{H}^{np}(K_n^{\wedge p})$, and by μ above, we can consider $\iota_n^{\otimes p}$ and $\iota_n^{\otimes p}T$ as representatives for classes of pointed maps $K_n^{\wedge p} \to K_{np}$. With this approach we have

Proposition 1.2.4. *The two maps $\iota_n^{\otimes p}$ and $\iota_n^{\otimes p}T : K_n^{\wedge p} \to K_{np}$ are homotopic relative to the basepoint.*

Proof. Recall that the map $T : K_n^{\wedge p} \to K_n^{\wedge p}$ permutes the p factors cyclically. This is also the case when just considering the np-skeleton of $K_n^{\wedge p}$, which by definition is $(S^n)^{\wedge p} \simeq S^{np}$. Note that T is cellular, so restricting it to the np-skeleton, we get a map T' between np-spheres. But we have that $T'^p = \mathrm{id}_{S^{np}}$, so $\deg T'^p = (\deg T')^p = 1$. Thus for p odd, T' has degree 1, and is homotopic to the identity.

For $p = 2$ both $\iota_n^{\otimes p}$ and $\iota_n^{\otimes p}T$ restrict to maps $S^{2n} \to K_{2n}$. Since by definition $\pi_{2n}(K(\mathbb{Z}_2, 2n)) = \mathbb{Z}_2$, either they are null-homotopic, or not. T' having degree ± 1 is not null-homotopic, so neither is T. Thus if $\iota_n^{\otimes p}$ is not null-homotopic then neither is $\iota_n^{\otimes p}T$, and of course if $\iota_n^{\otimes p}$ happens to be null-homotopic then so is $\iota_n^{\otimes p}T$. With the homotopy defined on the np-skeleton we may simply extend this to all of $I \times K_n^{\wedge p}$, by the extension lemma [Hat02] p. 348.

The homotopies may indeed be taken to be basepoint preserving in either case, as S^{pn} is simply connected as $n \geq 1$ and $p \geq 2$. □

This proposition can be said to carry the entire construction, as we will return to in section 3.1.

Now letting H denote the homotopy produced by the proposition with $\iota_n^{\otimes p}(x) = H(1, x)$, we shall see that this gives rise to a map on $\Gamma_{S^1} K_n$, where the $\mathbb{Z}_p$-action is given by rotating S^1 by $2\pi/p$.

First we note that since $H(0, x) = \iota_n^{\otimes p}T(x) = H(1, T(x))$, we get a map on the quotient $I \times K_n^{\wedge p}/(0, x) \sim (1, T(x))$. From the definition of $\Gamma_{S^1} K_n$, we

[5] The reduced cross product is described in [Hat02] p. 218-219.

see that it can be written as $I \times K_n^{\wedge p}/\!\sim$, where we make the identifications $(\frac{j}{p}, T^j(x))$ for $j = 0, 1, \ldots, p$. Here we have encoded an excess of relations, and actually have just $[0, \frac{1}{p}] \times K_n^{\wedge p}/(0, x) \sim'' (\frac{1}{p}, T(x))$. This we recognize as $\Gamma_{S^1} K_n$, by stretching $[0, \frac{1}{p}]$ to I.

Since the basepoint is preserved by H, we can now collapse this segment of $\Gamma_{S^1} K_n$ corresponding exactly to the section $S^1 \times \{x_0\}$ as described on page 4, and now get a map $\tilde{H}$ on $\Lambda_{S^1} K_n$. This is a subcomplex of $\Lambda_{S^\infty} K_n$, with the $\mathbb{Z}_p$ action on S^∞ being rotation by $2\pi/p$ in every coordinate. Note that S^1 is a subcomplex of S^∞ with coherent action. Now we wish to extend $\tilde{H}$ to a map λ defined on the whole complex.

We shall use Λ_{S^∞} a lot, and therefore abbreviate it to just Λ from here.

Proposition 1.2.5. *The map $\tilde{H}$ extends to a map $\lambda : \Lambda K_n \to K_{np}$, with the property that the restriction to the each fibre $K_n^{\wedge p}$ (cf. Lemma 1.2.3) is homotopic to $\iota_n^{\otimes p}$.*

Proof. $\Lambda K_n - \Lambda_{S^1} K_n$ has only cells of dimension greater than $np + 1$, as seen in section A.2. By the extension lemma[6], $\tilde{H}$ extends to a map λ since $\pi_i(K_{np}) = 0$ for $i > np$. The restriction of $\tilde{H}$ to the fibres $K_n^{\wedge p}$ is given by $\tilde{H}(t_0, x)$ for some fixed $t_0 \in I$. Thus such restrictions has the desired property, by definition of $\tilde{H}$.

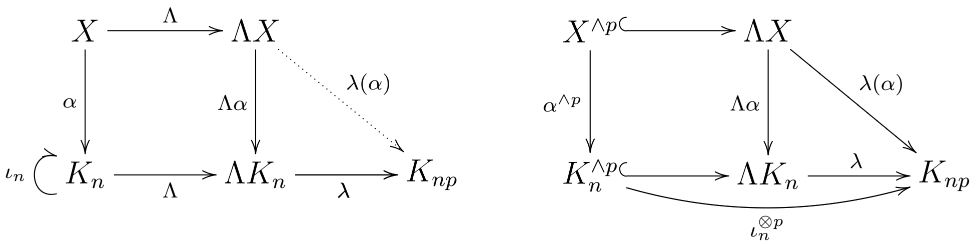
Now when extending $\tilde{H}$ to λ, this property also holds for the fibres not over $S^1/\mathbb{Z}_p$ as the base space L^∞ is connected, and the fibres therefore homeomorphic. We can simply choose a homeomorphism to a fibre over the $S^1/\mathbb{Z}_p$ where $\tilde{H}$ is defined, and the composition will then be homotopic to $\iota_n^{\otimes p}$. The whole thing is summed up in the following commutative diagram

$$\begin{array}{ccccccc}
K_n^{\wedge p} & \longrightarrow & \Gamma_{S^\infty} K_n & \longrightarrow & \Lambda K_n & & \\
\simeq \big\downarrow & \nearrow & \big\uparrow & & \big\uparrow & \searrow^{\lambda} & \\
K_n^{\wedge p} & \longrightarrow & \Gamma_{S^1} K_n & \longrightarrow & \Lambda_{S^1} K_n & \xrightarrow[\tilde{H}]{} & K_{np}
\end{array}$$

where the horizontal maps are, first inclusions of fibres, and then quotient maps to get Λ from Γ. The compositions across respectively top and bottom thus become the restrictions to the fibre. ☐

Now we extend this by the naturality coming from the functor Λ. For an element $\alpha \in H^n(X)$ we define $\lambda(\alpha)$ to be the map making the triangle in the first of the following diagrams commute.

[6] [Hat02] p. 348



I.e. $\lambda(\alpha) = (\Lambda\alpha)^*\lambda$. We note that $\lambda(\iota_n) = (\Lambda id_{K_n})^*\lambda = \lambda$, so this generalizes our construction λ, and also that by our bijection μ, we have $\lambda = \lambda^*(\iota_{np})$.

The second diagram shows that the restriction of $\lambda(\alpha)$ to the fibre $X^{\wedge p}$ is homotopic to $\alpha^{\times p}$. By the naturality of the cross product[7], we have that $\alpha^{\wedge p *}(\iota_n^{\times p}) = \alpha^*(\iota_n)^{\times p} = \alpha^{\times p}$

The following lemma suggest that our knowledge of $\lambda(\alpha)$ on the fibre, will be helpful later.

Proposition 1.2.6. *Let X be a CW-complex, such that the np-skeleton of ΛX is contained in $X^{\wedge p}$. A map $f : \Lambda X \to K_{np}$ is uniquely determined up to homotopy, by the restriction to the fibres $X^{\wedge p}$.*

Proof. f defines an element of $H^{np}(\Lambda X)$. Restricting f to $X^{\wedge p}$, we then define an element of $H^{np}(X^{\wedge p})$, and so restriction defines a map $H^{np}(\Lambda X) \to H^{np}(X^{\wedge p})$. By cellular cohomology we see that this map must be injective since the np-skeleton of ΛX is contained in $X^{\wedge p}$. Two cocycles assigning the same values to np-cells in $X^{\wedge p}$ will then also do this in ΛX, and are thus equal in $H^{np}(\Lambda X)$. □

1.2.3 Defining operations

All the preliminary work done, we now define some cohomology operations.

The map T from above is just the identity on the diagonal of $X^{\wedge p}$, so the embedding $S^\infty \times X \hookrightarrow S^\infty \times X^{\wedge p}$ with X sent to the diagonal, induce an inclusion $L^\infty \times X \hookrightarrow \Gamma_{S^\infty} X$.

Now composing with the quotient map $\Gamma_{S^\infty} X \to \Lambda X$, we get a map $\nabla : L^\infty \times X \to \Lambda X$ inducing $\nabla^* : H^*(\Lambda X) \to H^*(L^\infty \times X)$. Since $H^*(L^\infty)$ is finitely generated in each degree, and we are still working with $\mathbb{F}_p$ coefficients, the unreduced cross product gives an isomorphism like the reduced above did, and we get $H^*(L^\infty \times X) \simeq H^*(L^\infty) \otimes H^*(X)$.

[7][Hat02] p. 275, the topological Künneth formula gives a natural short exact sequence, where the first map is the cross product.

