Mapping Spaces, Centralizers, and p-Local Finite Groups of Lie Type

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PhD Thesis

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Abstract

We study the space of maps from the classifying space of a finite p-group to the Borel construction of a finite group of Lie type G in characteristic p acting on its building. The first main result is a description of the homology with $\mathbb{F}_p$-coefficients, showing that the mapping space, up to p-completion, is a disjoint union indexed over the group homomorphism up to conjugation of classifying spaces of centralizers of p-subgroups in the underlying group G. We complement this description by determining the actual homotopy groups of the mapping space. These results translate to descriptions of the space of maps between a finite p-group and the uncompleted classifying space of the p-local finite group coming from a finite group of Lie type in characteristic p, providing some of the first results in this uncompleted setting.

Resumé

Vi undersøger rummet af afbildninger fra det klassificerende rum af en endelig p-gruppe til Borel konstruktionen af en endelig gruppe G af Lie type i karakteristik p virkende på sin bygning. Det første hovedresultat er en beskrivelse af homologi med $\mathbb{F}_p$-koefficienter, der viser at afbildningsrummet op til p-fuldstændiggørelse er en disjunkt forening indekseret over gruppehomomorfier op til konjungering af klassificierende rum af centralisatorer af p-undergrupper i den underliggende gruppe G. Vi udvider denne beskrivelse ved at bestemme de egentlige homotopigrupper af afbildningsrummet. Disse resultater oversættes til beskrivelser af rummet af afbildninger mellem en endelig p-gruppe og et ikke fuldstændiggjort klassificerende rum for en p-lokal endelig gruppe stammende fra en endelig gruppe af Lie type i karakteristik p, og giver nogle af de første resultater i dette ikke-fuldstændiggjorte tilfælde.

Introduction

For two groups H and G the space of unpointed maps $\mathrm{Map}(BH, BG)$ between their classifying spaces encodes algebraic properties of the groups in a topological setting.

For discrete groups H and G the set of homotopy classes of maps $[BH, BG]$ is in bijection with $\mathrm{Rep}(H, G)$, the set of group homomorphisms modulo conjugation in the group G. A classical result states that there is a complementary description of the homotopy type of the space $\mathrm{Map}(BH, BG)$ in the form for a homotopy equivalence

$$\coprod_{\rho\in\mathrm{Rep}(H,G)} B(C_G(\rho(H))) \xrightarrow{\simeq} \mathrm{Map}(BH, BG) \ ,$$

where the left-hand side consists of the centralizers in the underlying group. For a group homomorphism $\rho\colon H \to G$ the homotopy equivalence from $B(C_G(\rho(H)))$ to $\mathrm{Map}(BH, BG)_{B\rho}$ is induced by the product map $B(C_G(\rho(H))) \cdot H \to G$ by taking the adjoint after going to classifying spaces.

Many generalizations of this result to the larger classes of groups have been proved. Among them, Dwyer–Zabrodsky [16] showed that the map

$$\coprod_{\rho\in\mathrm{Rep}(Q,G)} B(C_G(\rho(Q))) \xrightarrow[\text{mod-}p]{\text{strong}} \mathrm{Map}(BQ, BG) \ ,$$

defined in an analogous way to the classical result is a strong mod-p-equivalence, when p is a prime, Q is a finite p-group and G is a compact Lie group. Here, a strong mod-p-equivalence is a bijection on connected components — an isomorphism on the fundamental group for each component and an isomorphism in homology with coefficients in the finite field $\mathbb{F}_p$ for the lift to universal covers. Several authors have worked on relaxations of the conditions on the groups P and G as well as replacing them with their p-completion, e.g. Notbohm [30] and Broto–Kitchloo [7]. These results are often in the form of mod-p-equivalences, i.e. maps that induce isomorphism in homology with $\mathbb{F}_p$-coefficients.

The p-completion of a space was defined by Bousfield–Kan [6] and is a functor on spaces that embodies the properties detected by homology with $\mathbb{F}_p$-coefficients, in the sense that completion of a map is a homotopy equivalence if and only if it is a mod-p-equivalence. The p-completion comes equipped with a natural transformation from the identity functor to itself and for nice spaces — called p-good — the natural map $X \to X^\wedge_p$ is a mod-p-equivalence. Thus for p-good spaces the p-completion can be considered as a localization with respect to homology with $\mathbb{F}_p$-coefficients. It is worth noting that all spaces with finite fundamental group are p-good but even spaces like the wedge of two spheres S^1 is not p-good.

One of the first indications of what kind of information the p-completion of a finite group G contains comes from a result by Mislin [29], who showed that for a finite p-group Q the map

$$\mathrm{Rep}(Q, G) \xrightarrow{\cong} \left[BQ, BG^\wedge_p\right]$$

is a bijection. This result of Mislin is the essential component to show that the p-local structure of a group, i.e the conjugation action between p-subgroups, is witnessed by the p-completion of its classifying space. In this thesis the focus will be on a different space that similarly detects p-local information.

For a finite group G we have that $\mathcal{S}_p(G)$ is the poset of nontrivial p-subgroups of G ordered by inclusion. The G-conjugation action respects the inclusion making $\mathcal{S}_p(G)$ into a G-poset. One of the most important properties of the poset $\mathcal{S}_p(G)$, from the viewpoint of this thesis, is the idea proposed by Quillen [33] of considering $\mathcal{S}_p(G)$ as a generalization of the Tits building [40]. A building is an abstract simplicial complex with a very rigid structure. For groups acting simplicially on a building, the group itself has a BN-pair structure (also called a Tits system) and vice versa for a group G with a BN-pair there exists a building $\Delta(G, B)$ on which G acts. Thus, for groups with a BN-pair the associated building is a combinational way of encoding the group structure. A class of groups with a BN-pair of particular interest for this thesis are the finite groups of Lie type in characteristic p, since for such groups G the poset $\mathcal{S}_p(G)$ and the building $\Delta(G, B)$ are G-homotopy equivalent.

The geometric realization $|\mathcal{S}_p(G)|$ of the G-poset $\mathcal{S}_p(G)$ is a G-space. The main focus of the thesis is on the Borel construction of $\mathcal{S}_p(G)$, i.e. the space

$$|\mathcal{S}_p(G)|_{hG} = |\mathcal{S}_p(G)| \times_G EG = (|\mathcal{S}_p(G)| \times EG)/G$$

where EG is the total space for the universal bundle $EG \to BG$ for G. Like any G-invariant the map sending $|\mathcal{S}_p(G)|$ to a point will induce a map $|\mathcal{S}_p(G)|_{hG} \to BG$, called the Borel map, which is a fibration with fiber $|\mathcal{S}_p(G)|$. The collection of subgroups $\mathcal{S}_p(G)$ is ample in the sense of [17] when p divides the order of G, meaning that the Borel map is a mod-p-equivalence. In particular it induces a homotopy equivalence on the p-completions $(|\mathcal{S}_p(G)|_{hG})^\wedge_p \to BG^\wedge_p$, and thus by the result of Mislin [29] the space $|\mathcal{S}_p(G)|_{hG}$ also contains p-local information about the group G. Furthermore $|\mathcal{S}_p(G)|_{hG}$ could also encode more global information, which could prove relevant for the applications of the space in modular representation theory, among others the work of Grodal [20] on endotrivial kG-modules.

In this thesis we address the question of understanding the space $|\mathcal{S}_p(G)|_{hG}$ by studying the space of unpointed maps from a finite p-group to the Borel construction $|\mathcal{S}_p(G)|_{hG}$ in the case where G is a finite group of Lie type in characteristic p. The first main is result is in form of a mod-p-equivalence.

Theorem A. *Let G be finite group of Lie type in characteristic p, and let Q be a finite p-group. Then the Borel map $|\mathcal{S}_p(G)|_{hG} \to BG$ induces a mod-p-equivalence*

$$\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG}) \xrightarrow[p]{\backsim} \mathrm{Map}(BQ, BG).$$

The classical description of $\mathrm{Map}(BQ, BG)$ shows that this is a p-good space, and thus an equivalent formulation of the above theorem is that the map after p-completion

$$\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG})^\wedge_p \xrightarrow{\simeq} \mathrm{Map}(BQ, BG)^\wedge_p$$

is a homotopy equivalence. A consequence of this viewpoint is that the spaces $\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG})^\wedge_p$ and $\mathrm{Map}(BQ, (|\mathcal{S}_p(G)|_{hG})^\wedge_p)$ are homotopy equivalent, which will not be true in general when mapping into infinite complexes. In particular, Theorem A does not follow directly from the collection $\mathcal{S}_p(G)$ being ample.

An important corollary of Theorem A that mimics the classical description of homotopy classes of map as well more modern generalizations is:

Corollary B. *Let G be a finite group of Lie type in characteristic p, and let Q be a finite p-group. Then there is a bijection $[BQ, |\mathcal{S}_p(G)|_{hG}] \to \mathrm{Rep}(Q, G)$.*

In the thesis we also describe the homotopy type of the spaces appearing in the left hand side of Theorem A. For $f \in \mathrm{Rep}(Q, G)$ the corresponding component $\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG})$ is denoted $\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG})_f$. Drawing inspiration from the classical description of the mapping space between classifying spaces of groups, these spaces can be considered to be the homotopical analog of centralizers of p-order elements in the space $|\mathcal{S}_p(G)|_{hG}$.

Theorem C. *Let G be a finite group of Lie type in characteristic p. For Q a finite p-group and $f \in \mathrm{Map}(Q, G)$, let $|\mathcal{S}_p(G)|$ be a Q-space with the action induced by f. If the Lie rank of G is 1 we have that*

$$\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG})_1 \simeq B(B)$$
$$\mathrm{Map}(BQ, |\mathcal{S}_p(G)|)_f \simeq B(C_G(f(Q))), \quad f \neq 1$$

where B is a Borel subgroup of G. If the Lie rank of G is 2 we have that

$$\mathrm{Map}(BQ, |\mathcal{S}_p(G)|)_1 \simeq B(P_\alpha *_B P_\beta)$$
$$\mathrm{Map}(BQ, |\mathcal{S}_p(G)|)_f \simeq B(C_G(f(Q))), \quad f \neq 1$$

where B is a Borel subgroup of G and P_α and P_β are the maximal standard proper parabolic subgroups over B.

If G has Lie rank at least three, we have for f the trivial map

$$\pi_i(\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG})_1 \cong \pi_i(|\mathcal{S}_p(G)|), \quad i \geq 2$$
$$\pi_1(\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG})_1 \cong G$$

and for f nontrivial we have for all $i \geq 2$

$$\pi_i(\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG})_f) \cong \mathrm{Tor}_{p'}(\pi_i(|\mathcal{S}_p(G)|)^Q) \oplus \left(\pi_{i+1}(|\mathcal{S}_p(G)|)^Q \otimes (\mathbb{Z}_p/\mathbb{Z})\left[\frac{1}{p}\right]\right).$$

If the rank of G is at least 4 we have that

$$\pi_1(\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG})_f) \cong C_G(f(Q))$$

while for rank 3 we have the following short exact sequence:

$$0 \to \pi_2(|\mathcal{S}_p(G)|)^Q \otimes (\mathbb{Z}_p/\mathbb{Z})\left[\frac{1}{p}\right] \to \pi_1(\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG})_f) \to C_G(f(Q)) \to 0.$$

In the above Theorem we use the notation $\mathrm{Tor}_{p'}$ for the subgroup of the torsion group in an abelian group that is generated by elements of order prime to p.

The results in rank 1 and 2 are derived from the fact that in this case $|\mathcal{S}_p(G)|_{hG}$ is a classifying space of the Borel subgroup B in rank 1 and the given amalgamated product $P_\alpha *_B P_\beta$ in rank 2. In higher rank it is never the case that $|\mathcal{S}_p(G)|_{hG}$ is a classifying space of a discrete group.

As mentioned above, the space $|\mathcal{S}_p(G)|$ for a finite group of Lie type G in characteristic p is G-homotopy equivalent to the associated Tits building, and thus all the given results could equivalently be formulated in terms of the building. Furthermore according to the Solomon–Tits theorem[37], this implies that the geometric realization of $\mathcal{S}_p(G)$ for finite groups of Lie type in characteristic p has the homotopy type of a wedge of spheres, where the dimension of the spheres all are one less than the Lie rank of the underlying group. Hilton proved in [21] how the homotopy groups of arbitrary wedges of spheres can be presented in terms of homotopy groups of spheres and thus the same is true for the fixed points

$\pi_*(|\mathcal{S}_p(G)|)^Q$ from Theorem C. In particular one should not expect a general formula for the p prime part of the torsion groups of this group. Theorem C also concerns groups of the form $\pi_*(|\mathcal{S}_p(G)|)^Q \otimes (\mathbb{Z}_p/\mathbb{Z})[\frac{1}{p}]$. Here it is important that the group $(\mathbb{Z}_p/\mathbb{Z})[\frac{1}{p}]$ is divisible. The isomorphism type of an abelian group A tensored with a divisible group depends — by the classification of divisible groups — only on the free rank of the group or the dimension of the rationalization $A \otimes \mathbb{Q}$ as a vector space over $\mathbb{Q}$. This observation allows us to describe the divisible groups $\pi_*(|\mathcal{S}_p(G)|)^Q \otimes (\mathbb{Z}_p/\mathbb{Z})[\frac{1}{p}]$ from Theorem C in terms of dimensions of Lie algebras and more specifically in terms of the Lie operad.

The main application in the thesis is translating all results into the setting of p-local finite groups. Recall that a p-local finite group is a triple $(S, \mathcal{F}, \mathcal{L})$, where S is a finite p-group, $\mathcal{F}$ is a saturated fusion system on S and $\mathcal{L}$ a centric linking system associated to $\mathcal{F}$. A saturated fusion system on a finite p-group is a category with objects the subgroups of S and morphisms generalized conjugation maps that satisfy axioms of a similar nature to Sylow's theorems for groups. The axiomatic approach for fusion systems was originally presented by Puig in the nineties — he called them Frobenius categories — but the work was first published in [32]. By then the terminology and viewpoint of Broto–Levi–Oliver [9] had become standard in the community. They introduced the centric linking system, which is a lift of the category $\mathcal{F}_S(G)$ and by the work of Chermark [15] it is an invariant of the fusion system. In the thesis the geometric realization $|\mathcal{L}|$ is seen as the classifying space of the associated p-local finite group $(S, \mathcal{F}, \mathcal{L})$.

In [9] Broto–Levi–Oliver proved that for any p-local finite group $(S, \mathcal{F}, \mathcal{L})$ and finite p-group Q there is a bijection

$$[BQ, |\mathcal{L}|^\wedge_p] \cong \mathrm{Hom}(Q, S)/(\mathcal{F}\text{-conjugacy})$$

and each component of the mapping space $\mathrm{Map}(BQ, |\mathcal{L}|^\wedge_p)$ has the homotopy type of the p-completion of the classifying space of another p-local finite group, which can be seen as a generalization of centralizers to p-local finite groups.

A special case of p-local finite groups are those induced by the conjugation action of a finite group G on a Sylow-p-subgroup S. They are denoted $(S, \mathcal{F}_S(G), \mathcal{L}^c_S(G))$, and it was shown by Broto–Levi—Oliver in [8] that the space $|\mathcal{L}^c_S(G)|^\wedge_p$ is homotopy equivalent to $BG^\wedge_p$. In this thesis the focus is on on the case where G is a finite group of Lie type in characteristic p. Here the uncompleted space $|\mathcal{L}^c_S(G)|$ has the homotopy type of $|\mathcal{S}_p(\bar{G})|_{h\bar{G}}$ for the quotient group $\bar{G}$ of G modulo its center $\mathrm{Z}(G)$, thus opening for the possibility of detecting more than just the p-local information about G in the space $|\mathcal{L}^c_S(G)|$. This also allows a formulation of Theorem A, Corollary B and Theorem C for p-local finite groups of Lie type.

Theorem D. *Let G be finite group of Lie type in characteristic p, and let S be a Sylow-p-subgroup of G. Let $(S, \mathcal{F}_S(G), \mathcal{L}^c_S(G))$ be the p-local finite group associated to S, and Q be a finite p-group. The space* $\mathrm{Map}(BQ, |\mathcal{L}^c_S(G)|)$ *is then p-good and there is a homotopy equivalence*

$$\mathrm{Map}(BQ, |\mathcal{L}^c_S(G)|)^\wedge_p \xrightarrow{\simeq} \mathrm{Map}(BQ, BG)^\wedge_p.$$

Corollary E. *Let G be a finite group of Lie type in characteristic p, and let S be a Sylow-p-subgroup of G. Let $(S, \mathcal{F}_S(G), \mathcal{L}^c_S(G))$ be the p-local finite group associated to S, and Q be a finite p-group. Then there is a bijection $[BQ, |\mathcal{L}^c_S(G)|] \to \mathrm{Rep}(Q, G)$.*

Similarly we show results concerning the homotopy type of the components of the mapping space, and provide a complementing view of the results in [9] in an uncompleted setting.

Theorem F. *Let G be finite group of Lie type in characteristic p, and let S be a Sylow-p-subgroup of G. Let $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ be the p-local finite group associated to S. Let Q be a finite p-group and let $\mathrm{Map}(BQ, |\mathcal{L}_S^c(G)|)_f$ be the component corresponding to an $f \in \mathrm{Rep}(Q,G)$. Then the homotopy groups are given as follows.*

If the Lie rank of G is 1 we have that

$$\mathrm{Map}(BQ, |\mathcal{L}_S^c(G)|)_1 \simeq B(B/\,\mathrm{Z}(G))$$

$$\mathrm{Map}(BQ, |\mathcal{L}_S^c(G)|)_f \simeq B(C_G(f(Q))/\,\mathrm{Z}(G)), \quad f \neq 1$$

where B is a Borel subgroup of G. If The Lie rank of G is 2 we have that

$$\mathrm{Map}(BQ, |\mathcal{L}_S^c(G)|)_1 \simeq B((P_\alpha/\,\mathrm{Z}(G)) *_{(B/\,\mathrm{Z}(G))} (P_\beta/\,\mathrm{Z}(G))$$

$$\mathrm{Map}(BQ, |\mathcal{L}_S^c(G)|)_f \simeq B(C_G(f(Q))/\,\mathrm{Z}(G)), \quad f \neq 1$$

where B is a Borel subgroup of G and P_α and P_β are the maximal standard parabolic subgroups over B.

If G has Lie rank at least 3, then the component corresponding to the trivial map $\mathrm{Map}(BP, |\mathcal{L}_S^c(G)|)_1$ has the following homotopy groups:

$$\pi_1(\mathrm{Map}(BQ, |\mathcal{L}_S^c(G)|)_1) \cong G/\,\mathrm{Z}(G)$$

$$\pi_i(\mathrm{Map}(BQ, |\mathcal{L}_S^c(G)|)_1) \cong \pi_i(|\mathcal{S}_p(G)|), \quad i \geq 2.$$

If f is nontrivial we have for all $i \geq 2$ that

$$\pi_i(\mathrm{Map}(BQ, |\mathcal{L}_S^c(G)|)_f) \cong \mathrm{Tor}_{p'}(\pi_i(|\mathcal{S}_p(G)|^Q)) \oplus \left(\pi_{i+1}(|\mathcal{S}_p(G)|^Q \otimes (\mathbb{Z}_p/\mathbb{Z}) \left[\frac{1}{p}\right]\right).$$

Finally, the fundamental group fits into the following short exact sequence:

$$0 \to \pi_2(|\mathcal{S}_p(G)|)^Q \otimes (\mathbb{Z}_p/\mathbb{Z}) \left[\frac{1}{p}\right] \to \pi_1(\mathrm{Map}(BQ, |\mathcal{L}_S^c(G)|)_f) \to C_G(f(Q))/\,\mathrm{Z}(G) \to 0.$$

The results of the thesis could prove useful in modular representation theory. A more direct future direction for the work is seeing whether the given techniques can be used to describe other mapping spaces. In the thesis we only consider the situation of a finite p-group and a finite group of Lie type in characteristic p. In particular, the cross-characteristic case is still unexplored. The proof techniques are based heavily on $\mathcal{S}_p(G)$ being homotopy equivalent to a building, and thus also opens up the possibility that they could be used to describe the space of maps from a finite p-group to some Borel construction on a more general building. A similar idea in the setting of fusion systems is based on the fusion-chamber complex of Onofrei[31] which may be seen as a (B, N)-decomposition of a fusion system with an associated chamber complex. This covers examples such as the fusion system of a finite simple group of Lie type in characteristic p extended by diagonal and field automorphisms. The associated chamber complex will play the role of the Tits building in the case of a finite group of Lie type, and thus might be accessible using the techniques we present, and provide information about mapping spaces in the case of p-local finite groups with a fusion-chamber complex.

References

[1] Peter Abramenko and Kenneth S Brown. Buildings and groups. *Buildings: Theory and Applications*, pages 1–79, 2008.

[2] Michael Aschbacher, Radha Kessar, and Robert Oliver. *Fusion systems in algebra and topology.* Number 391. Cambridge University Press, 2011.

[3] David Balavoine. Deformations of algebras over a quadratic operad. *Contemporary Mathematics*, 202:207–234, 1997.

[4] Donald Anthony Barkauskas. *Centralizers in fundamental groups of graphs of groups*. PhD thesis, UNIVERSITY of CALIFORNIA at BERKELEY, 2003.

[5] Serge Bouc. Homologie de certains ensembles ordonnés. *CR Acad. Sci. Paris Sér. I Math*, 299(2):49–52, 1984.

[6] Aldridge Knight Bousfield and Daniel Marinus Kan. *Homotopy limits, completions and localizations*, volume 304. Springer Science & Business Media, 1972.

[7] Carles Broto and Nitu Kitchloo. Classifying spaces of kač-moody groups. *Mathematische Zeitschrift*, 240(3):621–649, 2002.

[8] Carles Broto, Ran Levi, and Bob Oliver. Homotopy equivalences of p-completed classifying spaces of finite groups. *Inventiones mathematicae*, 151(3):611–664, 2003.

[9] Carles Broto, Ran Levi, and Bob Oliver. The homotopy theory of fusion systems. *Journal of the American Mathematical Society*, 16(4):779–856, 2003.

[10] Kenneth S Brown. Euler characteristics of groups: Thep-fractional part. *Inventiones mathematicae*, 29(1):1–5, 1975.

[11] Kenneth S Brown. High dimensional cohomology of discrete groups. *Proceedings of the National Academy of Sciences*, 73(6):1795–1797, 1976.

[12] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[13] Kenneth S Brown. *Buildings*. Springer, 1989.

[14] Wojciech Chachólski and Jérôme Scherer. *Homotopy theory of diagrams*. Number 736. American Mathematical Soc., 2002.

[15] Andrew Chermak. Fusion systems and localities. *Acta mathematica*, 211(1):47–139, 2013.

[16] W Dwyer and Alexander Zabrodsky. Maps between classifying spaces. *Algebraic Topology Barcelona 1986*, pages 106–119, 1987.

[17] W. G. Dwyer. Homology decompositions for classifying spaces of finite groups. *Topology*, 36(4):783–804, 1997.

[18] Daniel Gorenstein, Richard Lyons, and Ronald Solomon. *The Classification of the Finite Simple Groups. Number 3. Part I. Almost Simple K-groups*, volume 40 of *Math. Surveys and Monog.* Amer. Math. Soc., Providence, RI, 1998.

[19] Jesper Grodal. Higher limits via subgroup complexes. *Annals of mathematics*, pages 405–457, 2002.

[20] Jesper Grodal. Endotrivial modules for finite groups via homotopy theory. *ArXiv e-prints*, August 2016.

[21] Peter John Hilton. On the homotopy groups of the union of spheres. *J. London Math. Soc.*, 30:154–172, 1955.

[22] Irving Kaplansky. *Infinite abelian groups*. Number 2. University of Michigan Press Ann Arbor, 1954.

[23] Chao Ku. A new proof of the solomon-tits theorem. *Proceedings of the American Mathematical Society*, 126(7):1941–1944, 1998.

[24] Jean Lannes. Sur les espaces fonctionnels dont la source est le classifiant d'un p-groupe abélien élémentaire. *Publications mathématiques de l'IHÉS*, 75(1):135–244, 1992.

[25] Assaf Libman and Antonio Viruel. On the homotopy type of the non-completed classifying space of a p-local finite group. In *Forum Mathematicum*, volume 21, pages 723–757, 2009.

[26] Gunter Malle and Donna Testerman. *Linear algebraic groups and finite groups of Lie type*, volume 133. Cambridge University Press, 2011.

[27] John Martino and Stewart Priddy. Stable homotopy classification of $bg_p^\wedge$. *Topology*, 34(3):633–649, 1995.

[28] Haynes Miller. The sullivan conjecture and homotopical representation theory. In *Proceedings of the International Congress of Mathematicians*, volume 1, page 2, 1986.

[29] Guido Mislin. On group homomorphisms inducing mod-p cohomology isomorphisms. *Commentarii Mathematici Helvetici*, 65(1):454–461, 1990.

[30] Dietrich Notbohm. Maps between classifying spaces. *Mathematische Zeitschrift*, 207(1):153–168, 1991.

[31] Silvia Onofrei. Saturated fusion systems with parabolic families. *arXiv:1008.3815*, 2010.

[32] Lluis Puig. Frobenius categories. *Journal of Algebra*, 303(1):309–357, 2006.

[33] Daniel Quillen. Homotopy properties of the poset of nontrivial p-subgroups of a group. *Advances in Mathematics*, 28(2):101 – 128, 1978.

[34] Christophe Reutenauer. Free lie algebras, volume 7 of london mathematical society monographs. new series, 1993.

[35] Hans Samelson. A connection between the whitehead and the pontryagin product. *American Journal of Mathematics*, 75(4):744–752, 1953.

[36] Jean Pierre Serre. *Trees Translated from the French by John Stillwell*. Springer, 1980.

[37] Louis Solomon. The steinberg character of a finite group with bn-pair. In *Theory of Finite Groups (Symposium, Harvard Univ., Cambridge, Mass., 1968)*, pages 213–221, 1969.

[38] Richard P Stanley. On the number of reduced decompositions of elements of coxeter groups. *European Journal of Combinatorics*, 5(4):359–372, 1984.

[39] Dennis Parnell Sullivan and Andrew Ranicki. *Geometric Topology: Localization, Periodicity and Galois Symmetry: the 1970 MIT Notes*, volume 8. Springer, 2005.

[40] Jacques Tits. On buildings and their applications. In *Proceedings of the International Congress of Mathematicians (Vancouver, BC, 1974)*, volume 1, pages 209–220, 1974.

MAPPING SPACES, CENTRALIZERS, AND p-LOCAL FINITE GROUP OF LIE TYPE

ISABELLE LAUDE

Abstract. We study the space of maps from the classifying space of a finite p-group to the Borel construction of a finite group of Lie type G in characteristic p acting on its building. The first main result is a description of the homology with $\mathbb{F}_p$-coefficients, showing that the mapping space, up to p-completion, is a disjoint union indexed over the group homomorphism up to conjugation of classifying spaces of centralizers of p-subgroups in the underlying group G. We complement this description by determining the actual homotopy groups of the mapping space. These results translate to descriptions of the space of maps between a finite p-group and the uncompleted classifying space of the p-local finite group coming from a finite group of Lie type in characteristic p, providing some of the first results in this uncompleted setting.

1. Introduction

For discrete groups H and G the space of unpointed maps between their classifying spaces $\mathrm{Map}(BH, BG)$ has the homotopy type of

$$\coprod_{\rho \in \mathrm{Rep}(H,G)} B(C_G(\rho(H))),$$

where $\mathrm{Rep}(H, G)$ denotes the set of group homomorphisms modulo conjugation in G. Generalizations of this result where H or G is allowed to be a compact Lie group or the classifying space BG is replaced by its p-completion have attracted significant attention. Notably in [16] Dwyer and Zabrodsky gave a mod-p description of the space $\mathrm{Map}(BQ, BG)$ for Q a finite p-group and G a compact Lie group in terms of centralizers in G, relaxations on H and G have been provided by several authors, e.g. Notbohm [30] and Broto–Kitchloo [7].

With a different approach — generalizing work of Mislin [29] — Broto, Levi, and Oliver showed in [9] that the space of maps $\mathrm{Map}(BQ, |\mathcal{L}|^\wedge_p)$ between the classifying space of a finite p-group Q and the p-completed classifying space $|\mathcal{L}|^\wedge_p$ of p-local finite groups $(S, \mathcal{F}, \mathcal{L})$ has a mod-p description in terms of centralizer fusion systems.

For a prime p we let $\mathcal{S}_p(G)$ denote the G-poset of finite non-trivial p-subgroups of G ordered by inclusion, where G acts by conjugation. This was originally introduced by Brown in [10, 11] to compute the Euler characteristic and cohomology of groups, and has since found many applications. The foundation is the work of Quillen in [33] where he introduced the idea of considering the poset $\mathcal{S}_p(G)$ as a generalization of a building. Notably in the case of a finite group of Lie type in characteristic p the poset $\mathcal{S}_p(G)$ is G-homotopy equivalent to the Tits building.

In this paper we look at the Borel construction $|\mathcal{S}_p(G)|_{hG}$. When p divides the order of G the collection $\mathcal{S}_p(G)$ of subgroups is ample (see [17]) meaning that after p-completion the Borel map from $|\mathcal{S}_p(G)|_{hG}$ to BG is a homotopy equivalence. The

p-local structure of the group G is detected by the p-completion of its classifying space according to Mislin's result [29], and similarly the space $|\mathcal{S}_p(G)|_{hG}$ encodes much of the p-local information about G as well as possibly some global information. The space $|\mathcal{S}_p(G)|_{hG}$ also shows up in modular representation theory, as in the work of Grodal [20] on endotrivial kG-modules.

In this paper we study the unpointed space of maps $\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG})$, for Q a finite p-group and G a finite group of Lie type in characteristic p. We will refer to maps inducing isomorphisms on homology with $\mathbb{F}_p$-coefficients as "mod-p-equivalences". Our first main result provides a mod-p description of the mapping space:

Theorem 1.1. *Let G be finite group of Lie type in characteristic p, and let Q be a finite p-group. Then the Borel map $|\mathcal{S}_p(G)|_{hG} \to BG$ induces a mod-p-equivalence*

$$\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG}) \xrightarrow[p]{\backsim} \mathrm{Map}(BQ, BG).$$

Both of the spaces in Theorem 1.1 are p-good and thus an equivalent formulation of the theorem is that after p-completion the map

$$\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG})^\wedge_p \xrightarrow{\simeq} \mathrm{Map}(BQ, BG)^\wedge_p$$

is a homotopy equivalence. Here we use the notion of Bousfield–Kan p-completion as defined in [6], but p-good spaces are exactly the spaces where the p-completion is localization with respect to $\mathbb{F}_p$-homology, so the statement could equivalently be given in terms of localizations.

Another consequence of the formulation of Theorem 1.1 in terms of p-completions is that the space $\mathrm{Map}(BQ, (|\mathcal{S}_p(G)|_{hG})^\wedge_p)$ is homotopy equivalent to the space where we move the completion outside the mapping space. This is not true in general when mapping into infinite complexes. This implies that Theorem 1.1 is a nontrivial extension of the result where we p-complete both targets of the mapping spaces.

An important corollary in itself is

Corollary 1.2. Let G be a finite group of Lie type in characteristic p, and let Q be a finite p-group. Then there is a bijection $[BQ, |\mathcal{S}_p(G)|_{hG}] \to \mathrm{Rep}(Q, G)$.

We can also determine the homotopy type of the spaces appearing in the left hand side of Theorem 1.1. For $f \in \mathrm{Rep}(Q, G)$ we determine the homotopy type of the corresponding component $\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG})$, denoted $\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG})_f$. Note that these spaces can be considered the homotopical analog of centralizers of p-order elements in the space $|\mathcal{S}_p(G)|_{hG}$.

Theorem 1.3. *Let G be a finite group of Lie type in characteristic p. For Q a finite p-group and $f \in \mathrm{Map}(Q, G)$, let $|\mathcal{S}_p(G)|$ be a Q-space with the action induced by f. If the Lie rank of G is 1 we have that*

$$\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG})_1 \simeq B(B)$$
$$\mathrm{Map}(BQ, |\mathcal{S}_p(G)|)_f \simeq B(C_G(f(Q))), \quad f \neq 1$$

where B is a Borel subgroup of G. If the Lie rank of G is 2 we have that

$$\mathrm{Map}(BQ, |\mathcal{S}_p(G)|)_1 \simeq B(P_\alpha *_B P_\beta)$$
$$\mathrm{Map}(BQ, |\mathcal{S}_p(G)|)_f \simeq B(C_G(f(Q))), \quad f \neq 1$$

where B is a Borel subgroup of G and P_α and P_β are the maximal standard proper parabolic subgroups over B.

If G has Lie rank at least three, we have for f the trivial map

$$\pi_i(\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG})_1 \cong \pi_i(|\mathcal{S}_p(G)|), \quad i \geq 2$$

$$\pi_1(\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG})_1 \cong G$$

and for f nontrivial we have for all $i \geq 2$

$$\pi_i(\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG})_f) \cong \mathrm{Tor}_{p'}(\pi_i(|\mathcal{S}_p(G)|)^Q) \oplus \left(\pi_{i+1}(|\mathcal{S}_p(G)|)^Q \otimes (\mathbb{Z}_p/\mathbb{Z}) \left[\frac{1}{p}\right]\right).$$

If the rank of G is at least 4 we have that

$$\pi_1(\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG})_f) \cong C_G(f(Q))$$

while for rank 3 we have the following short exact sequence:

$$0 \to \pi_2(|\mathcal{S}_p(G)|)^Q \otimes (\mathbb{Z}_p/\mathbb{Z}) \left[\frac{1}{p}\right] \to \pi_1(\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG})_f) \to C_G(f(Q)) \to 0.$$

In Theorem 1.3 we use the notation $\mathrm{Tor}_{p'}$ for subgroup the of the torsion group in an abelian group that is generated by elements of order prime to p.

A consequence of the Solomon–Tits theorems for spherical buildings [37] is that $|\mathcal{S}_p(G)|$ has the homotopy type of a wedge of spheres of dimension one less than the Lie rank of G. Hilton determined in [21] the homotopy groups of general wedges of spheres in terms of homotopy groups of spheres. Theorem 1.3 shows in the case where G has Lie rank at least three that the higher homotopy groups of $\mathrm{Map}(BQ, |\mathcal{S}_p(G)|_{hG})_f$ are given by Q-fixed points of the homotopy groups of $|\mathcal{S}_p(G)|$, and split as a finite group of order prime to p and a divisible group. Thus without a classification of homotopy groups of spheres no closed formula for the p-prime part of the Q-fixed points of $\pi_*(|\mathcal{S}_p(G)|)$ should be expected. In this paper we provide a description of the divisible part in terms of fixed points on a free Lie algebra.

A consequence of Theorem 1.3 is that the Borel map $|\mathcal{S}_p(G)|_{hG} \to BG$ induces homotopy equivalences on components corresponding to non-trivial maps in the case of a finite group of Lie type of Lie rank either 1 or 2. It follows from Hilton's work [21] that this will never be the case in higher rank.

Recall that a p-local finite group is a triple $(S, \mathcal{F}, \mathcal{L})$, where S is a finite group and $\mathcal{F}$ and $\mathcal{L}$ are categories encoding a generalized conjugation maps on subgroups of S. The category $\mathcal{F}$ is a saturated fusion system over S and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. Saturated fusion systems, called Frobenius categories, were originally defined by Puig in the nineties but his work was first published in [32]. An equivalent definition together with the definition of centric linking system is given by Broto–Levi–Oliver in [9].

In this article we will only work with a fusion system $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ coming from a Sylow-p-subgroup S of a finite group G and refer to chapter 15 for definitions. For p-local finite groups of this form, it was proved by Broto, Levi and Oliver in [8] that the realization $|\mathcal{L}_S^c(G)|$ and BG are homotopy equivalent after p-completion, and thus $|\mathcal{L}_S^c(G)|$ does contain the p-local information from G. By the work of Chermak [15] the non-completed space $|\mathcal{L}_S^c(G)|$ is an invariant of the fusion system $\mathcal{F}_S(G)$ and in this paper we prove for a finite group of Lie type G in characteristic p that the space $|\mathcal{L}_S^c(G)|$ is homotopy equivalent to $|\mathcal{S}_p(\bar{G})|_{h\bar{G}}$ for the quotient group $\bar{G}$ of G modulo its center $\mathrm{Z}(G)$. This allows for the possibility of the space $|\mathcal{L}_S^c(G)|$ containing global information about the ambient group G, and thus makes it relevant to compare the space of maps into it to the maps into the p-completion of the underlying group.