So for an element $\alpha \in H^n(X)$, we first get an element $\lambda(\alpha) \in H^{np}(\Lambda X)$, and then $\nabla^*(\lambda(\alpha)) \in H^{np}(L^\infty \times X)$. Due to the cross product isomorphism, this can then be written as

$$\nabla^*(\lambda(\alpha)) = \sum_i \omega_{(p-1)n-i} \otimes \theta_i(\alpha)$$

with ω_j a generator of $H^j(L^\infty)$ and $\theta_i(\alpha) \in H^{n+i}(X)$.

In order to get a unique element $\theta_i(\alpha)$ for any α, we have to choose the ω_j's in a consistent way. For $p = 2$ there is only one generator in each degree, and for odd p we choose ω_1 to be the class dual to the 1-cell in L^∞ with the normal cell structure, i.e. one cell in each dimension. We choose ω_2 to be $\beta\omega_1$, and from these two we define the rest to be ω_2^j in even degree $(2j)$, and $\omega_1\omega_2^j$ in odd degree $(2j+1)$.

Theorem 1.2.7. *The θ_i's define cohomology operations:*

$$\theta_i : H^n(-, \mathbb{F}_p) \Rightarrow H^{n+i}(-, \mathbb{F}_p)$$

Proof. For any space X, each θ_i defines a map $H^n(X, \mathbb{F}_p) \to H^{n+i}(X, \mathbb{F}_p)$. We now check naturality, and first note that for any map $f : X \to Y$, the following diagram commutes

$$\begin{array}{ccc} L^\infty \times X & \xrightarrow{id \times f} & L^\infty \times Y \\ \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle \nabla} \\ \Lambda X & \xrightarrow[\Lambda f]{} & \Lambda Y \end{array}$$

Since ∇ is a diagonal embedding, followed by a quotient map, and Λf is defined diagonally. That gives us commutativity of

$$\begin{array}{ccc} H^{np}(L^\infty \times X) & \xleftarrow{(id \times f)^*} & H^{np}(L^\infty \times Y) \\ \uparrow{\scriptstyle \nabla^*} & & \uparrow{\scriptstyle \nabla^*} \\ H^{np}(\Lambda X) & \xleftarrow[(\Lambda f)^*]{} & H^{np}(\Lambda Y) \end{array}$$

and so for $\alpha \in H^n(X)$ we have

$$\begin{aligned} \sum_i \omega_{n(p-1)-i} \otimes \theta_i(\alpha) &= \nabla^*(\lambda(\alpha)) \\ &= \nabla^*((\Lambda\alpha)^*\lambda(\iota_n)) \\ &= (id \times \alpha)^*(\nabla^*(\lambda(\iota_n))) \end{aligned}$$

Expanding this last expression through the cross product isomorphism finally yields $\sum_i \omega_{n(p-1)-i} \otimes \alpha^*\theta_i(\iota_n)$, and thus $\theta_i(\alpha^*(\iota_n)) = \alpha^*(\theta_i(\iota_n))$.

So for a fixed space X, the θ_i gives us the component at X, of a natural transformation, that we also denote θ_i. □

We can now verify that these cohomology operations are in fact the Steenrod operations.

Theorem 1.2.8. *In the case $p = 2$ we have $\theta_i = Sq^i$, the Steenrod squares. For p odd, we have $c \cdot \theta_{2i(p-1)} = P^i$ and $c \cdot \theta_{2i(p-1)+1} = \beta P^i$, the Steenrod powers, where c is a normalization constant.*

Proof. The first claim will be verified in the following section. The second will remain a claim in this thesis. □

1.3 Verification of properties

Although the ideas for verifying the properties listed in section 1.1, are basically the same in the case $p = 2$ and p odd, the odd case takes significantly more work than the even, and introduces new tools on the way. It is not "just similar". I have therefore chosen to do the even case by means that work in general, and branch of to this case as late as possible. This is done not to use up too much space, but at the same time leaves us with a solid foundation to take up the task of verifying the properties for the Steenrod powers at another time.

1.3.1 Basic properties

We shall first verify the properties (1.1)-(1.7), which by lack of better name, we shall call the basic properties.

For property (1.2) we need the following lemma, which describes cohomology operations in general.

Lemma 1.3.1. *Let $\iota \in H^m(K(G,m);G)$ be a fundamental class, and*

$$\Theta : H^m(-;G) \Rightarrow H^n(-;H)$$

a cohomology operation. The map φ given by mapping $\Theta \mapsto \Theta_{K(G,m)}(\iota)$, defines a bijection between the set of all such cohomology operations, and $H^n(K(G,m);H)$.

Proof. As noted in appendix A.2 we may assume that we work with CW-complexes for cohomology purposes, so we may use our natural bijection μ. By naturality of Θ we then have that $\Theta(\alpha) = \Theta(\alpha^*(\iota)) = \alpha^*(\Theta(\iota))$ for any CW-complex X and class $\alpha \in H^n(X; G)$. Thus φ is injective, as Θ is uniquely determined by $\Theta(\iota)$.

For surjectivity, let $\alpha \in H^n(K(G,m); H)$ be given. By μ we have $\alpha : K(G,m) \to K(H,n)$, and composition with α defines a natural transformation $\langle -, K(G,m)\rangle \Rightarrow \langle -, K(H,n)\rangle$, which by μ is also a transformation $\Theta : H^m(-; G) \Rightarrow H^n(-; H)$ with $\Theta(\iota) = \alpha$. To check naturality, we let $f : X \to Y$ be a map between CW-complexes, and note that the following diagram commutes

$$\begin{array}{ccc} \langle Y, K(G,m)\rangle & \xrightarrow{\quad\Theta\quad} & \langle Y, K(H,n)\rangle \\ \Big\downarrow {\scriptstyle f^*} & \begin{array}{ccc}\beta & \longmapsto & \alpha\circ\beta \\ \downarrow & & \downarrow \\ \beta\circ f & \longmapsto & \alpha\circ\beta\circ f\end{array} & \Big\downarrow {\scriptstyle f^*} \\ \langle X, K(G,m)\rangle & \xrightarrow[\quad\Theta\quad]{} & \langle X, K(H,n)\rangle \end{array}$$

simply by associativity of composition of maps. □

Now it is seen that property (1.2) holds since $K(G,m)$ is $(m-1)$-connected, and hence by the Hurewicz theorem, only the trivial cohomology operations can decrease dimension.

To show property (1.3) we will need some more work. First we consider the following diagram.

$$\begin{array}{ccccccc} \Lambda X & \xrightarrow{\Lambda(\Delta)} & \Lambda(X\times X) & \xrightarrow{\Lambda(\alpha\times\beta)} & \Lambda(K_n\times K_n) & \xrightarrow{\Lambda(\iota_n\otimes 1+1\otimes\iota_n)} & \Lambda K_n \\ & {\scriptstyle\Delta}\searrow & \Big\downarrow{\scriptstyle t} & & \Big\downarrow{\scriptstyle t} & \searrow{\scriptstyle \lambda(\iota_n\otimes 1+1\otimes\iota_n)} & \Big\downarrow{\scriptstyle\lambda} \\ & & \Lambda X\times\Lambda X & \xrightarrow[\Lambda\alpha\times\Lambda\beta]{} & \Lambda K_n\times\Lambda K_n & \xrightarrow[\lambda\otimes 1+1\otimes\lambda]{} & K_{np} \end{array}$$

In the right part of the diagram the unreduced cross product isomorphism and the bijection μ, is implicit in the naming of the maps, and the map $t : \Lambda(X\times X) \to \Lambda X \times \Lambda X$, is induced by

$$(s, x_1, y_1, \ldots, x_p, y_p) \mapsto (s, x_1, \ldots, x_p, s, y_1, \ldots, y_p)$$

where $s \in S^\infty$ and $x = (x_1, \ldots, x_p)$ lies in the first copy of X and $y = (y_1, \ldots, y_p)$ in the second. Δ denotes the diagonal map, and α and β are both classes in $H^n(X)$, and the maps in the diagram are those obtained through μ.

The left triangle and the center commutes by choice of t, and the upper right triangle commutes by the definition of λ. To see that the bottom right triangle commutes up to homotopy, we first note that the np-skeleton of $\Lambda(K_n \times K_n)$ is contained in $(K_n \times K_n)^{\wedge p}$, and apply proposition 1.2.6. By this, we only need to check commutativity when restricting to the fibre, but here we know that λ is homotopic to $\iota_n^{\otimes p}$. So we have

$$\begin{array}{ccc} (K_n \times K_n)^{\wedge p} & \xrightarrow{(\iota_n \otimes 1 + 1 \otimes \iota_n)^{\otimes p}} & K_{np} \\ \Big\downarrow t|_{(K_n \times K_n)^{\wedge p}} & \nearrow_{\iota_n^{\otimes p} \otimes 1 + 1 \otimes \iota_n^{\otimes p}} & \\ K_n^{\wedge p} \times K_n^{\wedge p} & & \end{array}$$

By the Künneth formula, the restriction of t induce the isomorphism

$$t|_{(K_n \times K_n)}{}^* : H^*(K_n)^{\otimes p} \otimes H^*(K_n)^{\otimes p} \to (H^*(K_n) \otimes H^*(K_n))^{\otimes p}$$

on cohomology, so the composition across the bottom of the diagram is just $(\iota_n \otimes 1)^{\otimes p} + (1 \otimes \iota_n)^{\otimes p}$. By definition the cup product of two classes $\alpha, \beta \in H^*(X)$, is $\alpha \smile \beta = \Delta^*(\alpha \times \beta)$, so pulling back by Δ an appropriate number of times we get

$$\begin{aligned} \Delta^*((\iota_n \otimes 1)^{\otimes p} + (1 \otimes \iota_n)^{\otimes p}) &= (\iota_n \smile 1)^p + (1 \smile \iota_n)^p \\ &= (\iota_n \smile 1 + 1 \smile \iota_n)^p \\ &= \Delta^*((\iota_n \otimes 1 + 1 \otimes \iota_n)^{\otimes p}) \end{aligned}$$

Where second equality is just the freshman's dream in the field $\mathbb{F}_p$. The homomorphism Δ^* is injective, so we conclude that the triangle commutes up to homotopy.

The map $\Lambda X \to K_{np}$ across the top of the previous diagram gives us the class $\lambda(\alpha + \beta)$, and across the bottom it gives $\lambda(\alpha) + \lambda(\beta)$. Since the two routes are homotopic these two classes are equal, and in conclusion each θ_i is additive. This gives us property (1.3).

Property (1.4) is also known as the Cartan formula, and we shall only consider the case $p = 2$ for this property. Much like above we will show this holds by considering the diagram

$$\begin{array}{ccccccc}
\Lambda X & \xrightarrow{\Lambda(\Delta)} & \Lambda(X \wedge X) & \xrightarrow{\Lambda(\alpha\wedge\beta)} & \Lambda(K_m \wedge K_n) & \xrightarrow{\Lambda(\iota_m \otimes \iota_n)} & \Lambda K_{m+n} \\
& \searrow^{\Delta} & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & \searrow^{\lambda(\iota_m \otimes \iota_n)} & \downarrow{\scriptstyle \lambda} \\
& & \Lambda X \wedge \Lambda X & \xrightarrow[\Lambda\alpha\wedge\Lambda\beta]{} & \Lambda K_m \wedge \Lambda K_n & \xrightarrow[\lambda(\iota_m)\otimes\lambda(\iota_n)]{} & K_{mp+np}
\end{array}$$

The only change from before, is that we now use the reduced cross product isomorphism, and that α lies in $H^m(X)$. All but the lower right triangle commutes by same arguments as above, and to check the last triangle, it will again suffice to look at the fibre. Here the two maps are homotopic to respectively $(\iota_m \otimes \iota_n)^{\otimes p}$ and $\iota_m^{\otimes p} \otimes \iota_n^{\otimes p}$. Applying Δ^* gives us cup products of the ι's, but in the case $p = 2$ this is commutative, so they are equal. Again by injectivity of Δ^*, the triangle therefore commutes up to homotopy. We identify the map $\Lambda X \to K_{mp+np}$ across the top of the diagram to represent $\lambda(\alpha \smile \beta)$, and across the bottom $\lambda(\alpha) \smile \lambda(\beta)$. From this we finally get

$$\begin{aligned}
\sum_i Sq^i(\alpha \smile \beta) \otimes \omega_{m+n-i} &= \nabla^*(\lambda(\alpha \smile \beta)) \\
&= \nabla^*(\lambda(\alpha)) \smile \nabla^*(\lambda(\beta)) \\
&= \sum_j Sq^j(\alpha) \otimes \omega_{m-j} \smile \sum_k Sq^k(\beta) \otimes \omega_{n-k} \\
&= \sum_i \left(\sum_{j+k=i} Sq^j(\alpha) \smile Sq^k(\beta) \right) \otimes \omega_{m+n-i}
\end{aligned}$$

Since $\omega_i = \omega_1^i$ in the case $p = 2$ we can collect the terms of the sums in the final equality. From this we conclude that property (1.4) holds.

Property (1.6) **is satisfied** since $\omega_i = 0$ for $i < 0$ implies that $\omega_{n-i} \otimes \theta_i = 0$ for $i > n$, and we may choose $\theta_i = 0$ in this case.

For property (1.7) **and** (1.11) we note that $\lambda(\alpha)$ restricts to $\alpha^{\times p}$ on the basepoint fibre. But on this fibre, the map ∇ coincides with Δ, so we get α^p when restricting. Restriction to fibre corresponds to fixing the class $\omega_0 = 1 \in H^0(L^\infty)$, and throwing away the other terms. So only option when restricting to the fibre is $1 \otimes \theta_{n(p-1)}(\alpha) = \alpha^p$, as we wanted.

To verify property (1.1) we begin by doing it in the special case $H^1(S^1)$. For this property we shall only consider $p = 2$.

Lemma 1.3.2. *Let α be a class in $H^1(S^1)$. Then $Sq^0(\alpha) = \alpha$.*

Proof. Let α be a generator of $H^1(S^1)$. To identify the element $\nabla^*(\lambda(\alpha)) \in H^2(\mathbb{R}\mathrm{P}^\infty \times S^1)$ we check, using cellular cohomology, what values this cocycle takes on the 2-cells of $\mathbb{R}\mathrm{P}^\infty \times S^1$. Using the smallest possible cell structure, there are only two such cells, namely $\mathbb{R}\mathrm{P}^2 \times \{s_0\}$ and $\mathbb{R}\mathrm{P}^1 \times S^1$, with s_0 the basepoint of S^1. $\lambda(\alpha)$ lies in $H^2(\Lambda S^1)$, and the entire $\mathbb{R}\mathrm{P}^\infty \times \{s_0\}$ is collapsed to a point in ΛS^1, and $\lambda(\alpha)$, being a 2-cocycle, is zero on a point. Applying the homomorphism ∇^* again gives us zero, so $\nabla^*(\lambda(\alpha))$ is zero on $\mathbb{R}\mathrm{P}^2 \times \{s_0\}$.

To evaluate $\nabla^*(\lambda(\alpha))$ on $\mathbb{R}\mathrm{P}^1 \times S^1$, we first note that ∇^* factors through $H^2(\Gamma_{S^\infty} S^1) \simeq \mathrm{Hom}(H_2(\Gamma_{S^\infty} S^1; \mathbb{F}_2)$. For the purpose of identifying 2-cycles, we might as well consider $H_2(\Gamma_{S^1} S^1)$ where it is more apparent that the element represented by $\mathbb{R}\mathrm{P}^1 \times S^1$ is the same as the element represented by a fibre $S^1 \wedge S^1$. This is since the union of these two subspaces in $\Gamma_{S^1} S^1$ is exactly the boundary of the interior 3-cell, when taken modulo 2, as illustrated on this figure

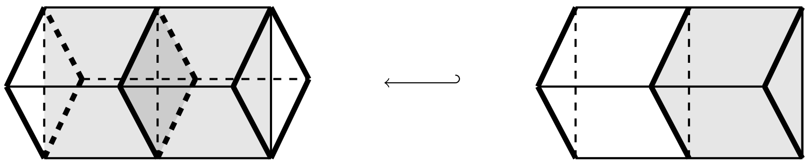


Figure 1.4: Bringing back the example used for figure 1.2 we insert S^1 for X, and get a nice picture of $\Gamma_{S^1,2} S^1$. The interior 3-cell pointed out to the right, with the two triangular ends constituting the fibre $S^1 \wedge S^1$, and the large rectangular backside being the $\mathbb{R}\mathrm{P}^1 \times S^1$.

So we may instead evaluate $\nabla^*(\lambda(\alpha))$ on $S^1 \wedge S^1$, where we can again just look at what happens with $\lambda(\alpha)$ in $H^2(\Lambda S^1)$. By construction this is $\alpha \times \alpha$ on fibres, but since α is a generator of $H^1(S^1)$, $\alpha \times \alpha$ is a generator of $H^2(S^1 \wedge S^1)$, and hence takes the value 1 on $S^1 \wedge S^1$.

In conclusion $\nabla^*(\lambda(\alpha)) = \omega_1 \times \alpha$, which expanded through the cross product isomorphism is just $\omega_1 \otimes \alpha$. So on $H^1(S^1)$ we have $Sq^0(\alpha) = \alpha$. □

property (1.5) **can now be verified** and thereafter property (1.1). The suspension map $\sigma : H^*(X) \to H^*(S^1 \wedge X)$ is given by $\sigma(\alpha) = \varepsilon \otimes \alpha$ for ε a generator of $H^1(S^1)$. So by the Cartan formula verified above, we have

$$\begin{aligned} Sq^i(\sigma(\alpha)) &= Sq^i(\varepsilon \otimes \alpha) \\ &= \sum_j Sq^j(\varepsilon) \otimes Sq^{i-j}(\alpha) \\ &= Sq^0(\varepsilon) \otimes Sq^i(\alpha) = \sigma(Sq^i(\alpha)) \end{aligned}$$

since $Sq^j(\varepsilon) = 0$ for $j > 1$ by property (1.6), $Sq^1(\varepsilon) = \varepsilon^2 = 0$ by property (1.7), and $Sq^0(\varepsilon) = \varepsilon$ by lemma 1.3.2. This shows that property (1.5) holds, and by this property Sq^0 is the identity on $H^n(S^n)$ for all $n > 0$.

property (1.1) **follows** now, since Sq^0 is also the identity on the fundamental classes ι_n as the n-skeleton of K_n is exactly S^n. This determines Sq^0 on all other classes as seen in the construction.

Property (1.9) **holds** by the same type of argument. Let ω be a generator of $H^1(\mathbb{R}\mathrm{P}^2)$, then $Sq^1(\omega) = \omega^2$ by property (1.7). The Bockstein β_2 on the generator is calculated in the following lemma.