The following statements are a translation of Theorems 1.1 and 1.3, as well as Corollary 1.2 into the language of p-local finite groups.

Theorem 1.4. *Let G be finite group of Lie type in characteristic p, and let S be a Sylow-p-subgroup of G. Let $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ be the p-local finite group associated to S, and Q be a finite p-group. The space $\mathrm{Map}(BQ, |\mathcal{L}_S^c(G)|)$ is then p-good and there is a homotopy equivalence*

$$\mathrm{Map}(BQ, |\mathcal{L}_S^c(G)|)^\wedge_p \xrightarrow{\simeq} \mathrm{Map}(BQ, BG)^\wedge_p.$$

Corollary 1.5. Let G be a finite group of Lie type in characteristic p, and let S be a Sylow-p-subgroup of G. Let $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ be the p-local finite group associated to S, and Q be a finite p-group. Then there is a bijection $[BQ, |\mathcal{L}_S^c(G)|] \to \mathrm{Rep}(Q, G)$.

Similarly we show results concerning the homotopy type of the components of the mapping space, and provide an uncompleted version of the results in [9].

Theorem 1.6. *Let G be finite group of Lie type in characteristic p, and let S be a Sylow-p-subgroup of G. Let $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ be the p-local finite group associated to S. Let Q be a finite p-group. Let $\mathrm{Map}(BQ, |\mathcal{L}_S^c(G)|)_f$ be the component corresponding to an $f \in \mathrm{Rep}(Q, G)$. Then the homotopy groups are given as follows.*

If the Lie rank of G is 1 we have that

$$\mathrm{Map}(BQ, |\mathcal{L}_S^c(G)|)_1 \simeq B(B/\mathrm{Z}(G))$$

$$\mathrm{Map}(BQ, |\mathcal{L}_S^c(G)|)_f \simeq B(C_G(f(Q))/\mathrm{Z}(G)), \quad f \neq 1$$

where B is a Borel subgroup of G. If The Lie rank of G is 2 we have that

$$\mathrm{Map}(BQ, |\mathcal{L}_S^c(G)|)_1 \simeq B((P_\alpha/\mathrm{Z}(G)) *_{(B/\mathrm{Z}(G))} (P_\beta/\mathrm{Z}(G))$$

$$\mathrm{Map}(BQ, |\mathcal{L}_S^c(G)|)_f \simeq B(C_G(f(Q))/\mathrm{Z}(G)), \quad f \neq 1$$

where B is a Borel subgroup of G and P_α and P_β are the maximal standard parabolic subgroups over B.

If G has Lie rank at least 3, then the component corresponding to the trivial map $\mathrm{Map}(BP, |\mathcal{L}_S^c(G)|)_1$ has the following homotopy groups:

$$\pi_1(\mathrm{Map}(BQ, |\mathcal{L}_S^c(G)|)_1) \cong G/\mathrm{Z}(G)$$

$$\pi_i(\mathrm{Map}(BQ, |\mathcal{L}_S^c(G)|)_1) \cong \pi_i(|\mathcal{S}_p(G)|), \quad i \geq 2.$$

If f is nontrivial we have for all $i \geq 2$ that

$$\pi_i(\mathrm{Map}(BQ, |\mathcal{L}_S^c(G)|)_f) \cong \mathrm{Tor}_{p'}(\pi_i(|\mathcal{S}_p(G)|)^Q)) \oplus \left(\pi_{i+1}(|\mathcal{S}_p(G)|)^Q \otimes (\mathbb{Z}_p/\mathbb{Z}) \left[\frac{1}{p}\right]\right).$$

Finally, the fundamental group fits into the following short exact sequence:

$$0 \to \pi_2(|\mathcal{S}_p(G)|)^Q \otimes (\mathbb{Z}_p/\mathbb{Z}) \left[\frac{1}{p}\right] \to \pi_1(\mathrm{Map}(BQ, |\mathcal{L}_S^c(G)|)_f) \to C_G(f(Q))/\mathrm{Z}(G) \to 0.$$

Now some words on the organization of the paper. After an overview of basic definitions and properties of the essential objects in the paper, we determine in Chapter 6 some homotopical properties of $|\mathcal{S}_p(G)|$ as well as the Borel construction $|\mathcal{S}_p(G)|_{hG}$ for the case of a finite group of Lie type of characteristic p. Most importantly we show that when the Lie rank is 1 (respectively 2) the space $|\mathcal{S}_p(G)|_{hG}$ is a classifying space of a finite group (respectively an amalgamated product of finite groups). As in the classical description of mapping spaces between classifying spaces of discrete groups, Theorem 1.1 and 1.3 reduces in this case to a question of comparing centralizers.

For a general amalgamated product $*_H G_i$ determining the centralizer of a subgroup of H is difficult. This question is discussed by Barkauskas in [4]. In this paper we restrict ourselves to the situation of $*_H G_i$ where the G_i's are subgroups of a finite group G with common intersection H. In chapter 7 we give conditions sufficient for the canonical map $*_H G_i \to G$ to induce isomorphisms on centralizers for some nontrivial subgroups of H. These conditions are in chapter 8 shown to be satisfied in the case of the standard maximal proper parabolic subgroups in a finite group of Lie type in characteristic p and Lie rank 2, which might be an interesting result in its own right.

Theorem 1.7. *For a finite group of Lie type G in characteristic p of rank 2, let S be a Sylow-p-subgroup of G, B a Borel subgroup containing S and P_α and P_β be the two standard maximal proper parabolic subgroups over B. Then the canonical map $P_\alpha *_B P_\beta \to G$ induces isomorphism*

$$C_{P_\alpha *_B P_\beta}(Q) \cong C_G(Q)$$

for any nontrivial subgroup Q of S.

From the centralizer comparison we deduce Theorem 1.1 and 1.3 in rank 1 and 2 in chapter 9. For higher rank the space $|\mathcal{S}_p(G)|_{hG}$ is never a classifying space of a discrete group, and so a different approach is needed in that case. In chapter 10 we look at general spaces of map into a Borel construction and show Theorem 1.1 in the higher rank case, using a spectral sequence argument depending heavily on the generalized Sullivan conjecture. It is worth noticing that the argument depends on the space $|\mathcal{S}_p(G)|$ being simply connected, which is not the case for G a finite group of Lie type of Lie rank 1 or 2. The proof of Theorem 1.3 in the high rank case is based on a Sullivan square argument. In chapters 11 and 12 we prove the necessary results to give the proof of Theorem 1.3 for G a finite group of Lie type of rank at least 3, which we present in chapter 13. Chapter 14 is devoted to the description in terms of Lie algebras of the divisible part of the higher homotopy groups in Theorem 1.3 for groups G of Lie rank at least 3. We end the paper by looking at applications of Theorem 1.1 and 1.3 for p-local finite groups of Lie type.

2. General definitions

For a group G, subgroup $H \le G$ and $g \in G$, we will use the following notation for the conjugate subgroup ${}^gH = \{ghg^{-1} \mid h \in H\}$ and write $c_g \colon H \to {}^gH$ for the associated conjugation map.

Definition 2.1.

- For subgroups P, Q of G the transporter set is

$$T_G(P,Q) = \{g \in G \mid {}^gP \le Q\}.$$

- Let G be a finite group. Then the transporter category $\mathcal{T}(G)$ of G is a category with objects all subgroups of G and morphism sets $T_G(P,Q)$ for $P,Q \leq G$
- For a subset $\mathcal{H}$ of subgroups in G, we let the transporter category $\mathcal{T}_{\mathcal{H}}(G)$ on $\mathcal{H}$ be the full subcategory of $\mathcal{T}(G)$ with objects in $\mathcal{H}$.
- For discrete groups H and G, the set $\mathrm{Rep}(H,G)$ is defined as

$$\mathrm{Hom}(H,G)/\{c_g \mid g \in G\}.$$

For a finitely generated abelian group A, we let $\mathrm{Tor}(A)$ denote the torsion subgroup, and for a prime p we let $\mathrm{Tor}_p(A)$ be the subgroup of $\mathrm{Tor}(A)$ generated by elements of order p. Similarly we let $\mathrm{Tor}_{p'}(A)$ be the subgroup of $\mathrm{Tor}(A)$ generated by elements of order prime to p.

3. Finite group of Lie type

In the mathematical community there are several slight variations on what is considered a finite group of Lie type. In this paper we will use the definition of a finite group of Lie type from [18], which is slightly different to the one in [26]. Thus, according to [18, Definition 2.2.1] a finite group of Lie type in characteristic p is the subgroup generated by p-order elements of the fixed points of a semisimple $\overline{\mathbb{F}}_p$-algebraic group under a Steinberg endomorphism.

We will now list some properties for finite groups of Lie type and also fix the notation for the rest of the paper. For explicit construction and proof of the statements, we refer to [18].

A central property of finite groups of Lie type in characteristic p is that they have a split BN-pair of characteristic p:

Definition 3.1. A split BN-pair of characteristic p on a group G is a pair of subgroups B and N, and a finite set $\mathcal{S}_W$ such that

- The groups B and N generate G.
- If we set $H = B \cap N$, then H is normal in N.
- The set $\mathcal{S}_W$ is a set of involutions on the group $W = N/H$ that generates W.
- For every $s \in \mathcal{S}_W$, and $w \in W$, we have that

$$sBw \subseteq BswB \cup BwB, \quad sBs \neq B.$$

- For $S = O_p(B)$ there is a decomposition $B = HS$.
- $\cap_{n \in N}{}^nB = H$.

The group B is called the Borel subgroup and W the Weyl group associated to the BN-pair. The Lie rank $\mathrm{Rk_L}(G)$ of a finite group of Lie type G is the cardinality of the set $\mathcal{S}_W$ of simple involutions generating the Weyl group. For a finite group of Lie type G in characteristic p, we also have that $S \in \mathrm{Syl_p}(G)$ and H is a group of order prime to p.

In fact the Weyl group W can be seen as a Weyl group of a root system Φ. In particular there is a basis of simple roots Δ, also known as the fundamental system, such that every vector of Φ is a linear combination of vectors from Δ with the coefficients either all nonnegative or all nonpositive. We will call the roots respectively positive and negative depending on the sign of their coefficients, and use the notation Φ^+ and Φ^- for the sets of positive roots and negative roots. The reflections corresponding to Δ will be identified with the set $\mathcal{S}_W$. For any $\gamma \in \Delta$ we let s_γ be the corresponding reflection in $\mathcal{S}_W$. Furthermore for each $\gamma \in \Phi$ there is a subgroup U_γ of G called the root subgroup of γ. Note that

$$S = \prod_{\gamma \in \Phi^+} U_\gamma.$$

A word of warning concerning the definition of root systems. Following the definitions from [18] we do not assume that a root system is crystallographic. In fact there are finite groups of Lie type with a non-crystallographic root system, e.g. the Ree groups of type ${}^2F_4(2^{2n+1})$.

A subgroup of G is called parabolic, if it contains a subgroup conjugate to B. For any $\gamma \in \Delta$ we will pick $t_\gamma \in N$, such that the image of t_γ in W is s_γ. For any subset $J \subset \Delta$ we let $N_J = \langle t_\gamma \mid \gamma \in J \rangle$ and the standard parabolic subgroup P_J be given as $P_J = BN_JB$. This is in particular a parabolic subgroup and every parabolic subgroup is conjugate to exactly one standard parabolic subgroup. Note that $P_\emptyset = B$ and $P_\Delta = G$. A standard proper parabolic subgroup P_J is maximal if $|\Delta| = |J| + 1$.

For the rest of the paper, we will always assume that a finite group of Lie type in characteristic p comes with a fixed split BN-pair of characteristic p, associated root system Φ and root subgroups with the properties described above.

4. Amalgamated product of groups

To make the paper more accessible, we will recall some definitions concerning amalgamated products of groups. The definitions coincide with the description in [4].

Let $(G_i)_{i \in I}$ be a family of groups. Assume that for a group H and $i \in I$ there exists an injective group homomorphism $f_i \colon H \to G_i$. Then the amalgamated product $*_H G_i$ is the direct limit of the G_i's over the f_i's. If we identify the group H and the G_i's with their image in $*_H G_i$, then every element in $*_H G_i$ is a word of the form

$$g = g_n \cdots g_1$$

where for all $1 \leq k \leq n$ we have that $g_k \in G_{i_k}$ for some $i_k \in I$. If $g \notin H$ we may assume that $g_k \in G_{i_k} \setminus H$ and $i_k \neq i_{k+1}$ for all $1 \leq k < n$. This is called the reduced form for g, and n is its length. For an element in H the reduced form is just the one letter word and the length defined to be zero. We call a reduced word $g_n \cdots g_1$ cyclically reduced if the length $n \geq 2$ and $i_n \neq i_1$.

The reduced form of an element $x \notin H$ is not unique. To get uniqueness we pick for each $i \in I$ a transversal T_i for the right cosets H in G_i. Then the normal form of a $g \in *_H G_i$ is a decomposition

$$g = h\tilde{g}_n \cdots \tilde{g}_1$$

where $h \in H$, and for $1 \leq k \leq n$ we have that $\tilde{g}_k \in T_{i_k}$ for some $i_k \in I$, where $i_k \neq i_{k+1}$ for all $1 \leq k < n$. Elements in B have normal form with $n = 0$. Every element in $*_H G_i$ has a unique normal form and n is called the length of g. In particular elements in B have length zero. A normal form is called cyclically reduced if the length $n \geq 2$ and $i_n \neq i_1$.

The process of transforming a reduced form of an element to its normal form does not change the length, so the length of an element in $*_H G_i$ may be determined from the reduced form. Furthermore if $g = g_n \cdots g_1$ is a reduced word, the normal form $g = h\tilde{g}_n \cdots \tilde{g}_1$ will also satisfy that $\tilde{g}_k$ and g_k both belong to the same group G_{i_k} for all $1 \leq k \leq n$. Thus the definition of being cyclically reduced for a reduced word and its normal form coincide.

5. Posets of subgroups and the transporter category

For a collection $\mathcal{C}$ of subgroups of a group G, which is closed under conjugation, we will consider it as a G-poset ordered by inclusion and with the conjugation action. Recall that the Borel construction $|\mathcal{C}|_{hG}$ is the G-coinvariant $|\mathcal{C}| \times_G EG$. The map sending $|\mathcal{C}|$ to a point, induces a map $|\mathcal{C}|_{hG} \to BG$. This map is the Borel fibration and the fiber is $|\mathcal{C}|$.

It is known that the homotopy type of Borel construction on G-poset of subgroups can be determined by a transporter category. This observation will be central in proving our main results and therefore we will include a proof for the sake of completeness. First the following definition:

Definition 5.1. For a finite group G we let $\mathcal{E}(G)$ be the category with object set G and exactly one morphism between each pair of objects. We will use the convention of denoting the morphism from g to h by (g, h), where $g, h \in G$. We equip $\mathcal{E}(G)$ with the G action induced by $g(h) = hg^{-1}$. Similarly we let $\mathcal{B}(G)$ be the category with one object $*_G$ and morphism set G.

We have that the geometric realization of $\mathcal{E}(G)$ is a model for EG, and the realization of $\mathcal{B}(G)$ is a model for BG. We will use this particular model in the rest of the paper. Furthermore we will use the notation $|g|$ and $|g, h|$ respectively to denote the image of an object g of $\mathcal{E}(G)$ and a morphism (h, g) in $\mathcal{E}(G)$ under the realization.

Lemma 5.2. Let G be a finite group and $S \in \mathrm{Syl}_\mathrm{p}(G)$. Let $\mathcal{C}$ be a collection of p-subgroups in G which is closed under conjugation in G. Then $|\mathcal{T}_{\mathcal{C}\cap S}(G)| \simeq |\mathcal{C}|_{hG}$ and the Borel map $|\mathcal{C}|_{hG} \to BG$ corresponds to the map realization of the simplicial map $\pi \colon \mathcal{T}_{\mathcal{C}\cap S}(G) \to \mathcal{B}(G)$ given by inclusion on the morphism sets.

Proof. By Sylows Theorems the poset $\mathcal{C}$ is the poset of all subgroups of G, which are conjugate to the object of $\mathcal{T}_{\mathcal{C}\cap S}(G)$. Thus it is well-known that $|\mathcal{T}_{\mathcal{C}\cap S}(G)| \simeq |\mathcal{C}|_{hG}$, but to prove the last part of the statement we need to look into the details of the homotopy equivalence. In this case we are using the description of the map from the proof of Lemma 1.2 in [8].

Let $\mathcal{C} \times_G \mathcal{E}(G)$ be the product category modulo the G-action induced by the free diagonal action of G on objects. Let $\mathcal{T}_\mathcal{C}(G)$ be the transporter category in G with object set the same as $\mathcal{C}$. Consider the map $\sigma \colon \mathcal{C} \times_G \mathcal{E}(G) \to \mathcal{T}_\mathcal{C}(G)$ given by $(P, g) \mapsto {}^gP$ on objects and $(P \hookrightarrow Q, (g, h)) \mapsto (c_{hg^{-1}} \colon {}^gP \to {}^hQ)$ on morphisms. We have that σ is a bijection on both objects and morphisms, and in particular a equivalence of categories. Furthermore we let $\iota \colon \mathcal{T}_{\mathcal{C}\cap S}(G) \to \mathcal{T}_\mathcal{C}(G)$ be the inclusion, which is also an equivalence of categories, where the essentially surjective condition is a consequence of Sylow's theorems.

We let $\pi_{\mathcal{C}\times_G\mathcal{E}(G)} \colon \mathcal{C} \times_G \mathcal{E}(G) \to \mathcal{B}(G)$ be the map given by $(P, g) \mapsto *_G$ and $(P \hookrightarrow Q, (g, h)) \mapsto hg^{-1}$. Note that this is a well-defined map of categories. In a similar way we let $\pi_{\mathcal{T}_\mathcal{C}(G)} \colon \mathcal{T}_\mathcal{C}(G) \to \mathcal{B}(G)$ and $\pi_{\mathcal{T}_{\mathcal{C}\cap S}(G)} \colon \mathcal{T}_{\mathcal{C}\cap S}(G) \to \mathcal{B}(G)$ be the maps induced by the natural inclusion of the morphism sets into G. Then the following diagram commutes:

$$\begin{array}{ccccc}
\mathcal{C} \times_G \mathcal{E}(G) & \xrightarrow{\sigma} & \mathcal{T}_{\mathcal{C}}(G) & \xleftarrow{\iota} & \mathcal{T}_{\mathcal{C}\cap S}(G) \\
& \searrow^{\pi_{\mathcal{C}\times_G\mathcal{E}(G)}} & \downarrow{\scriptstyle \pi_{\mathcal{T}_{\mathcal{C}}(G)}} & \swarrow_{\pi_{\mathcal{T}_{\mathcal{C}\cap S}(G)}} & \\
& & \mathcal{B}(G) & &
\end{array}$$

By applying geometric realization we obtain the right hand side of the following diagram:

$$\begin{array}{ccccccc}
|\mathcal{C}|_{hG} & \xleftarrow[\cong]{\tau} & |\mathcal{C} \times_G \mathcal{E}(G)| & \xrightarrow[\cong]{|\sigma|} & |\mathcal{T}_{\mathcal{C}}(G)| & \xleftarrow[\simeq]{|\iota|} & |\mathcal{T}_{\mathcal{C}\cap S}(G)| \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle |\pi_{\mathcal{C}\times_G\mathcal{E}(G)}|} & & \downarrow{\scriptstyle |\pi_{\mathcal{T}_{\mathcal{C}}(G)}|} & & \downarrow{\scriptstyle |\pi_{\mathcal{T}_{\mathcal{C}\cap S}(G)}|} \\
B(G) & = & B(G) & = & B(G) & = & B(G)
\end{array}$$

In this case geometric realization of $\mathcal{C} \times_G \mathcal{E}(G)$ commutes with both the product of categories as well as the quotient of the free G-action. Here, τ is the homeomorphism that embodies this. Let ψ be the map induced by $|\mathcal{C}| \mapsto *$. Then the left-hand square commutes as well, and the statement follows. $\square$

6. Transporter categories and finite groups of Lie type

Recall that for a finite group G the G-poset of nontrivial p-subgroups with conjugation action is denoted $\mathcal{S}_p(G)$. For a finite group of Lie type G in characteristic p we will show that the poset $\mathcal{S}_p(G)$ is G-homotopy equivalent to a subposet $\mathcal{D}_p(G)$. This allows us to calculate the homotopy type of $|\mathcal{S}_p(G)|_{hG}$ using a smaller transporter category as well as allowing us to identify $\mathcal{S}_p(G)$ with the Tits building associated to G.

We start out by recalling some definitions.

Definition 6.1. A p-subgroup P of a finite group G is called p-centric if $\mathrm{Z}(P) \in \mathrm{Syl_p}(C_G(P))$ or equivalently $C_G(P) \cong \mathrm{Z}(P) \times C'_G(P)$ for a (unique) subgroup $C'_G(P)$ of order prime to p.

Definition 6.2. We let $\mathcal{D}_p(G)$ denote the poset of principal p-radical of subgroups of G. That is, $P \in \mathcal{D}_p(G)$ if and only if P is p-centric p-subgroup of G and $N_G(P)/PC'_G(P)$ is p-reduced.

Recall that a group is p-reduced if it contains no nontrivial normal p-subgroup. For a Sylow-p-subgroup of G, we use the notation $\mathcal{T}^{cr}_S(G)$ for the transporter system $\mathcal{T}_{\mathcal{D}_p(G)\cap S}(G)$.

Definition 6.3. We let $\mathcal{B}_p(G)$ denote the poset of non-trivial p-subgroups in G which are p-radical in G, i.e. the quotient $N_G(P)/P$ is p-reduced.

We have that $\mathcal{D}_p(G)$ is a subposet of $\mathcal{B}_p(G)$ whereas the other inclusion does not hold in general, not even when restricting to the subposet of p-centric subgroups of $\mathcal{B}_p(G)$. A class for finite groups where the posets agree is the finite groups of Lie type in characteristic p, where the following lemma gives a useful description of it.

Lemma 6.4. Let G be a group of finite Lie type in characteristic p. Then as a G-poset

$$\mathcal{D}_p(G) = \mathcal{B}_p(G) = \{O_p(K) \mid K \text{ proper parabolic subgroups of } G\}.$$

Furthermore the map $N_G(-)$ from $\mathcal{D}_p(G)$ to the opposite poset of proper parabolic subgroups in G is a map of G-posets as well.

The rank of the posets is $\mathrm{Rk_L}(G) - 1$.

Proof. First we will prove the claim:

$$\mathcal{B}_p(G) = \{O_p(K) \mid K \text{ proper parabolic in } G\}.$$

A consequence of Borel-Tits theorem for finite groups of Lie type [18, Theorem 3.1.3] is that the set of p-radical subgroups of G is $\{O_p(K) \mid K \text{ parabolic in } G\}$. Note that $1 = O_p(G)$, and hence the non-trivial p-radical p-subgroups correspond to the proper parabolic subgroups.

To show the first equality, we show that if $P = O_p(K)$, where K is a proper parabolic subgroup in G, then P is p-centric. We have by [18, Theorem 2.6.5.] that $C_G(P) = Z(P)Z(G)$. Furthermore by [18, Proposition 2.5.9] we have that $\mathrm{Z}(G)$ is a subgroup of H as defined in [18, Theorem 2.3.4]. In particular it is a group of order prime to p, and hence we have that P is centric and $O_{p'}(C_G(P)) = Z(G)$.

To prove the first equality we have to show that for a p-group P we have that $O_p(N_G(P)/P) = 1$ implies $O_p(N_G(P)/PZ(G)) = 1$. Assume for the purpose of contradiction that $O_p(N_G(P)/PZ(G)) \neq 1$. Then there exists a normal p-subgroup of $N_G(P)/PZ(G)$. This subgroup has the form $K/PZ(G)$ where K is a normal subgroup of $N_G(P)$ containing $PZ(G)$, satisfying that there is some $n \in \mathbb{N}$ such that

$$p^n = |K/PZ(G)| = \frac{|K|}{|PZ(G)|} = \frac{|K|}{|P||Z(G)|}.$$

Note that $Z(G) \trianglelefteq K$ with index $|K : Z(G)| = |P|p^n$. As the index is a p-power, we have that $Z(G)$ is a normal p-complement, and hence $K = \tilde{K}Z(G)$ for any $\tilde{K} \in \mathrm{Syl_p}(K)$. We note that $\tilde{K}$ is normal in $\tilde{K}Z(G) = K$, and thus we conclude that K has only one Sylow-p-group. By assumption we have that $K \trianglelefteq N_G(P)$, so $N_G(P)$ acts on $\mathrm{Syl_p}(K)$ by conjugation. As $\mathrm{Syl_p}(K)$ consists only of $\tilde{K}$, we have that $\tilde{K}$ is normal in $N_G(P)$ as well. As $\tilde{K}$ is the only Sylow-p-subgroup of K and P is a p-subgroup of K, we have that $P \leq \tilde{K}$. Furthermore the index satisfies $|\tilde{K} : P| = p^n$. Thus we see that $\tilde{K}/P$ is a normal p-group of $N_G(P)/P$, and a contradiction emerges.

From our description of $\mathcal{D}_p(G)$ it now follows from [19, remark 4.3] that the correspondence between the poset $\mathcal{D}_p(G)$ and the opposite poset of proper parabolic subgroups of G given by $O_p(-)$ is a bijection of G-posets with inverse $N_G(-)$.

Recall that the rank of a poset is the length of a maximal chain in it. For a fixed BN-pair for G, any parabolic subgroup of G is conjugate to a standard parabolic subgroup containing B. As this correspondence preserves inclusion the rank of the poset of parabolic subgroups of G is the same as the rank of the poset of parabolic subgroups containing B. The posets consisting of subsets of Δ and parabolic subgroups of G containing B are isomorphic by the map $I \mapsto P_I$ for $I \subseteq \Delta$. Hence the rank of the poset of parabolic subgroups containing B is equal to the cardinality of Δ. Thus we conclude that the rank of the poset of proper parabolic subgoups is exactly $|\Delta| - 1 = \mathrm{Rk_L}(G) - 1$. □

Proposition 6.5. Let G be a finite group of Lie type in characteristic p and S a Sylow-p-subgroup. Then the inclusion map of $\mathcal{D}_p(G)$ into $\mathcal{S}_p(G)$ is a G-homotopy equivalence, and $|\mathcal{S}_p(G)|_{hG} \simeq |\mathcal{T}_S^{cr}(G)|$ where the Borel map $|\mathcal{S}_p(G)|_{hG} \to BG$ corresponds the the realization of the simpicial map $\mathcal{T}_S^{cr}(G) \to \mathcal{B}(G)$ induced by inclusion on morphism sets. Furthermore the map $\varphi\colon |\mathcal{S}_p(G)|_{hG} \to BG$ is a mod-p-equivalence and $|\mathcal{S}_p(G)|_{hG}$ is p-good.

Proof. By [5, Corollary p. 50] the collections of subgroups $\mathcal{B}_p(G)$ and $\mathcal{S}_p(G)$ are G-homotopy equivalent via the inclusion map. By Lemma 6.4 the collections $\mathcal{B}_p(G)$ and $\mathcal{D}_p(G)$ agree in this case. Then $|\mathcal{S}_p(G)|_{hG} \simeq |\mathcal{D}_p(G)|_{hG}$ and using Lemma 5.2, we conclude that $|\mathcal{D}_p(G)|_{hG} \simeq |\mathcal{T}_S^{cr}(G)|$ with the Borel map $|\mathcal{D}_p(G)|_{hG} \to BG$ corresponding to the simpicial map $\mathcal{T}_S^{cr}(G) \to \mathcal{B}(G)$ given by inclusion on morphism sets.

Since p divides the order of G, the collection $\mathcal{S}_p(G)$ of subgroups of G is ample in the sense of [17, Definition 1.2] (see [17, Remark 6.4] for a discussion of the various proofs of this fact). This means that the map $|\mathcal{S}_p(G)|_{hG} \to BG$ induced by sending $|\mathcal{S}_p(G)| \to *$ is a mod-p-equivalence. Since G is a finite group, the classifying space BG is p-good [2, III: Proposition 1.11]. This means that the completion map is a mod-p-equivalence. We have that p-completion is a functor that comes with a natural transformation from the identity to it. Hence we have the following commutative diagram:

$$\begin{array}{ccc}
|\mathcal{S}_p(G)|_{hG} & \xrightarrow[p]{\backsim} & BG \\
\big\downarrow & & \big\downarrow {\scriptstyle p\,\wr} \\
(|\mathcal{S}_p(G)|_{hG})^\wedge_p & \xrightarrow{\simeq} & BG^\wedge_p
\end{array}$$

Here two of the maps are mod-p-equivalences, and the third is a homotopy equivalence according to [6, Lemma I.5.5]. Since homotopy equivalences are mod-p-equivalences, we conclude that the last map is a mod-p-equivalence, and hence that $|\mathcal{S}_p(G)|_{hG}$ is p-good. □

The transporter category $\mathcal{T}_S^{cr}(G)$ in the case of a finite group of Lie type has a description in terms of the standard parabolic subgroups.

Corollary 6.6. Let G be a finite group of Lie type in characteristic p. For a subset $I \subset \Delta$ we set $U_J = O_p(P_J)$. Then $\mathcal{T}_S^{cr}(G)$ is the category with objects $\{U_J \mid J \subsetneq \Delta\}$ and $\mathrm{Mor}_{\mathcal{T}_S^{cr}(G)}(U_J, U_{J'}) = P_J$ if $J \subseteq J'$ and empty otherwise.

Proof. The objects of $\mathcal{T}_S^{cr}(G)$ are exactly the subgroups in $\mathcal{D}_p(G)$ which are contained in S. Under the bijection from Lemma 6.4 these are mapped to the proper parabolic subgroups of G containing B, i.e the subgroups of the form P_I for $I \subsetneq \Delta$. As $U_I = O_p(P_I)$ we get the description of the objects while the morphism set follows from by [19, equation (4.1)]. □

The standard parabolic subgroups are also essential in the definition of the Tits building $\Delta(G, B)$ for a group G with a BN-pair structure. The building is defined as the G-poset of G-cosets of standard proper parabolic subgroups ordered by reverse inclusion. The map sending a coset gP_J to ${}^gP_J$ preserves inclusion, and so an alternative description of the building $\Delta(G, B)$ is the poset of proper parabolic subgroups of G ordered with reverse inclusion. From this description we see directly that $|\mathcal{D}_p(G)|$ is G-homotopy equivalent to the Tits building $\Delta(G,B)$ for G. From Proposition 6.5 we get directly:

Corollary 6.7. For a finite group G of Lie type in characteristic p the posets $\mathcal{S}_p(G)$ and $\Delta(G,B)$ are G-homotopy equivalent.

The Weyl group of G is finite, thus by [13, Chapter V 3] we have that the poset of parabolic subgroups of G is a spherical building. Hence by the Solomon–Tits Theorem [13, Theorem VI.5.2] we conclude that $|\mathcal{S}_p(G)|$ is homotopy equivalent to a wedge of spheres. The dimension of the spheres is the rank of the poset of proper parabolic subgroups of G, which is $\mathrm{Rk_L}(G)-1$.

The Sylow-p-subgroup S of G acts naturally on the poset $\mathcal{S}_p(G)$ by restricting the G-action, thus inducing an S action on the space $|\mathcal{S}_p(G)|$. We will now show the following equivariant refinement of the Solomon–Tits Theorem, where we keep track of the S-action.

In the proof we use the description of the Tits building as the flag complex of the incident geometry of the maximal parabolic subgroups of G, where two maximal parabolic subgroup are incident if their intersection is parabolic. Note that in this description of the building $\Delta(G,B)$, the poset of proper parabolic ordered with reverse inclusion is the set of simplices of $\Delta(G,B)$ ordered by inclusion, and thus the set of parabolic subgroups corresponds to the barycentric subdivision of $\Delta(G,B)$ and in particular the realizations agree.

Lemma 6.8. Let G be a finite group of Lie type in characteristic p of rank $\mathrm{Rk_L}(G)=n$ and let $S\in\mathrm{Syl_p}(G)$. Then $|\mathcal{S}_p(G)|$ with the induced action is S-homotopy equivalent to $\bigvee_S S^{n-1}$, where the S action on the wedge sum of spheres is induced by the left action on the set of indices.

Proof. By Corollary 6.7 it is suffient to detect the induced S action on the realization of the building $\Delta(G,B)$. We first note that we may assume that $B=N_G(S)$. Let C be a fundamental chamber corresponding to all maximal parabolic subgroups of G containing B. Recall that the fundamental chamber is an $(n-1)$-simplex with vertices corresponding to the maximal parabolic subgroups over B. The G-action in $\Delta(G,B)$ is given by the conjugation action on maximal parabolic subgroups, hence for $g\in G$ we let ${}^gC$ be the simplex consisting of all maximal parabolic subgroups containing ${}^gB$.

Let $W=N/H$ be the Weyl group from the BN-pair of G. Then $\Sigma=\bigcup_{n\in N}{}^nC$ is a fundamental apartment and $\Delta(G,B)=\bigcup_{u\in S}{}^u\Sigma$, where for $u\in S$ we have that ${}^u\Sigma=\bigcup_{n\in N}{}^{un}C$.

For any $x\in\Delta(G,B)$ we let S_x be the stabilizer subgroup for the action of S on x. For $u\in S$ we set

$$\mathrm{fix}({}^u\Sigma)=\bigcup_{\tilde u\in S\setminus\{u\}}{}^u\Sigma\cap{}^{\tilde u}\Sigma.$$

A point in ${}^u\Sigma$ that is fixed by some nontrivial $\tilde u\in S$ is a point in $\mathrm{fix}({}^u\Sigma)$. To identify this set as the set points in the chamber ${}^u\Sigma$ with nontrivial stabilizer subgroup, we observe using [23, Lemma 4] that

$$\mathrm{fix}({}^u\Sigma)=\bigcup_{\tilde u\in S\setminus\{1\}}{}^u\Sigma\cap{}^{\tilde u}({}^u\Sigma)\subset\bigcup_{\tilde u\in S\setminus\{1\}}\{y\in{}^u\Sigma\mid\tilde u\in S_y\}=\{y\in{}^u\Sigma\mid S_y\neq 1\}.$$

Furthermore as each intersection ${}^{u}\Sigma \cap {}^{\tilde{u}}\Sigma$ for $\tilde{u} \in S$ is a subcomplex consisting of a union of chambers, the same is true for $\operatorname{fix}({}^{u}\Sigma)$. Based on [23, Lemma 1 and Proposition 3] we see that

$$\operatorname{fix}({}^{u}\Sigma) = \bigcup_{\tilde{u} \in S \setminus \{u\}} {}^{u}\Sigma \cap {}^{\tilde{u}}\Sigma \simeq \bigvee_{\tilde{u} \in S \setminus \{u\}} {}^{u}\Sigma \cap {}^{\tilde{u}}\Sigma$$

where all the wedge summands are contractible, and thus the subspace $\operatorname{fix}({}^{u}\Sigma)$ is contractible as well.

Then for $u, v \in S$ we observe that

$$({}^{v} \operatorname{fix}({}^{u}\Sigma)) = \bigcup_{\tilde{u} \in S \setminus \{u\}} {}^{vu}\Sigma \cap {}^{v\tilde{u}}\Sigma = \bigcup_{\tilde{u} \in S \setminus \{vu\}} {}^{vu}\Sigma \cap {}^{v\tilde{u}}\Sigma = \operatorname{fix}({}^{vu}\Sigma).$$

By combining [23, Lemma 1 and Proposition 3] we get that the map

$$\Delta(G, B) = \bigcup_{u \in S} {}^{u}\Sigma \to \bigvee_{u \in S} {}^{u}\Sigma / \operatorname{fix}({}^{u}\Sigma) \tag{1}$$

is a homotopy equivalence. As ${}^{v} \operatorname{fix}({}^{u}\Sigma) = \operatorname{fix}({}^{vu}\Sigma)$ for any $u, v \in S$ we have an induced S-action on $\bigvee_{u \in S} {}^{u}\Sigma / \operatorname{fix}({}^{u}\Sigma)$. This action freely permutes wedge summands corresponding to the left S-action on the indexing set. Note that with the induced S-action the map (1) is an S-map.