Lemma 1.3.3. *Let X be a $K(\mathbb{Z}_m, 1)$. The Bockstein homomorphism*

$$\beta_m \ : \ H^n(X; \mathbb{Z}_m) \ \to \ H^{n+1}(X; \mathbb{Z}_m)$$

is an isomorphism for n odd, and zero for n even.

Proof. Consider the long exact sequence in cohomology related to the coefficient sequence

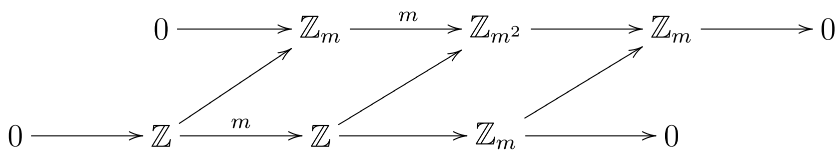
$$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}_m \longrightarrow 0 \tag{1.14}$$

and the associated Bockstein $\tilde{\beta} : H^n(X; \mathbb{Z}_m) \to H^{n+1}(X; \mathbb{Z})$. This gives the upper row of the following diagram

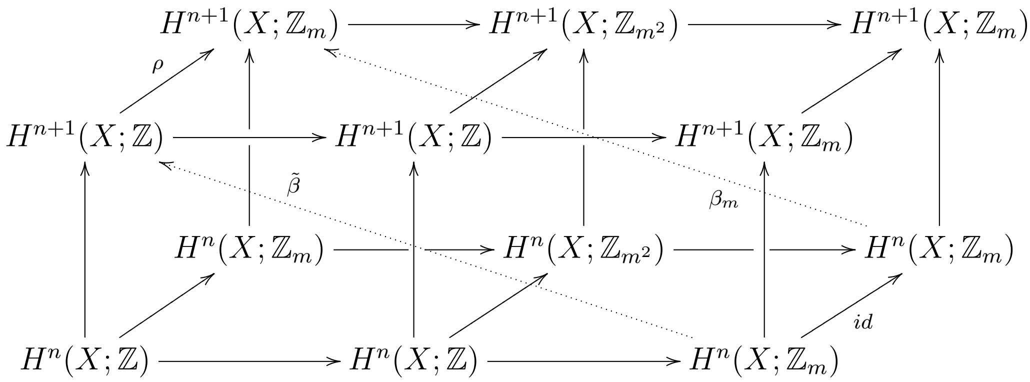
$$\begin{array}{ccccccc} H^n(X;\mathbb{Z}) & \xrightarrow{\rho} & H^n(X;\mathbb{Z}_m) & \xrightarrow{\tilde{\beta}} & H^{n+1}(X;\mathbb{Z}) & \xrightarrow{m} & H^{n+1}(X;\mathbb{Z}) \\ & & & \searrow^{\beta_m} & \downarrow \rho & & \\ & & & & H^{n+1}(X;\mathbb{Z}_m) & & \end{array}$$

where ρ is just the induced map from reduction modulo m. We show that the triangle commutes, since we can then insert the cohomology (well-known) for X and verify the statement.

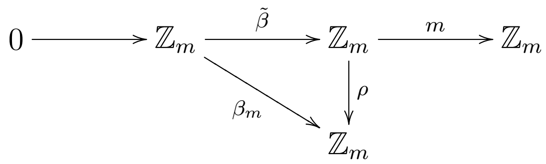
There is a natural map from the short exact sequence (1.14) to the one for β_m given by reduction modulo m and m^2.



When determining the connecting homomorphisms for the long exact sequences we get $\beta_m = \tilde{\beta}\rho$ from this, as it induces ρ in one end, and the identity in the other. Sketched for convenience.



Knowing that the triangle commutes, we insert for n odd and get



Here we see that $\tilde{\beta}$ is surjective by exactness, and ρ being induced by reduction modulo m is also surjective. We conclude that β_m is surjective and hence an isomorphism when n is odd. For n even, we see that β_m factors through zero, and hence is zero. □

By this $\beta_2(\omega_1) = \omega_2$ as there is only this one generator, so Sq^1 and β_2 coincide on ω_1. By property (1.5) they agree on all suspensions of ω_1. Now since $\mathbb{R}\mathrm{P}^2$ is the 2-skeleton of $\mathbb{R}\mathrm{P}^\infty$, which again is a $K(\mathbb{Z}_2, 1)$, we may suspend $\mathbb{R}\mathrm{P}^2$ n times, to get the $n+2$ skeleton of K_{n+1}. Then β_2 agrees with Sq^1 on the fundamental class, and we are done.

1.3.2 The Adem relations

In this section, the general approach of section 1.2.1 will pay off. The property (1.8) is called the Adem relations, and we will show that this holds on ι_n, and thereby on all classes by naturality.

We will first try to naively iterate our construction of the operations by considering the composition

$$L^\infty \times L^\infty \times K_n \xrightarrow{id\times\nabla} L^\infty \times \Lambda K_n \xrightarrow{\nabla} \Lambda(\Lambda K_n) \xrightarrow{\lambda(\lambda)} K_{np^2}$$

This yields

$$\begin{aligned}(id\times\nabla)^*(\nabla^*(\lambda(\lambda))) &= (id\times\nabla)^*\left(\sum_i \omega_{np(p-1)-i}\otimes\theta_i(\lambda)\right)\\ &= \sum_i \omega_{np(p-1)-i}\otimes\nabla^*\theta_i(\lambda^*(\iota_{np}))\\ &= \sum_i \omega_{np(p-1)-i}\otimes\theta_i(\nabla^*\lambda_c(\iota_n))\\ &= \sum_i \omega_{np(p-1)-i}\otimes\left(\sum_j \omega_{n(p-1)-j}\otimes\theta_j(\iota_n)\right)\\ &= \sum_{i,j} \omega_{np(p-1)-i}\otimes\theta_i(\omega_{n(p-1)-j}\otimes\theta_j(\iota_n))\end{aligned}$$

Assuming $p=2$ we have the Cartan formula, which then gives

$$\sum_{i,j}\omega_1^{2n-i}\otimes Sq^i(\omega_1^{n-j}\otimes Sq^j(\iota_n)) = \sum_{i,j}\omega_1^{2n-i}\otimes\sum_k Sq^k(\omega_1^{n-j})\otimes Sq^{i-k}Sq^j(\iota_n)$$

We can compute the $Sq^k(\omega_1^{n-j})$ terms by introducing the total Steenrod square, $Sq := \sum_i Sq^i$. By property (1.6) only finitely many of the squares are non-zero on a given class, so the total square is well-defined. Rewriting the Cartan formula in terms of Sq then gives $Sq(\alpha\smile\beta) = Sq(\alpha)\smile Sq(\beta)$. Now for $\alpha\in H^1(X;\mathbb{Z}_2)$ we then get

$$Sq(\alpha) = \alpha+\alpha^2 = \alpha(1+\alpha)$$

and further

$$\sum_i Sq^i(\alpha^n) = Sq(\alpha^n) = Sq(\alpha)^n = \alpha^n(1+\alpha)^n = \sum_i\binom{n}{i}\alpha^{n+i}$$

So for each i we have $Sq^i(\alpha^n) = \binom{n}{i}\alpha^{n+i}$. From this we get $Sq^k(\omega_1^{n-j}) = \binom{n-j}{k}\omega_1^{n-j+k}$, and defining $n-j+k = 2n-l$ we end up with

$$\begin{aligned}
\sum_{i,j} \omega_1^{2n-i} \otimes \sum_k Sq^k(\omega_1^{n-j}) \otimes Sq^{i-k}Sq^j(\iota_n) \\
&= \sum_{i,j,k} \binom{n-j}{k} \omega_1^{2n-i} \otimes \omega_1^{n-j+k} \otimes Sq^{i-k}Sq^j(\iota_n)) \\
&= \sum_{i,j,l} \binom{n-j}{n+j-l} \omega_1^{2n-i} \otimes \omega_1^{2n-l} \otimes Sq^{i+l-n-j}Sq^j(\iota_n))
\end{aligned} \tag{1.15}$$

This involves $Sq^{i+l-n-j}Sq^j$, mixed square factors that we wish to investigate, but for now we cannot say much more about it. We need another way to arrive at something similar, so we shall make a small detour.

The group $\mathbb{Z}_p \times \mathbb{Z}_p$ acts freely on $S^\infty \times S^\infty$, each $\mathbb{Z}_p$ rotating the corresponding S^∞ just as above. It also acts on $X^{\wedge p^2}$, if we write points as (x_{ij}) $1 \le i, j \le p$, and let first factor permute the i-index cyclically, and the second factor permutes the j-index likewise. This action fixes the basepoint, and thus we have the functor $\Lambda' := \Lambda_{S^\infty \times S^\infty, \mathbb{Z}_p \times \mathbb{Z}_p, p^2}$. We can now more or less repeat the construction made in section 1.2.2.

Consider $\iota_n^{\otimes p^2} : K_n^{\wedge p^2} \to K_{np^2}$. The action of $\mathbb{Z}_p \times \mathbb{Z}_p$ restricts to $S^1 \times \{s_0\}$ and to $\{s_0\} \times S^1$, and here it is just a $\mathbb{Z}_p$-action on S^1. Let T_i be the map that permutes just the index i of (x_{ij}) cyclically, corresponding to the restriction of the action. Similar for the index j.