According to [23, Proposition 1 and 3] the space $\Sigma / \operatorname{fix}(\Sigma)$ is homotopic to an $(n-1)$-sphere. For a fixed choice of homotopy equivalence this extends to a homotopy equivalence $\bigvee_{u \in S} ({}^{u}\Sigma / \operatorname{fix}({}^{u}\Sigma)) \to \bigvee_{u \in S} S^{n-1}$. If we let $\bigvee_{u \in S} S^{n-1}$ be an S-space where the S action on the wedge sum of spheres is induced by the left action on the set of indices, then the composite

$$f \colon \Delta(G, B) \to \bigvee_{u \in S} S^{n-1}$$

is an S-map and a homotopy equivalence.

To see that the map f is in fact an S-homotopy equivalence, it is sufficient to show, by the equivariant Whitehead theorem, that the map on Q-fixed points for any subgroup $Q \leq S$ is a homotopy equivalence. For a non-trivial Q the Q-fixed point on $\bigvee_{u \in S} S^{n-1}$ is the wedging point, so we have to show that $\Delta(G, B)^{Q}$ is contractible.

For any $u \in S$ let $\operatorname{fix}_{Q}({}^{u}\Sigma)$ be the Q-fixed points in the chamber ${}^{u}\Sigma$. Then $\operatorname{fix}_{Q}({}^{u}\Sigma)$ is a subcomplex of $\Delta(G, B)^{Q}$ and $\Delta(G, B)^{Q} = \bigcup_{u \in S} \operatorname{fix}_{Q}({}^{u}\Sigma)$. So by [23, Lemma 1] it is sufficient to prove that for any subset X of elements of S we have that $\bigcap_{x \in X} \operatorname{fix}_{Q}({}^{x}\Sigma) \simeq *$. Using [23, Lemma 4] we see that

$$\operatorname{fix}_{Q}({}^{x}\Sigma) = \bigcap_{q \in Q} {}^{qx}\Sigma$$

and thus

$$\bigcap_{x \in X} \operatorname{fix}_{Q}({}^{x}\Sigma) = \bigcap_{x \in X} \bigcap_{q \in Q} {}^{qx}\Sigma.$$

Since conjugation with an element of S is a homeomorphism on a G-simplicial complex the above intersection is homotopy equivalent to an intersection $\bigcap_{x \in \tilde{X}} {}^{x}\Sigma$, where $\tilde{X}$ is a non-empty subset of S containing 1. Thus by [23, Proposition 3] it is contractible. □

The proof in [23] shows that the fixed points of the fundamental apartment of any nontrivial subset Q of S consists of entire chambers and is closed with respect to the order on Weyl group W, i.e if ${}^wC \in \Sigma^Q$ and $v \leq w$ then ${}^vC \in \Sigma^Q$. By [1, Corollary 1.75 and 6.47] the opposite chamber to C is ${}^{w_0}C$ where $w_0 \in W$ is the unique longest element. The stabilizer of ${}^{w_0}C$ is ${}^{w_0}B$. According to [18, Theorem 2.3.8(d)] the intersection $B \cap {}^{w_0}B$ equals H, a maximal torus, and thus a p'-group. In particular for any p-subgroup Q of B we have that $Q \nsubseteq {}^{w_0}B$. Thus, the chamber ${}^{w_0}C$ is not fixed by Q. The proof [23] uses this observation as it shows that removing the chamber in Σ^Q of the form ${}^wC$, where w has maximal length in the Weyl group, does not affect the homotopy type and thus Σ^Q is homotopic to the chamber C which is contractible.

It is worth noting that the result in [23] does not provide a canonical contraction Σ^Q onto C corresponding to the fact that there does not always exist a unique minimal gallery of chambers between C and ${}^wC$ for a $w \in W$. This is illustrated in the following example.

Example 6.9. By [1, Corollary 1.75] a minimal gallery corresponds to a reduced decomposition of elements of the Weyl group in terms of a fixed set of simple reflections that generates W. By [18, Proposition 2.3.2] there exists a group G of finite Lie type with Weyl group W of type B_3. In particular note that G has rank 3. Every element of W is a linear transformation of $\mathbb{R}^3$ given by a permutation of the basis vectors possibly composed with multiplication with -1. Then the set of linear transformations $\{213, 132, 12\bar{3}\}$ generates W, where the bar indicates sign change of particular basis vector. By calculations from [38] the element $312 \in W$ has length 3 and can be written as both (132)(213)(132) and (213)(132)(213). Note that the unique element of maximal length is $\bar{1}\bar{2}\bar{3}$ and this has length 9.

An alternative proof of the fact that Σ^Q is homotopic to a point in the building $\Delta(G, B)$ may be found in [33, Theorem 3.1 part (b)]. Here the building $\Delta(G, B)$ is equipped with a metric and the contraction given along unique geodesics.

Corollary 6.10. Let G be a finite group of Lie type in characteristic p and $S \in \mathrm{Syl_p}(G)$. Let P be a finite p-group and $\varphi \in \mathrm{Hom}(P, G)$ be non-trivial. Then φ induces a P action on $|\mathcal{S}_p(G)|$, and with respect to this action the fixed points $|\mathcal{S}_p(G)|^P$ are contractible.

Proof. According to Lemma 6.8 the space $|\mathcal{S}_p(G)|$ is S-homotopy equivalent to $X = \bigvee_S S^{\mathrm{Rk_L}(G)-1}$ where S acts by left multiplication on the index set. In particular for any non-trivial subgroup of S acting on X the only fixed point is the wedging point of X. An S-homotopy equivalence restricts to a homotopy equivalence on fixed points for any subgroup of S and thus the fixed points of $|\mathcal{S}_p(G)|$ for a nontrivial subgroup $Q \leq S$ must be contractible. Let $\varphi \in \mathrm{Hom}(P, G)$ be non-trivial. By Sylow's theorems we have that $\varphi(P)$ is conjugate to a non-trivial subgroup of S, and as the conjugation map on $\mathcal{S}_p(G)$ induces a homotopy equivalence on $|\mathcal{S}_p(G)|$, we now conclude that $|\mathcal{S}_p(G)|^{\varphi(P)}$ is contractible. □

Lemma 6.11. Let G be a finite group of Lie type in characteristic p and assume that the Lie rank of G is at least 3. Then $|\mathcal{S}_p(G)|$ is $(\mathrm{Rk_L}(G) - 2)$-connected, $\pi_{\mathrm{Rk_L}(G)-1}(|\mathcal{S}_p(G)|) \cong \mathbb{Z}^{|S|}$ and all of the higher homotopy groups $\pi_n(|\mathcal{S}_p(G)|)$ for $n \geq 2$ are finitely generated abelian groups.

Proof. Recall that by Lemma 6.8 the space $|\mathcal{S}_p(G)|$ is homotopy equivalent to a wedge of spheres of dimension $\mathrm{Rk}_{\mathrm{L}}(G)-1$. We note that $|\mathcal{S}_p(G)|$ is connected and by using the wedge point as base point, we have that $|\mathcal{S}_p(G)|$ is a pointed S-space. By a special case of the Hilton–Milnor Theorem [21, Corollary 4.10] we see that

$$\pi_n(|\mathcal{D}_p(G)|) \cong \sum_{k=1}^{\infty} \sum_{Q(k,|S|)} \pi_n(S^{k(\mathrm{Rk}_{\mathrm{L}}(G)-2)+1}). \tag{2}$$

Here, Q is the Necklace polynomial and as function it is given using the Möbius function μ by

$$Q(k,r) = \frac{1}{k}\sum_{d|k} \mu(d) r^{k/d}.$$

The value of the function $Q(k,r)$ is the number of Lyndon words of length k in an alphabet with r letters. In particular Q has non-negative values for $k, r > 0$ and positive values when $r > 1$. Note that $Q(1,r) = r$.

The dimension of the spheres in (2) are increasing. As a sphere of dimension n is $(n-1)$-connected, we see that for a fixed n only a finite number of them have nontrivial n'th homotopy groups. Since homotopy groups of spheres are all finitely generated abelian, we conclude that the same holds for the space $|\mathcal{S}_p(G)|$. An n-sphere is $(n-1)$-connected. The lowest dimensional spheres (2) are of dimension $\mathrm{Rk}_{\mathrm{L}}(G)-1$, and hence we conclude that $|\mathcal{S}_p(G)|$ is $(\mathrm{Rk}_{\mathrm{L}}(G)-2)$-connected.

For the $(\mathrm{Rk}_{\mathrm{L}}(G)-1)$th homotopy group, we see that

$$\pi_{\mathrm{Rk}_{\mathrm{L}}(G)-1}(|\mathcal{S}_p(G)|) \cong \prod_{Q(1,|S|)} \pi_{\mathrm{Rk}_{\mathrm{L}}(G)-1}(S^{\mathrm{Rk}_{\mathrm{L}}(G)-1}) \cong \mathbb{Z}^{|S|}.$$

□

From Lemma 6.8 we see that the homotopical properties of $|\mathcal{S}_p(G)|$ are vastly different depending on whether the Lie rank of G is 1, 2 or at least 3. This is reflected in the homotopy type of $|\mathcal{T}_S^{cr}(G)| \simeq |\mathcal{S}_p(G)|_{hG}$, and so the three cases will now be handled separately.

6.1. **The homotopy type of transporter systems in rank 1.**

The rank 1 case can easily be determined by from the description of $\mathcal{T}_S^{cr}(G)$.

Proposition 6.12. Let G be a finite group of Lie type of rank 1. The category $\mathcal{T}_S^{cr}(G)$ consists of only one object, namely S, with morphism set B, and so in particular $|\mathcal{T}_S^{cr}(G)| \simeq B(B)$. In addition $B(B) \simeq |\mathcal{S}_p(G)|_{hG}$ and the inclusion $B(B) \to BG$ corresponds to the Borel map $|\mathcal{S}_p(G)|_{hG} \to BG$ under the given homotopy equivalences.

Proof. By Corollary 6.6 the category $\mathcal{T}_S^{cr}(G)$ has one object S and morphism set $N_G(S) = B$ and thus $|\mathcal{T}_S^{cr}(G)| \simeq B(B)$.

The homotopy equivalence $|\mathcal{S}_p(G)|_{hG} \simeq |\mathcal{T}_S^{cr}(G)|$ from Proposition 6.5, comes from the upper zig-zag from the diagram in Lemma 5.2 together with the inclusion of $\mathcal{D}_p(G) \to \mathcal{S}_p(G)$. Thus, the map induced by $|\mathcal{S}_p(G)| \to *$ corresponds to $|\pi_{\mathcal{T}_S^{cr}(G)}|$ under the homotopy equivalence $|\mathcal{S}_p(G)|_{hG} \simeq |\mathcal{T}_S^{cr}(G)|$. Finally, we note that $|\pi_{\mathcal{T}_S^{cr}(G)}|$ is exactly the map from $B(B) \to B(G)$ induced by the inclusion $B \to G$. □

6.2. **The homotopy type of transporter systems in rank 2.** In this part we restrict ourself to the case where G be a finite group of Lie type of Lie rank 2 in characteristic p. The goal is to determine the homotopy type of $|\mathcal{S}_p(G)|_{hG} \simeq |\mathcal{D}_p(G)|_{hG}$. We will use the following notation:

Let Φ be the root system of G associated to the BN-pair and let $\Delta = \{\alpha, \beta\}$ be the corresponding set of simple roots. Similarly let $\{s_\alpha, s_\beta\}$ be the corresponding set of reflections generating the Weyl group and t_α, t_β be the corresponding lifts to N. Note that the Weyl group W in this case is a finite dihedral group. We denote by P_α and P_β the standard parabolic subgroups of G associated to the subsets $\{s_\alpha\}$ and $\{s_\beta\}$ of Δ. Then the Borel subgroup B is a subgroup of both P_α and P_β.

We have by Lemma 6.4 that the G-poset $\mathcal{D}_p(G)$ is isomorphic to the inverse poset of proper parabolic subgroups of G, which in our case consists of the G-conjugates of B, P_α and P_β. To determine the homotopy type of the transporter system we will study the situation in more generality in the following proposition.

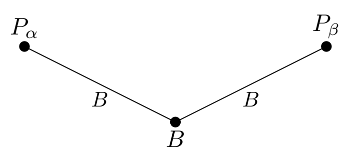
Proposition 6.13. Let G be a finite group generated by two subgroups P_α and P_β. Let $B = P_\alpha \cap P_\beta$. Assume that P_α, P_β and B are all self-normalizing and mutually non-conjugate. Let $\mathcal{D}$ be the poset of G-conjugates of P_α, P_β and B ordered by inclusion. Then $\mathcal{D}$ is a G-poset by the conjugation action and with respect to this we have $|\mathcal{D}|_{hG} \simeq B(P_\alpha *_B P_\beta)$.

Using the model $|\mathcal{E}G|$ for EG, we let the class $[B, |1|] \in |\mathcal{D}|_{hG} = (|\mathcal{D}| \times EG)/P_\alpha *_B P_\beta$ be the base point. Then under the given homotopy equivalence the Borel map $|\mathcal{D}|_{hG} \to BG$ is induced by the map of fundamental groups $\phi\colon P_\alpha *_B P_\beta \to G$ which is the inclusion on all elements in P_α and P_β. Furthermore the kernel of $\phi\colon P_\alpha *_B P_\beta \to G$ is a finitely generated free group.

Proof. We want to construct a contractible space X with a $P_\alpha *_B P_\beta$ covering space action, such that the orbit space can be identified with $|\mathcal{D}|_{hG}$.

Serre defined a graph of groups (G, T) in [36, I Chapter 4.4 Definition 8] as a graph with groups associated to vertices as well as edges, together with injective group homomorphisms such that a group associated to an edge can be seen as a subgroup of the groups associated to its vertices.

We will first consider the following graph of groups (G, T)



By [36, I Chapter 4.4 example c] the graph (G, T) has fundamental group $\tilde{G} := P_\alpha *_B P_\beta$. Let $\tilde{T}$ be the Bass–Serre covering tree for (G, T) (see [36, I Chapter 5.3]). Then the vertex set $\tilde{T}$ is the set of cosets gP_α, gP_β and gB for $g \in \tilde{G}$. We denote the vertices corresponding to the cosets by $[gP_\alpha], [gP_\beta]$ and $[gB]$. The set of edges is given in the following way: For $\gamma \in \{\alpha, \beta\}$ there is an edge connecting $[gB]$ and $[hP_\gamma]$ if there exists a $k \in \tilde{G}$ such that $gB = kB$ and $hP_\gamma = kP_\gamma$. The $\tilde{G}$ action is induced by the natural action on the cosets.

We claim that the tree $\tilde{T}$ is locally finite, i.e that every vertex of $\tilde{T}$ is contained in a finite number of edges. First consider a vertex of type $[gP_\gamma]$ for $\gamma \in \{\alpha, \beta\}$ and $g \in \tilde{G}$. Then any adjacent vertex is of the form $[hB]$ such that $hP_\gamma = gP_\gamma$. Thus $h = gk$ for some $k \in P_\gamma$. As P_γ is finite, the vertex $[gP_\gamma]$ has finite degree. Likewise if we consider a vertex $[gB]$ for some $g \in \tilde{G}$, then any adjacent vertex is of the form $[kP_\gamma]$ for some $\gamma \in \{\alpha, \beta\}$ and $k \in \tilde{G}$ satisfying that $gB = kB$. Then $g = kb$ for some $b \in B$, but since $B \leq P_\gamma$, we have that $gP_\gamma = kP_\gamma$. Thus, the vertex $[gB]$ is only connected to $[gP_\alpha]$ and $[gP_\beta]$.

Note that the $\tilde{G}$-conjugates of P_α, P_β and B are exactly the stabilizer subgroups of vertices in $\tilde{T}$. In particular $\tilde{T}$ is a locally finite tree with finite stabilizer groups of vertices, and thus the action of $\tilde{G}$ on $\tilde{T}$ is properly discontinuous in the sense that every point in $\tilde{G}$ has a neighborhood U such that $U \cap g(U)$ is non empty for only finitely many $g \in \tilde{G}$. The action is not a covering space action, since it is not free.

Consider the canonical group homomorphism $\phi\colon \tilde{G} \to G$ which is the inclusion on all elements in P_α and P_β. As G is generated by the subgroups P_α and P_β, the map is surjective. Let $K := \ker(\phi)$. We have that $\tilde{G}$ acts on $\tilde{T}$, and so by restriction the subgroup K acts on $\tilde{T}$ as well. We claim that the K action is free. To see this we note that ϕ is injective on the subgroups P_α, P_β and B. Since ϕ is a group homomorphism, we deduce that ϕ is injective on all $\tilde{G}$-conjugates of P_α, P_β and B, and so all of these groups have trivial intersections with K. Note that the $\tilde{G}$-conjugates of P_α, P_β and B are exactly the stabilizer subgroups of vertices in $\tilde{T}$, and hence K acts freely on the vertices. As the $\tilde{G}$- action on $\tilde{T}$ is without edge-inversion, we conclude that K acts freely on $\tilde{T}$. Furthermore by [36, I Chapter 4 Theorem 8] every torsion element of $\tilde{G}$ is contained in a conjugate of P_α or P_β, and thus we conclude that K is a free group.

Recall that EG is a contractible space, on which G acts freely and properly discontinuously. Consider the space $\tilde{T} \times EG$ with the $\tilde{G}$-action given by $g(t,x) = (gt, \phi(g)x)$ for $G \in \tilde{G}$ and $(t,x) \in \tilde{T} \times EG$. We see that the $\tilde{G}$-action is free and properly discontinuous and the quotient space $(\tilde{T} \times EG)/\tilde{G}$ is a classifying space for $\tilde{G}$.

To identify the quotient space $(\tilde{T} \times EG)/\tilde{G}$ with $|\mathcal{D}|_{hG}$ recall that since K is normal in $\tilde{G}$, we have that

$$(\tilde{T} \times EG)/\tilde{G} \cong ((\tilde{T} \times EG)/K)/G \cong (\tilde{T}/K)_{hG}.$$

Hence it is enough to identify the quotient space $\tilde{T}/K$ with $|\mathcal{D}|$.

First we define a map on vertices $\pi\colon \operatorname{Vert} \tilde{T} \to \operatorname{Vert} |\mathcal{D}|$ given by $[gQ] \mapsto {}^{\phi(g)}Q$ for $Q \in \{P_\alpha, P_\beta, B\}$. Note that this is well-defined as ${}^{\phi(g)}Q$ does not depend on the representative for gQ. Any pair of adjacent vertices in $\tilde{T}$ is of the form $[gB]$ and $[gP_\gamma]$ for some $\gamma \in \{\alpha, \beta\}$. As $B \leq P_\gamma$, we have that ${}^{\phi(g)}B \leq {}^{\phi(g)}P_\gamma$, hence $\pi[gB]$ and $\pi[gP_\gamma]$ are adjacent in $|\mathcal{D}|$. Since $|\mathcal{D}|$ is without multiple edges we get a unique extension $\pi\colon \tilde{T} \to |\mathcal{D}|$. Thus, the resulting map is onto. We want to identify the orbit space $\tilde{T}/K$ with $|\mathcal{D}|$ via π. The map π is onto and identifies K-orbits, and so it induces an onto map $\tilde{T}/K \to |\mathcal{D}|$.

To see that $\tilde{T}/K \cong |\mathcal{D}|$, we assume that $\pi[gQ] = \pi[hR]$ for some $h, g \in \tilde{G}$ and $Q, R \in \{P_\alpha, P_\beta, B\}$. Then ${}^{\phi(g)}Q = {}^{\phi(h)}R$, and hence Q and R are conjugate in G and must agree by assumption. Furthermore $\phi(h^{-1}g) \in N_G(Q) = Q$, and so $\phi(g) = \phi(h)q$ for some $q \in Q$. As $\phi(q) = q$ we see that there exists a $k \in K$ such that $kg = hq$. Then

$$[hR] = [hQ] = [hqQ] = [kgQ] = k[gQ] \in K[gQ].$$

Thus, π is exactly the map identifying K-orbits, and hence $\tilde{T}/K \cong |\mathcal{D}|$.

We now have that $|\mathcal{D}|_{hG} \simeq (\tilde{T} \times EG)/(\tilde{G})$ where $\tilde{T} \times EG$ is the universal cover for $|\mathcal{D}|_{hG}$. Using covering theory this implies that every element of $g \in \pi_1(|\mathcal{D}|_{hG}, [B, |1|])$ is represented by the image of a path in $\tilde{T} \times EG$ from $g([B], |1|)$ to $([B], |1|)$ under the covering map. As the fundamental group is the amalgamated product $\tilde{G}$ it is generated by elements in P_α and P_β, and hence we may restrict ourselves to the case of an element $g \in P$ where P is either P_α or P_β. Then $[gP] = [P]$, and so there exist edges in $\tilde{T}$ from $[B]$ to $[P]$ and from $[gB]$ to $[P]$. We denote these by e_1 and e_g respectively. The following diagram represents a path from $g([B], |1|) = ([gB], |g^{-1}|)$ to $([B], |1|)$:

$$([gB], |g^{-1}|) \xrightarrow{(e_g, \mathrm{id})} ([P], |g^{-1}|) \xrightarrow{(\mathrm{id}, |g^{-1}, 1|)} ([P], |1|) \xrightarrow{(e_1, \mathrm{id})} ([B], |1|)$$

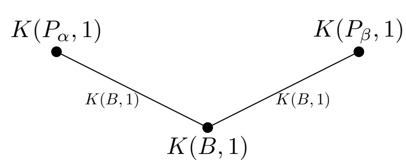
This path is mapped to the following loop in $|\mathcal{D}|_{hG}$ by the covering map:

$$\begin{array}{ccccccc} & & [P, |g^{-1}|] & \xrightarrow{[\mathrm{id}, |g^{-1}, 1|]} & [P, |1|] & & \\ & \nearrow^{[{}^gB \leq P, \mathrm{id}]} & & & & \searrow^{[B \leq P, \mathrm{id}]} & \\ [{}^gB, |g^{-1}|] & & & = & & & [B, |1|] \end{array}$$

and hence this loop is a representative for $g \in \pi_1(|\mathcal{D}|_{hG}, [B, |1|])$. Recall that the Borel map $|\mathcal{D}|_{hG} \to BG$ can be identified with the realization of the simplical map $\pi \colon \mathcal{D} \times_G \mathcal{E}(G) \to \mathcal{B}(G)$ given by $(P, g) \mapsto *_G$ and $(P \leq Q, (g, h)) \mapsto hg^{-1}$. Thus, for any $g \in P_\alpha \cup P_\beta$ the loop representing g in $\pi_1(|\mathcal{D}|_{hG}, [B, |1|])$ is mapped to the loop at $*_G$ along the edge $|g|$ in BG, which is the standard generator for the class $g \in \pi_1(BG, *_G) = G$.

As G is a finite group, we have that $|\mathcal{D}|$ has only a finite number of vertices and edges. The free group K acts freely on the tree $\tilde{T}$ with orbit space $|\mathcal{D}|$, and so by [36, I Chapter 3 Theorem 4'] we conclude that K is a finitely generated free group. $\square$

An alternative proof of Proposition 6.13 comes from the work of Libman and Viruel in [25], as we can identify $|\mathcal{D}|_{hG}$ with the homotopy colimit over the subdivision category of the following tree of Eilenberg-Maclane spaces:



The fact that $|\mathcal{D}|_{hG}$ is homotopy equivalent to $B(P_\alpha *_B P_\beta)$ follows directly from [25, Proposition 4.3]. We have chosen to present the argument based on graph of groups given in the proof of Proposition 6.13, as it allows us to detect what happens to the fundamental group under the Borel map as well proves interesting facts about the kernel of $\phi \colon P_\alpha *_B P_\beta \to G$ using the work of Serre.

For G a finite group of Lie type of rank 2 in characteristic p the Weyl group is a dihedral group. Assume that the Weyl group is dihedral of order $2L$. The next corollary shows that the kernel of a map $\phi \colon P_\alpha *_B P_\beta \to G$ essentially is the lift of the relation $(s_\alpha s_\beta)^L$ in the Weyl group to $P_\alpha *_B P_\beta$. We will need to keep track of the different lifts in $P_\alpha *_B P_\beta$ of the longest word in W. These will be the two alternating words in $\{t_\alpha, t_\beta\}$ of length L and we will now introduce a notation allowing us to denote the lift starting or ending with a particular element. For the alternating word of length L we let

$$_{\alpha}n = \underbrace{t_\alpha t_\beta \cdots}_{\text{length } L}, \quad n_\alpha = \underbrace{\cdots t_\beta t_\alpha}_{\text{length } L}, \quad _{\beta}n = \underbrace{t_\beta t_\alpha \cdots}_{\text{length } L}, \quad n_\beta = \underbrace{\cdots t_\alpha t_\beta}_{\text{length } L}.$$

In particular $_{\alpha}n = n_\alpha$ when L is odd and $_{\alpha}n = n_\beta$ when L is even.

Corollary 6.14. *Let G be a finite group of Lie type of rank 2 in characteristic p for which the Weyl group is isomorphic to D_{2L}. Let $\phi\colon P_\alpha *_B P_\beta \to G$ be the canonical map. Then the kernel of ϕ is a free group generated by the set*

$$\{sn_\beta \phi((t_\alpha t_\beta)^{-L})_{\alpha}ns^{-1} \mid s \in S\}$$

where $\phi((t_\alpha t_\beta)^{-L}) \in B$. In particular all generators are reduced words in $P_\alpha *_B P_\beta$ of length $2L$.

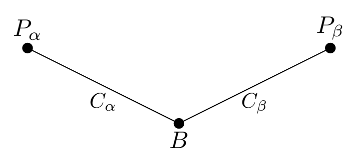
Proof. By [18, Theorem 2.6.5] we have that the subgroups P_α, P_β and B are all self-normalizing and non-conjugate. Furthermore, from the definition of standard parabolic subgroups it follows that the intersection $P_\alpha \cap P_\beta$ is the Borel subgroup B. Thus we are in the situation presented in Proposition 6.13. In particular we have that the kernel K is a free group and it acts freely on the covering tree $\tilde{T}$.

We have observed that the orbit space $\tilde{T}/K$ can be identified with $|\mathcal{D}|$, where $\mathcal{D}$ is the poset of G-conjugates of P_α, P_β and B ordered by opposite inclusion. In this case it is exactly the poset of proper parabolic subgroups of G ordered by opposite inclusion. Recall that $\mathcal{D}$ is isomorphic to the poset of simplices of the building $\Delta(G, B)$ ordered by inclusion. Thus $|\mathcal{D}|$ is a subdivision of the realization of the building, and hence $|\mathcal{D}|$ is homotopy equivalent to $\bigvee_{|S|} S^1$ by the Solomon–Tits theorem (see Lemma 6.8). By the topological version of [36, Theorem 4'] we have that the set of basis elements for K is in bijection with a set of generators of the fundamental group of $|\mathcal{D}|$, and so the basis for K consists of $|S|$ elements.

The element $\phi((t_\alpha t_\beta)^{-L})$ belongs to N and is mapped to the trivial element under the quotient map to the Weyl group, and thus belongs to H, a subgroup of B. For $s \in S$ we let k_s be the element $sn_\beta\phi((t_\alpha t_\beta)^{-L})_{\alpha}ns^{-1} \in P_\alpha *_B P_\beta$. We note that $\phi((t_\alpha t_\beta)^{-L}) = \phi(n_\beta^{-1}{}_{\alpha}n^{-1})$, and so k_s belongs to the kernel K. As k_s is a reduced word of length $2L$, it is nontrivial in $P_\alpha *_B P_\beta$. To determine that $\{k_s \mid s \in S\}$ is actually a generating set for K we will identify a tree of representatives inside $\tilde{T}$.

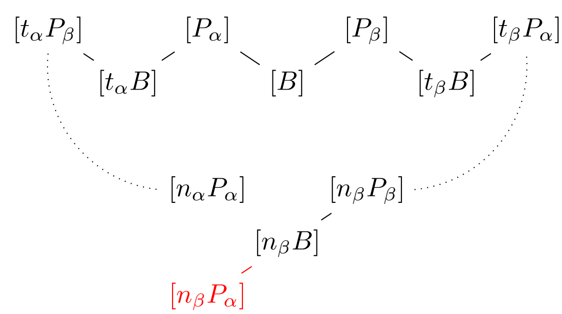
Recall that in the building $\Delta(G, B)$ the standard choice of fundamental chamber $\tilde{C}$ can be identified with the simplex:

For the complex $|\mathcal{D}|$ we let C_α be the edge between B and P_α and C_β be the edge between B and P_β. Set C to be the union of these:



Then C is the subdivision of fundamental chamber $\tilde{C}$ in $\Delta(G, B)$. Moreover, $\Sigma = \cup_{n \in N} {}^{n}C$ is the subdivision of the associated fundamental apartment, and

$|\mathcal{D}| = \cup_{s \in S} {}^s\Sigma$. The Solomon–Tits theorem implies that we obtain a maximal tree in $\Delta(G, B)$ by removing all the chambers opposite to $\tilde{C}$, i.e. all chambers of the form ${}^{sn_0}\tilde{C}$ where $n_0 \in N$ is the preimage of the longest word in the Weyl group. Similarly we obtain a maximal tree $T_{\mathcal{D}}$ in $|\mathcal{D}|$ by removing the set $\{{}^{sn_\beta}C_\alpha \mid s \in S\}$. Let $\tilde{T}_{\mathcal{D}}$ be the lift of $T_{\mathcal{D}}$ to $\tilde{T}$ such that the edge C_α is the edge between $[P_\alpha]$ and $[B]$. The fundamental chamber Σ is lifted to the following graph, where the marked edge corresponds to the edge ${}^{n_\beta}C_\alpha$ not contained in $T_{\mathcal{D}}$.



The lift of the other part of $T_{\mathcal{D}}$ is obtained by the action of S on the above graph. We may equip $\tilde{T}$ with an orientation of the edges consisting of all edges from $[gB]$ to $[gP_\gamma]$ for $g \in P_\alpha *_B P_\beta$ and $\gamma \in \{\alpha, \beta\}$. According to[36, Theorem 4' (a)] we obtain a basis for K by taking the $k \in K$ for which there exist edges in $\tilde{T}$ that start in $T_{\mathcal{D}}$ and end in $kT_{\mathcal{D}}$ with the given orientation of $\tilde{T}$. By considering the graph $|\mathcal{D}|$ we see that the edges connecting $\tilde{T}_{\mathcal{D}}$ and a vertex not in the tree of representatives are exactly the edges between $[sn_\beta B]$ and $[sn_\beta P_\alpha]$ and those between $[sn_\alpha B]$ and $[sn_\alpha P_\alpha]$ for $s \in S$. Under the choice of orientation only the first type will belong to the orientation, and so [36, Theorem 4' (a)] implies that it is sufficient to show that for $s \in S$ we have

$$k_s \cdot [sn_\alpha P_\alpha] = [sn_\beta P_\alpha] \tag{3}$$

as vertices of $\tilde{T}$.

For this purpose we investigate the following claim: For every alternating word w in t_α and t_β inside $P_\alpha *_B P_\beta$, the product of $\tilde{w}w$, where $\tilde{w}$ is the reverse word, is an element of $B \cap N \leq P_\alpha *_B P_\beta$. This follows by induction on the length of w. For length 0 we have that $w = \tilde{w} = 1$, and hence the claim is clear. Let w be a word of positive length and write $w = vt_\gamma$. We assume that the claim holds for v. As $\tilde{w}w = t_\gamma \tilde{v} v t_\gamma$, where $\tilde{v}v \in B \cap N$, this is an element of $P_\gamma \cap N$. The map ϕ is injective on P_γ, and so it is enough to show that $\phi(\tilde{w}w) \in B \cap N$. Since the Weyl group is dihedral, the images of t_α and t_β have order two in W, which implies that the image of $\phi(\tilde{w}w)$ is mapped to the trivial element in W. Thus, $\phi(\tilde{w}w)$ belongs to the kernel of $N \to W$, which is exactly $B \cap N$. Note that n_α is the reverse word of ${}_\alpha n$, and so ${}_\alpha n n_\alpha \in B \cap N \leq P_\alpha$ by the above claim. Since $\phi((t_\alpha t_\beta)^{-N}) \in B$, we get

$$k_s(sn_\alpha P_a) = sn_\beta \phi((t_\alpha t_\beta)^{-L})_\alpha n s^{-1} sn_\alpha P_\alpha = sn_\beta P_\alpha$$

and thus showing the equality in (3).

Hence we conclude that $\{k_s \mid s \in S\}$ is a basis for K, and as the basis found in this way consists of $|S|$ elements, we furthermore conclude that all the k_s's are distinct. $\square$

Corollary 6.15. Let g be a finite group of Lie type of Lie rank 2 in characteristic p. Then $|\mathcal{D}_p(G)|_{hG} \simeq B(P_\alpha *_B P_\beta)$. In particular $\pi_1(|\mathcal{D}_p(G)|_{hG}) \simeq P_\alpha *_B P_\beta$.

Furthermore the Borel fibration $|\mathcal{D}_p(G)|_{hG} \mapsto BG$ corresponds to the canonical map $\phi\colon P_\alpha *_B P_\beta \to G$ under this identification.

Proof. As shown in Lemma 6.4 we have that $\mathcal{D}_p(G)$ is equivalent to the G-poset $\mathcal{D}$ consisting of G-conjugates of P_α, P_β and B with inverse inclusion. As $\mathcal{D}$ is a poset of rank 1 it is isomorphic to its opposite, and so we may identify it with its opposite. This puts us in the situation given in Proposition 6.13.

Recall that [18, Theorem 2.3.4] shows that the group G has a BN-pair. It follows from the definition of BN-pairs that G is generated by B and the preimages of s_α and s_β in N. By construction of the standard parabolic subgroups (see [18, Definition 2.6.4]) we have that these preimages belong to P_α and P_β respectively, and hence G is generated by P_α and P_β as well. By [18, Theorem 2.6.5] we have that the subgroups P_α, P_β and B are all self-normalizing and non-conjugate. Furthermore the definition of standard parabolic subgroups implies that the intersection $P_\alpha \cap P_\beta$ is the Borel subgroup B. Thus we conclude from Proposition 6.13 that $|\mathcal{D}_p(G)|_{hG}$ has the homotopy type of $B(P_\alpha *_B P_\beta)$ and that the Borel map $|\mathcal{D}_p(G)|_{hG} \to BG$ is induced by the canonical map $\phi\colon P_\alpha *_B P_\beta \to G$. □

Proposition 6.16. Let G be a finite group of Lie type of Lie rank 2 in characteristic p. Then $|\mathcal{S}_p(G)|_{hG} \simeq B(P_\alpha *_B P_\beta)$ and the Borel fibration $|\mathcal{S}_p(G)|_{hG} \mapsto BG$ corresponds to the canonical map $\phi\colon P_\alpha *_B P_\beta \to G$ under this identification.

Proof. Recall that in this case the inclusion of $\mathcal{D}_p(G)$ into $\mathcal{S}_p(G)$ is a G-equivalence by Proposition 6.5, and induces a homotopy equivalence $|\mathcal{D}_p(G)|_{hG} \to |\mathcal{S}_p(G)|_{hG}$, and thus the result follows by Corollary 6.15. □

6.3. **Lie Rank ≥ 3.** We will finish the discussion of the homotopy type of $|\mathcal{S}_p(G)|_{hG}$ by looking at the case of a finite group of Lie type of rank at least 3.

Lemma 6.17. Let G be a finite group of Lie type in characteristic p of rank at least 3. Then $\pi_1(|\mathcal{S}_p(G)|_{hG}) \cong G$ and $\pi_k(|\mathcal{S}_p(G)|_{hG}) \cong \pi_k(|\mathcal{S}_p(G)|)$.

Proof. Note that for the Borel fibration

$$|\mathcal{S}_p(G)| \to |\mathcal{S}_p(G)|_{hG} \to BG$$

the fiber $|\mathcal{S}_p(G)|$ is by Proposition 6.8 homotopy equivalent to a wedge of $(\mathrm{Rk}_{\mathrm{L}}(G)-1)$-dimensional spheres. In particular it is simply connected. Thus the result comes directly from the long exact sequence in homotopy. □

The space $|\mathcal{S}_p(G)|$ is a wedge of spheres with homotopy groups given as in (2). In particular $|\mathcal{S}_p(G)|$ will have nontrivial homotopy groups in arbitrarily large dimension, when the Lie rank of G is a least 3. A direct consequence of Lemma 6.17 is that in the case of finite groups of Lie type in characteristic p of Lie rank at least 3, the space $|\mathcal{S}_p(G)|_{hG}$ will never be a classifying space for a discrete group.

7. Centralizers and amalgamated products of groups

In the following we consider a group G with a family $(G_i)_{i\in I}$ of subgroups. Let H be the common intersection $\cap_{i\in I} G_i$. The goal is to compare $C_{*_H G_i}(x)$ with $C_G(x)$ for any $x \in H$. The canonical map $\phi\colon *_H G_i \to G$ is the inclusion on H and in particular induces a map on centralizers $C_{*_H G_i}(x) \to C_G(x)$ for any $x \in H$. Our goal is find conditions for this map to be injective respectively surjective.

For a word $g = g_n \cdots g_1 \in *_H G_i$ we let $g_{[k,\ell]} = g_k \cdots g_\ell \in *_H G_i$ for any $1 \leq \ell \leq k \leq n$.

Definition 7.1. A subgroup S of H has finite transporter length in $*_H G_i$, if there exists an $L \in \mathbb{N}$, such that for any reduced word $g = g_L \cdots g_1 \in *_H G_i$ of length L and $x \in S$ satisfying ${}^{g_{[k,1]}}x \in S$ for all $1 \leq k \leq L$, we have that x is the trivial element.

When talking about a subgroup with finite transporter length, we let L be the smallest natural number that satisfies the condition. Note that the definition of finite transporter length says that any non-trivial element of S after conjugating with a reduced word of length L, will no longer be an element of S. A very restrictive condition on the subgroup S, that implies finite transporter length in $*_H G_i$, is that for every word g in $*_H G_i$ of length L, for some L, the intersection $S \cap {}^g S$ is trivial. In general an equivalent condition is

Lemma 7.2. A subgroup S of H has finite transporter length in $*_H G_i$ if and only if for all words $\tilde{g} \in *_H G_i$ of length L we have that

$$S \cap {}^{\tilde{g}_{[L,1]}}S \cap {}^{\tilde{g}_{[L,2]}}S \cap \cdots \cap {}^{\tilde{g}_{[L,L]}}S = 1.$$

Proof. The lemma follows from noticing that for any $x \in S$ and $g = g_L \cdots g_1 \in *_H G_i$, the condition ${}^{g_{[k,1]}}x \in S$ for all $1 \leq k \leq L$ is equivalent to

$${}^g x \in S \cap {}^{g_{[L,1]}}S \cap {}^{g_{[L,2]}}S \cap \cdots \cap {}^{g_{[L,L]}}S.$$

□

For subgroups with finite transporter length the centralizer in $*_H G_i$ has a simple form:

Proposition 7.3. Let S be a subgroup of H, such that S has finite transporter length in $*_H G_i$ and ${}^g S \cap H \leq S$ for any $g \in \cup_{i \in I} G_i$

Let $x \in S$ be nontrivial. Then any $g \in C_{*_H G_i}(x)$ either is in H or is of the form $g = \tilde{g}^{-1} g_0 \tilde{g}$ where $\tilde{g} = g_k \cdots g_1$ is a reduced word in $*_H G_i$ satisfying the further condition that g is a reduced word of length $2k + 1 < L$.

Proof. Let $x \in S$ and let $g \in C_{*_H G_i}(x)$. We assume that $g \notin H$. Every element in the amalgamated product $*_H G_i$ has a normal form, so in particular we may assume $g = h\tilde{g}_n \cdots \tilde{g}_1$ is the normal form for g. To simplify the argument we set $g_k = \tilde{g}_k$ for $1 \leq k \leq n-1$ and $g_n = h\tilde{g}_n$. With this, we then have $g = g_n \ldots g_1$.

Simplifying further, we set $x_i = {}^{g_{[k,1]}}x$ for $1 \leq k \leq n$ and $x_0 = x$. Note that $x_n = x$ as well. By [4, Proposition 2.9] we have that $x_k \in H$ for all $k = 1, \ldots, n$. As $x_1 = {}^{g_1}x$ is an element of ${}^{g_1}S \cap H$ our assumptions imply that $x_1 \in S$ and thus it follows that $x_k \in S$ for all $k = 0, \ldots, n$.

The case when $n = 0$ corresponds to $g \in H$ whereas $n = 1$ corresponds to $g \in G_i$ for some $i \in I$, where g is of the form $\tilde{g}^{-1} g \tilde{g}$ with $\tilde{g}$ being trivial. We will first show that if $n > 1$ and g is not cyclically reduced, then x is trivial. We assume that $n > 0$ and g is not cyclically reduced. For any $K \in \mathbb{N}$ the concatenated word g^K is a reduced word of length Kn and for $1 \leq \ell \leq Kn$ we have that ${}^{g^K_{[\ell,1]}}x = x_{\ell \mod n} \in S$. By choosing K such that $Kn \geq L$ we see that according to definition of finite transporter length this implies that x is trivial.

Now assume that $n > 1$ and g is cyclically reduced. By the previous observation we have that $x_k \in S$ for all $0 \leq k \leq n$. Thus by definition of finite transporter length we conclude that $n < L$ or x is trivial. Words of length two cannot be cyclically reduced, so we can assume that $n \geq 3$.

As before $x_1 = {}^{g_1}x \in S$. From the assumption $g \in C_{*_H G_i}(x)$ we get that $g_1 g_n \cdots g_2 \in C_{*_H G_i}(x_1)$. We note that $g_1 g_n \in G_{i_n}$, and hence $g_1 g_n \cdots g_{n-1}$ is an element in the amalgamated product of shorter length. If $g_1 g_n \in G_{i_n} \setminus H$ we are in the case where $g_n g_1 \cdots g_{n-1}$ has length $n-1$. We observe that $g_n g_1 \cdots g_{n-1}$ is not cyclically reduced. Since $n - 1 > 1$ we have by the previous observation that $x_1 = 1$, and thus $x = 1$. So we can assume that $g_1 g_n = b_1 \in H$. Now $b_1 g_{n-1} \cdots g_2 \in C_{*_H G_i}(x_1)$ has length $n-2$ and by repeated uses of the previous argument we get that either x is trivial or there exist $b_0, \ldots, b_k \in H$ where $n = 2k+1$ such that $b_0 = 1$ and $b_{\ell-1} g_{n+1-\ell} = g_\ell{}^{-1} b_\ell$ for $1 \leq \ell \leq k$. Thus,

$$\begin{aligned} g &= g_n \cdots g_1 = b_0 g_n \cdots g_1 = g_1^{-1} b_1 g_{n-1} \cdots g_1 = g_1^{-1} g_2^{-1} b_2 g_{n-3} \cdots g_1 \\ &= (g_k \cdots g_1)^{-1} (b_k g_{k+1})(g_k \cdots g_1) \end{aligned}$$

We note that $b_k g_{k+1}$ is an element of $G_{i_{k+1}} \setminus H$, and so up to a re-indexing of the above expression, we have shown that g is of the desired form. □

Note that we have not assumed that for $i, j \in I$ the intersection $G_i \cap G_j$ is equal to H. If there exists a $g \in (G_i \cap G_j) \setminus H$ then the map $\phi \colon *_H G_i \to G$ is not injective, but under our assumption on the subgroup S they will not be letters in centralizers for nontrivial elements in S.

Lemma 7.4. Let S be a subgroup of H, such that S has finite transporter length in $*_H G_i$ and ${}^gS \cap H \leq S$ for any $g \in \cup_{i \in I} G_i$.

If $g \in (G_i \cap G_j) \setminus H$ for $i, j \in I$ with $i \neq j$, then ${}^gS \cap H = 1$. If $x \in S$ and $\tilde{g} = g_n \ldots g_1 \in C_{*_H G_i}(x)$ is a word in normal form where some $g_k \in (G_i \cap G_j) \setminus H$ for $i, j \in I$ with $i \neq j$, then $x = 1$.

Proof. Assume that there exists a $g \in (G_i \cap G_j) \setminus H$ for $i, j \in I$ with $i \neq j$. If we pick a $x \in S$ such that ${}^gx \in H$, then by assumption ${}^gx \in S$. Let $\tilde{g} = \tilde{g}_N \cdots \tilde{g}_1$ where $\tilde{g}_k = g$ for k odd and $\tilde{g}_k = g^{-1}$ for k even. Then $\tilde{g}$ is a reduced word in $*_H G_i$ and for all $1 \leq k \leq L$ we have that ${}^{g_{[k,1]}}x$ is either x or ${}^gx$ and in particular elements of S. As S has finite transporter length, we conclude that x is trivial.

If $x \in S$ and $\tilde{g} = g_n \ldots g_1 \in C_{*_H G_i}(x)$ is a word in normal form, then we see using the argument from Proposition 7.3 that ${}^{g_{[k,1]}}x \in {}^{g_k}S \cap H$ for all $1 \leq k \leq n$. Thus, if it satisfies that $g_k \in (G_i \cap G_j) \setminus H$ for $i, j \in I$ with $i \neq j$, then $x = 1$. □

Theorem 7.5. *Let S be a subgroup of H, such that S has finite transporter length in $*_H G_i$ and ${}^gS \cap H \leq S$ for any $g \in \cup_{i \in I} G_i$. Then the canonical map $\phi \colon *_H G_i \to G$ induces an injective map on $C_{*_H G_i}(x) \to C_G(x)$ for any nontrivial $x \in S$.*

Proof. Let $x \in S$. As $S \leq H$ we have that $\phi(x) = x$ and ϕ does induce a map from $C_{*_H G_i}(x)$ to $C_G(x)$.

To prove that ϕ is injective on the centralizers, we consider $g \in C_{*_H G_i}(x)$ and assume $\phi(g) = 1$. By Proposition 7.3 we have that either $g \in H$ or $g = \tilde{g}^{-1} g_0 \tilde{g} \in *_H G_i$ is a reduced word of length $2k+1$. Here, $\tilde{g} = g_1 \cdots g_k$ for some $k \geq 0$ is a reduced word of length k and $g_0 \in G_i$ for some $i \in I$. The map ϕ restricted to H is the inclusion and in particular injective. Thus, if $g \in H$ with $\phi(g) = 1$ we have that

$g = 1$. If $g = \tilde{g}^{-1}g_0\tilde{g}$ we see that

$$1 = \phi(g) = \phi(\tilde{g}^{-1}g_0\tilde{g}) = \phi(\tilde{g})^{-1}\phi(g_0)\phi(\tilde{g}) = \phi(g_k)^{-1}\cdots\phi(g_1)^{-1}\phi(g_0)\phi(g_1)\cdots\phi(g_k)$$

and hence $1 = \phi(g_1)\cdots\phi(g_k)\phi(g_k)^{-1}\cdots\phi(g_1)^{-1} = \phi(g_0)$. As ϕ is injective on G_i for all $i \in I$, we conclude that $g_0 = 1$ which implies that $g = 1$. Thus we conclude that ϕ is injective on $C_{*_H G_i}(x)$. □

Corollary 7.6. Let S be a subgroup of H, such that ${}^gS \cap H \leq S$ for any $g \in \cup_{i\in I}G_i$. If for all words $\tilde{g} \in *_H G_i$ of length L we have that

$$S \cap {}^{\tilde{g}_{[L,1]}}S \cap {}^{\tilde{g}_{[L,2]}}S \cap \cdots \cap {}^{\tilde{g}_{[L,L]}}S = 1.$$

then the map $\phi\colon *_H G_i \to G$ induces an injective map on $C_{*_H G_i}(x) \to C_G(x)$ for any nontrivial $x \in S$.

Proof. This is immediate by combining Theorem 7.5 with Lemma 7.2. □

The above Theorem implies that we may consider $C_{*_H G_i}(x)$ as a subgroup of $C_G(x)$ for $x \in S$ under certain conditions. Note that the map being surjective on centralizers is a stronger statement than $\phi\colon *_H G_i \to G$ itself being surjective. In this case there exists a $\tilde{g} \in \phi^{-1}(g)$ for $g \in C_G(x)$, but we might have that x and ${}^{\tilde{g}}x$ are not identified in $*_H G_i$ and thus $\tilde{g}$ is not an element of $C_{*_H G_i}(x)$. To get an isomorphism, we therefore have to check surjectivity as well. The following proposition will tackle the problem in more generality by giving a condition on when conjugacies in G can be lifted to $*_H G_i$.

Proposition 7.7. Let $(G_i)_{i\in I}$ be a family of subgroups of a group G , and let $H = \cap_{i\in I}G_i$. Let $\phi\colon *_H G_i \to G$ be the canonical map, and let P and Q be subgroups of H. Then the map

$$\phi\colon T_{*_H G_i}(P,Q) \to T_G(P,Q)$$

is surjective if and only if for every $g \in T_G(P,Q)$ there exists a $\tilde{g} = \tilde{g}_n \cdots \tilde{g}_1 \in *_H G_i$ with $\phi(\tilde{g}) = g$ and ${}^{\phi(\tilde{g}_{[k,1]})}P \leq H$ for all $1 \leq k \leq n$.

Proof. Assume ϕ is surjective on the transporter set. Let $g \in T_G(P,Q)$ and $\tilde{g} \in T_{*_H G_i}(P,Q)$ with $\phi(\tilde{g}) = g$. Pick an $x \in P$ and set $y = {}^{\tilde{g}}x \in H$. If $\tilde{g} = \tilde{g}_n \cdots \tilde{g}_1 \in *_H G_i$ is in normal form, then

$$\tilde{g}_n \cdots \tilde{g}_1 x = y\tilde{g}_n \cdots \tilde{g}_1.$$

The right hand side is in normal form, while the left hand side is not if x is non-trivial. Every element in $*_H G_i$ has a unique normal form, so the normal form of the left hand side is just the right hand side. Just as in the proof of [4, Proposition 2.9] this implies that ${}^{\tilde{g}_{[k,1]}}x$ are elements of H (seen as a subgroup of $*_H G_i$) for all $1 \leq k \leq n$. In particular they are identified with their image under ϕ.

Conversely let $g \in T_G(P,Q)$, and assume that there exists $\tilde{g} = \tilde{g}_n \cdots \tilde{g}_1 \in *_H G_i$ with $\phi(\tilde{g}) = g$ and ${}^{\phi(\tilde{g}_{[k,1]})}P \leq H$ for all $1 \leq k \leq n$. We need to show that for every $x \in P$ we have that ${}^{\tilde{g}}x = {}^{g}x$ in $*_H G_i$. This follows directly from the following claim.

Let $\tilde{g}_{[0,1]} = 1$. We want to show that for all $0 \leq k \leq n$ we have ${}^{\tilde{g}_{[k,1]}}x = {}^{\phi(\tilde{g}_{[k,1]})}x$ is in H (seen as a subgroup of $*_H G_i$). For $k = 0$ the statement is clear, and we assume it holds for a given $0 \leq k < n$. We have that ${}^{\tilde{g}_{[k+1,1]}}x = {}^{\tilde{g}_{k+1}}({}^{\tilde{g}_{[k,1]}}x)$. Here ${}^{\tilde{g}_{[k,1]}}x \in H$ and $\tilde{g}_{k+1} \in G_i$ for some $i \in I$. As $H \leq G_i$ the conjugation ${}^{\tilde{g}_{k+1}}({}^{\tilde{g}_{[k,1]}}x)$

takes place inside G_i. The map ϕ is injective on G_i, and hence we have that ${}^{\tilde{g}_{k+1}}({}^{\tilde{g}_{[k,1]}}x)$ is identified with its image under ϕ. Thus,

$${}^{\tilde{g}_{[k+1,1]}}x = {}^{\tilde{g}_{k+1}}({}^{\tilde{g}_{[k,1]}}x) = \phi({}^{\tilde{g}_{k+1}}({}^{\tilde{g}_{[k,1]}}x)) = {}^{\tilde{g}_{k+1}}({}^{\phi(\tilde{g}_{[k,1]})}x) = {}^{\phi(\tilde{g}_{[k+1,1]})}x.$$

We therefore conclude that ${}^{\tilde{g}_{[k+1,1]}}x = {}^{\phi(\tilde{g}_{[k+1,1]})}x \in H \leq *_H G_i$. □

Note that the condition for the trivial group reduces to the fact that ϕ itself is surjective. The following proposition will provide a sufficient condition for the map on transporter sets to be surjective for nontrivial subgroups.

Proposition 7.8. Let $(G_i)_{i\in I}$ be a family of subgroups of a group G, and let $H = \cap_{i\in I} G_i$. Let $\phi\colon *_H G_i \to G$ be the canonical map. Assume there exists a subgroup S of H, such that for any $g \in G$ there exists a $\tilde{g} = \tilde{g}_n \cdots \tilde{g}_1 \in *_H G_i$ with $\phi(\tilde{g}) = g$ and

$$S \cap {}^gS = S \cap {}^{\phi(\tilde{g}_{[n,1]})}S \cap {}^{\phi(\tilde{g}_{[n,2]})}S \cap \cdots \cap {}^{\phi(\tilde{g}_{[n,n]})}S.$$

Then for any non-trivial subgroups $P, Q \leq S$ the map

$$\phi\colon T_{*_H G_i}(P,Q) \to T_G(P,Q)$$

is surjective.

Proof. Let P and Q be non-trivial subgroups of S and let $g \in T_G(P,Q)$. Pick a $\tilde{g} = \tilde{g}_n \cdots \tilde{g}_1 \in *_H G_i$ satisfying the conditions in the Proposition. According to Proposition 7.7 we need to show that ${}^{\phi(\tilde{g}_{[k,1]})}P \leq H$ for all $1 \leq k \leq n$. For $k = n$ we see

$${}^{\phi(\tilde{g}_{[n,1]})}P = {}^{\phi(\tilde{g})}P = {}^gP \leq S \leq H$$

We fix a $k \in \{1, \ldots, n-1\}$. Now

$${}^{\phi(\tilde{g})}P \leq S \cap {}^gS \leq {}^{\phi(\tilde{g}_{[n,k+1]})}S$$

and thus ${}^{\phi(\tilde{g}_{[k,1]})}P = {}^{\phi(\tilde{g}_{[n,k+1]}^{-1}\tilde{g})}P \leq S \leq H$. □

Corollary 7.9. Let $(G_i)_{i\in I}$ be a family of subgroups of a group G, and let H be the intersection $\cap_{i\in I} G_i$. Let $\phi\colon *_H G_i \to G$ be the canonical map. Assume there exists a subgroup S of H, such that for any $g \in G$ there exists a $\tilde{g} = \tilde{g}_n \cdots \tilde{g}_1 \in *_H G_i$ with $\phi(\tilde{g}) = g$ and

$$S \cap {}^gS = S \cap {}^{\phi(\tilde{g}_{[n,1]})}S \cap {}^{\phi(\tilde{g}_{[n,2]})}S \cap \cdots \cap {}^{\phi(\tilde{g}_{[n,n]})}S.$$

Then for any non-trivial subgroups $P \leq S$ the maps

$$\phi\colon N_{*_H G_i}(P) \to N_G(P), \quad \phi\colon C_{*_H G_i}(P) \to C_G(P)$$

are surjective.

Proof. Setting $P = Q$ in Proposition 7.8 is sufficient to ensure that ϕ is surjective on normalizers. For centralizers, recall that for subgroups P and Q of H, they can be identified with their image under ϕ. Then, for any $g \in T_{*_H G_i}(P,Q)$ the conjugation maps c_g and $c_{\phi(g)}$ agree as maps between subgroups of $*_H G_i$. So the map of normalizers restricts to a map of centralizers. In particular, it is surjective when the map on normalizers is surjective. □

8. Centralizers and finite groups of Lie type

In this section we will show how the centralizers of p-subgroups in a finite group of Lie type of rank 1 or 2 can be be determined using the proper parabolic subgroups.

8.1. **The rank one case.** For a finite group of Lie type in characteristic p of rank 1, we let B be a Borel subgroup of G and $S := O_p(B)$. Note that $S \in \mathrm{Syl_p}(G)$.

Lemma 8.1. Let G be a finite Lie group of rank 1 in characteristic p. For any nontrivial subgroup $P \leq S$ and $g \in G$ we have that if ${}^gP \leq S$, then $g \in B$.

Proof. Let P be a nontrivial subgroup of S and let $g \in G$ such that ${}^gP \leq S$. As we assume G is a finite group of Lie type of rank 1, the Weyl group is isomorphic to $\mathbb{Z}/2$. Let $t \in N \leq G$ be a preimage of the generator of the Weyl group. By the Bruhat decomposition for G [18, Proposition 2.3.5], we have that $G = B \sqcup BtB$. Assume, for the purpose of contradiction, that $g \in BtB$, i.e that there exist $b, \tilde{b} \in B$ such that $g = bt\tilde{b}$. Since S is normal and a Sylow-p-subgroup of B, it consists of all p-elements of B. Thus, we have that ${}^gP \leq S \cap {}^gS$, and

$$ {}^gP \leq {}^b\left(S \cap {}^{t\tilde{b}}S\right) = {}^b\left(S \cap {}^tS\right). $$

In the case of rank 1, we have that t is the longest word in the Weyl group, and hence by [18, Theorem 2.3.8 (c)] we have that $S \cap {}^tS = 1$. This implies that P is trivial, which is a contradiction. Thus, we conclude that $g \notin BtB$ and by the Bruhat decomposition we get that $g \in B$. □

Corollary 8.2. Let G be a finite group of Lie type of rank 1 in characteristic p. For any nontrivial p-subgroup P of S we have that $C_B(P) = C_G(P)$.

Proof. As B is a subgroup of G, the inclusion induces a injective map $C_B(P) \to C_G(P)$. By Lemma 8.1 this is surjective as well, when P is nontrivial. □

Corollary 8.3. Let G be a finite group of Lie type of rank 1 in characteristic p. Let $\iota\colon B \to G$ be the inclusion. For any p-group P we have that $\beta_P\colon \mathrm{Rep}(P, B) \to \mathrm{Rep}(P, G)$ given by $[\rho] \mapsto [\iota \circ \rho]$ is a bijection that is natural in P.

Proof. Let P be a p-group, and β_P be given as above. Clearly β_P is well-defined. As S is a Sylow-p-subgroup of both B and G we have that any image of P in either group is conjugate to a subgroup of S. This implies that every class in $\mathrm{Rep}(P, B)$ as well as $\mathrm{Rep}(P, G)$ has a representative in $\mathrm{Hom}(P, S)$. The map β_P is the identity on representatives in $\mathrm{Hom}(P, S)$ and thus surjective by the above remark.

To see that is it injective as well, we consider $[\rho_1], [\rho_2] \in \mathrm{Rep}(P, B)$ where $\rho_1, \rho_2 \in \mathrm{Hom}(P, S)$ and assume that $[\rho_1] = [\rho_2] \in \mathrm{Rep}(P, G)$. Then there exists a $g \in G$, such that $\rho_1 = c_g \circ \rho_2$. If ρ_2 is the trivial map, then the same is true for ρ_1 and clearly $[\rho_1] = [\rho_2] \in \mathrm{Rep}(P, B)$. If we assume that ρ_2 is nontrivial, then the subgroup $P' := \rho_2(P)$ is a nontrivial subgroup of S such that ${}^g(P') = \rho_1(P)$ is a subgroup of S as well. By Lemma 8.1 this implies that $g \in B$, and thus $[\rho_1] = [\rho_2] \in \mathrm{Rep}(P, B)$.

It follows immediately from the definition of β_P that precomposing with any homomorphism of p-groups makes this natural in P. □

8.2. **Centralizers of amalgamated products of parabolic subgroups in finite groups of Lie type.** In this part we restrict ourselves to the case where G is a finite group of Lie type in characteristic p of rank 2. We let $\Delta = \{\alpha, \beta\}$ be a basis of simple roots for the root system for the Weyl group of G and for all $\gamma \in \Delta$ we let $P_\gamma = P_{\{\gamma\}}$ be the associated parabolic subgroup. Then B is a subgroup of P_γ for all $\gamma \in \Delta$ and we can consider the amalgamated product $P_\alpha *_B P_\beta$. Note that in this case $B = P_\alpha \cap P_\beta$.

Our goal is to compare $C_{P_\alpha *_B P_\beta}(Q)$ with $C_G(Q)$ for any non-trivial subgroup Q in $S \in \mathrm{Syl_p}(G)$. This is motivated by the fact that $|\mathcal{S}_p(G)|_{hG}$ in this case is a classifying space for $P_\alpha *_B P_\beta$.

First, we give a description of the words in $P_\alpha *_B P_\beta$.

Lemma 8.4. For $\gamma \in \Delta$ let U_γ be the root subgroup associated to γ. Then the set $\{1, t_\gamma x \mid x \in U_\gamma\}$ is a transversal for the cosets $B\backslash P_\gamma$.

Proof. For any $\gamma \in \Delta$ recall that the parabolic subgroup satisfies $P_\gamma = B \cup Bt_\gamma B$. A variation on the Bruhat normal form [18, Theorem 2.3.5] implies that every element in P_γ is an element of B or can be written uniquely as $bt_\gamma x$ where $b \in B$ and $x \in S_\gamma$, where the group S_γ by [18, Theorem 2.3.8] is a product of root subgroups of positive roots that are made negative by the action of s_γ. According to [18, Theorem 1.8.3] we have that S_γ is exactly the root subgroup U_γ. Thus, the set $\{1, t_\gamma x \mid x \in U_\gamma\}$ is a set of representatives of the cosets $B\backslash P_\gamma$. □

If we use the transversals for the cosets from the above lemma to define the normal form of an element, then every $g \in P_\alpha *_B P_\beta$ has a unique form

$$bt_{\gamma_k} x_{\gamma_k} \cdots t_{\gamma_1} x_{\gamma_1}$$

where $b \in B$ and for $1 \leq i \leq k$ we have that $\gamma_i \in \Delta$, $x_{\gamma_i} \in U_{\gamma_i}$ and $\gamma_i \neq \gamma_{i+1}$.

Recall that the Weyl group is a dihedral group in the case of a finite group of Lie type of Lie rank 2. When the Weyl group W is a dihedral group of order $2L$, the following proposition gives an alternative form of elements of length at most L. It shows that the Bruhat normal form in G lifts to $P_\alpha *_B P_\beta$. The proof is similar to the process for finding the Bruhat normal form for an element, but care needs to be taken to ensure that the identifications still hold in the amalgamated product.

Proposition 8.5. Let G be a finite group of Lie type of Lie rank 2. Let L be the length of the longest word in the Weyl group W, i.e $W \cong D_{2L}$. For $g = g_k \cdots g_1$ a reduced word in $P_\alpha *_B P_\beta$ of length $k \leq L$, we let $n = t_k \cdots t_1$ be the word in t_α and t_β of length k such that t_i and g_i belongs to the same parabolic subgroup P_i. Let w be the image of n in the Weyl group.

Then there exists a $b \in B$ and $x \in \prod_{\{\gamma \in \Phi^+ \mid w.\gamma \in \Phi^-\}} U_\gamma$ such that

$$g = bnx \in P_\alpha *_B P_\beta.$$

Proof. Recall that the root system Φ is the disjoint union of the positive roots Φ^+ and the negative roots Φ^-. An element $w \in W$ permutes the roots, and for convenience we let

$$\Phi(w) = \{\gamma \in \Phi^+ \mid w.\gamma \in \Phi^-\}.$$

If $w = s_n \cdots s_1$ is a reduced expression and for $1 \leq i \leq n$ we fix $\gamma_i \in \Delta$ such that $s_i = s_{\gamma_i}$, then according to [18, Theorem 1.8.3 (e)] we have

$$\Phi(w) = \{\gamma_1, s_1 \cdots s_k.\gamma_{k+1} \mid 1 \leq k \leq n-1\}.$$

In particular if a word of the form $\tilde{w}s_{\tilde{\gamma}}$ is reduced, the above description implies that

$$\Phi(\tilde{w}s_{\tilde{\gamma}}) = s_{\tilde{\gamma}}.\Phi(\tilde{w}) \cup \{\tilde{\gamma}\}.$$

For $k = 0$ the claim is immediate, as elements of length zero are in B. Assume that for some $k \leq N$ we have that every element in $P_\alpha *_B P_\beta$ of length $k-1$ has the given form. Let g be a word of length k. Writing g in normal form, we have that

$$g = \tilde{g} t_{\tilde{\gamma}} x_{\tilde{\gamma}}$$

where $\tilde{g}$ has length $k-1$ and $\tilde{\gamma} \in \Delta$ and $x_{\tilde{\gamma}} \in U_{\tilde{\gamma}}$. Using our assumption on $\tilde{g}$, we have that there exists an alternating word $\tilde{n}$ in t_α and t_β of length $k-1$ with image $\tilde{w} \in W$, such that

$$\tilde{g} = b\tilde{n}x, \quad b \in B, x \in \prod_{\gamma \in \Phi(\tilde{w})} U_\gamma.$$

Furthermore, the word $\tilde{n}$ does not end in $t_{\tilde{\gamma}}$. Thus, $\tilde{n}t_{\tilde{\gamma}}$ must be an alternating word of t_α, t_β of length k. The image of $\tilde{n}t_{\tilde{\gamma}}$ in the Weyl group is $\tilde{w}s_{\tilde{\gamma}}$. This is a reduced word, and thus the length with respect to the basis Δ is also k. According to [18, Theorem 1.8.3 (d)] this implies that $\tilde{w}.\tilde{\gamma} \in \Phi^+$. In particular we have that $\tilde{\gamma} \notin \Phi(\tilde{w})$. The only positive root made negative by s_γ is γ itself by [18, Theorem 1.8.3 (c)], and so

$$s_{\tilde{\gamma}}.\Phi(\tilde{w}) \subset \Phi^+.$$

Recall that conjugating a root subgroup U_γ by $t_{\tilde{\gamma}}$ is described by the $s_{\tilde{\gamma}}$-action on the associated root following [18, Theorem 2.3.8 (b)]. Thus, we have that

$$\left(\prod_{\gamma \in \Phi(\tilde{w})} U_\gamma \right) t_{\tilde{\gamma}} = t_{\tilde{\gamma}} \left(\prod_{\gamma \in \Phi(\tilde{w})} U_{s_{\tilde{\gamma}}.\gamma} \right).$$

The product $\prod_{\gamma \in \Phi(\tilde{w})} U_{s_{\tilde{\gamma}}.\gamma}$ is a product of root subgroups over positive roots, and thus a subset of B. In particular we have that there exists $\tilde{x} \in \prod_{\gamma \in \Phi(\tilde{w})} U_{s_{\tilde{\gamma}}.\gamma}$, such that $xt_{\tilde{\gamma}} = t_{\tilde{\gamma}}\tilde{x}$. This is an equation of elements of B, and hence we also have that $xt_{\tilde{\gamma}} = t_{\tilde{\gamma}}\tilde{x} \in P_\alpha *_B P_\beta$. We conclude that as elements of $P_\alpha *_B P_\beta$ we have that

$$g = b\tilde{w}t_\gamma \tilde{x}x.$$

Here $\tilde{x}x$ is an element in the product of root subgroups corresponding to roots in set $s_{\tilde{\gamma}}.\Phi(\tilde{w}) \cup \{\tilde{\gamma}\}$. The proposition now follows, since we observed that $\Phi(\tilde{w}s_{\tilde{\gamma}}) = s_{\tilde{\gamma}}.\Phi(\tilde{w}) \cup \{\tilde{\gamma}\}$. □

Injectivity. First we will use the result from chapter 7 on generalized amalgamated products to prove that the induced map between the two centralizers is injective.

Proposition 8.6. Let G be a finite group of Lie type of Lie rank 2 of characteristic p. Then S (seen as a subgroup of $P_\alpha *_B P_\beta$) has finite transporter length as defined in Definition 7.1 with L the maximal length of a word in the Weyl group W.

Proof. Let g be an element of $P_\alpha *_B P_\beta$ of length L. Then it has a Bruhat form in $P_\alpha *_B P_\beta$ in the sense of Proposition 8.5, i.e. there exist $b, \tilde{b} \in B$ such that $g = bn\tilde{b}$ where n is a reduced word in t_α, t_β of length L. In particular $\phi(n) \in N \leq G$ and under the map to the Weyl group it is mapped to the longest element w_0.

We want to show that the intersection $S \cap {}^gS$ is trivial in $P_\alpha *_B P_\beta$. The map ϕ is injective on B. As this intersection in contained in B, it is suffient to show that the image is trivial in G. Using the fact that S is normalized by B, we see that

$$\phi(S \cap {}^gS) = S \cap {}^{\phi(g)}S = {}^b\left(S \cap {}^{\phi(n)}S\right).$$

We observed that $\phi(n)$ is mapped to the longest element in W, and hence it follows directly by [18, Theorem 2.3.8 (c)] that the given intersection is trivial in G. We see that

$$S \cap {}^{\tilde{g}_{[L,1]}}S \cap {}^{\tilde{g}_{[L,2]}}S \cap \cdots \cap {}^{\tilde{g}_{[L,L]}}S \leq S \cap {}^gS = 1$$

and conclude by Lemma 7.2 that S has finite transporter length in $P_\alpha *_B P_\beta$. □

Using Theorem 7.5 we will now provide a description of the centralizers in $P_\alpha *_B P_\beta$.

Corollary 8.7. Let G be a finite group of Lie type of Lie rank two. Then the canonical map $\phi\colon P_\alpha *_B P_\beta \to G$ induces an injective map on $C_{P_\alpha *_B P_\beta}(x) \to C_G(x)$ for any non-trivial $x \in S$.

Proof. We have that S is a normal Sylow-p-subgroup of B, and thus it consists of exactly all elements of p-power order in B. For any $g \in P_\alpha$ or P_β we have that ${}^gS \cap B$ is a p-subgroup of B, and in particular we see that ${}^gS \cap B \leq S$.

By Proposition 8.6, S has finite transporter length in $P_\alpha *_B P_\beta$, and hence the statement now follows directly from Theorem 7.5. □

Surjectivity. To solve the question concerning surjectivitity, we will use the decomposition of S into root subgroups in connection with the action of the Weyl groups on the roots. The following lemma is about general root systems, but will be central to our further arguments.

Lemma 8.8. Let Φ be a root system, W its Weyl group and Δ a basis for Φ. Let $w = s_k \cdots s_1$ where $s_i \in \{s_\alpha \mid \alpha \in \Delta\}$. If the given expression for w is reduced, then $\omega.\Phi^+ \cap \Phi^+ \subset s_k \cdots s_2.\Phi^+ \cap \Phi^+$.