Analogous to proposition 1.2.4 we now get that $\iota_n^{\otimes p^2}$ and $\iota_n^{\otimes p^2} T_i$ are homotopic. The two homotopies are basepoint preserving, and thus defines an extension of $\iota_n^{\otimes p^2}$ to the union of $\Lambda_{S^1 \times \{s_0\}, \mathbb{Z}_p \times \mathbb{Z}_p, p^2} K_n$ and $\Lambda_{\{s_0\} \times S^1, \mathbb{Z}_p \times \mathbb{Z}_p, p^2} K_n$. From here we attach cells of dimension greater than $np^2 + 1$ to get to $\Lambda_{S^\infty \times S\infty, \mathbb{Z}_p \times \mathbb{Z}_p, p^2} K_n$, so analogous to proposition 1.2.5, we can extend further to a map $\lambda' : \Lambda' K_n \to K_{np^2}$. This has the property that the restriction to the fibres $K_n^{\wedge p^2}$ is homotopic to $\iota_n^{\otimes p^2}$.

We shall also need $\nabla' : L^\infty \times L^\infty \times X \to \Lambda' X$, the analogue map to ∇. This is the obvious construction, namely the quotient of the diagonal embedding $S^\infty \times S^\infty \times X \to S^\infty \times S^\infty \times X^{\wedge p^2}$ composed with the quotient map $\Gamma_{S^\infty \times S^\infty, \mathbb{Z}_p \times \mathbb{Z}_p, p^2} X \to \Lambda' X$.

Examining $\Lambda' X$ and $\Lambda(\Lambda X)$, we see that they may be considered as quotients of respectively $S^\infty \times S^\infty \times X^{p^2}$ and $S^\infty \times (S^\infty \times X^p)^p$, where we first identify all points having at least one X-coordinate equal to the basepoint x_0, and then factor out the action by respectively $\mathbb{Z}_p \times \mathbb{Z}_p$ and the wreath product $\mathbb{Z}_p \wr \mathbb{Z}_p$. As described above $\mathbb{Z}_p \times \mathbb{Z}_p$ acts diagonally, and $\mathbb{Z}_p \wr \mathbb{Z}_p$

acts as follows. The $\mathbb{Z}_p^p$ part acts by letting each factor act diagonally on the corresponding factor of $(S^\infty \times X^p)^p$ by rotation in first coordinate and permuting the p X factors. The quotient $\mathbb{Z}_p$ then acts by rotating the first S^∞, and permuting the p factors of $(S^\infty \times X^p)^p$.

This is a more direct approach, as opposed to our previous construction where we first make smashes of the products, then factor out by the group actions, and finally again identify points in the basepoint section, which we did in order to factor through the spaces given by the Γ-functors. We needed those in the construction to get the first extension of $\iota_n^{\otimes p}$ as a homotopy, but having done that we can now disband that approach for the new one.

The point of this, is that we want to be able to define a map $u : \Lambda' K_n \to \Lambda(\Lambda K_n)$ such that the square of the following diagram commutes

$$\begin{array}{ccccc} L^\infty \times L^\infty \times K_n & \xrightarrow{\nabla'} & \Lambda' K_n & \xrightarrow{\lambda'} & K_{np^2} \\ \downarrow{\scriptstyle id\times\nabla} & & \downarrow{\scriptstyle u} & \nearrow{\scriptstyle \lambda(\lambda)} & \\ L^\infty \times \Lambda K_n & \xrightarrow[\nabla]{} & \Lambda(\Lambda K_n) & & \end{array}$$

Where the lower route through the diagram is seen to be the composition we worked at earlier.

Now $\mathbb{Z}_p \times \mathbb{Z}_p$ lies as a subgroup in $\mathbb{Z}_p \wr \mathbb{Z}_p$, which we obtain if we restrict the action of the quotient $\mathbb{Z}_p$ on $\mathbb{Z}_p^p$ to the diagonal $\mathbb{Z}_p$ where it is trivial. The map $v : S^\infty \times S^\infty \times X^{p^2} \to S^\infty \times (S^\infty \times X^p)^p$ given by

$$(s, t, x_{11}, \ldots, x_{pp}) \mapsto (s, t, x_{11}, \ldots, x_{1p}, \ldots, t, x_{p1}, \ldots, x_{pp})$$

then induce a map on the spaces, where we quotient out the basepoint relation, and then the action of $\mathbb{Z}_p \times \mathbb{Z}_p$. If we compose this with the quotient map to the space where we factor out by the entire $\mathbb{Z}_p \wr \mathbb{Z}_p$ action, we get our map u. It is straight forward to check that the square above commutes, and instead we turn our attention to the triangle. If we can show that it commutes up to homotopy we get a different point of view at the map we worked with before.

As before we can apply proposition 1.2.6 and restrict to the fibre $K_n^{\wedge p^2}$, where we then have up to homotopy

$$\begin{array}{ccc} K_n^{\wedge p^2} & \xrightarrow{\iota_n^{\otimes p^2}} & K_{np^2} \\ \downarrow{\scriptstyle u|_{K_n^{\wedge p^2}}} & \nearrow{\scriptstyle (\iota_n^{\otimes p})^{\otimes p}} & \\ (K_n^{\wedge p})^{\wedge p} & & \end{array}$$

The restriction of u induce the right isomorphism on cohomology just as t did when verifying property (1.3), and we conclude that the diagram commutes up to homotopy.

We shall need the following diagram also

$$\begin{array}{ccc} L^\infty \times L^\infty \times K_n & \xrightarrow{\nabla'} & \Lambda' K_n \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} \\ L^\infty \times L^\infty \times K_n & \xrightarrow[\nabla']{} & \Lambda' K_n \end{array}$$

where τ_1 switches the L^∞ factors, and τ_2 switches the two $\mathbb{Z}_p$ factors in the action on the space. The diagram commutes since ∇' maps the K_n diagonally into the smash $K_n^{\wedge p^2}$. Switching the $\mathbb{Z}_p$'s correspond to interchanging the indices on the $K_n^{\wedge p^2}$ coordinates (x_{ij}), which is a permutation consisting of $p(p-1)/2$ transpositions of K_n factors. Thus restricting to fibres, the class $\iota_n^{\otimes p^2}$ is sent to $(-1)^{np(p-1)/2}\iota_n^{\otimes p^2}$ by an argument similar to that of proposition 1.2.4. As we have seen λ' is uniquely determined by the restriction to the fibre, so we also get $\tau_2^*\lambda'^*(\iota_{np^2}) = (-1)^{np(p-1)/2}\lambda'^*(\iota_{np^2})$. Now set $k = np(p-1)/2$, and note that class $\nabla'^*\lambda'^*(\iota_{np^2}) \in H^*(L^\infty \times L^\infty \times K_n)$ can be written as $\sum_{r,s} \omega_r \otimes \omega_s \otimes \varphi_{rs}$ by the Künneth formula. From the square above we then get

$$\begin{aligned} \sum_{r,s} \omega_r \otimes \omega_s \otimes \varphi_{rs} &= \nabla'^*\lambda'^*(\iota_{np^2}) \\ &= (-1)^k \nabla'^*\tau_2^*\lambda'^*(\iota_{np^2}) \\ &= (-1)^k \tau_1^*\nabla'^*\lambda'^*(\iota_{np^2}) \\ &= (-1)^k \sum_{r,s} (-1)^{rs}\omega_s \otimes \omega_r \otimes \varphi_{rs} \end{aligned}$$

τ_1 switches the L^∞ factors, and this gives us the extra sign since the cross product is graded commutative. Switching r and s in the summation, then gives $\varphi_{rs} = (-1)^{k+rs}\varphi_{sr}$.

Returning to the calculation (1.15) we started earlier in the case $p = 2$, the above result is just $\varphi_{rs} = \varphi_{sr}$. So we may switch i and l, and for every choice of these, and all $n \geq 1$ we obtain

$$\sum_j \binom{n-j}{n+j-l} Sq^{i+l-n-j}Sq^j(\iota_n) = \sum_j \binom{n-j}{n+j-i} Sq^{l+i-n-j}Sq^j(\iota_n)$$

Now set $n = 2^r - 1 + s$ and $l = n + s$ (note that $j = k + s$ with these choices). Then $\binom{n-j}{n+j-l} = \binom{2^r-1-(j-s)}{j-s} \equiv 0 \bmod 2$, unless $j = s$, by the following lemma.

Lemma 1.3.4. *Let r, x be integers. For fixed r we have that $\binom{2^r-1-x}{x} \equiv 0$ mod 2, for all $x > 0$.*

Proof. With the convention that the binomial coefficient is zero when it does not make combinatorial sense, we may assume that $2x < 2^r$. We now write out the binomial coefficient $\binom{2^r-1-x}{x} = \frac{(2^r-2x)\cdots(2^r-(x+1))}{x!}$, and compare 2-powers in nominator and denominator.

In any sequence of x consecutive natural numbers there is at least $\sum_i \lfloor \frac{x}{2^i} \rfloor$ 2's, and among $1, \ldots, x$ there is exactly this number. We simply count the even numbers, the 4-even, 8-even and so on. Note that if 2^k is the largest 2-power occurring in $1, \ldots, x$ then $2 \cdot 2^k$ will be in $x+1, \ldots, 2x$.

Pair off a single 2 from the all even numbers in one sequence with a single 2 from the evens of the other, then pair the remaining 2 from the 4-evens, and so on. After k steps we will have depleted the 2's among $1, \ldots, x$, and will still have a 2 left among $x+1, \ldots, 2x$.