Proof. Let $w = s_k \cdots s_1$ be reduced. Recall that an expression in terms of elements in the Weyl group associated to the basis elements from Δ is reduced if it is of minimal length. Then $\tilde{w} = s_k \cdots s_2$ is reduced as well, and has length $k-1$. By [18, Theorem 1.8.3(d)] we have that $\tilde{w}.\alpha \in \Phi^+$, and thus

$$w.\alpha = (\tilde{w}s_1).\alpha = \tilde{w}.(-\alpha) = -\tilde{w}.\alpha \in \Phi^-.$$

The only positive root made negative by s_α is α itself by [18, Theorem 1.8.3(c)], and so $s_\alpha.\Phi^+ = \Phi^+ \setminus \{\alpha\} \cup \{-\alpha\}$. From this it follows that

$$w.\Phi^+ = (\tilde{w}s_1).\Phi^+ = \tilde{w}.(\Phi^+ \setminus \{\alpha\} \cup \{-\alpha\}) = \tilde{w}.(\Phi^+ \setminus \{\alpha\}) \cup \{w.\alpha\}.$$

Thus we conclude that

$$w.\Phi^+ \cap \Phi^+ = \tilde{w}.(\Phi^+ \setminus \{\alpha\}) \cap \Phi^+ \subset \tilde{w}.\Phi^+ \cap \Phi^+.$$

□

Lemma 8.9. Let G be a finite group of Lie type in characteristic p. Consider any $n \in N$ of the form $n = t_k \cdots t_1$ with $t_i \in \{t_\alpha \mid \alpha \in \Delta\}$ for $1 \leq i \leq k$. Let $n_\ell = t_k \cdots t_\ell$ for $1 \leq \ell \leq k$. If the corresponding element $w = s_k \cdots s_1 \in W$ is reduced, then

$$S \cap {}^nS = S \cap {}^{n_1}S \cap {}^{n_2}S \cap \cdots \cap {}^{n_k}S.$$

Proof. By [18, Theorem 2.3.6] the group S decomposes as the product of the root subgroups corresponding to the positive roots, i.e $S = \prod_{\gamma \in \Phi^+} U_\gamma$. Note that the product can be taken in any given order. The element n is mapped to w in the Weyl group $W = N/(N \cap B)$. Recall that the action of n on a root subgroup U_γ is described by the w-action on the associated root, in the form of ${}^nU_\gamma = U_{w.\gamma}$ by [18, Theorem 2.3.7 (b)]. From this we get that

$${}^nS = \prod_{\gamma \in \Phi^+} {}^nU_\gamma = \prod_{\gamma \in \Phi^+} U_{w.\gamma}$$

In the decomposition of S we may assume that the ordering of the roots is such that

$$^{n}S = \left(\prod_{\gamma\in w.\Phi^+\cap\Phi^+} U_\gamma\right)\left(\prod_{\gamma\in w.\Phi^+\cap\Phi^-} U_\gamma\right).$$

Let S_- be the Sylow-p-subgroup of the opposite Borel group B_-. Then S_- is the product of all the root subgroup associated to negative roots Φ^-. By [18, Theorem 2.3.8(c)] we have that $S\cap S_- = 1$, and hence

$$S\cap {}^{n}S = \prod_{\gamma\in w.\Phi^+\cap\Phi^+} U_\gamma.$$

Let $w_2 = s_k\cdots s_2$. By Lemma 8.8 we have that $w.\Phi^+\cap\Phi^+ \subset w_2.\Phi^+\cap\Phi^+$ and therefore

$$S\cap {}^{n}S = \prod_{\gamma\in w.\Phi^+\cap\Phi^+} U_\gamma \subset \prod_{\gamma\in w_2.\Phi^+\cap\Phi^+} U_\gamma \subset {}^{n_2}S.$$

Thus, $S\cap {}^{n}S = S\cap {}^{n}S\cap {}^{n_2}S$. By repeated application of this argument, the lemma now follows. □

Corollary 8.10. Let G be a finite group of Lie type in characteristic p of Lie rank 2. Let $\phi\colon P_\alpha *_B P_\beta \to G$ the canonical map. For any $g\in G$ there exists a word $\tilde{g} = \tilde{g}_k\cdots\tilde{g}_1 \in P_\alpha *_B P_\beta$ with $\phi(\tilde{g}) = g$ and

$$S\cap {}^{g}S = S\cap {}^{\phi(\tilde{g}_{[k,1]})}S\cap {}^{\phi(\tilde{g}_{[k,2]})}S\cap\cdots\cap {}^{\phi(\tilde{g}_{[k,k]})}S.$$

Proof. Let $g\in G$. By the Bruhat normal form for G there exist $b,\tilde{b}\in B$ and $n\in N$ such that $g = bn\tilde{b}$. As N is generated by t_α ,t_β and $N\cap B$, we may write this as $n = t_k\cdots t_1$ where $t_i\in\{t_\alpha, t_\beta\}$ for $1\le i\le k$. Without loss of generality we may assume that the corresponding expression $s_k\cdots s_1\in W$ is reduced.

Now we consider $\tilde{g} = bt_k\cdots t_1\tilde{b}\in P_\alpha *_B P_\beta$. Observe that $\phi(\tilde{g}) = g$. Since S is normal in B the desired result now follows from Lemma 8.9. □

Corollary 8.11. Let G be a finite group of Lie type in characteristic p of Lie rank two. Then for every subgroup $P, Q\le S$ the canonical map $\phi\colon P_\alpha *_B P_\beta\to G$ induces a surjective map on

$$T_{P_\alpha *_B P_\beta}(P,Q)\to T_G(P,Q),\quad N_{P_\alpha *_B P_\beta}(P)\to N_G(P)$$
$$C_{P_\alpha *_B P_\beta}(P)\to C_G(P).$$

Proof. The group G is generated by P_α and P_β, and so the map ϕ itself is surjective. Thus, in the induced map for P the trivial element is surjective.

Let P and Q be non-trivial subgroups of S. Then the maps on transporter set, normalizers as well as centralizers is surjective by Proposition 7.8 and Corollary 7.9 in connection with Corollary 8.10. □

Based on the previous results we are now able to provide the desired description of the centralizers of non-trivial p-subgroups of a finite group of Lie type of rank 2.

Theorem 8.12. *Let G be a finite group of Lie type in characteristic p of Lie rank 2. Let Q be a nontrivial p-subgroup of S. Then the canonical map $\phi\colon P_\alpha *_B P_\beta\to G$ induces an isomorphism on $C_{P_\alpha *_B P_\beta}(Q)\to C_G(Q)$.*

Proof. As Q is a non-trivial subgroup of S, we get using Corollary 8.7 and 8.11 that the map $\phi\colon C_{P_\alpha *_B P_\beta}(x) \to C_G(x)$ is an isomorphism for $x \in Q \setminus \{1\}$. This implies

$$C_{P_\alpha *_B P_\beta}(Q) = \bigcap_{x\in Q\setminus\{1\}} C_{P_\alpha *_B P_\beta}(x) \cong_\phi \bigcap_{x\in Q\setminus\{1\}} C_G(x) = C_G(Q),$$

which completes the proof. □

Note that for the trivial group the corresponding map is ϕ itself, which is surjective but not injective.

Lemma 8.13. Let G be a finite group of Lie type in characteristic p of Lie rank 2 and $\phi\colon P_\alpha *_B P_\beta \to G$ the canonical map. Then for any finite p-group P the map $F_P\colon \operatorname{Rep}(P, P_\alpha *_B P_\beta) \to \operatorname{Rep}(P, G)$ given by $[\rho] \mapsto [\phi \circ \rho]$ is a bijection that is natural in P with respect to homeomorphisms.

Furthermore, any class in both $\operatorname{Rep}(P, P_\alpha *_B P_\beta)$ and $\operatorname{Rep}(P, G)$ has a representative in $\operatorname{Hom}(P, S)$ and restricting F_P to such representatives is induced by the identity on $\operatorname{Hom}(P, S)$.

Proof. For every $\rho\colon P \to P_\alpha *_B P_\beta$ and $g \in P_\alpha *_B P_\beta$ we have that $\phi(c_g\rho) = c_{\phi(g)}(\phi\rho)$. Thus, the given map $\operatorname{Rep}(P, P_\alpha *_B P_\beta) \to \operatorname{Rep}(P, G)$ is well-defined.

As P is a p-group and $S \in \operatorname{Syl_p}(G)$ we conclude by Sylow's theorems that every $[\rho] \in \operatorname{Rep}(P, G)$ has a representative satisfying that $\rho(P) \le S$. Likewise consider a $\rho\colon P \to P_\alpha *_B P_\beta$. Then the image $\rho(P)$ is a finite group, and by [36, I 4.3] we have that $\rho(P)$ is conjugate to a subgroup of P_α or P_β. We have that $\rho(P)$ is a p-group and S is a Sylow-p-subgroup of both P_α and P_β, and hence it follows by Sylow's theorems that $\rho(P)$ is conjugate in $P_\alpha *_B P_\beta$ to a subgroup of S. In particular every class $[\rho] \in \operatorname{Rep}(P, P_\alpha *_B P_\beta)$ has a representative with $\rho(P) \le S$. Furthermore, ϕ is the identity on S, and so the map is question is induced by the identity on $\operatorname{Hom}(P, S)$. From this is follows clearly that the map is surjective.

To prove injectivity consider $[\rho], [\tilde\rho] \in \operatorname{Rep}(P, P_\alpha *_B P_\beta)$ with $\rho, \tilde\rho \in \operatorname{Hom}(P, S)$, such that $[\rho] = [\tilde\rho] \in \operatorname{Rep}(P, S)$. Then there exists a $g \in G$ such that $c_g\rho = \tilde\rho$. If ρ is the trivial map, then clearly $[\rho] = [\tilde\rho] \in \operatorname{Rep}(P, P_\alpha *_B P_\beta)$, and hence we may assume ρ is non-trivial.

Now $g \in T_G(\rho(P), S)$ with $\rho(P)$ a non-trivial subgroup of S. According to Proposition 7.8 in connection with Corollary 8.10 the map $T_{P_\alpha *_B P_\beta}(\rho(P), S) \to T_G(\rho(P), S)$ induced by ϕ is surjective. Thus, there exists a $\tilde g \in P_\alpha *_B P_\beta$ such that ${}^{\tilde g}\rho(P) \le S \le P_\alpha *_B P_\beta$ and $\phi(\tilde g) = g$. We conclude that $c_{\tilde g}\rho = c_g\rho = \tilde\rho$ as maps from P to $P_\alpha *_B P_\beta$. In particular $[\rho] = [\tilde\rho] \in \operatorname{Rep}(P, P_\alpha *_B P_\beta)$ and hence the map is injective. □

9. Mapping spaces and finite groups of Lie type of low rank

First a general lemma, which is useful in the case where $|\mathcal{S}_p(G)|_{hG}$ is the classifying space of a discrete group. Recall that for discrete groups H and G, the set $\operatorname{Rep}(H, G)$ is defined as $\operatorname{Hom}(H, G)/\{c_g \mid g \in G\}$. It is a classical result that for discrete groups H and G there is a bijection $\tilde\chi_{H,G}\colon \operatorname{Rep}(H, G) \to [BH, BG]$ given by sending the class $[\rho]$ to the class $[B\rho]$ for $\rho\colon H \to G$. This extends to a homotopy equivalence:

$$\chi_{H,G}\colon \coprod_{\rho\in\operatorname{Rep}(H,G)} BC_G(\rho(H)) \to \operatorname{Map}(BH, BG). \tag{4}$$

See [7, Proposition 7.1] for a proof. This decomposition allows for a way of checking whether a map $\mathrm{Map}(BH, BG) \to \mathrm{Map}(BH, B\tilde{G})$ induced by a group homomorphism in $\mathrm{Hom}(G, \tilde{G})$ is a mod-p-equivalence or a homotopy equivalence.

Lemma 9.1. Let H, G and $\tilde{G}$ be discrete groups, and $\phi \in \mathrm{Hom}(G, \tilde{G})$. Consider the map $\Phi\colon \mathrm{Map}(BH, BG) \to \mathrm{Map}(BH, B\tilde{G})$ induced by post-composing with $B\phi$. With respect to the map $\tilde{\chi}$ the map Φ on the set of components is just the map $\mathrm{Rep}(H, G) \to \mathrm{Rep}(H, \tilde{G})$ given by $[\rho] \mapsto [\phi \circ \rho]$. Furthermore for any $\rho\colon H \to G$ we have that Φ restricted to

$$\mathrm{Map}(BH, BG)_{B\rho} \to \mathrm{Map}(BH, B\tilde{G})_{B(\phi\rho)}$$

is identified with the restriction of $B\phi$ to

$$BC_G(\rho(H)) \to BC_{\tilde{G}}(\phi\rho(H))$$

under (4).

Proof. The statement on the set of components is that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Rep}(H,G) & \xrightarrow{\tilde{\chi}_{H,G}} & [BH, BG] \\ \downarrow{\scriptstyle \phi \circ -} & & \downarrow{\scriptstyle \Phi} \\ \mathrm{Rep}(H,\tilde{G}) & \xrightarrow{\tilde{\chi}_{H,\tilde{G}}} & [BH, B\tilde{G}] \end{array}$$

By checking the definitions the diagram commutes since B is a functor on the category of discrete groups.

For any $\rho\colon H \to G$ to show that Φ on the component $\mathrm{Map}(BH, BG)_{B\rho}$ is identified with the restriction of $B\phi$ to

$$BC_G(\rho(H)) \to BC_{\tilde{G}}(\phi\rho(H))$$

under (4), we need to look at how (4) is constructed.

For the homeomorphism $\rho\colon H \to G$ we consider the product map

$$(incl, \rho)\colon C_G(\rho(H)) \times H \to G.$$

We then have the following commutative diagram:

$$\begin{array}{ccc} C_G(\rho(H)) \times H & \xrightarrow{(incl,\rho)} & G \\ \downarrow{\scriptstyle \phi \times \mathrm{id}} & & \downarrow{\scriptstyle \phi} \\ C_{\tilde{G}}(\phi \circ \rho(H)) \times H & \xrightarrow{(incl,\phi\circ\rho)} & \tilde{G} \end{array}$$

After applying the functor B and taking adjoints we get the commutative diagram:

$$\begin{array}{ccc} B(C_G(\rho(H))) & \xrightarrow{Ad(B(incl,\rho))} & \mathrm{Map}(BH, BG)_{B\rho} \\ \downarrow{\scriptstyle B\phi} & & \downarrow{\scriptstyle B\phi \circ -} \\ B(C_{\tilde{G}}(\phi \circ \rho(P))) & \xrightarrow{Ad(B(incl,\phi\circ\rho))} & \mathrm{Map}(BH, B\tilde{G})_{B(\phi\circ\rho)} \end{array}$$

According to [7, Proposition 7.1] the horizontal maps are the homotopy equivalence from (4) restricted to the component associated to ρ and $\phi \circ \rho$. The right vertical map is Φ on these components, and thus it is identified with the restriction of $B\phi$ to

$$BC_G(\rho(H)) \to BC_{\tilde{G}}(\phi\rho(H)).$$

□

In the case where G is a finite group of Lie type in characteristic p of Lie rank either 1 or 2, we have proven that the space $|\mathcal{S}_p(G)|_{hG}$ is a classifying space of a discrete group, which allows us to prove our main theorems in this situation.

Theorem 9.2. *Let G be a finite group of Lie type of rank 1 in characteristic p. For any p-subgroup P the map $\Phi\colon \mathrm{Map}(BP, |\mathcal{S}_p(G)|_{hG}) \to \mathrm{Map}(BP, BG)$ induced by the Borel map $|\mathcal{S}_p(G)|_{hG} \to BG$ is a mod-p-equivalence.*

Moreover, the map (4) induces homotopy equivalences

$$\mathrm{Map}(BP, |\mathcal{S}_p(G)|_{hG}) \simeq \coprod_{\rho \in \mathrm{Rep}(P,B)} BC_B(\rho(P))$$

as well as $\mathrm{Map}(BP, BG) \simeq \coprod_{\rho \in \mathrm{Rep}(P,G)} BC_G(\rho(P))$ and the map Φ agrees with the map induced by inclusion $\iota\colon B \to G$ under the given identifications. Furthermore the map Φ is a homotopy equivalence on all components corresponding to $\rho \in \mathrm{Rep}(P, B)$ with $\rho(P)$ non-trivial.

Proof. Let P be a p-group. To prove that Φ is a mod-p-equivalence we need to show that it is a bijection on components and that it induces an equivalence on $\mathbb{F}_p$-homology on the components.

According to Proposition 6.12 we have that $|\mathcal{S}_p(G)|_{hG} \simeq B(B)$ and the Borel map $|\mathcal{S}_p(G)|_{hG}$ corresponds to the inclusion $\iota\colon B \to G$. Thus we have that Φ is post composition with $B\iota$. Moreover, the components of $\mathrm{Map}(B(P), |\mathcal{S}_p(G)|_{hG})$ are in bijection with $\mathrm{Rep}(P, B)$ by sending $[\rho]$ to the component containing $B\rho$. Similarly we get that the set of components of $\mathrm{Map}(B(P), B(G))$ is identified with $\mathrm{Rep}(P, G)$. Furthermore the induced map of Φ on π_0 under this identification is exactly $\beta_P\colon \mathrm{Rep}(P, B) \to \mathrm{Rep}(P, G)$ by Lemma 9.1. Thus we conclude by Corollary 8.3 that Φ is a bijection on π_0.

To tackle the components themselves, we have by Lemma 9.1 that for any $\rho\colon P \to B$ the restriction of Φ on component

$$\mathrm{Map}(BP, B(B))_{B\rho} \to \mathrm{Map}(BP, BG)_{B(\iota\rho)}$$

with respect to the homotopy equivalences is the map

$$B\iota\colon BC_B(\rho(P)) \to BC_G(\iota\rho(P)).$$

By Corollary 8.2 we conclude that this is a homotopy equivalence when $\rho(P)$ is nontrivial. If $\rho(P) = 1$ then the corresponding map is simply $B\iota$, which is a mod-p-equivalence by Lemma 6.5. In both cases we have that the map induces isomorphisms on $\mathbb{F}_p$-homology and by the above this is exactly Φ on the particular component. □

Theorem 9.3. *Let G be a finite group of Lie type of rank 2 in characteristic p. For any p-subgroup P the map $\Phi\colon \mathrm{Map}(BP, |\mathcal{S}_p(G)|_{hG}) \to \mathrm{Map}(BP, BG)$ induced by $|\mathcal{S}_p(G)|_{hG} \to BG$ is a mod-p-equivalence.*

Moreover there is a commutative diagram

$$\begin{array}{ccc} \coprod_{\rho\in \mathrm{Rep}(P,P_\alpha *_B P_\beta)} B_{P_\alpha *_B P_\beta}(\rho(P)) & \xrightarrow{\simeq} & \mathrm{Map}(BP, |\mathcal{S}_p(G)|_{hG}) \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\Phi} \\ \coprod_{\rho\in \mathrm{Rep}(P,G)} BC_G(\rho(P)) & \xrightarrow{\simeq} & \mathrm{Map}(BP, BG) \end{array}$$

such that the map Φ agrees with the map induced by $\phi\colon P_\alpha *_B P_\beta \to G$ under the given identifications. Furthermore the map Φ is a homotopy equivalence on all components corresponding to a $\rho \in \mathrm{Rep}(P, P_\alpha *_B P_\beta)$ with $\rho(P)$ non-trivial.

Proof. By Corollary 6.16 we have that $|\mathcal{S}_p(G)|_{hG} \simeq B(P_\alpha *_B P_\beta)$ and that the Borel fibration $|\mathcal{S}_p(G)|_{hG} \to BG$ corresponds to the canonical map $\phi\colon P_\alpha *_B P_\beta \to G$ on the fundamental groups.

Using the homotopy equivalence (4) we get a commutative diagram as in the theorem, where according to Lemma 9.1 the map of components corresponds to $\mathrm{Rep}(P, P_\alpha *_B P_\beta) \to \mathrm{Rep}(P,G)$ given by $[\rho] \mapsto [\phi \circ \rho]$, and for any $\rho\colon P \to P_\alpha *_B P_\beta$ we have Φ restricted to

$$\mathrm{Map}(BP, |\mathcal{S}_p(G)|_{hG})_{B\rho} \to \mathrm{Map}(BP, BG)_{B(\phi\rho)}$$

is identified with the restriction of $B\phi$ to

$$BC_{P_\alpha *_B P_\beta}(\rho(P)) \to BC_G(\phi\rho(P)).$$

By Lemma 8.13 the map on the set of components is a bijection. For the individual components the map is a homotopy equivalence when ρ is non-trivial by Theorem 8.12, whereas by Lemma 6.5 it is a mod-p-equivalence in the case of the trivial map. □

10. Mapping spaces and finite groups of Lie type of high rank

In the previous section we showed that the Borel map $|\mathcal{S}_p(G)|_{hG} \to BG$ induces a mod-p-equivalence from $\mathrm{Map}(BP, |\mathcal{S}_p(G)|_{hG})$ to $\mathrm{Map}(BP, BG)$ for a finite p-group P and finite group of Lie type G in characteristic p and of rank 1 or 2. We will in this section show that this result holds groups of higher rank. The argument will be of a very different nature, since $|\mathcal{S}_p(G)|_{hG}$ will not be a classifying space of discrete group.

10.1. **The mod-p-homology type of** $\mathrm{Map}(BP, X_{hG})$**.** To begin we focus on the more general case of the space of maps $\mathrm{Map}(BP, X_{hG})$ where X is a G-space and P is a finite p-group. For any G-space the homotopy orbit space X_{hG} is the Borel construction

$$EG \times_G X = (EG \times X)/\langle (v,x) - g(v,x) \mid (v,x) \in EG \times X, g \in G\rangle.$$

The homotopy orbit space fits into the Borel fibration:

$$X \longrightarrow X_{hG} \xrightarrow{\pi} BG$$

Here the map $\pi\colon X_{hG} \to BG$ is induced by sending X to a point. The homotopy fixed point space X^{hG} is dually defined as $\mathrm{Map}_G(EG, X)$, i.e the space of equivariant maps, and is homotopy equivalent to the space of sections for the Borel fibration $\pi\colon X_{hG} \to BG$.

We will now study the induced map $\Phi\colon \mathrm{Map}(BP, X_{hG}) \to \mathrm{Map}(BP, BG)$. By [2, corollary 1.6] we have that $[BP, BG]$ is in bijection with $\mathrm{Rep}(P, G)$, and so every component of $\mathrm{Map}(BP, BG)$ contains an element of the form Bf for some $f \in \mathrm{Hom}(P, G)$. We let $\mathrm{Map}(BP, X_{hG})_f$ denote the space of maps $\tilde{f}\colon BP \to X_{hG}$ such that $\pi \circ \tilde{f} \simeq f$, i.e $\mathrm{Map}(BP, X_{hG})_f$ is the preimage of the component $\mathrm{Map}(BP, BG)_f$ under Φ.

For a G-space Y the G-coinvariant is Y_G is the orbit space Y/G. We will use the notation $[y]_G$ for the image of $y \in Y$ under the quotient map $Y \to Y_G$.

Lemma 10.1. Let P be a finite group and let $f\colon P \to G$ be a group homomorphism. For any G-space X we consider X as a P-space via f. Then the restriction $\Phi\colon \mathrm{Map}(BP, X_{hG})_f \to \mathrm{Map}(BP, BG)_f$ is a fibration with fiber homotopy equivalent to X^{hP}.

The map $X^{hP} \to \mathrm{Map}(BP, X_{hG})_f$ is given by sending a $g \in X^{hP}$ to the map $BP \to X_{hG}$ defined by identifying BP with EP/P and sending $[v]_P \mapsto [Ef(v), g(v)]_G$ for any $v \in EP$.

Proof. As the Borel map $\pi\colon X_{hG} \to BG$ is a fibration the induced map

$$\Phi\colon \mathrm{Map}(BP, X_{hG})_f \to \mathrm{Map}(BP, BG)_f$$

is a fibration as well.

To understand the fiber, we will use the approach of [7, Lemma 7.4]. Recall that we are using the model $|\mathcal{E}(G)|$ for EG, where $\mathcal{E}(G)$ is defined in Definition 5.1. The construction is functorial in G, and induces a map $Ef\colon EP \to EG$. Note that this map is P-equivariant in the sense that for all $q \in P$ and $v \in EP$ we have that $Ef(qv) = f(q)Ef(v)$. Let $\psi\colon X_{hP} \to X_{hG}$ be the map given by $\psi[v, x]_P = [Ef(v), x]_G$. Since the P action on X is induced by f, the map ψ is easily seen to be well-defined.

Then the following diagram commutes:

$$\begin{array}{ccc} X_{hP} & \xrightarrow{\psi} & X_{hG} \\ \downarrow{\scriptstyle proj} & & \downarrow{\scriptstyle \pi} \\ BP & \xrightarrow{Bf} & BG \end{array}$$

By [7, Lemma 7.4] it is enough to show that it is a homotopy pullback. In this case the fiber over f can be identified with the space of sections of the homotopy pullback, which is homotopy equivalent to X^{hP}. The homotopy is given by sending $g \in X^{hP} = \mathrm{Map}_P(EP, X)$ to the section of $X_{hP} \to BP$ defined by $[v]_P \mapsto [v, g(v)]_P$ using the model EP_P for BP.

As the map π is a fibration, the inclusion of the pullback of the diagram into the homotopy pullback is a homotopy equivalence, and it is therefore sufficient to give a homotopy equivalence from $EP \times_P X$ to the pullback:

$$BP \times_{BG} X_{hG} = \{(a, b) \in BP \times X_{hG} \mid Bf(a) = \pi(b)\}.$$

Let $h\colon X_{hG} \to BP \times_{BG} X_{hG}$ be the map given by the universal property of the pullback. Using the model EP_P for BP as well as $EG \times_G X$ for X_{hG} we get a description of the pullback as

$$BP \times_{BG} X_{hG} = \{([v]_P, [Ef(v), x]_G) \in BP \times X_{hG} \mid v \in EP, x \in X\}.$$

This description shows that h is a homotopy equivalence by providing an inverse $BP \times_{BG} X_{hG} \to X_{hP}$ given by $([v]_P, [Ef(v), x]_G) \mapsto [v, x]_P$. $\square$

Proposition 10.2. Let X be a G-space. Let P be a finite p-group and let $f\colon P \to G$ be a group homomorphism. Then X is a P-space via f. Assume that X^P is $\mathbb{F}_p$-acyclic. If X is a simply-connected finite G-CW-complex then the preimage $\mathrm{Map}(BP, X_{hG})_f$ is path-connected and the restriction of Φ to

$$\Phi\colon \mathrm{Map}(BP, X_{hG})_f \to \mathrm{Map}(BP, BG)_f$$

is a mod p-equivalence.

Proof. We have that the map $X^P \to *$ is a mod-p-equivalence. By the property of the Bousfield-Kan p-completion [6, Lemma I 5.5] this implies that the induced map $(X^P)^\wedge_p \to *^\wedge_p$ is a homotopy equivalence. Since X is a finite P-CW complex and P a finite group, it follows from the Generalized Sullivan conjecture (proved in the 80's see [24] and [28]) that the map $(X^P)^\wedge_p \to (X^\wedge_p)^{hP}$ is a weak homotopy equivalence.

Since X is simply-connected we conclude by [16, Lemma 2.4] that the natural map $X^{hP} \to (X^\wedge_p)^{hP}$ induces an isomorphism on homology with $\mathbb{F}_p$ coefficients. Both homotopy equivalences as well as weak homotopy equivalences induce isomorphism on homology with $\mathbb{F}_p$-coefficients and so we have that $H_*(X^{hP}; \mathbb{F}_p) \cong H_*(*; \mathbb{F}_p)$. In the fibration

$$X^{hP} \to \mathrm{Map}(BP, X_{hG})_f \to \mathrm{Map}(BP, BG)_f$$

the base space is path-connected and the homology of the fiber is $H_*(*; \mathbb{F}_p)$. In particular the fiber is path-connected as well, and hence by the long exact sequence in homotopy for fibrations we conclude that $\mathrm{Map}(BP, X_{hG})_f$ is path-connected. It now follows that the fundamental group of the base space acts trivially on the $\mathbb{F}_p$-homology of the fiber. The Serre spectral sequence for $\mathbb{F}_p$-homology

$$E^2_{p,q} = H_p(\mathrm{Map}(BP, BG)_f; H_q(*; \mathbb{F}_p)) \Longrightarrow H_{p+q}(\mathrm{Map}(BP, X_{hG})_f; \mathbb{F}_p)$$

degenerates to the zero line. The identification of the second page of the spectral sequence is induced by Φ so the collapse shows that Φ is a mod p-equivalence. □

The mapping space $\mathrm{Map}(BP, BG)$ contains the component containing $B1$, where $1\colon P \to G$ is the trivial map. The previous proposition requires that $X^P = X$ is contractible, which is too restrictive for our purpose. Thus we now tackle this component separately.

Proposition 10.3. Let G be finite group and X a finite G-CW-space. For a finite group P we let $1\colon P \to G$ denote the trivial group homomorphism. Let $\mathrm{Map}(BP, X_{hG})_1$ be the preimage of $\mathrm{Map}(BP, BG)_{B1}$ under Φ. Then the map $X_{hG} \to \mathrm{Map}(BP, X_{hG})$ given by sending $x \in X_{hG}$ to the constant map at x is a weak homotopy equivalence and furthermore the diagram

$$\begin{array}{ccc} X_{hG} & \longrightarrow & BG \\ \downarrow & & \downarrow \\ \mathrm{Map}(BP, X_{hG})_1 & \xrightarrow{\Phi} & \mathrm{Map}(BP, BG)_{B1} \end{array}$$

commutes and the right vertical map is a homotopy equivalence. In particular we have that $\Phi\colon \mathrm{Map}(BP, X_{hG})_1 \to \mathrm{Map}(BP, BG)_{B1}$ is a mod-p-equivalence if and only if the map $X_{hG} \to BG$ is.

Proof. Consider the fibration from Lemma 10.1

$$X^{hP} \longrightarrow \mathrm{Map}(BP, X_{hG})_1 \xrightarrow{\pi} \mathrm{Map}(BP, BG)_1$$

where X has the trivial P action. Then we have that $X^{hP} \cong \mathrm{Map}(BP, X)$ and the map $X^{hP} \to \mathrm{Map}(BP, X_{hG})$ coincide with postcomposition with $X \to X_{hG}$. For a space Y we let $c_Y : Y \to \mathrm{Map}(BP, Y)$ be the map given by sending a point of $y \in Y$ to the constant map $BP \to Y$ with image y. We get the induced commutative diagram:

$$\begin{array}{ccccc} X & \longrightarrow & X_{hG} & \longrightarrow & BG \\ \downarrow{\scriptstyle c_X} & & \downarrow{\scriptstyle c_{X_{hG}}} & & \downarrow{\scriptstyle c_{BG}} \\ \mathrm{Map}(BP, X) & \longrightarrow & \mathrm{Map}(BP, X_{hG})_1 & \xrightarrow{\Phi} & \mathrm{Map}(BP, BG)_1 \end{array}$$

The left vertical map is a homotopy equivalence by the Sullivan's conjecture in the trivial action case, see [28, Example 1] for a formulation in this language. Likewise the map $Y_{BG} \colon BG \to \mathrm{Map}(BP, BG)_{B1}$ is a homotopy equivalence by a classical result, see [7, Proposition 7.1]. The upper sequence is the Borel fibration. Both fibrations admit a long exact sequence in homotopy and the given maps induces a map between them. By successive applications of the 5-lemma we deduce that $c_{X_{hG}} \colon X_{hG} \to \mathrm{Map}(BP, X_{hG})_1$ is a weak homotopy equivalence. A weak homotopy equivalence induces isomorphisms on homology and cohomology with any coefficients, and is in particular a mod-p-equivalence. The rest of the proposition now follows from the commutative diagram. □

Theorem 10.4. *Let G be a finite group and X a finite G-CW-space which is simply connected. Let P be a finite p-group and assume that for any non-trivial $\varphi \in \mathrm{Hom}(P, G)$ the induced P action on X satisfies that the set of fixed-points X^P is $\mathbb{F}_p$-acyclic. If the map $X_{hG} \to BG$ is a mod-p-equivalence, then the same is true for the induced map $\Phi \colon \mathrm{Map}(BP, X_{hG}) \to \mathrm{Map}(BP, BG)$.*

Proof. We need to check that the preimage under Φ of each path-component of $\mathrm{Map}(BP, BG)$ is path-connected and the restriction of Φ to each component is a mod-p-equivalence. As every component of $\mathrm{Map}(BP, BG)$ contains an element of the form Bf for some $f \in \mathrm{Hom}(P, G)$ the theorem follows from Propositions 10.2 and 10.3. □

10.2. **Mod-p-equivalence for finite groups of Lie type in high rank.** Now we will restrict our focus to the G-CW-complex $\mathcal{S}_p(G)$, where G is a finite group of Lie type in characteristic p and apply the previous results.

Theorem 10.5. *Let G be a group of finite Lie type in characteristic p of Lie rank at least 3. For any p-subgroup P of G the map $\Phi \colon \mathrm{Map}(BP, |\mathcal{S}_p(G)|_{hG}) \to \mathrm{Map}(BP, BG)$ induced by $|\mathcal{S}_p(G)|_{hG} \to BG$ is a mod-p-equivalence.*

Proof. As the group G is finite the realization of a poset of subgroups $|\mathcal{S}_p(G)|$ is a finite G-CW-complex. From Lemma 6.11 we have that $|\mathcal{S}_p(G)|$ is $(\mathrm{Rk_L}(G) - 2)$-connected. In particular we have by the assumptions that $|\mathcal{S}_p(G)|$ is simply connected. From Corollary 6.10 we conclude that $|\mathcal{S}_p(G)|^{\varphi(P)}$ is contractible for any nontrivial $\varphi \in \mathrm{Hom}(P, G)$, and in particular $\mathbb{F}_p$-acyclic.

The theorem now follows directly from Theorem 10.4 using Lemma 6.5. □

Corollary 10.6. let G be a finite group of Lie type in characteristic p of rank at least 3. For a finite p-group P there is a bijection $[BP, |\mathcal{S}_p(G)|_{hG}] \to \mathrm{Rep}(P, G)$ induced by the Borel map $|\mathcal{S}_p(G)|_{hG} \to BG$.

Proof. A mod-p-equivalence induces a bijection on components, and so the corollary follows directly from Theorem 10.5 by identifying the components of $\mathrm{Map}(BP, BG)$ with $\mathrm{Rep}(P, G)$ □

11. Homotopy group of homotopy fixed points applied to localizations.

In this chapter we will — for a nilpotent space X and a finite p-group P — determine the homotopy groups of $(X^{\wedge}_{\ell})^{hP}$ for a prime ℓ different from p, as well as $(X_{\mathbb{Q}})^{hP}$ in terms of the homotopy groups of X. Here, ${}^{\wedge}_{\ell}$ is the ℓ-completion in the sense of Bousfield–Kan [6, Chapter VI].

Recall that the action of a group π on a group G is called nilpotent if there exists a finite sequence of subgroups

$$1 = G_1 \subset G_2 \subset \cdots G_{n-1} \subset G_n = G$$

such that the following conditions hold:

- The subgroup G_j is closed under the action of π.
- The subgroup G_j is normal in G_{j+1} and the quotient G_{j+1}/G_j is abelian.
- The induced action of π on the quotient G_{j+1}/G_j is trivial.

If the conjugation action on a group is nilpotent, the group is called nilpotent. For abelian groups this trivially holds, and they are all nilpotent.

A connected pointed space X is nilpotent, if the action of the fundamental group $\pi_1(X)$ on all the homotopy groups is nilpotent. In particular all simply-connected spaces are nilpotent.

Proposition 11.1. Let p be prime, and P be a finite p-group, and let X be a connected P-space satisfying that $\pi_t(X)$ is a uniquely p-divisible abelian group for all $t > 0$. Then we have that $\pi_t(X^{hP}) \cong \pi_t(X)^P$ for all t and this isomorphism is natural in X.

Proof. The result follows using the homotopical homotopy fixed point spectral sequence. Recall that this spectral sequence is a special case of the spectral sequence for homotopy groups of homotopy limits [6, Ch. XI §7].

For the P-space X we may consider the associated diagram $\underline{X}\colon I \to \mathcal{T}op$, where I is the one-object category with morphism set P and $\mathcal{T}op$ the category of topological spaces. The homotopy limit of the diagram $\underline{X}$ is X^{hP}.

For any t we have that $\pi_t(X)$ with induced P-action is a P-module. We let $\mathcal{A}b$ be the category of abelian groups with group homomorphisms. Let $\pi_t(\underline{X})$ denote the associated diagram $I \to \mathcal{A}b$ given by mapping I to $\pi_t(X)$. Now the homotopical spectral sequence for homotopy limits applied to $\underline{X}$ reduces to a spectral sequence satisfying

$$E_2^{s,t} \cong \lim_I{}^s(\pi_t(\underline{X})) \Longrightarrow \pi_{t-s}((X)^{hP}).$$

As in [6, Ch. XI 6.3] we may identify $\lim_I^s(\pi_t(\underline{X}))$ with $H^s(P, \pi_t(X))$ the classical group cohomology of P with coefficients in $\pi_t(X)$.

Recall that being a uniquely p-divisible module means that every p-power is invertible in the module. In particular the order of the group $|P|$ is invertible in $\pi_t(X)$ for all t. Thus, by [12, Corollary 10.2] all cohomology groups $H^s(P, \pi_t(X))$ for $s > 0$ vanish, and hence the E_2-page is trivial expect for the 0'th row, where for all t

$$E_2^{0,t} \cong H^0(P, \pi_t(X)) = \pi_t(X)^P.$$

The differentials in the spectral sequence have the form $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$. Thus, the above spectral sequence collapses to the 0'th row and we conclude that $\pi_t((X)^{hP}) \cong \pi_t(X)^P$ for all t.

Lastly we observe that the spectral sequence as well as all the used isomorphisms are natural in X, from which we get the naturality of the above isomorphism. $\square$

For a nilpotent group N the classifying space $K(N,1)$ is an example of a connected and nilpotent space. Using the notation from [6, Ch. VI] we define for a nilpotent group N the Ext-completion as well as the Hom-completion to be

$$\operatorname{Ext}(\mathbb{Z}_{p^\infty}, N) = \pi_1(K(N,1)_p^\wedge)$$
$$\operatorname{Hom}(\mathbb{Z}_{p^\infty}, N) = \pi_2(K(N,1)_p^\wedge)$$

For N abelian this definition coincides with the classical one, if we let $\mathbb{Z}_{p^\infty}$ denote the Sylow-p-subgroup of $\mathbb{Q}/\mathbb{Z}$. A concrete description of the group $\mathbb{Z}_{p^\infty}$ can be given as the quotient of the subgroup

$$\left\{ \frac{a}{p^n} \mid a \in \mathbb{Z}, n \geq 0 \right\}$$

over the integers.

The completion map $K(N,1) \to K(N,1)_p^\wedge$ induces the associated completion map on the fundamental groups $N \to \operatorname{Ext}(\mathbb{Z}_{p^\infty}, N)$. A nilpotent group N is called Ext-p-complete if the completion map $N \to \operatorname{Ext}(\mathbb{Z}_{p^\infty}, N)$ is an isomorphism and if $\operatorname{Hom}(\mathbb{Z}_{p^\infty}, N) = 0$.

As remarked in [6, Ch. VI] a nilpotent group N that is Ext-p-complete, is uniquely ℓ-divisible for any prime ℓ different from p, but no proof is provided. For completeness, we include a proof of this statement here.

Lemma 11.2. Let p be a prime and N an Ext-p-complete nilpotent group. Then N is uniquely ℓ-divisible for any prime $\ell \neq p$.

Proof. First we consider the case where N is abelian. By definition of Ext-p-completeness we have that the map $N \to \operatorname{Ext}(\mathbb{Z}_{p^\infty}, N)$ is an isomorphism. Hence it is sufficient to prove that $\operatorname{Ext}(\mathbb{Z}_{p^\infty}, N)$ is uniquely ℓ-divisible. Note that Ext does in this case agree with the classical definition as a right derived functor of $\operatorname{Hom}(\mathbb{Z}_{p^\infty}, -)$.

For a prime $\ell \neq p$ let the multiplication by ℓ on $\mathbb{Z}_{p^\infty}$ be denoted by $(\cdot\ell)$ and note that this is an isomorphism. Thus, the induced map $\operatorname{Ext}((\cdot\ell))$ on $\operatorname{Ext}(\mathbb{Z}_{p^\infty}, N)$ is an isomorphism as well. To identify the morphism $\operatorname{Ext}((\cdot\ell))$, first observe that for any group homomorphism $f\colon \mathbb{Z}_{p^\infty} \to B$ of abelian groups, we have that $\ell f = f \circ (\cdot\ell)$. By the definition of Ext as the derived functor of Hom we conclude that the automorphism $\operatorname{Ext}((\cdot\ell))$ of $\operatorname{Ext}(\mathbb{Z}_{p^\infty}, N)$ is in fact multiplication by ℓ. As this is an isomorphism we conclude that $\operatorname{Ext}(\mathbb{Z}_{p^\infty}, N)$ is uniquely ℓ-divisible.

For the general case of a nilpotent group recall that one of the definitions of nilpotent group is that the upper central series of N stabilizes at N. By [6, Ch. VI, 3.4 (ii)] we have that all abelian quotients of the upper central series are Ext-p-complete and thus by the previous result uniquely ℓ-divisible. Using the fact the the upper central series stabilizes at N it now follows by the 5-lemma that N is uniquely ℓ-divisible. $\square$

Proposition 11.3. Fix a prime p. Let P be a p-group and X be a connected nilpotent P-space with abelian fundamental group. Then for all t we have that

$$\pi_t((X_{\mathbb{Q}})^{hP}) \cong \pi_t(X_{\mathbb{Q}})^P.$$

If furthermore the fundamental group of X is finitely generated, then for any prime ℓ different from p the homotopy groups of $(X^\wedge_\ell)^{hP}$ satisfy that for all t

$$\pi_t((X^\wedge_\ell)^{hP}) \cong \pi_t(X^\wedge_\ell)^P.$$

Proof. For the rationalization $X_{\mathbb{Q}}$ the homotopy groups satisfy that $\pi_t(X_{\mathbb{Q}}) \cong \pi_t(X) \otimes \mathbb{Q}$ according to [6, Ch V Proposition 3.1]. In particular $X_{\mathbb{Q}}$ is connected. Since $\mathbb{Q}$ is uniquely p-divisible, we have in this case that the homotopy groups of $X_{\mathbb{Q}}$ are all uniquely p-divisible abelian groups. The statement now follows from Proposition 11.1.

Now let ℓ be a prime different from p and consider the ℓ-completion $X^\wedge_\ell$. We have that the space $X^\wedge_\ell$ is connected. The higher homotopy groups of $X^\wedge_\ell$ are all abelian. Based on [6, Ch. VI 5.2] the fundamental group $\pi_1(X^\wedge_\ell)$ is isomorphic to $\pi_1(X) \otimes \mathbb{Z}_\ell$, where $\mathbb{Z}_\ell$ denotes the ℓ-adic integers. In particular under the above assumptions the fundamental group of $X^\wedge_\ell$ is abelian.

Furthermore we have that $X^\wedge_\ell$ is $\mathbb{Z}_\ell$-complete, and hence by [6, Ch. VI 5.4] the homotopy groups $\pi_t(X^\wedge_\ell)$ are Ext-ℓ-complete as well. As $\ell \neq p$ we see from Lemma 11.2 that these groups are uniquely p-divisible for all $t > 0$. Now the second part of the statement likewise follows from Proposition 11.1. □

Lemma 11.4. Let X and Y be connected pointed nilpotent spaces, and $f\colon X \to Y$ a map between them. Then the following diagram commutes, where the the horizontal maps are the canonical maps associated to the rationalization.

$$\begin{array}{ccc} \pi_*(X) \otimes \mathbb{Q} & \xrightarrow{\cong} & \pi_*(X_{\mathbb{Q}}) \\ \downarrow{\scriptstyle (f_*)\otimes 1_{\mathbb{Q}}} & & \downarrow{\scriptstyle (f_{\mathbb{Q}})_*} \\ \pi_*(Y) \otimes \mathbb{Q} & \xrightarrow{\cong} & \pi_*(Y_{\mathbb{Q}}) \end{array}$$

If furthermore all homotopy groups of X and Y are finitely generated abelian groups, then for a prime ℓ the associated diagram for ℓ-completion commutes as well.

$$\begin{array}{ccc} \pi_*(X) \otimes \mathbb{Z}_\ell & \xrightarrow{\cong} & \pi_*(X^\wedge_\ell) \\ \downarrow{\scriptstyle (f_*)\otimes 1_{\mathbb{Z}_\ell}} & & \downarrow{\scriptstyle (f^\wedge_\ell)_*} \\ \pi_*(Y) \otimes \mathbb{Z}_\ell & \xrightarrow{\cong} & \pi_*(Y^\wedge_\ell) \end{array}$$

Proof. Using the funtoriality and naturality of rationalization of nilpotent spaces from [6] we obtain the following commutative diagram where ϕ is the natural transformation from the identity functor to the rationalization functor.

$$\begin{array}{ccc} \pi_*(X) & \xrightarrow{\phi(X)_*} & \pi_*(X_{\mathbb{Q}}) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle (f_{\mathbb{Q}})_*} \\ \pi_*(Y) & \xrightarrow{\phi(Y)_*} & \pi_*(Y_{\mathbb{Q}}) \end{array}$$

Recall [6, Ch. V 2.5] that for a group H the natural map $H \to H \otimes \mathbb{Q}$ is universal for maps from H to $\mathbb{Q}$-nilpotent groups. Using the universal property in connection with the above diagram, the diagram factors as:

$$\begin{array}{ccccc} \pi_*(X) & \longrightarrow & \pi_*(X)\otimes\mathbb{Q} & \longrightarrow & \pi_*(X_{\mathbb{Q}}) \\ \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle (f_*)\otimes 1_{\mathbb{Q}}} & & \Big\downarrow{\scriptstyle (f_{\mathbb{Q}})_*} \\ \pi_*(Y) & \longrightarrow & \pi_*(Y)\otimes\mathbb{Q} & \longrightarrow & \pi_*(Y_{\mathbb{Q}}) \end{array}$$

By [6, V Proposition 3.1 (ii)] the right hand horizontal maps are exactly the canonical isomorphism associated to rationalization and the first part of the lemma follows directly.

For the second part of the lemma let ℓ be a prime number. Then the natural transformation ψ from the ℓ-completion gives rise to a similar diagram.

$$\begin{array}{ccc} \pi_*(X) & \xrightarrow{\psi(X)_*} & \pi_*(X_\ell^\wedge) \\ \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle (f_\ell^\wedge)_*} \\ \pi_*(Y) & \xrightarrow{\psi(Y)_*} & \pi_*(Y_\ell^\wedge) \end{array} \tag{5}$$

Now recall that $\mathbb{Z}_\ell$-completion for finitely generated groups agrees with the ℓ-profinite completion. In particular for a finitely generated abelian group A the ℓ-completion is isomorphic to $A\otimes\mathbb{Z}_\ell$. The natural map $A\to A\otimes\mathbb{Z}_\ell$ is universal for maps from A to ℓ-profinite groups.

For any nilpotent connected space Z with finitely generated homotopy groups there is a sequence of isomorphisms

$$\pi_t(Z)\otimes\mathbb{Z}_\ell \to \mathrm{Ext}(\mathbb{Z}_{\ell^\infty},\pi_t(Z)) \to \pi_t(Z_\ell^\wedge) \tag{6}$$

by [6, Ch. VI 5]. Using this isomorphism for the spaces X and Y, we see that all the homotopy groups are $\mathbb{Z}_\ell$-complete, and the universal property factors the diagram (5) in a similar fashion as before:

$$\begin{array}{ccccc} \pi_*(X) & \longrightarrow & \pi_*(X)\otimes\mathbb{Z}_\ell & \longrightarrow & \pi_*(X_\ell^\wedge) \\ \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle (f_*)\otimes 1_{\mathbb{Z}_\ell}} & & \Big\downarrow{\scriptstyle (f_\ell^\wedge)_*} \\ \pi_*(Y) & \longrightarrow & \pi_*(Y)\otimes\mathbb{Z}_\ell & \longrightarrow & \pi_*(Y_\ell^\wedge) \end{array}$$

The right hand horizontal maps coincide with (6) and are thus both isomorphisms. □

Proposition 11.5. Fix a prime p. Let P be a finite p-group and X be a connected nilpotent P-space with abelian fundamental group. Let $X_{\mathbb{Q}}$ be a P-space with the induced P-action. Then for any t the group P acts on the t'th homotopy group of both X and $X_{\mathbb{Q}}$ and the fixed points satisfy

$$\pi_t(X_{\mathbb{Q}})^P \cong \pi_t(X)^P\otimes\mathbb{Q}.$$

Let ℓ be a prime different from p. Assume that all homotopy groups of X are finitely generated abelian. Then the P-fixed points of the homotopy groups of both X and $X_\ell^\wedge$ satisfy

$$\pi_t(X_\ell^\wedge)^P \cong \pi_t(X)^P\otimes\mathbb{Z}_\ell$$

for all t.

Proof. Recall that $X_{\mathbb{Q}}$ is a P-space with the P-action induced by the one on X. In particular if for $g \in P$ the action on X is $\varphi(g) \in \mathrm{Aut}(X)$, then the induced action on $X_{\mathbb{Q}}$ by g is $\varphi(g)_{\mathbb{Q}} \in \mathrm{Aut}(X_{\mathbb{Q}})$. By Lemma 11.4 the associated action on homotopy groups satisfies

$$\begin{array}{ccc} \pi_t(X) \otimes \mathbb{Q} & \xrightarrow{\cong} & \pi_t(X_{\mathbb{Q}}) \\ \big\downarrow{\scriptstyle (\varphi(g)_*)\otimes 1_{\mathbb{Q}}} & & \big\downarrow{\scriptstyle (\varphi(g)_{\mathbb{Q}})_*} \\ \pi_t(X) \otimes \mathbb{Q} & \xrightarrow{\cong} & \pi_t(X_{\mathbb{Q}}), \end{array}$$

where the vertical maps are the same isomorphism. As g only acts on the first factor on the left hand side, we conclude that the g-fixed points $\pi_t(X_{\mathbb{Q}})^g$ are isomorphic to $\pi_t(X)^g \otimes \mathbb{Q}$ and the isomorphism is induced by restriction of the canonical map $\pi_t(X) \otimes \mathbb{Q} \to \pi_t(X_{\mathbb{Q}})$.

The set of rational numbers $\mathbb{Q}$ is flat as a $\mathbb{Z}$-module, and thus tensoring with $\mathbb{Q}$ commutes with finite intersection. In particular

$$\pi_t(X)^P \otimes \mathbb{Q} = \left(\bigcap_{g \in P} \pi_t(X)^g \right) \otimes \mathbb{Q} \cong \bigcap_{g \in P} \left(\pi_t(X)^g \otimes \mathbb{Q} \right).$$

For all $g \in P$ the maps from $\pi_t(X)^g \otimes \mathbb{Q} \to (\pi_t(X_{\mathbb{Q}}))^g$ are compatible and together with the above observation they induce an isomorphism

$$\pi_t(X)^P \otimes \mathbb{Q} \cong \pi_t(X_{\mathbb{Q}})^P.$$

In the case of ℓ-completion $X^\wedge_\ell$ for ℓ a prime different from p the P-action is induced by the ℓ-completion functor. For a $g \in P$ the action on the homotopy groups of $X^\wedge_\ell$ is given by $(\varphi(g)^\wedge_\ell)_* \in \mathrm{Aut}(\pi_*(X^\wedge_\ell)$. Using the fact that the homotopy groups of X are finitely generated abelian we get by Lemma 11.4 for any t the commutative diagram

$$\begin{array}{ccc} \pi_t(X) \otimes \mathbb{Z}_\ell & \xrightarrow{\cong} & \pi_t(X^\wedge_\ell) \\ \big\downarrow{\scriptstyle (\varphi(g)_*)\otimes 1_{\mathbb{Z}_\ell}} & & \big\downarrow{\scriptstyle (\varphi(g)^\wedge_\ell)_*} \\ \pi_t(X) \otimes \mathbb{Z}_\ell & \xrightarrow{\cong} & \pi_t(X^\wedge_\ell), \end{array}$$

where the two isomorphisms agree. As before this implies that $\pi_t(X)^g \otimes \mathbb{Z}_\ell \cong \pi_t(X^\wedge_\ell)^g$ with the isomorphism induced by the canonical map $\pi_t(X) \otimes \mathbb{Z}_\ell \cong \pi_t(X^\wedge_\ell)$. By observing that the ℓ-adic integers $\mathbb{Z}_\ell$ is torsion free and thus flat over $\mathbb{Z}$, the argument now follows the form of the rationalization case. □

Corollary 11.6. Fix a prime p. Let P be a finite p-group and X be a connected nilpotent P-space with abelian fundamental group. Then for any t the homotopy groups of $(X_{\mathbb{Q}})^{hP}$ satisfies

$$\pi_t((X_{\mathbb{Q}})^{hP}) \cong \pi_t(X)^P \otimes \mathbb{Q}.$$

Let ℓ be a prime different for p. If all homotopies of X are finitely generated abelian, then the homotopy groups of $(X^\wedge_\ell)^{hP}$

$$\pi_t((X^\wedge_\ell)^{hP}) \cong \pi_t(X)^P \otimes \mathbb{Z}_\ell$$

for all t.

Proof. The statement follows from Proposition 11.3 and Proposition 11.5. □

12. Sullivan square with missing p-term and divisible groups

In [39] Sullivan presented his arithmetic square providing a connection between localization at the rational numbers and completion at primes. For a finitely generated abelian group A he showed that the commuting square

$$\begin{array}{ccc} A & \longrightarrow & \prod_{p \text{ prime}} A \otimes \mathbb{Z}_p \\ \downarrow & & \downarrow i \\ A \otimes \mathbb{Q} & \xrightarrow{j} & \left(\prod_{p \text{ prime}} A \otimes \mathbb{Z}_p\right) \otimes \mathbb{Q} \end{array}$$

with $i((g_p \otimes z_p)_p) = (g_p \otimes z_p)_p \otimes 1$ and $j(g \otimes q) = (g \otimes 1)_p \otimes q$ is a fiber square, i.e we have the following exact sequence

$$0 \longrightarrow A \longrightarrow (A \otimes \mathbb{Q}) \oplus \left(\prod_{p \text{ prime}} A \otimes \mathbb{Z}_p\right) \xrightarrow{i-j} \left(\prod_{p \text{ prime}} A \otimes \mathbb{Z}_p\right) \otimes \mathbb{Q} \longrightarrow 0$$

We will now fix a prime p and consider the square with missing p-term in the upper right corner:

$$\begin{array}{ccc} A & \longrightarrow & \prod_{\ell \text{ prime}, \ell \neq p} A \otimes \mathbb{Z}_\ell \\ \downarrow & & \downarrow i \\ A \otimes \mathbb{Q} & \xrightarrow{j} & \left(\prod_{\ell \text{ prime}} A \otimes \mathbb{Z}_\ell\right) \otimes \mathbb{Q} \end{array}$$

In the following lemma we identify the kernel and cokernel of the restriction of $i - j$ to this situation.

Lemma 12.1. For a finitely generated abelian group A and a prime p we consider the map

$$f_A \colon (A \otimes \mathbb{Q}) \oplus \left(\prod_{\substack{\ell \text{ prime} \\ \ell \neq p}} A \otimes \mathbb{Z}_\ell \right) \to (A \otimes \mathbb{Q}_p) \oplus \left(\left(\prod_{\substack{\ell \text{ prime} \\ \ell \neq p}} A \otimes \mathbb{Z}_\ell \right) \otimes \mathbb{Q} \right)$$

given by

$$(g \otimes q) \oplus (g_\ell \otimes z_\ell)_{\ell \neq p} \mapsto (g \otimes q) \oplus ((g \otimes 1)_{\ell \neq p} \otimes q - (g_\ell \otimes z_\ell)_{\ell \neq p} \otimes 1).$$

Then $\ker(f_A) = \operatorname{Tor}_{p'}(A)$ and $\operatorname{coker}(f_A) = A \otimes (\mathbb{Z}_p/\mathbb{Z})[\frac{1}{p}]$.

Proof. Let A be a finitely generated abelian group. The map f_A is the restriction of $i - j$ from the Sullivan arithmetic square to a subgroup, so the kernel is the intersection between the kernel of $i - j$ and the domain of f_A. By the exactness of the associated short exact sequence, $\ker(i - j)$ is isomorphic to A via the map

$$a \mapsto (a \otimes 1) \oplus (a \otimes 1)_\ell \in (A \otimes \mathbb{Q}) \oplus \left(\prod_{\ell \text{ prime}} A \otimes \mathbb{Z}_\ell \right).$$

For any $a \in A$ the element $(a \otimes 1) \oplus (a \otimes 1)_\ell$ belongs to the domain of f_A if and only if $a \otimes 1 = 0 \in A \otimes \mathbb{Z}_p$. Note that the kernel of $A \hookrightarrow A \otimes \mathbb{Z}_p$ is $\operatorname{Tor}_{p'}(A)$, and hence we can identify the kernel f_A with the p-prime part of $\operatorname{Tor}(A)$.

We have that $A \cong \bigoplus_1^k A_k$ where each A_k is a cyclic group. By the distributive laws we have that $\operatorname{coker}(f_k) \cong \bigoplus_1^k \operatorname{coker}(f_{A_k})$. For finite groups the target is zero, and thus $\operatorname{coker}(f_A) \cong \operatorname{coker}(f_{\mathbb{Z}})^{\operatorname{Rank}(A)}$. Thus, it suffices to consider the case where $A = \mathbb{Z}$.

We first observe that every element of $(y_\ell)_{\ell \neq p} \in \left(\prod_{\ell \text{ prime}, \ell \neq p} \mathbb{Z}_\ell\right) \otimes \mathbb{Q}$ is in the target of the map $i - j$. As $i - j$ is surjective there exists an element $(q, (x_\ell)_\ell) \in \mathbb{Q} \oplus \left(\prod_{p \text{ prime}} \mathbb{Z}_p\right)$ with $(i-j)(q,(x_\ell)_\ell) = (y_\ell)$. Note that $f_{\mathbb{Z}}(q,(x_\ell)_{\ell \neq p}) = (q,(y_\ell)_{\ell \neq p})$. Then every element of $\operatorname{coker}(f_{\mathbb{Z}})$ has a representative of the form $(y, 0)$ with $y \in \mathbb{Q}_p$. Thus, the projection on the first factor induces a surjective map from $\operatorname{coker}(f_{\mathbb{Z}})$ to $\mathbb{Q}_p$. To identify the kernel we observe that for any $q \in \mathbb{Q}$ we have $f_{\mathbb{Z}}(q, 0) = (q, q)$, and hence $(q, 0) = (0, -q)$ as elements of $\operatorname{coker}(f_{\mathbb{Z}})$. The element $(q, 0)$ is zero in $\operatorname{coker}(f_{\mathbb{Z}})$ exactly when $q \in \mathbb{Q} \cap \mathbb{Z}_\ell$ for all primes $\ell \neq p$, i.e. when

$$q \in \bigcap_{\ell \neq p} \mathbb{Q} \cap \mathbb{Z}_\ell = \bigcap_{\ell \neq p} \mathbb{Z}_{(\ell)} = \mathbb{Z}\left[\frac{1}{p}\right].$$

We conclude that $\operatorname{coker}(f_{\mathbb{Z}}) \cong \mathbb{Q}_p/(\mathbb{Z}[\frac{1}{p}]) = (\mathbb{Z}_p/\mathbb{Z})[\frac{1}{p}]$. The general result follows from this. ∎

To further understand the cokernel of Lemma 12.1 we will look at $(\mathbb{Z}_p/\mathbb{Z})[\frac{1}{p}]$. The following lemma is the first important steep.

Lemma 12.2. Let p be a prime number. There is a group isomorphism

$$\mathbb{Q}/\left(\mathbb{Z}\left[\frac{1}{p}\right]\right) \cong \bigoplus_{\substack{\ell \text{ prime} \\ \ell \neq p}} \mathbb{Z}_{\ell^\infty}$$

given by identifying the class represented by $\frac{a}{\ell^k}$ in the group $\mathbb{Z}_{\ell^\infty}$ with the class represented by $\frac{a}{\ell^k}$ in $\mathbb{Q}/\left(\mathbb{Z}\left[\frac{1}{p}\right]\right)$ for every prime ℓ different from p.

Proof. The abelian group $\mathbb{Q}/\mathbb{Z}$ is a torsion group and thus it is isomorphic to the direct sum of its unique Sylow-ℓ-subgroups, where ℓ ranges over all primes. More explicitly the isomorphism

$$\bigoplus_{\ell \text{ prime}} \mathbb{Z}_{\ell^\infty} \cong \mathbb{Q}/\mathbb{Z}$$

is given by identifying the class represented by $\frac{a}{\ell^k}$ where $a \in \mathbb{Z}$ and $k \geq 0$ in $\mathbb{Z}_{\ell^\infty}$ with the corresponding class epresented by $\frac{a}{\ell^k}$ in $\mathbb{Q}/\mathbb{Z}$. By Noether's isomorphism theorem we conclude that

$$\mathbb{Q}/\left(\mathbb{Z}\left[\frac{1}{p}\right]\right) \cong (\mathbb{Q}/\mathbb{Z})/\left(\mathbb{Z}\left[\frac{1}{p}\right]/\mathbb{Z}\right) \cong \left(\bigoplus_{\ell \text{ prime}} \mathbb{Z}_{\ell^\infty}\right)/\mathbb{Z}_{p^\infty} \cong \bigoplus_{\substack{\ell \text{ prime} \\ \ell \neq p}} \mathbb{Z}_{\ell^\infty}.$$

∎

Proposition 12.3. Let p be a prime number. The group $(\mathbb{Z}_p/\mathbb{Z})[\frac{1}{p}]$ is a divisible abelian group and isomorphic to

$$\left(\bigoplus_{\substack{\ell \text{ prime} \\ \ell \neq p}} \mathbb{Z}_{\ell^\infty}\right) \bigoplus \left(\bigoplus_{\mathbb{R}} \mathbb{Q}\right).$$

Proof. Recall that the quotient is $(\mathbb{Z}_p/\mathbb{Z})[\frac{1}{p}]$ which is isomorphic to $\mathbb{Q}_p/\mathbb{Z}[\frac{1}{p}]$.

First consider the p-adic numbers $\mathbb{Q}_p$ under addition. This is a divisible group with no torsion, and thus by the structure theorem for divisible groups [22, Theorem 4] it is a vector space over $\mathbb{Q}$. The cardinality of $\mathbb{Q}_p$ is $|\mathbb{R}|$, and hence the dimension $\dim_{\mathbb{Q}}(\mathbb{Q}_p)$ must be infinite as well. To determine the dimension explicitly note that

$$|\mathbb{R}| = |\mathbb{Q}_p| = |\mathbb{Q}| \cdot \dim_{\mathbb{Q}}(\mathbb{Q}_p) = \max\{|\mathbb{Q}|, \dim_{\mathbb{Q}}(\mathbb{Q}_p)\},$$

and observe that $\dim_{\mathbb{Q}}(\mathbb{Q}_p)$ must be $|\mathbb{R}|$.

Considering $\mathbb{Q}$ as a subring of $\mathbb{Q}_p$ gives rise to a choice of a $\mathbb{Q}$-basis for $\mathbb{Q}_p$ where the subgroup $\mathbb{Z}[\frac{1}{p}]$ is contained in only one factor of $\mathbb{Q}$. Now,

$$\mathbb{Q}_p/\mathbb{Z}\left[\frac{1}{p}\right] \cong \left(\mathbb{Q}/\mathbb{Z}\left[\frac{1}{p}\right]\right) \bigoplus \left(\bigoplus_{\mathbb{R}} \mathbb{Q}\right)$$

and the result follows from Lemma 12.2. ☐

Corollary 12.4. Let p be a prime number and A a finitely generated abelian group. If A is a torsion group, then $A \otimes (\mathbb{Z}_p/\mathbb{Z})[\frac{1}{p}] = 0$ and if $\mathrm{Rank}(A) \geq 1$ then

$$A \otimes (\mathbb{Z}_p/\mathbb{Z})\left[\frac{1}{p}\right] \cong \left(\bigoplus_{\substack{\ell \text{ prime} \\ \ell \neq p}} (\mathbb{Z}_{\ell^\infty})^{\mathrm{Rank}(A)}\right) \bigoplus \left(\bigoplus_{\mathbb{R}} \mathbb{Q}\right).$$

Proof. By observing that the tensor product of a torsion group and a divisible group is trivial, the corollary follows from Proposition 12.3. ☐

Divisible groups are injective $\mathbb{Z}$-modules, therefore exact sequences of abelian groups with a divisible factor behave nicely.

Lemma 12.5. Let

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} A_4 \xrightarrow{f_4} A_5$$

be an exact sequence of abelian groups. If A_2 is a divisible group, then

$$A_3 \cong \mathrm{coker}(f_1) \oplus \ker(f_4).$$

Proof. Using the isomorphism theorem we observe that

$$\mathrm{im}(f_2) \cong A_2/\ker(f_2)$$

and as A_2 is divisible, the same is true for $\mathrm{im}(f_2)$. Thus, $\mathrm{im}(f_2)$ is a direct factor of A_3 and

$$A_3 \cong \mathrm{im}(f_2) \oplus (A_3/\mathrm{im}(f_2)).$$

Using the exactness in connection with isomorphism theorem we identify the summands as

$$A_3/\mathrm{im}(f_2) \cong A_3/\ker(f_3) \cong \mathrm{im}(f_3) \cong \ker(f_4)$$

as well as

$$\mathrm{im}(f_2) \cong A_2/\ker(f_2) \cong A_2/\mathrm{im}(f_1) = \mathrm{coker}(f_1).$$

☐

13. The homotopy type of the components of $\mathrm{Map}(BP, X_{hG})$

Recall that for G a finite group and X a G-CW-space we are studying the fibration

$$\Phi\colon \mathrm{Map}(BP, X_{hG}) \to \mathrm{Map}(BP, BG)$$

induced by the Borel map $X_{hG} \to EG$. Let $f\colon P \to G$ be a non-trivial group homomorphism. We consider X to be a P-space via the induced action. If X is a simply connected finite CW-complex and the fixed points X^P are contractible then according to Proposition 10.2 the preimage of $\mathrm{Map}(BP, BG)_f$ under Φ is connected. As before we call this component $\mathrm{Map}(BP, X_{hG})_f$. The following theorem will provide a description of the homotopy groups under further assumptions on X.

Theorem 13.1. *Let G be a finite group and X a finite G-CW-complex. Assume that X is simply-connected and all the higher homotopy groups $\pi_i(X)$ for $i \geq 2$ are finitely generated.*

Let P be a finite p-group and $f \in \mathrm{Map}(P, G)$ be nontrivial. Let X be a P-space via f, and assume that $X^P \simeq *$. Then for all $i \geq 2$

$$\pi_i(\mathrm{Map}(BP, X_{hG})_f) \cong \mathrm{Tor}_{p'}(\pi_i(X)^P) \oplus \left(\pi_{i+1}(X)^P \otimes (\mathbb{Z}_p/\mathbb{Z})\left[\frac{1}{p}\right]\right)$$

and for the fundamental group we have the following short exact sequence:

$$0 \to \pi_2(X)^P \otimes (\mathbb{Z}_p/\mathbb{Z})\left[\frac{1}{p}\right] \to \pi_1(\mathrm{Map}(BP, X_{hG})_f) \to C_G(f(P)) \to 0$$

Thus, if X is 2-connected we have that

$$\pi_1(\mathrm{Map}(BP, X_{hG})_f) \cong C_G(f(P)).$$

Proof. Given that X is a 1-connected space, it is also nilpotent. As all the homotopy groups are finitely generated the arithmetic square from [6, Ch. VI, Lemma 8.1]

$$\begin{array}{ccc} X & \longrightarrow & \prod_{\ell \text{ prime}} X_\ell^\wedge \\ \downarrow & & \downarrow \\ X_\mathbb{Q} & \longrightarrow & \left(\prod_{\ell \text{ prime}} X_\ell^\wedge\right)_\mathbb{Q} \end{array}$$

is a homotopy pullback.

Recall that the homotopy fixed-points functor $(-)^{hP}$ is the homotopy limit over the diagram associated to the group P. By the Fubini Theorem[14, 31.5] for homotopy limits, the homotopy fixed-point functor preserves homotopy pullback, and thus we get the following homotopy pullback diagram.

$$\begin{array}{ccc} X^{hP} & \longrightarrow & \left(\prod_{\ell \text{ prime}} X_\ell^\wedge\right)^{hP} \\ \downarrow & & \downarrow \\ (X_\mathbb{Q})^{hP} & \longrightarrow & \left(\left(\prod_{\ell \text{ prime}} X_\ell^\wedge\right)_\mathbb{Q}\right)^{hP} \end{array}$$

The homotopy pullback square induces a long exact sequence in homotopy

$$\cdots \xrightarrow{f_{k+1}} \pi_{k+1}\left(\left(\left(\prod_{\ell \text{ prime}} X_\ell^\wedge\right)_{\mathbb{Q}}\right)^{hP}\right) \longrightarrow \pi_k(X^{hP})$$

$$\longrightarrow \pi_k\left(\left(\prod_{\ell \text{ prime}} X_\ell^\wedge\right)^{hP}\right) \bigoplus \pi_k\left((X_{\mathbb{Q}})^{hP}\right)$$

$$\xrightarrow{f_k} \pi_k\left(\left(\left(\prod_{\ell \text{ prime}} X_\ell^\wedge\right)_{\mathbb{Q}}\right)^{hP}\right) \longrightarrow \cdots$$

To use this sequence to determine the homotopy groups of X^{hP} we first need to identify the other groups in the sequence.

We first observe that

$$\begin{aligned}\left(\prod_{\ell \text{ prime}} X_\ell^\wedge\right)^{hP} &= \mathrm{Map}_P(EP, \prod_{\ell \text{ prime}} X_\ell^\wedge) \\ &\simeq \prod_{\ell \text{ prime}} \mathrm{Map}_P(EP, X_\ell^\wedge) = \prod_{\ell \text{ prime}} (X_\ell^\wedge)^{hP}.\end{aligned}$$

As homotopy groups of a product are products of the homotopy groups of factors, we conclude that

$$\pi_k\left(\left(\prod_{\ell \text{ prime}} X_\ell^\wedge\right)^{hP}\right) \cong \prod_{\ell \text{ prime}} \pi_k((X_\ell^\wedge)^{hP}).$$

To identify the homotopy groups of $\pi_k((X_\ell^\wedge)^{hP})$ we need to treat the case where $\ell = p$ on this own. Recall that the space X is a finite CW-complex on which the finite p-group P acts cellulary. Using the generalized Sullivan conjecture, we infer that there are weak homotopy equivalences

$$(X_p^\wedge)^{hP} \longleftarrow (X^P)_p^\wedge \longrightarrow *_p^\wedge \simeq *.$$

Thus the homotopy groups of $(X_p^\wedge)^{hP}$ vanish. For ℓ a prime different from p, we use Proposition 11.6 to conclude that

$$\pi_k\left(\left(\prod_{\ell \text{ prime}} X_\ell^\wedge\right)^{hP}\right) \cong \prod_{\ell \text{ prime}, \ell \neq p} \pi_k(X)^P \otimes \mathbb{Z}_\ell. \tag{7}$$

A direct application of Proposition 11.6 implies that

$$\pi_k\left((X_{\mathbb{Q}})^{hP}\right) \cong \pi_k(X)^P \otimes \mathbb{Q}. \tag{8}$$

Likewise the space $\prod_{\ell \text{ prime}} X_\ell^\wedge$ is 1-connected, and so by a similar argument as above be see that

$$\pi_k\left(\left(\left(\prod_{\ell \text{ prime}} X_\ell^\wedge\right)_{\mathbb{Q}}\right)^{hP}\right) \cong \pi_k\left(\left(\prod_{\ell \text{ prime}} X_\ell^\wedge\right)\right)^{P} \otimes \mathbb{Q} \cong$$

$$\left(\prod_{\ell \text{ prime}} \pi_k(X_\ell^\wedge)\right)^{P} \otimes \mathbb{Q} \cong \left(\prod_{\ell \text{ prime}} \pi_k(X_\ell^\wedge)^P\right) \otimes \mathbb{Q}$$

By Proposition 11.5 we see that

$$\pi_k\left(\left(\left(\prod_{\ell \text{ prime}} X_\ell^\wedge\right)_{\mathbb{Q}}\right)^{hP}\right) \cong \left(\prod_{\ell \text{ prime}} \pi_k(X)^P \otimes \mathbb{Z}_\ell\right) \otimes \mathbb{Q}. \tag{9}$$

We observe that all the groups in (9) are divisible and thus for all $k \geq 2$ we can apply Lemma 12.5 on part of the long exact sequence in homotopy to see that

$$\pi_k(X^{hP}) \cong \operatorname{coker}(f_{k+1}) \oplus \ker(f_k).$$

To determine the fundamental group of X^{hP}, we first recall that the fundamental group $\pi_1(X)$ is trivial, and thus the same is true for the P-fixed point sets. Thus we have that the beginning of the long exact sequence is of the form:

$$\cdots \xrightarrow{f_2} \left(\prod_{\ell \text{ prime}} \pi_2(X)^P \otimes \mathbb{Z}_\ell\right) \otimes \mathbb{Q} \longrightarrow \pi_1(X^{hP}) \longrightarrow 0$$

Thus, by the isomorphism theorem and exactness of the sequence we conclude:

$$\pi_1(X^{hP}) \cong \operatorname{coker}(f_2).$$

To identify the maps f_k we first notice that all the isomorphisms in (7), (8) and (9) are based on the description of the homotopy groups of homotopy fixed point space in terms of actual fixed points from Proposition 11.1. Thus the given long exact sequence is the restriction of the long exact sequence for the arithmetic square for X to the subgroups given in (7), (8) and (9). The map in question is just like the map from Sullivan's arithmetic square of groups given by the difference of rationalization and formal completion, and thus the restriction f_k equals $f_{\pi_k(X)^P}$ from Lemma 12.1. Using Lemma 12.1 we conclude that for $k \geq 2$ we have that

$$\pi_k(X^{hP}) \cong \left(\pi_{k+1}(X)^P \otimes (\mathbb{Z}_p/\mathbb{Z})\left[\frac{1}{p}\right]\right) \bigoplus \operatorname{Tor}_{p'}(\pi_k(X)^P).$$

as well as

$$\pi_1(X^{hP}) \cong \pi_2(X)^P \otimes (\mathbb{Z}_p/\mathbb{Z})\left[\frac{1}{p}\right].$$

Recall that as shown in Lemma 10.1 the space $\operatorname{Map}(BP, X_{hG})_f$ fits into a fibration

$$X^{hP} \to \operatorname{Map}(BP, X_{hG})_f \to \operatorname{Map}(BP, BG)_f.$$

The space $\operatorname{Map}(BP, BG)_f$ is known to be homotopic to the classifying space of the group $C_G(f(P))$. We now deduce directly from the long exact sequence associated to the fibration that for all $k \geq 2$ we have that

$$\pi_k(\operatorname{Map}(BP, X_{hG})_f) \cong \pi_k(X^{hP}).$$

and the fundamental group fits into the following short exact sequence

$$0 \to \pi_1(X^{hP}) \to \pi_1(\mathrm{Map}(BP, X_{hG})_f) \to C_G(f(P)) \to 0.$$

In particular if X is 2-connected, we have that

$$\pi_1(\mathrm{Map}(BP, X_{hG})_f) \cong C_G(f(P))$$

□

An application of Theorem 13.1 allows for a description of the homotopy groups of the components of $\mathrm{Map}(BP, |\mathcal{S}_p(G)|_{hG})$ in the case where G is a finite group of Lie type and of Lie rank at least three. Recall that in this case, there is a bijection $[BP, |\mathcal{S}_p(G)|_{hG}]$ with $\mathrm{Rep}(P, G)$ according to Corollary 10.6.