We now claim that there are at least the same 2's among $2^r - 2x, \ldots, 2^r - (x+1)$ as there are among $x+1, \ldots, 2x$, if $r > k$. If $x + i = 2^m x'$ for $1 \leq i \leq x$ and maximal m such that x' is integer, then $2^r - (x+i) = 2^m(2^{r-m} - x')$. But since $r > k \geq m-1$ this shows the claim. The condition $r > k$ is always satisfied in our case since $2^{k+1} \leq 2x < 2^r$.

In conclusion there are strictly more 2's in the nominator than in the denominator of $\frac{(2^r-2x)\cdots(2^r-(x+1))}{x!}$, so we are done. □

So with the choices above we get

$$\begin{aligned} Sq^i Sq^s(\iota_n) &= \sum_j \binom{2^r - 1 + s - j}{2^r - 1 + s + j - i} Sq^{i+s-j} Sq^j(\iota_n) \\ &= \sum_j \binom{2^r - 1 + s - j}{i - 2j} Sq^{i+s-j} Sq^j(\iota_n) \end{aligned}$$

Finally, if we show that

$$\binom{2^r - 1 + s - j}{i - 2j} \equiv \binom{s - 1 - j}{i - 2j} \bmod 2, \qquad \text{for } i < 2s, \tag{1.16}$$

we have shown that the Adem relations hold on the classes ι_{2^r-1+s}. This implies that they hold on all classes in same dimensions, and by property (1.5) they hold for all classes since we can just suspend a class an appropriate number of times to a dimension of this form.

If $i < 2j$ then both sides of (1.16) is zero. Otherwise we have $2j \leq i < 2s$ so that $s - j - 1 \geq 0$. If we apply Vandermonde's identity to the left side we then get

$$\binom{2^r - 1 + s - j}{i - 2j} = \binom{s - j - 1}{i - 2j} + \sum_{m=1}^{2^r} \binom{2^r}{m} \binom{s - j - 1}{i - 2j - m}$$

Now $\binom{2^r}{m} \equiv 0 \bmod 2$, when $0 < m < 2^r$. Choosing r sufficiently large will give us $\binom{s-j-1}{i-2j-m} = 0$ when $m = 2^r$. So we have the congruence we wanted, and have shown all the properties we set out to do. This finishes the verification of our construction, and we shall now have a look at an application.

Chapter 2

Applications

2.1

First example will not use the full power of the squares and in particular the corollary could be achieved without them

Example 2.1.1 Consider the complex projective space $\mathbb{C}\mathrm{P}^2$, which we give the CW-structure as in [Bre93] example 8.9, i.e.

$$\mathbb{C}\mathrm{P}^2 \simeq \mathbb{C}\mathrm{P}^1 \cup_h D^4 \simeq S^2 \cup_h D^4$$

with slight abuse of notation regarding the map $h : S^3 \to S^2$. It is given by mapping the complex coordinates (z_0, z_1) where $|z_0^2| + |z_1^2| = 1$, to the homogeneous complex coordinates $(z_0 : z_1 : 0)$. We know that $\mathbb{C}\mathrm{P}^2 \not\simeq S^2 \vee S^4$ since one has trivial cup product and the other does not, so h is non-trivial. As choice of notation suggests it must then be the Hopf map.

We know that $H^*(\mathbb{C}\mathrm{P}^2; \mathbb{F}_2) \simeq \mathbb{F}_2[x]/(x^3)$ with $|x| = 2$, and by property (1.7), $Sq^2(x) = x^2 \neq 0$. Now consider the suspension $\Sigma\mathbb{C}\mathrm{P}^2 \simeq S^3 \cup_{\Sigma h} D^5$. From (1.5) we know that Sq^2 commutes with suspension, so also $Sq^2\sigma(x) = \sigma(Sq^2 x) = \sigma(x^2) \neq 0$. Now if Σh is homotopically trivial then $\Sigma\mathbb{C}\mathrm{P}^2 \simeq S^3 \vee S^5$, and we get from naturality of Sq^2 that the following diagram commutes

$$\begin{array}{ccc} H^3(S^3;\mathbb{F}_2) & \xrightarrow{Sq^2} & H^5(S^3;\mathbb{F}_2) = 0 \\ \downarrow & & \downarrow \\ H^3(S^3 \vee S^5;\mathbb{F}_2) & \xrightarrow{Sq^2} & H^3(S^3 \vee S^5;\mathbb{F}_2) \end{array}$$

The vertical maps are induced by the projection onto S^3, so the left one is an isomorphism. Thus $Sq^2 : H^3(S^3 \vee S^5; \mathbb{F}_2) \to H^5(S^3 \vee S^5; \mathbb{F}_2)$ is zero, which contradicts our previous calculation, so Σh is non-trivial. The same argument can be made for any suspension $\Sigma^k h : S^{3+k} \to S^{2+k}$.

Corollary 2.1.2. *$\pi_{k+1}(S^k) \neq 0$ for all $k \geq 2$.*

Proof. From the example above, we have at least one non-zero class in $\pi_{k+1}(S^k)$ for any $k \geq 2$. □

Second example is a bit more generic for the squares

Example 2.1.3 With the squares we can show that $\mathbb{C}\mathrm{P}^4/\mathbb{C}\mathrm{P}^2$ is not homotopy equivalent to $S^8 \vee S^6$, even though $H^*(\mathbb{C}\mathrm{P}^4/\mathbb{C}\mathrm{P}^2; R) \simeq H^*(S^8 \vee S^6; R)$ for any coefficient ring R. From cellular cohomology we know that both have cohomology groups R, in degree 0,6 and 8, and both must have trivial product structures, by degree considerations.

We know that $H^*(\mathbb{C}\mathrm{P}^4; \mathbb{F}_2) \simeq \mathbb{F}_2[x]/(x^5)$ for $|x| = 2$, so $H^6(\mathbb{C}\mathrm{P}^4/\mathbb{C}\mathrm{P}^2; \mathbb{F}_2)$ is generated by x^3, and $H^8(\mathbb{C}\mathrm{P}^4/\mathbb{C}\mathrm{P}^2; \mathbb{F}_2)$ by x^4. Now by property (1.4), the Cartan formula and (1.1) we have

$$\begin{aligned} Sq^2(x^3) &= Sq^0(x^2)Sq^2(x) + Sq^1(x^2)Sq^1(x) + Sq^2(x^2)Sq^0(x) \\ &= x^2 \smile x^2 + x \smile Sq^2(x^2) \\ &= x^4 + x \smile (Sq^0(x)Sq^2(x) + Sq^1(x)Sq^1(x) + Sq^2(x)Sq^0(x)) \\ &= x^4 + x \smile (x \smile x^2 + x^2 \smile x) \\ &= 3x^4 = x^4 \neq 0 \end{aligned}$$

On the other hand we have a diagram similar to the previous example

$$\begin{array}{ccc} H^6(S^6; \mathbb{F}_2) & \xrightarrow{Sq^2} & H^8(S^6; \mathbb{F}_2) = 0 \\ \downarrow & & \downarrow \\ H^6(S^6 \vee S^8; \mathbb{F}_2) & \xrightarrow{Sq^2} & H^8(S^6 \vee S^8; \mathbb{F}_2) \end{array}$$

showing that $Sq^2 : H^6(S^6 \vee S^8; \mathbb{F}_2) \to H^8(S^6 \vee S^8; \mathbb{F}_2)$ is zero. Hence $\mathbb{C}\mathrm{P}^4/\mathbb{C}\mathrm{P}^2 \not\simeq S^6 \vee S^8$.

These were both fairly simple examples, and they were all I could fit in here. There is a nice example in [Hat] (theorem 1.40) calculating homotopy groups of spheres. That one shows more of the powerful tool they are, but it is rather long.

Chapter 3

Discussion

3.1 Key points in construction

This section is meant to be a brief comment on why what we have done actually works.

First thing to notice is the important role the fibres from lemma 1.2.3 play, in giving the construction the appropriate properties, and letting us verify these. Beginning our construction on the fibre with the cross product power of the fundamental class, and extending from there give us just the necessary control of the construction. That depends on the Borel construction (in our case $S^\infty \times_G X^{\wedge p}$) and the subspaces and quotients of the Borel construction made on the original space, and this construction can be generalized a lot.

The second thing is something already hinted at during the construction. What gets the whole thing going is proposition 1.2.4, the first extension from the map defined on the fibre. The subsequent extensions were obtained more or less for free through standard obstruction theory.

Other references[1] typically make the construction of the Steenrod operations on cochain level, instead of on space level as we have done here. One advantage of that approach, is the transparency of the origin of the operations. In [Bre93] it is claimed, that the Steenrod operations owe their existence to the fact that the cup product is not graded commutative on cochain level. It is illustrated by noting, that the first extension of a diagonal approximation (playing the same role as $\iota_n \otimes \iota_n$ in the case $p = 2$) can be taken to be trivial if the cup product were graded commutative. It is then claimed that this would lead to properties for the operations, which are

[1] E.g. [Bre93] chapter VI 16

known not to be true.

We can rediscover this in our construction as well. The corresponding claim is that the homotopy $\iota_n^{\otimes p} T \sim \iota_n^{\otimes p}$ is not an equality. Consider the composition

$$L^1 \times K_n \hookrightarrow \underbrace{L^\infty \times K_n \xrightarrow{\phi} \Gamma_{S^\infty} K_n \longrightarrow \Lambda K_n}_{\nabla} \xrightarrow{\lambda} K_{np}$$

where we recall that ϕ was induced by $id \times d$ mapping K_n to the diagonal of $K_n^{\wedge p}$ by d. Had the homotopy been an equality, then for purpose of the above composition, the action of $\mathbb{Z}_p$ on $S^\infty \times K_n^{\wedge p}$ could be taken to be trivial on the second factor. In particular $\nabla^*(\lambda(\iota_n)) = (id_{L^\infty} \times d)^*(\iota_n^{\otimes p}) = 1 \otimes \iota_n^p$ when expanded through the Künneth formula. But then

$$\theta_i(\iota_n) = \begin{cases} \iota_n^p & i = n \\ 0 & i \neq n \end{cases}$$

and we know this is not the case. If we accept the claim that the non-triviality of the homotopy correspond to the non-triviality of the chain homotopy given in [Bre93], this shows that the existence of Steenrod operations is an obstruction to a graded commutative cup product at cochain level. The other way around as claimed in [Bre93] is not evident from this.

Finally there is the correspondence between cohomology and maps into Eilenberg-MacLane spaces, which we use in our construction all the time. But with regard to the other references, this seems to be more of a convenient thing to exploit, than a necessity for the construction.

3.2 Further development

Aside from the obvious task of verifying the properties of the Steenrod powers, I see at least two interesting further developments to pursue.

The first one is connected to the discussion above, namely to establish in what other contexts we might expect to find similar operations, and maybe even find sufficient and necessary conditions for their existence. Some of the key points noticed in the construction we have made here might provide a clue to this. For example [Lur] shares some of the traits.

It can be shown that it is in fact possible to make the cup product graded commutative on cochain level if we consider cohomology with coefficients in a field of characteristic zero. As an example of the consequence of this, we may consider singular cohomology with coefficients in $\mathbb{Q}$. From lemma 1.3.1 we see that the possible cohomology operations would correspond to elements of $H^m(K(\mathbb{Q},n);\mathbb{Q})$. It can be shown[2] that

$$H^*(K(\mathbb{Q},n);\mathbb{Q}) \simeq \begin{cases} \mathbb{Q}[x] & |x| = n \text{ odd} \\ \Lambda_{\mathbb{Q}}[x] & |x| = n \text{ even} \end{cases}$$

So for n even there are only the zero and the identity operations, and for n odd there are the exponentiation operations as well (with respect to cup product). These are however not stable (i.e. commuting with suspension) as cup products of positive dimensional classes are zero in any suspension. Simply consider the relative cup product in the following setting

$$\begin{array}{ccc} H^k(\Sigma X, CX_+) \times H^l(\Sigma X, CX_-) & \xrightarrow{\smile} & H^{k+l}(\Sigma X, CX_+ \cup CX_-) \\ \big| \simeq & & \big| \simeq \\ \tilde{H}^k(\Sigma X, *) \times \tilde{H}^k(\Sigma X, *) & \xrightarrow[\smile]{} & \tilde{H}^{k+l}(\Sigma X / \Sigma X) \end{array}$$

with any coefficient ring. CX_+ and CX_- are the two cones plus a small strip so that they cover the suspension ΣX. The pairs are all good, so the vertical maps are isomorphisms, and the bottom map shows that the cup product vanishes on a suspension.

This shows that there are then no stable operations at all, when taking cohomology with coefficients in $\mathbb{Q}$. In particular there are nothing similar to the Steenrod operations as we also argued would be the case above.

On a related note, one could ask if there are any other stable cohomology operations for $\mathbb{F}_p$-cohomology, than those we already have mentioned in this project. Steenrod himself showed, that the Steenrod operations are uniquely determined by the properties (1.1) - (1.5) and respectively (1.6) - (1.7) or (1.10) - (1.11), for squares respectively powers.[3] But could there be others?

This also bring us to the second point to pursue. Namely the Steenrod algebra, as mentioned already in the introduction, along with the structure and applications of this. We define the Steenrod algebra $\mathcal{A}_2$ as the $\mathbb{F}_2$-algebra which is a quotient of polynomials in the non-commuting variables $Sq^1, Sq^2, \ldots$ by the two-sided ideal generated by the Adem relations. Of course there are similar Steenrod algebras $\mathcal{A}_p$, for p odd. In this case in the variables are $\beta_p, P^1, P^2, \ldots$, and the ideal is generated by the Adem relations and the relation $\beta_p^2 = 0$.

For every space X, we then have that $H^*(X;\mathbb{F}_p)$ is a module over the Steenrod algebra $\mathcal{A}_p$, and examining this further will let us exploit the Steen-

[2] By use of the Serre spectral sequence [Hat] chapter 1, prop. 1.20.

[3] In [Ste62] chapter VIII, paragraph 3.

rod operations to a much higher extend, and put further constraints on topological spaces. But more to the point of the question above, the introduction of *admissible monomials*, and *excess* of those, leads to a theorem originally by J. P. Serre,[4] which has as a consequence that all stable cohomology operations with $\mathbb{F}_p$ coefficients are polynomials in the Steenrod operations.

This connects the study of stable cohomology operations to that of the Steenrod algebra. A common approach to ring theory, is to study the modules over a ring instead of the ring itself, which suggests that the study of modules over the Steenrod algebra is worth a visit. For example [Sch94] looks interesting.

[4] Theorem top of p. 500 [Hat02], proved in [Hat] section 1.3.

Appendix A

Loose ends

A.1 Preliminary functors revisited

The first part of the construction presented a few functors, which were used throughout the project. It is easily checked that they are in fact functors, but it was only claimed so to make the exposition a bit smoother. Here we will make some remarks on the would be functors, but not show them to be so in full detail.

The first construction made was F_n mapping a pointed space (X, x_0) to $X^{\wedge n}$. For a map $f : (X, x_0) \to (Y, y_0)$ we then define $F_n f([(x, \ldots, x)]) = [(f(x), \ldots, f(x))]$. This is easily seen to be well-defined and functorial.

Second construction was $Y \times_G -$ mapping a pointed G-space (X, x_0) to $(Y \times X)/G$, whenever Y was a Hausdorff G-space with free action, and x_0 was fixed by G. For a G-equivariant map $f : (X, x_0) \to (X', x_0')$, we then let $Y \times_G f = \bar{f} : Y \times_G X \to Y \times_G X'$ be given by $\bar{f}([y, x]) = [y, f(x)]$. We here check that this is well-defined, i.e. that if $(x, y) = (g.y, g.x)$ then $(y, f(x)) = (g.y, f(g.x)) = (g.y, g.f(x))$, since f is equivariant.

Clearly id_X is sent to $id_{Y \times_G X}$ and compositions are preserved, so we have defined a functor.

Third and final construction was to pass to the quotient Λ_Y from Γ_Y. But quotients are functorial, so having noted that the above construction is a functor, this one is too.

Also we postponed the proof of lemma 1.2.3, which we will include here.

proof of lemma 1.2.3. Clearly we get an induced map on the quotients since the projection commutes with the group action, and we check that this is a fibre bundle.

First we show that the quotient map $p : Y \to Y/G$ is a fibre bundle with fibre G. For a point $[y] \in Y/G$, we construct an open neighborhood as follows. Pick a representative y for $[y]$. For each $g \in G$ we can choose an open neighborhood U_{gy} of gy in Y, such that $U_{gy} \cap U_{hy} = \emptyset$ for all $h \neq g \in G$. This we can do since the action of G on Y is free, so $h \neq g$ implies $hy \neq gy$, and these distinct points can be separated as Y is Hausdorff. Now as p is open and G is finite, $U_{[y]} := \bigcap_{g\in G} p(U_{gy})$ is an open neighborhood around $[y]$ in Y/G, and $p^{-1}(U_{[y]}) = \coprod_{g\in G} gU_{[y]} \simeq G \times U_{[y]}$. So on $U_{[y]}$ we have the trivialization we wanted, showing that $p : Y \to Y/G$ is a fibre bundle with fibre G.

Equivalently we can form a local section on $U_{[y]}$. Note that $U_{[y]}$ is of the form $Z_{[y]}/G$ for an open set $Z_{[y]} \subset Y$, and consider one such section $s : Z_{[y]}/G \to Y$. This we will now use to define our homeomorphism for the local trivialization of $\overline{\pi} : Y \times_G X \to Y/G$.

We have that $\overline{\pi}^{-1}(Z_{[y]}/G) = Z_{[y]} \times_G X$, and define a map $h : Z_{[y]}/G \times X \to Z_{[y]} \times_G X$, by mapping $([z], x) \mapsto [s([z]), x]$. This is well defined since s only depends on the class of z, and it is continuous since both s and the quotient map $q : Y \times X \to Y \times_G X$ are so. Note that first coordinate of q is just p.

Now assume that $[s([z]), x] = [s([z']), x']$. In particular there exists a $g \in G$ such that $s([z]) = gs([z'])$ and $x = gx'$, and if we apply p to this first equation we get $[z] = [z']$ since s is a section for p. G acts freely on Y so this implies that $g = e$, the identity, and further $x = x'$. Hence h is injective.

Finally we show that h is surjective, so we let $[z, x] \in Z_{[y]} \times_G X$ be given. Since $ps = id$ we know that there exists a $g \in G$ such that $gs([z]) = z$. Now $h([z], g^{-1}x) = [s([z]), g^{-1}x] = [z, x]$, and h is surjective. h is also an open map, since both s and q are open, so we conclude that h is a homeomorphism, and it is evident from the above, that the fibre for $\overline{\pi}$ is indeed X.