Theorem 13.2. *Let G be a finite group of Lie type in characteristic p of Lie rank at least three. Let P be a finite p-group and $f \in \mathrm{Map}(P, G)$.*

For f the trivial map the homotopy groups are

$$\pi_i(\mathrm{Map}(BP, |\mathcal{S}_p(G)|_{hG})_1 \cong \pi_i(|\mathcal{S}_p(G)|), \quad i \geq 2$$

$$\pi_1(\mathrm{Map}(BP, |\mathcal{S}_p(G)|_{hG})_1 \cong G$$

and for f nontrivial we have for all $i \geq 2$

$$\pi_i(\mathrm{Map}(BP, |\mathcal{S}_p(G)|_{hG})_f) \cong \mathrm{Tor}_{p'}(\pi_i(|\mathcal{S}_p(G)|)^P)) \oplus \left(\pi_{i+1}(|\mathcal{S}_p(G)|)^P \otimes (\mathbb{Z}_p/\mathbb{Z}) \left[\frac{1}{p}\right]\right).$$

If the rank of G is at least four we have that

$$\pi_1(\mathrm{Map}(BP, |\mathcal{S}_p(G)|_{hG})_f) \cong C_G(f(P))$$

whereas for rank three we have the following short exact sequence:

$$0 \to \pi_2(|\mathcal{S}_p(G)|)^P \otimes (\mathbb{Z}_p/\mathbb{Z}) \left[\frac{1}{p}\right] \to \pi_1(\mathrm{Map}(BP, |\mathcal{S}_p(G)|_{hG})_f) \to C_G(f(P)) \to 0.$$

Proof. Note that since G is a finite group the G-poset $\mathcal{S}_p(G)$ is finite and thus its realization $|\mathcal{S}_p(G)|$ is a finite G-CW-complex.

For the trivial map $1 \in \mathrm{Hom}(P, G)$ we have according to Proposition 10.3 that the evaluation map $\mathrm{Map}(BP, |\mathcal{S}_p(G)|_{hG}) \to |\mathcal{S}_p(G)|_{hG}$ is a weak equivalence. The homotopy groups of $|\mathcal{S}_p(G)|_{hG}$ are determined in Lemma 6.17.

We conclude by Lemma 6.11 that $|\mathcal{S}_p(G)|$ is simply connected with all higher homotopy groups finitely generated. Using Lemma 6.10 we see that for a non-trivial $f \in \mathrm{Hom}(P, G)$ the fixed points $|\mathcal{S}_p(G)|^P$ are contractible. The description of homotopy groups now follows from Theorem 13.1. □

14. Calculating fixed points on homotopy groups over divisible groups

In the main theorem we get fixed points of homotopy groups $\pi_*(|\mathcal{S}_p(G)|)$ tensored with different divisible groups. The key observation is that tensoring with a divisible group eliminates all torsion, which will allow for a description using the free Lie ring construction.

Recall that in the case of a finite group of Lie type G in characteristic p we have that $|\mathcal{S}_p(G)|$ is S-homotopy equivalent to $\bigvee_S S^{\mathrm{Rk_L}(G)-1}$ where $S \in \mathrm{Syl_p}(G)$ and S-acts on the on the index set. As we only consider P-fixed points of $\pi_*(|\mathcal{S}_p(G)|)$

for P a subgroup of S, the situation can be studied in a more general setting. After the general discussion we return to $|\mathcal{S}_p(G)|$ at the end of the chapter.

For a finite group G and $r \geq 1$ we define X_G^r to be $\bigvee_G S^{r+1}$, i.e a wedge of $|G|$ spheres of dimension $r+1$, with the G-action given by multiplication on the left of the index set. We will now study $\pi_*(X_G^r)^H \otimes D$ for any subgroup $H \leq G$ and divisible group D.

First we fix the notation for the following. Let A be a $\mathbb{Z}$-graded abelian group A. Recall that the suspension map $s\colon A \to sA$ is given by $s(A)_k = A_{k-1}$. For $n \in \mathbb{Z}$ we define a degree n-Lie bracket on A a bilinear map $[\cdot,\cdot]\colon A_k \times A_\ell \to A_{k+\ell+n}$ satisfying:

- For $x \in A_p, y \in A_q$
$$[x,y] = (-1)^{n+1}(-1)^{pq}[y,x]$$
- For $x \in A_p, y \in A_q, z \in A_r$
$$(-1)^{pr}[[x,y],z] + (-1)^{pq}[[y,z],x] + (-1)^{qr}[[z,x],y] = 0$$

A $\mathbb{Z}$-graded abelian group A with a degree 0-Lie bracket is a graded Lie ring.

If A has a degree n-Lie bracket $[\cdot,\cdot]\colon A_k \times A_\ell \to A_{k+\ell+n}$, then the induced bracket
$$[s(x), s(y)] := (-1)^k s[x,y]$$
for $x \in A_k$ and $y \in A_\ell$ is a degree $(n-1)$-Lie bracket $[\cdot,\cdot]\colon (sA)_k \times (sA)_\ell \to (sA)_{k+\ell-1}$ on the suspension group sA. This makes the suspension map into a bijection of $\mathbb{Z}$-graded abelian groups with a degree n-Lie bracket and $\mathbb{Z}$-graded abelian groups with a degree $(n-1)$-Lie bracket. The inverse s^{-1} is given as follows. For a $\mathbb{Z}$-graded abelian group B with a degree $(n-1)$-Lie bracket let $s^{-1}(B)$ be the graded abelian group with $s^{-1}(B_k) = B_{k+1}$ and Lie bracket

$$[s^{-1}(x), s^{-1}(y)] := (-1)^{1-k} s^{-1}([x,y]) \tag{10}$$

for $x \in B_k$ and $y \in B_\ell$. A graded Lie ring A of degree n is a $\mathbb{Z}$-graded abelian group, such that $s^n A$ is a graded Lie ring. We have argued that this is equivalent to having a degree n-Lie bracket as defined above. These definitions are slight variations on the standard definitions for a degree n-Lie bracket. We have chosen them as they apply more directly to homotopy groups.

Recall that for a pointed simply-connected CW-complex X the homotopy groups $\pi_*(X)$ with the Whitehead product is a $\mathbb{Z}$-graded abelian group with a degree -1 Lie bracket. The desuspension $s^{-1}(\pi_*(X))$ can be identified with the homotopy groups $\pi_*(\Omega X)$ of the loop space ΩX with the Samelson product where the equation (10) was proven by Samelson in [35]. In particular the homotopy groups of the loop space ΩX is a $\mathbb{Z}$-graded Lie ring.

Let G be a finite group. We let $\mathbb{Z}G$ denote the free abelian group generated by G considered as a set. We write elements of $\mathbb{Z}G$ as formal sums $\sum_{g\in G} n_g g$ for $n_g \in \mathbb{Z}$. Note that $\mathbb{Z}G$ is the additive part of the group ring $\mathbb{Z}[G]$ and thus has a natural G-action coming from the multiplication in the group ring. Furthermore we will consider $\mathbb{Z}G$ to be a $\mathbb{Z}$-graded abelian group by the trivial grading giving every element degree zero.

For a $\mathbb{Z}$-graded abelian group A we let $Lie(A)$ be the free graded Lie ring on A. Note that even in the case where A is a free abelian group in each degree, the free graded Lie ring $Lie(A)$ may not be torsion free, as the product $[x,x]$ of elements of even degree is 2-torsion. For this reason it is more manageable to change to a rational setting. We let $\mathbb{Q}G$ denote the left regular $\mathbb{Q}$ representation, i.e. a G-$\mathbb{Q}$-vector space with basis indexed by G and G-action induced by left multiplication on the basis elements. Note that it agrees with the additive group of the group ring $\mathbb{Q}G$ with the induced G-action. Similarly we will consider $\mathbb{Q}G$ to be a $\mathbb{Z}$-graded vector space concentrated in degree zero. For a $\mathbb{Z}$-graded vector space V we let $Lie(V)$ be the free graded Lie algebra on V.

Theorem 14.1. *Let G be a finite group and r be an integer such that $r \notin \{1,3,7\}$. For $g \in S$ let $\iota_g\colon S^{r+1} \to X_G^r$ be the inclusion into the g'th factor of the wedge sum and $x_g \in \pi_{r+1}(X_G^r)$ be the corresponding class in homotopy. Let $s\left(Lie\left(s^r(\mathbb{Z}G)\right)\right)$ be given the left G-action induced by the diagonal action on bracketed expressions.*

Then the canonical map

$$\varphi\colon s\left(Lie\left(s^r(\mathbb{Z}G)\right)\right) \to \pi_*\left(X_G^r\right)$$

given by $g \mapsto x_g$ is a G-equivariant inclusion of graded abelian groups with a degree (-1) Lie bracket, such that the induced map

$$\overline{\varphi}\colon s\left(Lie\left(s^r(\mathbb{Z}G)\right)\right)/\mathrm{Tor}(s\left(Lie\left(s^r(\mathbb{Z}G)\right)\right)) \to \pi_*\left(X_G^r\right)/\mathrm{Tor}\left(\pi_*\left(X_G^r\right)\right) \tag{11}$$

is a G-equivariant isomorphism of graded abelian groups with a degree (-1) Lie bracket.

Proof. The map φ from $s\left(Lie\left(s^r(\mathbb{Z}S)\right)\right)$ to $\pi_*(X_G^r)$ given by $g \mapsto x_g$ is clearly an injective map of $\mathbb{Z}$-graded abelian groups with a degree (-1) Lie bracket. The induced S action on $\pi_*(X_G^r)$ is the diagonal action on Whitehead products in the sense that for $g \in S$, $x \in \pi_k(X_G^r)$ and $y \in \pi_\ell(X_G^r)$ we have that

$$[g.x, g.y] = g.[x,y].$$

In particular φ is a S-equivariant map.

As φ is an injective group homomorphism the map $\overline{\varphi}$ obtained by factoring out the torsion elements will be injective as well. To prove subjectivity we want to detect generators of the quotient $\pi_*(X_G^r)/\mathrm{Tor}(\pi_*(X_G^r))$ using Hilton's Theorem in [21].

Recall that Hilton defines basic products recursively as successive Whitehead products of elements of the form x_g subject to some conditions. A basic product is a suitably bracketed word in the x_g's. In this sense a basic product of length k is an element of $\pi_{kr+1}(X_G^r)$. The number of basic products of length k is $Q(k,|G|) = \frac{1}{k}\sum_{d|k}\mu(d)|G|^{k/d}$ where μ is the Möbius function. Using Hilton's Theorem in the form of [21, Corollary 4.10] we have that for any $n > 1$

$$\pi_n\left(X_G^r\right) \cong \sum_{k=1}^{\infty} \sum_{Q(k,|G|)} \pi_n(S^{kr+1})$$

where the isomorphism is given by composing elements in $\pi_n(S^{kr+1})$ with the associated basic product in $\pi_{kr+1}(X_G^r)$.

Recall that the group $\pi_k(S^n)$ is a torsion group except for the case $\pi_n(S^n) \cong \mathbb{Z}$ generated by $[1_{S_n}]$ and for n even $\pi_{2n-1}(S^n) \cong \mathbb{Z} \oplus T$ where T is a torsion group. If $n \notin \{2,4,8\}$ the Whitehead product $[[1_{S_n}],[1_{S_n}]]$ will generate the free summand. For a sphere S^{kr+1} corresponding to a basic product $b \in \pi_{kr+1}(X_G^r)$ these specific generators for free summands correspond — under the embedding from the Hilton Theorem — to b and $[b,b]$. Both b and $[b,b]$ are in the image of $\overline{\varphi}$ and thus the map is surjective. □

The above theorem is an integral version of the well-known fact that for $r \geq 1$ the map

$$s\left(Lie\left(s^r(\mathbb{Q}G)\right)\right) \to \pi_*(X_G^r) \otimes \mathbb{Q} \tag{12}$$

given by $g \mapsto x_g \otimes 1$ where x_g is the class corresponding to the inclusion of the g'th wedge summand, is a G-equivariant isomorphism of graded $\mathbb{Q}$-vector spaces with a degree (-1) Lie bracket. The main difference in proving this statement compared to Theorem 14.1 is that the Hopf invariant one maps do not complicate the matter as $[[1_{S_n}],[1_{S_n}]] \otimes 1$ generates $\pi_{2n-1}(S^n) \otimes \mathbb{Q}$ for n even.

Corollary 14.2. Let G be a finite group and r be an integer such that $r \notin \{1,3,7\}$. Then for a subgroup $H \leq G$ and a divisible group D we have an isomorphism of $\mathbb{Z}$-graded abelian groups

$$\pi_*(X_G^r)^H \otimes D \cong s\left(Lie\left(s^r(\mathbb{Z}G)\right)^H\right) \otimes D$$

induced by the map $\varphi \otimes 1$, with φ as defined in Theorem 14.1.

Proof. By observing that tensoring with a divisible group eliminates torsion in connection with the fact that it is left exact, we have that for any abelian group A the quotient map induces an isomorphism $A \otimes D$ to $(A/\operatorname{Tor}(A)) \otimes D$. Using the isomorphism (11) from Theorem 14.1 we get that

$$\begin{array}{ccc}
\pi_*(X_G^r) \otimes D & \xrightarrow{\cong} & \pi_*(X_G^r)/\operatorname{Tor}(\pi_*(X_G^r)) \otimes D \\
\uparrow{\scriptstyle \varphi\otimes 1} & & \uparrow{\scriptstyle \overline{\varphi}\otimes 1} \\
s\left(Lie\left(s^r(\mathbb{Z}G)\right)\right) \otimes D & \xrightarrow{\cong} & s\left(Lie\left(s^r(\mathbb{Z}G)\right)\right)/\operatorname{Tor}(s\left(Lie\left(s^r(\mathbb{Z}G)\right)\right)) \otimes D,
\end{array}$$

is a commuting square of S-equivariant isomorphisms of graded abelian groups. Thus $\varphi \otimes 1$ restricts to an isomorphism on the H-fixed points. □

In the case of the rational numbers the corollary simplifies to the following.

Corollary 14.3. Let G be a finite group and $r \geq 1$. Then for any subgroup $H \leq G$ there is an isomorphism of $\mathbb{Z}$-graded $\mathbb{Q}$-vector spaces with a degree (-1) Lie bracket

$$\pi_*(X_G^r)^H \otimes \mathbb{Q} \cong s\left(Lie\left(s^r(\mathbb{Q}G)\right)^H\right).$$

Proof. The map (12) gives rise to a S-equiviariant isomorphism $s\left(Lie\left(s^r(\mathbb{Q}G)\right)\right)$ to $\pi_*(X_G^r) \otimes \mathbb{Q}$ that restricts to H-fixed points. □

The stated results until now have been in terms of isomorphisms between $\mathbb{Z}$-graded abelian groups with a degree -1 Lie bracket. If we for a divisible group D only are interested in the structure of $\pi_*(X_G^r)^H \otimes D$ as a $\mathbb{Z}$-graded abelian group, the situation simplifies. By the classification of divisible groups [22, Theorem 4] a divisible groups D is isomorphic to $\bigoplus_{p \text{ prime}} \mathbb{Z}_{p^\infty}^{\alpha_p} \oplus \mathbb{Q}^\beta$ where the cardinals α_p and β are uniquely determined. For a finitely generated abelian group A the tensor product $A \otimes D$ is divisible and isomorphic to $\bigoplus_{p \text{ prime}} \mathbb{Z}_{p^\infty}^{\tilde{\alpha}_p} \oplus \mathbb{Q}^{\tilde{\beta}}$ with cardinals $\tilde{\alpha}_p = \operatorname{Rank}(A)\alpha_p$ and $\tilde{\beta} = \operatorname{Rank}(A)\beta$, where $\operatorname{Rank}(A)$ is the free rank of A. The free rank of an abelian group A coincides with the dimension over $\mathbb{Q}$ of the rational vector space $A \otimes \mathbb{Q}$. Thus the question of determining the isomorphism type of $A \otimes D$ reduces to determining dimension in a rational vector space.

Note that Corollary 14.3 reduces the question of determining the homotopy groups of the homotopy fixed points of the rationalization $|\mathcal{S}_p(G)|_{\mathbb{Q}}$ to a calculation of fixed points of a free graded Lie algebra.

The free graded Lie algebra can be expressed in terms of the Lie operad $\mathcal{L}ie$, and this reformulation allows for a more concrete description of the fixed points.

Recall that a Lie monomial in an alphabet $\mathcal{A}$ is a finite bracketed expression of letters from $\mathcal{A}$. A particular example is $\ell(a_1, \dots, a_n)$ given inductively by $\ell(a_1) = a_1$ and $\ell(a_1, \dots, a_n) = [\ell(a_1, \dots, a_{n-1}), a_n]$. The $\mathbb{Q}$-vector space with basis Lie monomials in $a_1, \dots, a_n$, which are linear in the generators, is the n'th component of the Lie operad and is denoted $\mathcal{L}ie(n)$. Let Σ_n be the symmetric group on n elements. We have that $\mathcal{L}ie(n)$ is a right Σ_n-module where the action for $f \in \mathcal{L}ie(n)$ and $\sigma \in \Sigma_n$ is given by

$$\sigma f(a_1, \dots, a_n) = f(a_{\sigma(1)}, \dots, a_{\sigma(n)}).$$

For a $\mathbb{Q}$-vector space V and $n \geq 1$ we let $V^{\otimes n}$ be a left Σ_n-module with the following action

$$\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

Then the free Lie algebra over V can be described as

$$Lie(V) = \bigoplus_{n \in \mathbb{N}} \mathcal{L}ie(n) \otimes_{\Sigma_n} V^{\otimes n}.$$

If V is a $\mathbb{Z}$-graded rational vector space, the free graded Lie algebra over V can only be given in terms of the Lie operad in the case where V is trivially graded. In the trivially graded case where $V_0 = V$, we have that $Lie(s^r(V))$ depends on whether or not r is odd or even. In the case where r is even we recover the ungraded case as

$$Lie(s^r(V)) = \bigoplus_{n \in \mathbb{N}} \mathcal{L}ie(n) \otimes_{\Sigma_n} (s^r(V))^{\otimes n}.$$

Looking at the graded component of the module $Lie(s^r(V))$ when $r > 0$ we have that for $k \in \mathbb{N}$

$$Lie(s^r(V))_{kr} = \mathcal{L}ie(k) \otimes_{\Sigma_k} (s^r(V))^{\otimes k}$$

and all other components are trivial. In the case for r is odd we have by [3, 1.6.2]

$$Lie(s^r(V) = \bigoplus_{n \in \mathbb{N}} \mathcal{L}ie(n) \otimes_{\Sigma_n} \left((s^r(V)^{\otimes n} \otimes \mathbf{sign}_n \right).$$

Here $V^{\otimes n} \otimes \mathbf{sign}_n$ is isomorphic to $V^{\otimes n}$ as $\mathbb{Z}$-graded vector spaces over $\mathbb{Q}$, but the action of the symmetric group Σ_n takes the sign of the permutation into account by having $\sigma \in \Sigma_n$ act by

$$\sigma(v_1 \otimes \cdots \otimes v_n) = \mathrm{Sign}(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

To reduce the calculation of fixed points in a free Lie algebra to the the components, we first need a technical lemma. Recall that for a ring R and a R-module M on which a group G acts, the G-invariants M^G are the G-fixed points while the G-coinvariants are $M/\langle gm - m \rangle$. Thus M_G is the maximal G-invariant quotient of M. For $m \in M$ let $[m]_G$ be the image under the quotient map $M \twoheadrightarrow M_G$. The norm map on M given by $m \mapsto \sum_{g \in G} gm$ induces a map

$$G\uparrow \colon M_G \to M^G$$

by $[m]_G \mapsto \sum_{g\in G} gm$. As M^G is a submodule of M the composition

$$G\downarrow\colon M^G \hookrightarrow M \twoheadrightarrow M_G$$

is well-defined and both compositions are multiplications by the order of G. In particular if G is invertible in R, the G-invariants and coinvariants are isomorphic.

Lemma 14.4. Let R be a ring and let M and N be R-modules. Let G and H be finite groups such that $G\times H$ acts on both M and N. Assume that H acts trivially on M and that $|H|$ is invertible in R. Then the map

$$f\colon M\otimes_G N^H \to (M\otimes_G N)^H$$

given by $f(m\otimes_G n) = m\otimes_G n$ is an isomorphism.

Proof. Since the order of H is invertible in R, we have that H-invariants and coinvariants are isomorphic. Thus, using this fact we reduce the question to involve only coinvariants. Note that for $R[G]$-modules M and N we have that $M\otimes_G N$ is the G-coinvariant $(M\otimes N)_G$. Let the map g be defined by the commutative diagram

$$\begin{array}{ccc} (M\otimes N^H)_G & \xrightarrow{f} & ((M\otimes N)_G)^H \\ {\scriptstyle (1\otimes H\uparrow)_G}\uparrow & & \downarrow{\scriptstyle \frac{1}{|H|}\cdot H\downarrow} \\ (M\otimes N_H)_G & \xrightarrow{g} & ((M\otimes N)_G)_H \end{array}$$

Then g is given by $[m\otimes [n]_H]_G \mapsto [[m\otimes n]_G]_H$ and factors through

$$(M\otimes N_H)_G \to ((M\otimes N)_H)_G \to ((M\otimes N)_G)_H.$$

As the action of H is trivial on M, the first map is an isomorphism while the second map is an isomorphism since the actions of H and G commute. □

Corollary 14.5. Let G be a finite group and $r\geq 1$. Then for a subgroup $H\leq G$ and for $k\in\mathbb{N}$

$$\pi_{kr+1}(X^r_G)^H\otimes\mathbb{Q}\cong$$
$$\begin{cases} \mathcal{L}ie(k)\otimes_{\Sigma_k}\left(\mathbb{Q}^{\otimes k}\right)^H & \text{, if } r \text{ is even} \\ \mathcal{L}ie(k)\otimes_{\Sigma_k}\left(\left(\mathbb{Q}S^{\otimes k}\right)^H\otimes \mathbf{sign}_k\right) & \text{, if } r \text{ is odd} \end{cases}$$

and $\pi_n(X^r_G)^H\otimes\mathbb{Q}$ is trivial for all n not of the form $n=kr+1$ for some $k\in\mathbb{N}$.

Proof. By Corollary 14.3 the homotopy groups satisfy

$$\pi_n(X^r_G)^H\otimes\mathbb{Q} = \left(Lie\left(s^r(\mathbb{Q}G)\right)^H\right)_{n-1}$$

for $n\in\mathbb{N}$. Here the right hand side is the set of fixed points of a free graded Lie algebra on a graded vector space concentrated in one degree. Note that H acts on the graded components of the free Lie algebra $Lie(s^r(\mathbb{Q}G))$. The graded components can be described in terms of the Lie operad:

$$Lie\left(s^r(\mathbb{Q}G)\right)_{kr}\cong\begin{cases} \mathcal{L}ie(k)\otimes_{\Sigma_k}\left(\mathbb{Q}G^{\otimes k}\right) & \text{, if } r \text{ is even} \\ \mathcal{L}ie(k)\otimes_{\Sigma_k}\left(\left(\mathbb{Q}G^{\otimes k}\right)\otimes \mathbf{sign}_k\right) & \text{, if } r \text{ is odd} \end{cases}$$

and $Lie\left(s^r(\mathbb{Q}G)\right)_n$ is trivial for n not of the form kr for $k\in\mathbb{N}$. For $k\in\mathbb{N}$ let H act trivially on $\mathcal{L}ie(k)$ where the action on $\left(\mathbb{Q}G^{\otimes k}\right)$ and $\left(\left(\mathbb{Q}G^{\otimes k}\right)\otimes\mathbf{sign}_k\right)$ is induced by the H-action on $\mathbb{Q}G$. Then the given isomorphisms of $Lie\left(s^r(\mathbb{Q}G)\right)_{kr}$ are H-equivariant and hence induce an isomorphism on H-fixed points. We have that the H-action on $\mathbb{Q}S^{\otimes k}$ as well as $\left((\mathbb{Q}S^{\otimes k}) \otimes \mathbf{sign}_k\right)$ commutes with the Σ_k-action. Since the order of H is invertible over $\mathbb{Q}$ the statement follows from Lemma 14.4. □

Recall that the necklace polynomial provides a formula for the dimension of the n-graded parts of the free Lie algebra $Lie(s^r(V))$ in the case where r is even. This allows for explicit upper and lower bounds for the dimension over $\mathbb{Q}$ of

$$\pi_{kr+1}(X_G^r)^P \otimes \mathbb{Q}$$

in the case where r is even. The necklace polynomial is given as

$$Q(w,k) = \frac{1}{w}\sum_{d|w} \mu(d) k^{w/d}.$$

where μ denotes the Möbius function. The degree w-part of $Lie(V)$ has dimension $Q(w, \dim_{\mathbb{Q}}(V))$ over $\mathbb{Q}$.

Proposition 14.6. For a finite group G with subgroup H, and $r > 1$ and even, we have for $k \in \mathbb{N}$

$$Q(k, [G:H]) \leq \dim_{\mathbb{Q}}(\pi_{kr+1}(X_G^r)^H \otimes \mathbb{Q}) \leq Q(k, |G|).$$

If H is a proper subgroup of G, then $\pi_{kr+1}(X_G^r)^H \otimes \mathbb{Q}$ is non-trivial for all $k \in N$.

Proof. When H is a subgroup of G there exist inclusions of rational vector spaces:

$$((\mathbb{Q}G)^H)^{\otimes k} \hookrightarrow (\mathbb{Q}G^{\otimes k})^H \hookrightarrow \mathbb{Q}G^{\otimes k}.$$

The vector spaces have a compatible action of Σ_k and thus give rise to inclusions

$$\mathcal{L}ie(k) \otimes_{\Sigma_k} ((\mathbb{Q}G)^H)^{\otimes k} \hookrightarrow \mathcal{L}ie(k) \otimes_{\Sigma_k} (\mathbb{Q}G^{\otimes k})^H \hookrightarrow \mathcal{L}ie(k) \otimes_{\Sigma_k} \mathbb{Q}G^{\otimes k}$$

By noticing that that first and last vector spaces are the degree k pieces of the free Lie algebra on $(\mathbb{Q}G)^H$ and $\mathbb{Q}G$ respectively, their dimension can be calculated using the necklace polynomial. The dimension over $\mathbb{Q}$ of $(\mathbb{Q}G)^H$ and $\mathbb{Q}G$ is $[G:H]$ and $|G|$ and thus the degree-k pieces have dimensions $Q(k, [G:H])$ and $Q(k, |G|)$. The statement now follows by Corollary 14.5 identifying $\mathcal{L}ie(k) \otimes_{\Sigma_k} (\mathbb{Q}G^{\otimes k})^H$ and $\pi_{kr+1}(X_G^r)^H \otimes \mathbb{Q}$.

The last statement follows from observing that the necklace polynomial $Q(k, [G : H])$ has positive values whenever the index $[G:H] > 1$. □

To further understand the the fixed points of the rational homotopy groups $\pi_*(X_G^r) \otimes \mathbb{Q}$ using Corollary 14.5 we need to determine the fixed points of a subgroup on the iterated tensor products of the regular representation.

Proposition 14.7. Let G be a finite group and H a subgroup of G. For any $n \in \mathbb{N}$ let H act diagonally on G^n by left multiplication. Then the map

$$\varphi\colon H\backslash G^n \to (\mathbb{Q}G^{\otimes n})^H$$

given by

$$\varphi[g_1 \dots, g_n]_H = \sum_{h \in H} hg_1 \otimes \cdots \otimes hg_n$$

is well-defined and the image is a $\mathbb{Q}$-basis for $(\mathbb{Q}G^{\otimes n})^H$. In particular

$$\dim_{\mathbb{Q}}((\mathbb{Q}G^{\otimes n})^H) = \frac{|G|^n}{|H|}.$$

Proof. It follows directly from the definition of φ that it is a well-defined map with image in the H-fixed points of $\mathbb{Q}G^{\otimes n}$. To see that is a basis for $(\mathbb{Q}G^{\otimes n})^H$ we reduce to a vector space that is G-equviariant to $\mathbb{Q}G^{\otimes n}$.

Recall that for a finite-dimensional rational G-representation V with basis B we have that $\operatorname{Ind}_1^G \operatorname{Res}_1^G V = \mathbb{Q}G \otimes \bigoplus_B \mathbb{Q}$. According to [12, Proposition III (5.6)(a)] this implies that the map

$$f\colon \mathbb{Q}G \otimes \bigoplus_B \mathbb{Q} \to \mathbb{Q}G \otimes V$$

given by $g \otimes 1_v \to g \otimes gv$ for $g \in G$ and $v \in B$ is a G-equivariant isomorphism. In particular for $n \geq 1$ and $V = (\mathbb{Q}G)^{\otimes n-1}$ we have a G-isomorphism

$$f\colon \mathbb{Q}G \otimes \bigoplus_{G^{n-1}} \mathbb{Q} \xrightarrow[\cong]{} (\mathbb{Q}G)^{\otimes n}$$

given by

$$g \otimes 1_{(g_1,\dots,g_{n-1})} \to g \otimes gg_1 \otimes \cdots \otimes gg_{n-1}$$

for $g, g_1, \dots, g_{n-1} \in G$.

Let the tensor product $\bigoplus_{G^{n-1}} \mathbb{Q}G$ be equipped with the diagonal G-action. Then the map

$$g\colon \bigoplus_{G^{n-1}} \mathbb{Q}G \to \mathbb{Q}G \otimes \bigoplus_{G^{n-1}} \mathbb{Q}$$

given by $g((x)_{(g_1,\dots g_{n-1})}) = x \otimes 1_{(g_1,\dots g_{n-1})}$ is a G-equivariant isomorphism.

For a subgroup H of G the H-fixed points are preserved under a G-equivariant isomorphism and hence we have that

$$\bigoplus_{G^{n-1}} (\mathbb{Q}G)^H \xrightarrow[\cong]{f\circ g} \left((\mathbb{Q}G)^{\otimes n}\right)^H .$$

Consider an element $\sum_{g\in G} q_g g \in \mathbb{Q}G$ where $q_g \in \mathbb{Q}$ for all $g \in G$. Note that it is a H-fixed point if and only if the coefficients a_g are constant for each right coset of $H\backslash G$. Thus the map

$$\alpha\colon \mathbb{Q}[H\backslash G] \to \mathbb{Q}G$$

given by $\alpha(Hg) = \sum_{h\in H} hg$ has image $\mathbb{Q}G^H$ and will be a basis for $\mathbb{Q}G^H$. Its restriction to the image is an isomorphism. Thus by combining the three given maps we get a monomorphism

$$\beta\colon \bigoplus_{G^{n-1}} \mathbb{Q}[H\backslash G] \to (\mathbb{Q}G)^{\otimes n}$$

that for any $g, g_1, \dots, g_{n-1} \in G$ sends the basis vector $(Hg)_{(g_1,\dots,g_{n-1})}$ to

$$\beta(Hg_{(g_1,\dots,g_{n-1})}) = \sum_{h\in H} (hg \otimes hgg_1 \otimes \cdots \otimes hgg_{n-1}) .$$

The image of β is the H-fixed points of $(\mathbb{Q}G)^{\otimes n}$, and $\beta(Hg_{(g_1,\dots,g_{n-1})})$ is a basis for $g, g_1, \dots, g_{n-1} \in G$. By noticing that $\beta(Hg_{(g_1,\dots,g_{n-1})}) = \varphi[g, gg_1, \dots gg_{n-1}]_H$ the statement now follows. □

While $(\mathbb{Q}G^H)^{\otimes n}$ is a subspace of $(\mathbb{Q}G^{\otimes n})^H$, they only agree by the above Proposition when $n = 1$ or H is the trivial group. Furthermore it is worth noticing that the basis for $(\mathbb{Q}G^{\otimes n})^H$ is given by the norm map on the basis of $\mathbb{Q}G^{\otimes n}$, when $\mathbb{Q}G^{\otimes n}$ is seen as an H-module.

Corollary 14.8. Let G be a finite group and H be a subgroup of G. Then for $r \geq 1$ we have

$$\pi_{r+1}(X_G^r)^H \otimes \mathbb{Q} \cong \mathbb{Q}[H\backslash G].$$

Proof. By setting $k = 1$ in Corollary 14.5 we conclude that $\pi_{r+1}(X_G^r)^H \otimes \mathbb{Q}$ is isomorphic to the fixed points $\mathbb{Q}G^H$ independently of whether r is even or odd. The fixed points are now determined by Proposition 14.7. □

To understand the group $\pi_*(X_G^r)^H \otimes \mathbb{Q}$ for higher degrees using the Lie operad we need to understand how Σ_n acts on $(\mathbb{Q}G^{\otimes n})^H$ for bigger values of n. The following Proposition describes for a $\sigma \in \Sigma_n$ the fixed points for σ on the basis for $(\mathbb{Q}G^{\otimes n})^H$ given in Proposition 14.7.

Proposition 14.9. Let G be a finite group and H a subgroup of G. For $n \geq 1$ let Σ_n act on G^n by

$$\sigma(g_1, \ldots, g_n) = (g_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(n)}),$$

for any $\sigma \in \Sigma_n$ and $(g_1, \ldots, g_n) \in G^n$. Let H act diagonally on G^n by multiplication from the left. Then the Σ_n-action on G^n induces an action of Σ_n on the orbit set $H\backslash G^n$.

For a $\sigma \in \Sigma_n$ with cycle type $(\ell_1, \ldots, \ell_k)$ and standard form

$$\sigma = \big(x_1, \sigma(x_1), \ldots, \sigma^{\ell_1 - 1}(x_1)\big) \cdots \big(x_k, \sigma(x_k), \ldots, \sigma^{\ell_k - 1}(x_k)\big)$$

define the set

$$G_\sigma = G^k \times \{h \in H \mid h^{\gcd(\ell_1, \ldots, \ell_k)} = 1\}.$$

Let H act on G_σ by

$$\tilde{h}.(a_1, \ldots a_k, h) = (\tilde{h}a_1, \ldots, \tilde{h}a_k, \tilde{h}h\tilde{h}^{-1})$$

for $\tilde{h} \in H$ and $(a_1, \ldots a_k, h) \in G_\sigma$. Then the map

$$\psi \colon H\backslash G_\sigma \to (H\backslash G^n)^\sigma$$

given by setting $\psi[a_1, \ldots a_k, h] \in H\backslash G^n$ to be the class $[g_1, \ldots, g_n]$ where

$$g_{\sigma^j(x_i)} = h^j a_i$$

is a well-defined bijection.