□

A.2 CW structure of spaces involved

In section 1.2.2 we make the assumption that the space we construct the Steenrod operations on has a CW structure. This we do in order to apply the bijection $\langle X, K(\mathbb{Z}_p, n)\rangle \simeq H^n(X, \mathbb{Z}_p)$.

As shown in Hatcher, we may replace a space X by a homotopy equivalent one X' without changing the cohomology groups. By naturality of the cup product also the ring structure is preserved. In particular the cohomology ring is unaffected when we replace our pointed spaces by CW-approximations.

Having done this, it is fairly easy to check, that for a CW complex X with basepoint x_0 as zero cell, also $\Gamma_Y X$ and $\Lambda_Y X$ carries a CW structure whenever Y is a CW complex such that also Y/G has a CW-structure. We note that both these spaces arise from a series of product and quotient constructions both of which are given natural CW-structures as described in [Hat02] p. 8 and p. 524-525.

A.3 Finite generation of $H^k(K_n)$

Finite generation in each degree is one of the prerequisites for the cross product to give an isomorphism. This does not hold for arbitrary spaces, and the observant reader will have noticed that the notation $\alpha^{\times p}$ has been used in place of $\alpha^{\otimes p}$ otherwise used when handling the ι_n's. These are classes in $H^*(K_n)$, and hence we must show that $H^k(K_n)$ is finitely generated for all k and n.

Lemma A.3.1. *If $H_k(A;\mathbb{Z})$ and $H_{k-1}(A;\mathbb{Z})$ are finitely generated, then $H^k(A;\mathbb{Z}_p)$ is finitely generated.*

Proof. Assume that $H_k(A;\mathbb{Z})$ and $H_{k-1}(A;\mathbb{Z})$ are finitely generated. Then also $H_k(A;\mathbb{Z}) \otimes \mathbb{Z}_p$ and

$$\mathrm{Tor}^{\mathbb{Z}}(H_{n-1}(A;\mathbb{Z}),\mathbb{Z}_p) \simeq \mathrm{Ker}(H_{n-1}(A;\mathbb{Z}) \xrightarrow{p} H_{n-1}(A;\mathbb{Z}))$$

are so, and by the universal coefficient theorem for homology[1], $H_k(A;\mathbb{Z}_p)$ is finitely generated. Now by the universal coefficient theorem for cohomology $H^k(A;\mathbb{Z}_p) \simeq \mathrm{Hom}_{\mathbb{Z}_p}(H_k(A;\mathbb{Z}_p),\mathbb{Z}_p)$, but this is finitely generated as it is just the dual vector space to the finitely generated $H_k(A;\mathbb{Z}_p)$. □

Thus having reduced our task to showing that homology is finitely generated, we proceed by induction on n. The case $n = 1$ is known, as this is just an infinite lens space, which have homology groups $\mathbb{Z}, \mathbb{Z}_p$ and 0, in 0'th, odd and even degrees. For $n \geq 2$ we have to use somewhat more advanced tools. One approach to this is to use Serre's spectral sequence for homology[2], which

[1] [Hat02] p. 264.
[2] [Hat].

we shall have to assume is well-known. Consider the path-loop fibration

$$\begin{array}{ccccc} \Omega K(\mathbb{Z}_p, n) & \longrightarrow & PK(\mathbb{Z}_p, n) & \longrightarrow & K(\mathbb{Z}_p, n) \\ \| & & \| & & \\ K(\mathbb{Z}_p, n-1) & & pt. & & \end{array}$$

and note that $\pi_1(K(\mathbb{Z}_p, n)) = 0$ since $n \geq 2$.

Inductively we assume that $H_*(K(\mathbb{Z}_p, n-1); \mathbb{Z})$ is finitely generated in every degree. We know that the sequence converges to the homology of a point since the pathspace is contractible. So on the E^∞-page we get filtrations of zero in every diagonal except in total degree zero, where we get $\mathbb{Z}$. We show by induction on k that $H_k(K(\mathbb{Z}_p, n); \mathbb{Z})$ is finitely generated for all k. $K(\mathbb{Z}_p, n)$ is connected as $n > 0$, so the base case is fine.

Now assume inductively that $H_m(K(\mathbb{Z}_p, n); \mathbb{Z})$ is finitely generated for all $m < k$. By the splitting given by universal coefficient theorem we get that

$$\begin{aligned} E^2_{s,t} &= H_s(K(\mathbb{Z}_p, n); H_t(K(\mathbb{Z}_p, n-1); \mathbb{Z})) \\ &\simeq H_s(K(\mathbb{Z}_p, n); \mathbb{Z}) \otimes H_t(K(\mathbb{Z}_p, n-1); \mathbb{Z}) \\ &\qquad \oplus \operatorname{Tor}^{\mathbb{Z}}(H_{s-1}(K(\mathbb{Z}_p, n); \mathbb{Z}), H_t(K(\mathbb{Z}_p, n-1); \mathbb{Z})) \end{aligned}$$

so all the E^2 terms with $s < k$ are finitely generated, since each factor in the tensor product is, by the two induction hypothesis. Then also the E^r terms with $s < k$ are finitely generated for any $r > 2$ as they are quotients of the corresponding E^2 terms.

We know that $E^\infty_{k,0} = 0$, and we can now recover $E^2_{k,0} = H_k(K(\mathbb{Z}_p, n); \mathbb{Z}_p)$ from the differentials, all with images in E^*-terms with $s < k$.

Concretely consider $E^{r+1}_{k,0} = \operatorname{Ker} d_r \subseteq E^r_{k,0}$, and the standard short exact sequence

$$0 \longrightarrow E^{r+1}_{k,0} \longrightarrow E^r_{k,0} \xrightarrow{d_r} \operatorname{Im} d_r \longrightarrow 0 \tag{A.1}$$

The d_r differential lands in $E^r_{k-r,r-1}$, so by the induction hypothesis ($k - r < k$), the image is finitely generated. From (A.1) and the following lemma, we conclude that $E^{r+1}_{k,0}$ is finitely generated if and only if $E^r_{k,0}$ is. By downward induction we get $E^2_{k,0}$ finitely generated as we wanted.

Lemma A.3.2. *Given a short exact sequence of abelian groups*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

with C finitely generated, then A is finitely generated if and only if B is.

Proof. First assume that B is finitely generated. Then also $A \simeq \mathrm{Im}\alpha \subseteq B$ is finitely generated. On the other hand, assume A is finitely generated. Every element $b \in B$ represents a class of $B/\mathrm{Ker}\beta \simeq C$. Elements of each class differ only by a finitely generated element as $\mathrm{Im}\alpha \simeq \mathrm{Ker}\beta$, and each class can be presented in the finitely many generators of C. □

Thus also $H_*(K(\mathbb{Z}_p, n); \mathbb{Z})$ is finitely generated in every degree, and by induction we conclude that $H_*(K(\mathbb{Z}_p, n); \mathbb{Z})$ is finitely generated in every degree for all n. By lemma A.3.1 we conclude that also $H^*(K(\mathbb{Z}_p, n); \mathbb{Z}_p)$ is finitely generated in every degree.

A.4 Note on the cross product

A few places in the construction it has been stated that something is obtained by expanding through the cross product isomorphism. This we shall elaborate a bit on here.

Let A, B, C, D be CW-complexes, such that at least A and C have finitely generated cohomology in every degree. Consider cohomology with coefficients in some field k, such that the cross product gives an isomorphism. For maps $f : A \to C$ and $g : B \to D$ and arbitrary $n \geq 0$, we get a map $(f \times g)^* : H^n(C \times D; k) \to H^n(A \times B; k)$, and by naturality of the cross product, a commuting square

$$\begin{array}{ccc} H^*(C \times D; k) & \xrightarrow{(f\times g)^*} & H^*(A \times B; k) \\ \times \uparrow \simeq & & \simeq \uparrow \times \\ H^*(C; k) \otimes H^*(D; k) & \longrightarrow & H^*(A; k) \otimes H^*(B; k) \end{array}$$

The claim implicitly made during the construction, is that the bottom map of this square is $H^*(f) \otimes H^*(g)$. We show this by verifying that the square actually commutes with this choice. It is enough to check for elements of the form $c \otimes d$ with $c \in H^i(C; k)$ and $d \in H^j(D; k)$, since these generate $H^i(C; k) \otimes H^j(D; k)$, and we then extend linearly to get to $H^*(C; k) \otimes H^*(D; k)$. The cellular cross product for cohomology maps $c \otimes d$ to $c \times d \in H^{i+j}(C \times D; k)$ where $(c \times d)(e^k \times e^l) = c(e^k)d(e^l)$ for all $k, l \geq 0$. Here we use the convention that a cellular cochain in degree n is defined on the whole complex, but only supported on n-cells. Then $H^{i+j}(f \times g)(c \times d) = H^i(f)(c) \times H^j(g)(d) \in H^{i+j}(A \times B; k)$.[3]

Going the other way around the square gives us first

$$(H^*(f) \otimes H^*(g))(c \otimes d) = H^i(f)(c) \otimes H^j(g)(d)$$

which is then mapped to $H^i(f)(c) \times H^j(g)(d)$ by the cross product. This shows the claim.

[3]Although this looks quite innocent, it is a nontrivial fact proved in [Hat02] Lemma 3B.2

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