Finally, the number of σ-fixed points of $H\backslash G^n$ is

$$|(H\backslash G^n)^\sigma| = \frac{|G_\sigma|}{|H|}.$$

Proof. First we first prove that ψ is well-defined. Note that every number $1 \leq m \leq n$ is of the form $\sigma^j(x_i)$ for a unique $i \in \{1, \ldots k\}$ where j is only uniquely determined modulo ℓ_i. As $h^{\ell_i} = 1$ the element $g_{\sigma^j(x_i)}$ does not depend on the choice of j. To see that it does not depend on the representative in $H\backslash G_\sigma$ consider $\tilde{h} \in H$ and $(a_1, \ldots a_k, h) \in G_\sigma$ and let $\psi[a_1, \ldots a_k, h]$ respectively $\psi[\tilde{h}.(a_1, \ldots a_k, h)]$ be given by classes $[g_1, \ldots, g_n]$ and $[\tilde{g}_1, \ldots, \tilde{g}_n]$. Now

$$\tilde{g}_{\sigma^j(x_i)} = (\tilde{h}h\tilde{h}^{-1})^j \tilde{h}a_i = \tilde{h}h^j a_i = \tilde{h}g_{\sigma^j(x_i)}$$

and $[g_1, \ldots, g_n]$ and $[\tilde{g}_1, \ldots, \tilde{g}_n]$ are the same class in $H\backslash G^n$.

For $[a_1, \ldots a_k, h] \in H\backslash G_\sigma$ let the image $\psi[a_1, \ldots a_k, h]$ be represented by $[g_1, \ldots g_n]$. Then the image of $(g_1, \ldots, g_n)$ under σ is given by

$$\sigma(g_1, \ldots, g_n)_{\sigma^j(x_i)} = g_{\sigma^{j-1}(x_i)} = h^{j-1}a_i = h^{-1}g_{\sigma^j(x_i)} = h^{-1}(g_1, \ldots, g_n)_{\sigma^j(x_i)}.$$

Hence $\psi[a_1, \dots a_k, h]$ is fixed by σ.

To show that ψ is in fact a bijection, we now construct an inverse. An element $[g_1 \dots, g_n] \in H\backslash G^n$ is fixed by σ if there exists a unique $h \in H$ such that

$$g_{\sigma^j(x_i)} = hg_{\sigma^{j-1}(x_i)} \tag{13}$$

for all $i \in \{1, \dots, k\}$ and $j \in \{0, \dots, \ell_k - 1\}$. By combining equation (13) for each orbit of σ we conclude that $h^{\ell_i} = 1$ for all $i \in \{1, \dots, k\}$. Thus $(g_{x_1}, \dots g_{x_n}, h)$ is an element of G_σ. Furthermore by changing the representative for $[g_1 \dots, g_n] \in H\backslash G^n$ by $\tilde{h} \in H$ to $(\tilde{h}g_1, \dots \tilde{h}g_n)$ we get the element $\tilde{h}.(g_{x_1}, \dots g_{x_n}, h)$ and this defines a map $(H\backslash G^n)^\sigma \to H\backslash G_\sigma$. One easily sees that this is an inverse to ψ.

By observing that the H action on G_σ is free, the number of σ-fixed points follows from the established bijection. □

Note that the above proposition for the trivial group is that an element of G^n is a σ-fixed point if and only if it is constant on its σ-orbits.

The calculation of fixed points allows us to determine the dimension of the graded component in degree two of $Lie(\mathbb{Q}G)^H$ when $\mathbb{Q}G$ is concentrated in a fixed even degree.

Lemma 14.10. Let G be a finite group with subgroup H. Let $\sigma \in \Sigma_2$ be the two-cycle (12). We have that $\mathcal{L}ie(2) \otimes_{\Sigma_2} (\mathbb{Q}G \otimes \mathbb{Q}G)^H$ has a $\mathbb{Q}$-basis corresponding to representatives of σ-orbits of $(\mathbb{Q}G \otimes \mathbb{Q}G)^H$ of length two and thus the dimension over $\mathbb{Q}$ is

$$\frac{|G|(|G| - |\{h \in H \mid h^2 = 1\}|)}{2|H|}.$$

Proof. We have that $\mathcal{L}ie(2) \otimes_{\Sigma_2} (\mathbb{Q}G \otimes \mathbb{Q}G)^H$ is a subspace of the degree-2 component of the free Lie algebra $Lie(\mathbb{Q}G)$. By [34, Theorem 4.9] the Hall polynomials form a basis for $Lie(\mathbb{Q}G)$ and in particular for the degree-2 component $\mathcal{L}ie(2) \otimes_{\Sigma_2} (\mathbb{Q}G \otimes \mathbb{Q}G)$ the basis is of the form $[g_1, g_2]$ where $g_1 < g_2$ for some total order on G.

Recall that from Proposition 14.7 we have that $(\mathbb{Q}G^{\otimes 2})^H$ has a $\mathbb{Q}$-basis given by

$$\varphi[g_1, g_2]_H = \sum_{h \in H} hg_1 \otimes hg_2$$

where $[g_1, g_2]_H \in H\backslash G^2$. The corresponding element in $Lie(\mathbb{Q}G)$ is $\sum_{h \in H} [hg_1, hg_2]$, and ranging over $[g_1, g_2]_H \in H\backslash G^2$ we get a generating set for $\mathcal{L}ie(2) \otimes_{\Sigma_2} (\mathbb{Q}G \otimes \mathbb{Q}G)^H$. If $[g_1, g_2]_H \in H\backslash G^2$ is fixed by the two-cycle $\sigma = (12) \in \Sigma_2$ we have that $\sum_{h \in H} [hg_1, hg_2]$ factors as sums $[x, y] + [y, x]$ and possibly a factor of the form $[x, x]$, and hence the sum $\sum_{h \in H} [hg_1, hg_2] = 0$ in the graded Lie algebra $Lie(\mathbb{Q}G)$. Similarly if the σ-orbit of $[g_1, g_2]_H$ has length two the corresponding images $\varphi[g_1, g_2]_H$ and $\varphi[g_2, g_1]_H$ in $Lie(\mathbb{Q}G)$ are linearly dependent. Thus we get a generating set for $\mathcal{L}ie(2) \otimes_{\Sigma_2} (\mathbb{Q}G \otimes \mathbb{Q}G)^H$ by picking a representative for each σ-orbit of $H\backslash G^2$ of length two and looking at the corresponding element of $Lie(\mathbb{Q}G)$. Each image of each such representative $[g_1, g_2]_H$ is a linear combination of basis vectors in $Lie(\mathbb{Q}G)$ and if two such representatives $[g_1, g_2]_H$ and $[\tilde{g}_1, \tilde{g}_2]_H$ include the same basis vector, then there exists an $h \in H$ such that $[hg_1, hg_2] = [\tilde{g}_1, \tilde{g}_2]$ or $[hg_2, hg_1] = [\tilde{g}_1, \tilde{g}_2]$, which is a contradiction. Thus we conclude that the image of $[g_1, g_2]_H$ and $[\tilde{g}_1, \tilde{g}_2]_H$ in $Lie(\mathbb{Q}G)$ is a linear combination over disjoint sets of basis vectors, and hence we conclude that the given generating set of $\mathcal{L}ie(2) \otimes_{\Sigma_2} (\mathbb{Q}G \otimes \mathbb{Q}G)^H$ is in fact a basis.

Finally, the number of σ-orbits in $H\backslash G^2$ of length two is

$$\frac{|H\backslash G^2| - |(H\backslash G^2)^\sigma|}{2}$$

which reduces to the formula using Proposition 14.9. □

In Theorem 13.2 the statement for finite groups of Lie type of Lie rank at least three includes the groups $\pi_i(|\mathcal{S}_p(G)|)^P \otimes (\mathbb{Z}_p/\mathbb{Z})[\frac{1}{p}]$, where $(\mathbb{Z}_p/\mathbb{Z})[\frac{1}{p}]$ is a divisible group. We are now able to determine the groups up to isomorphism in terms of dimensions in a free Lie algebra.

Proposition 14.11. Let G be a finite group of Lie type in characteristic p of Lie rank at least three. Let P be a finite p-group and $f \in \mathrm{Map}(P, G)$ be nontrivial. If n is of the form $n = k(\mathrm{Rk_L}(G) - 2) + 1$ for some $k \in \mathbb{N}$ then

$$\pi_n(|\mathcal{S}_p(G)|)^P \otimes (\mathbb{Z}_p/\mathbb{Z})\left[\frac{1}{p}\right] \cong \left(\bigoplus_{\substack{\ell \text{ prime} \\ \ell \neq p}} (\mathbb{Z}_{\ell^\infty})^{\alpha_i}\right) \bigoplus \left(\bigoplus_{\mathbb{R}} \mathbb{Q}^{\alpha_i}\right)$$

where

$$\alpha_n = \dim_{\mathbb{Q}}\left(\left(Lie\left(s^{\mathrm{Rk_L}(G)-2}(\mathbb{Q}S)\right)^P\right)_{n-1}\right).$$

In fact for $n = k(\mathrm{Rk_L}(G) - 2) + 1$ for a $k \in \mathbb{N}$ we have that

$$\alpha_n = \begin{cases} \dim_{\mathbb{Q}}\left(\mathcal{L}ie(k) \otimes_{\Sigma_k} \left(\mathbb{Q}^{\otimes k}\right)^H\right) & \text{, if } \mathrm{Rk_L}(G) \text{ is even} \\ \dim_{\mathbb{Q}}\left(\mathcal{L}ie(k) \otimes_{\Sigma_k} \left(\left(\mathbb{Q}S^{\otimes k}\right)^H \otimes \mathbf{sign}_k\right)\right) & \text{, if } \mathrm{Rk_L}(G) \text{ is odd} \end{cases}$$

For n not of the form $n = k(\mathrm{Rk_L}(G) - 2) + 1$ for some $k \in \mathbb{N}$ then $\pi_n(|\mathcal{S}_p(G)|)^P \otimes (\mathbb{Z}_p/\mathbb{Z})\left[\frac{1}{p}\right]$ is trivial.

Proof. By Lemma 6.11 the homotopy groups $\pi_n(|\mathcal{S}_p(G)|)$ are finitely generated abelian groups, so the same holds for the P-fixed points. By Corollary 12.4 the group $\pi_n(|\mathcal{S}_p(G)|)^P \otimes (\mathbb{Z}_p/\mathbb{Z})\left[\frac{1}{p}\right]$ is determined by the free rank of $\pi_n(|\mathcal{S}_p(G)|)^P$. The free rank coincides with the dimension of $\pi_i(|\mathcal{S}_p(G)|)^P \otimes \mathbb{Q}$ as a rational vector space. Using the notation from this chapter we have following Lemma 6.8 that $|\mathcal{S}_p(G)|$ is S-homotopy equivalent to $X_S^{\mathrm{Rk_L}(G)-2}$, where $S \in \mathrm{Syl_p}(G)$. We may assume that P is a subgroup of S and then $\pi_n(|\mathcal{S}_p(G)|)^P$ and $\pi_n(X_S^{\mathrm{Rk_L}(G)-2})^P$ are isomorphic as groups. Using Corollary 14.3 we observe that $\pi_n(|\mathcal{S}_p(G)|)^P \otimes \mathbb{Q}$ isomorphic to $\left(Lie\left(s^{\mathrm{Rk_L}(G)-2}(\mathbb{Q}S)\right)^P\right)_{n-1}$. We use Corollary 14.5 to give the alternative description using the Lie operad as well as to observe that the dimension of the vector space is trivial for all n not of the form $n = k(\mathrm{Rk_L}(G) - 2) + 1$. □

Corollary 14.12. Let G be a finite group of Lie type in characteristic p of Lie rank at least four and even. Let P be a finite p-group and $f \in \mathrm{Map}(P, G)$ be nontrivial. Assume that $f(P)$ is not a Sylow-p-subgroup of G. If i of the form $i = k(\mathrm{Rk_L}(G) - 2) + 1$ for some $k \in \mathbb{N}$ then

$$\pi_i(|\mathcal{D}_p(G)|)^P \otimes (\mathbb{Z}_p/\mathbb{Z})\left[\frac{1}{p}\right]$$

is non-trivial.

Proof. Follows directly from Proposition 14.11 and Corollary 14.6. □

15. Applications to p-local finite groups of Lie type.

First we recall a few definitions concerning p-local finite groups.

Definition 15.1. For a group G and a p-subgroup S of G for a prime p, the fusion system $\mathcal{F}_S(G)$ on S generated by G is the category of all subgroups of S with morphisms

$$\mathrm{Mor}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q) = \{c_g \mid g \in T_G(P,Q)\}$$

for $P,Q \leq S$.

The standard definition of the fusion system $\mathcal{F}_S(G)$ is the restriction to the case where G is a finite group and $S \in \mathrm{Syl_p}(G)$. Then the fusion system $\mathcal{F}_S(G)$ is saturated (see [9, Definition 1.2]).

Definition 15.2. Let G be a finite group and $S \in \mathrm{Syl_p}(G)$.

- A subgroup $P \leq S$ is $\mathcal{F}_S(G)$-centric if it is p-centric, i.e. $\mathrm{Z}(P) \in \mathrm{Syl_p}(C_G(P))$ or equivalently $C_G(P) \cong \mathrm{Z}(P) \times C'_G(P)$ for a (unique) subgroup $C'_G(P)$ of order prime to p.
- A subgroup $P \leq S$ is radical in $\mathcal{F}_S(G)$ if $N_G(P)/(P \cdot C_G(P))$ is p-reduced, i.e if $O_p(N_G(P)/(P \cdot C_G(P))) = 1$.
- The centric linking system associated to $\mathcal{F}_S(G)$ is denoted $\mathcal{L}^c_S(G)$ and is a category with objects the $\mathcal{F}_S(G)$-centric subgroups of S and morphism sets
$$\mathrm{Mor}_{\mathcal{L}^c_S(G)}(P,Q) = T_G(P,Q)/C'_G(P).$$
The centric radical linking system $\mathcal{L}^{cr}_S(G)$ is the restriction of $\mathcal{L}^c_S(G)$ to the full subcategory of $\mathcal{F}_S(G)$-centric-radical subgroups.
- The p-local finite group associated to the Sylow-p-subgroup S in G, is the triple $(S, \mathcal{F}_S(G), \mathcal{L}^c_S(G))$.

Note that the above definitions are a special case of $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroups for general fusion systems in the case of fusion systems coming from a finite group.

Definition 15.3. Let G_1 and G_2 be groups and for a prime p let $S_1 \leq G_1$ and $S_2 \leq G_2$ be p-subgroups. An isomorphism $\varphi\colon S_1 \to S_2$ is fusion preserving if for all subgroups P and Q of S_1 and $\alpha \in \mathrm{Hom}(P,Q)$ it satisfies

$$\alpha \in \mathrm{Hom}_{G_1}(P,Q) \iff (\varphi|_Q)\alpha(\varphi|_P)^{-1} \in \mathrm{Hom}_{G_2}(\varphi(P),\varphi(Q))$$

Note that a fusion preserving isomorphism will induce an isomorphism of categories from $\mathcal{F}_{G_1}(S_1)$ to $\mathcal{F}_{G_2}(S_2)$. We have the following lemma

Lemma 15.4. Let G be a finite group of Lie type of rank 1 in characteristic p with Borel subgroup B. Then the inclusion $\iota\colon B \to G$ induces an equivalence of categories $\mathcal{F}_S(B) = \mathcal{F}_S(G)$ for $S = O_p(B)$.

Proof. It was essentially proven by [27], see [2, Proposition III: 1.15] for details, that for the map $f\colon G \to \tilde{G}$ between finite groups, the restriction to Sylow-p-subgroups is fusion preserving if and only if the induced map $\mathrm{Rep}(P,G) \to \mathrm{Rep}(P,\tilde{G})$ is a bijection for all p-groups P. From Corollary 8.3 we conclude that the inclusion $B \to G$ is fusion preserving and thus $\mathcal{F}_S(B) = \mathcal{F}_S(G)$. □

Lemma 15.5. Let G be a finite group of Lie type of rank 2 in characteristic p. Let P_α and P_β be the maximal proper parabolic subgroups over a Borel subgroup B. Then the map $\phi\colon P_\alpha *_B P_\beta \to G$ induces an isomorphism between the categories $\mathcal{F}_S(P_\alpha *_B P_\beta)$ and $\mathcal{F}_S(G)$ for $S = O_p(B)$.

Proof. The argument from [2, III Proposition 1.15] may be applied to the infinite group $P_\alpha *_B P_\beta$ and in connection with Corollary 8.13 we conclude that the map $\phi\colon P_\alpha *_B P_\beta \to G$ induces a isomorphism of categories between $\mathcal{F}_S(P_\alpha *_B P_\beta)$ and $\mathcal{F}_S(G)$. □

The centric-radical linking system is a quotient of the centric-radical transporter system. By Proposition 6.5 the Borel construction $|\mathcal{S}_p(G)|_{hG}$ we have been studying up to this point is homotopy equivalent to the centric-radical transporter system. This observation allow us to translate the previous results into the setting of p-local finite groups.

Theorem 15.6. *Let G be a finite group of Lie type in characteristic p with trivial center and let P be a finite p-group. Then $\mathcal{T}_S^{cr}(G) = \mathcal{L}_S^{cr}(G)$ and the map from $\mathrm{Map}(BP, |\mathcal{L}_S^{cr}(G)|)$ to $\mathrm{Map}(BP, BG)$ induced by the simplicial map from $\mathcal{L}_S^{cr}(G)$ to G given by inclusion on the morphism sets is a mod-p-equivalence.*

Proof. The set of $\mathcal{F}_S(G)$-centric-radical subgroups of S coincides with the intersection $\mathcal{D}_p(G) \cap S$, where $\mathcal{D}_p(G)$ is defined in Definition 6.2. Following the proof of Lemma 6.4 we have for any $\mathcal{F}_S(G)$-centric-radical subgroup Q that $C'_G(Q) = \mathrm{Z}(G)$, and hence in our case $C'_G(Q) = 1$. By definition, the categories $\mathcal{T}_S^{cr}(G)$ and $\mathcal{L}_S^{cr}(G)$ agree. From Proposition 6.5 we have that $|\mathcal{T}_S^{cr}(G)| \simeq |\mathcal{S}_p(G)|_{hG}$, and so the result follows from the three cases from Theorem 9.2, 9.3 and 10.5, as we identify the Borel map $|\mathcal{S}_p(G)|_{hG} \to BG$ with the indicated simpicial map $\mathcal{T}_S^{cr}(G) \to \mathcal{B}(G)$ □

To understand the space of maps $\mathrm{Map}(BP, |\mathcal{L}_S^{cr}(G)|)$ for a general finite group G of Lie type in characteristic p, the strategy is to reduce to the trivial center case by the quotient map $G \to G/\mathrm{Z}(G)$. For general non-abelian groups the quotient $G/\mathrm{Z}(G)$ need not have trivial center, but in the case of finite groups of Lie type they do.

The quotient map for a central subgroup of order prime to p behaves nicely with respect to p-fusion as shown in the following results.

Proposition 15.7. Let G be a finite group and p a prime number. Let $H \leq \mathrm{Z}(G)$ be a group of order prime to p and let $\pi\colon G \to G/H$ be the associated quotient map. For a subgroup P of G set $\pi(P) = PH/H$. Then π induces a bijection of posets

$$\pi\colon \{P \leq G \mid P \text{ is a } p\text{-group}\} \to \{\bar{P} \leq G/H \mid \bar{P} \text{ is a } p\text{-group}\}$$

where both sets are ordered by inclusion. For p-subgroups P and Q of G the group H acts on the transporter set $T_G(P,Q)$ by multiplication on the right and the quotient map induces bijection on transporter sets

$$T_{\pi(G)}(\pi(P), \pi(Q)) = T_G(P,Q)/H.$$

For p-subgroups P the quotient map also induces isomorphisms on normalizers as well as centralizers

$$N_{\pi(G)}(\pi(P)) = \pi(N_G(P)) = N_G(P)/H$$
$$C_{\pi(G)}(\pi(P)) = \pi(C_G(P)) = C_G(P)/H.$$

Proof. Let P be a p-subgroup of G. Then the image $\pi(P)$ is a p-subgroup of G/H with the same order as P. For a p-subgroup $\bar{P}$ in G/H the preimage $\pi^{-1}(\bar{P})$ is a subgroup of G of order $|\bar{P}| \cdot |H|$. Then a Sylow-p-subgroup S of $\pi^{-1}(\bar{P})$ has the same order as $\bar{P}$. The subgroup S is normal in $\langle S, H\rangle$. This is a group of order at least $|\bar{P}| \cdot |H|$ and thus we conclude that $\langle S, H\rangle = \pi^{-1}(\bar{P})$ and that S is normal in $\pi^{-1}(\bar{P})$. In particular S is the unique Sylow-p-subgroup of $\pi^{-1}(\bar{P})$. By observing that $\pi^{-1}(\bar{P}) = SH$ it follows easily that the map $\bar{P} \mapsto \mathrm{Syl}_{\mathrm{p}}(\pi^{-1}(\bar{P}))$ is an inverse to π on the set of p-subgroups.

Let P and Q be p-subgroups of G. Since H is a subgroup of the center of G we have that π maps $T_G(P,Q)$ to a subset of $T_{\pi(G)}(\pi(P), \pi(Q))$ and it induces an injective map

$$T_G(P,Q)/H \to T_{\pi(G)}(\pi(P), \pi(Q)).$$

To verify that this is a bijection assume that $gH \in T_{\pi(G)}(\pi(P), \pi(Q))$. Then

$${}^gP \leq ({}^gP)H = {}^g(PH) \leq QH.$$

Since P is a p-subgroup we have that ${}^gP$ is contained in a Sylow-p-subgroup of QH. By a similar argument we have that Q is the unique Sylow-p-subgroup of QH and we therefore conclude that $g \in T_G(P,Q)$. For $P = Q$ one sees easily that the map of normalizers is a group homomorphism as well.

For a p-subgroup P of G the inclusion $C_G(P)/H \leq C_{G/H}(\pi(P))$ follows from $H \leq \mathrm{Z}(G)$. For the other direction let $gH \in C_{G/H}(\pi(P))$. The fact that gH centralizes $\pi(P)$ together with H being central implies that for any $q \in P$ we have that $gqg^{-1}q^{-1} \in H$. Since $gH \in N_{G/H}(\pi(P))$ the previous argument implies that $g \in N_G(P)$ and thus $gqg^{-1}q^{-1} \in P$. As the intersection $H \cap P$ is trivial we conclude that $gqg^{-1}q^{-1} = 1$ and $g \in C_G(P)$. □

Corollary 15.8. Let G be a finite group and p a prime number. Let $H \leq \mathrm{Z}(G)$ be a group of order prime to p and $\pi\colon G \to G/H$ the associated quotient map. Let $S \in \mathrm{Syl}_{\mathrm{p}}(G)$. Then $\pi(S) = SH/H \in \mathrm{Syl}_{\mathrm{p}}(G/H)$ and the restriction $\pi\colon S \to \pi(S)$ is a fusion preserving isomorphism.

Proof. Since the order of $\pi(S)$ equals the order of S the map π is an isomorphism. Furthermore, as H is a group of order prime to p, we have that $\pi(S)$ is a Sylow-p-subgroup of G/H. The definition of $\pi\colon S \to \pi(S)$ being fusion preserving reduces to

$$c_g \in \mathrm{Hom}_G(P,G) \iff c_{\pi(g)} \in \mathrm{Hom}_{G/H}(\pi(P), \pi(Q)).$$

By Proposition 15.7 we have that

$$g \in T_G(P,Q) \iff \pi(g) \in T_{G/H}(\pi(P), \pi(Q))$$

which implies the statement for the associated conjugation maps. □

To fully be able to restrict to the trivial center case, we also need to understand how the quotient map of a central p-prime group behaves on the mapping space $\mathrm{Map}(BP, BG)$ for a finite p-group P.

Lemma 15.9. Let G be a finite group and p a prime number. For a normal subgroup H of order prime to p the quotient map $\pi\colon G \to G/H$ induces a mod-p-equivalence $BG \to B(G/H)$.

Proof. Using the model EG/H for BH we let BH be equipped with the induced free G/H-action. Then $BH_{h(G/H)}$ is homotopy equivalent to BG and the associated Borel fibration can be identified with

$$BH \longrightarrow BG \xrightarrow{B\pi} B(G/H).$$

Here H is an group of order prime to p, and so the order $|H|$ is invertible in the finite field $\mathbb{F}_p$. Thus the higher homology groups $H_k(BH;\mathbb{F}_p)$ for $k>0$ vanish according to [12, III Corollary 10.2]. As BH is connected, we have that $H_0(BH;\mathbb{F}_p)=\mathbb{F}_p$. In particular the action of G/H on the homology of BH with $\mathbb{F}_p$-coefficients is trivial and the resulting Serre spectral sequence collapses to the zeroth line showing that $B\pi$ is a mod-p-equivalence. □

Proposition 15.10. Let G be a finite group and p a prime number. Let $H \leq \mathrm{Z}(G)$ be a group of order prime to p and $\pi\colon G \to G/H$ the associated quotient map. Then for any finite p-group P postcompostion with π induces a bijection $\mathrm{Rep}(P,G) \to \mathrm{Rep}(P,G/H)$. Furthermore the map $\mathrm{Map}(BP,BG) \to \mathrm{Map}(BP,B(G/H))$ induced by π is a mod-p-equivalence.

Proof. According to Corollary 15.8 the quotient map π is fusion preserving, and thus based on [2, III Proposition 1.15] we conclude that the induced map $\mathrm{Rep}(P,G) \to \mathrm{Rep}(P,G/H)$ is a bijection. Recall that

$$\mathrm{Map}(BP,BG) \simeq \coprod_{\rho\in\mathrm{Rep}(P,G)} BC_G(\rho(P))$$

as well as

$$\mathrm{Map}(BP,B(G/H)) \simeq \coprod_{\rho\in\mathrm{Rep}(P,G/H)} BC_{G/H}(\rho(P)).$$

As described in Lemma 9.1, post composition with the map π under these homotopy equivalences corresponds to the induced map $\mathrm{Rep}(P,G) \to \mathrm{Rep}(P,G/H)$ on components and for a component associated to a map $\rho \in \mathrm{Rep}(P,G)$ we have the following map $BC_G(\rho(P)) \to BC_{G/H}(\pi(\rho(P)))$. Using Corollary 15.7 this corresponds to the map induced by the quotient map $C_G(\rho(P)) \to C_G(\rho(P))/H$ which is a mod-p-equivalence by Proposition 15.9. □

We are now ready to extend Proposition 15.6 to general finite groups of Lie type.

Theorem 15.11. *Let G be a finite group of Lie type in characteristic p and P a finite p-group. Then the inclusion of morphism sets induces a mod-p-equivalence* $\mathrm{Map}(BP,|\mathcal{L}_S^{cr}(G)|) \to \mathrm{Map}(BP,B(G/\mathrm{Z}(G)))$.

Proof. For a finite group G of Lie type in characteristic p it follows from [18, Theorem 2.6.6] that $G/\mathrm{Z}(G)$ is also a finite group of Lie type in characteristic p. Note that according to [18, Theorem 2.6.6(c)] the group $G/\mathrm{Z}(G)$ has a trivial center. Furthermore by [18, Proposition 2.5.9] we have that $\mathrm{Z}(G)$ is a subgroup of H as defined in [18, Theorem 2.3.4]. In particular it is a group of order prime to p. Thus we have by Corollary 15.8 that the quotient map from G to $G/\mathrm{Z}(G)$ is fusion preserving. By the uniqueness of centric linking systems it induces an equivalence of categories on the subcategories $\mathcal{L}_S^{cr}(G) \cong \mathcal{L}_S^{cr}(G/\mathrm{Z}(G))$. Note that this equivalence of categories is given by the map on objects as well as morphisms specified in Proposition 15.7. If we consider the following composition

$$\mathrm{Map}(BP,|\mathcal{L}_S^{cr}(G)|) \longrightarrow \mathrm{Map}(BP,|\mathcal{L}_S^{cr}(G/\mathrm{Z}(G))|) \longrightarrow \mathrm{Map}(BP,B(G/\mathrm{Z}(G)))$$

where the first map is induced by the equivalence of categories, while the second is the mod-p-equivalence from Theorem 15.6, then the composition is a mod-p-equivalence and it is the map induced by inclusion of morphism sets of $\mathcal{L}_S^{cr}(G)$ into $G/\operatorname{Z}(G)$. □

Corollary 15.12. Let G be a finite group of Lie type in characteristic p and let P be a finite p-group. Then there exists a zig-zag of mod-p-equivalences.

$$\begin{array}{ccc} \operatorname{Map}(BP, |\mathcal{L}_S^{cr}(G/\operatorname{Z}(G))|) & \to & \operatorname{Map}(BP, B(G/\operatorname{Z}(G))) \\ \uparrow & & \uparrow \\ \operatorname{Map}(BP, |\mathcal{L}_S^{cr}(G)|) & & \operatorname{Map}(BP, BG) \end{array}$$

Proof. This follows from Theorem 15.11 Proposition 15.10. □

Corollary 15.13. Let G be a finite group of Lie type in characteristic p and let P be a finite p-group. Then there is a bijection between $[BP, |\mathcal{L}_S^{cr}(G)|]$ and $\operatorname{Rep}(P, G)$.

Proof. A mod-p-equivalence induces an bijection on connected components, and hence the zig-zag of maps from Corollary 15.12 induces a bijection between the components of $\operatorname{Map}(BP, |\mathcal{L}_S^{cr}(G)|)$ and $\operatorname{Map}(BP, BG)$, which may be identified with $[BP, |\mathcal{L}_S^{cr}(G)|]$ and $\operatorname{Rep}(P, G)$ respectively. □

The stated results have so far only been about the centric radical linking system associated to the fusion system $\mathcal{F}_S(G)$. Generally the centric linking system has been the object of focus when working with p-local finite groups. Furthermore for a fusion system of the form $\mathcal{F}_S(G)$ where $S \in \operatorname{Syl_p}(G)$ there exists a linking system on every collection $\mathcal{H}$ of subgroups of S that satisfies the following conditions:

- $\mathcal{H}$ is closed under G-conjugation and taking overgroups
- $\mathcal{H}$ contains all $\mathcal{F}_S(G)$-centric radical subgroups
- $\mathcal{H}$ is a subset of the set of all $\mathcal{F}_S(G)$-quasicentric subgroups of S.

Recall that in this case a subgroup P of S is $\mathcal{F}_S(G)$-quasicentric if and only if $O^p(C_G(P))$ has order prime to p. An explicit construction is setting $\mathcal{L}_{\mathcal{H}}(G)$ to be the category with objects $\mathcal{H}$ and morphism sets

$$\operatorname{Mor}_{\mathcal{L}_{\mathcal{H}}(G)}(P, Q) = T_G(P, Q)/O^p(C_G(P))$$

for $P, Q \in \mathcal{H}$.

Corollary 15.14. Let G be a finite group of Lie type in characteristic p. Let $S \in \operatorname{Syl_p}(G)$ and P be a finite p-group. Then for any collection $\mathcal{H}$ of subgroups of S satisfying the above conditions the space $\operatorname{Map}(BP, |\mathcal{L}_{\mathcal{H}}(G)|)$ is p-good and there is a homotopy equivalence

$$\operatorname{Map}(BP, |\mathcal{L}_{\mathcal{H}}(G)|)^\wedge_p \xrightarrow{\simeq} \operatorname{Map}(BP, BG)^\wedge_p.$$

Proof. By the definition of the category $\mathcal{L}_{\mathcal{H}}(G)$ it contains $\mathcal{L}_S^{cr}(G)$ as a full subcategory. They are both linking systems associated to the saturated fusion system $\mathcal{F}_S(G)$, and thus by [2, III: Proposition 4.20] the inclusion $\mathcal{L}_S^{cr}(G) \hookrightarrow \mathcal{L}_{\mathcal{H}}(G)$ induces a homotopy equivalence of spaces $\iota\colon |\mathcal{L}_S^{cr}(G)| \hookrightarrow |\mathcal{L}_{\mathcal{H}}(G)|$. By extending the zig-zag from Corollary 15.12, we get a zig-zag of mod-p-equivalences connecting the spaces

$\mathrm{Map}(BP, |\mathcal{L}_{\mathcal{H}}(G)|)$ and $\mathrm{Map}(BP, BG)$. Hence using [6, I Lemma 5.5] we conclude that the two spaces are homotopy equivalent after p-completion.

The space $\mathrm{Map}(BP, BG)$ is a finite disjoint union of classifying spaces of finite groups. Since classifying spaces of finite groups are p-good, the same holds for $\mathrm{Map}(BP, BG)$. Applying the p-completion functor to the zig-zag together with the natural transformation from the identify functor to p-completion, the resulting commutative diagram implies that all spaces are p-good by a repeated argument similar to the proof of Proposition 6.5. $\square$

Note that Corollary 15.14 covers the case of a centric linking system $\mathcal{L}_S^c(G)$. Thus we have that for a finite group of Lie type G in characteristic p and a finite p-group P the spaces of maps $\mathrm{Map}(BP, |\mathcal{L}_S^c(G)|)$ and $\mathrm{Map}(BP, BG)$ have isomorphic homology with $\mathbb{F}_p$ coefficients.

Since the homotopy type of $|\mathcal{L}_{\mathcal{H}}(G)|$ does not depend on the collection of subgroups $\mathcal{H}$, we will only state the description of the homotopy groups for the components of $\mathrm{Map}(BP, |\mathcal{L}_S^c(G)|)$.

Theorem 15.15. *Let G be a finite group of Lie type in characteristic p, and let S be a Sylow-p-subgroup of G. Let $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ be the p-local finite group associated to S and let P be a finite p-group. Then $[BP, |\mathcal{L}_S^c(G)|] \cong \mathrm{Rep}(P, G)$. Let $\mathrm{Map}(BP, |\mathcal{L}_S^c(G)|)_f$ be the component corresponding to $f \in \mathrm{Rep}(P, G)$. Then the homotopy groups are given by:*

If $\mathrm{Rk_L}(G) = 1$ then

$$\mathrm{Map}(BP, |\mathcal{L}_S^c(G)|)_1 \simeq B(B/\,\mathrm{Z}(G))$$

$$\mathrm{Map}(BP, |\mathcal{L}_S^c(G)|)_f \simeq B(C_G(f(P))/\,\mathrm{Z}(G)), \quad f \neq 1$$

where B is a Borel subgroup of G. If $\mathrm{Rk_L}(G) = 2$ then

$$\mathrm{Map}(BP, |\mathcal{L}_S^c(G)|)_1 \simeq B((P_\alpha/\,\mathrm{Z}(G)) *_{(B/\,\mathrm{Z}(G))} (P_\beta/\,\mathrm{Z}(G))$$

$$\mathrm{Map}(BP, |\mathcal{L}_S^c(G)|)_f \simeq B(C_G(f(P))/\,\mathrm{Z}(G)), \quad f \neq 1$$

where B is a Borel subgroup of G and P_α and P_β are the maximal standard parabolic subgroups over B.

Moreover, if G has Lie rank at least 3, then the component corresponding to the trivial map $\mathrm{Map}(BP, |\mathcal{L}_S^c(G)|)_1$ has the following homotopy groups:

$$\pi_1(\mathrm{Map}(BP, |\mathcal{L}_S^c(G)|)_1) \cong G/\,\mathrm{Z}(G)$$

$$\pi_i(\mathrm{Map}(BP, |\mathcal{L}_S^c(G)|)_1) \cong \pi_i(|\mathcal{S}_p(G)|), \quad i \geq 2.$$

Furthermore for all nontrivial $f \in \mathrm{Rep}(P, G)$ and all $i \geq 2$ we have that

$$\pi_i(\mathrm{Map}(BP, |\mathcal{L}_S^c(G)|)_f) \cong \mathrm{Tor}_{p'}(\pi_i(|\mathcal{S}_p(G)|)^P) \oplus \left(\pi_{i+1}(|\mathcal{S}_p(G)|)^P \otimes (\mathbb{Z}_p/\mathbb{Z}) \left[\frac{1}{p}\right]\right).$$

Finally, the fundamental group fits into the following short exact sequence:

$$0 \to \pi_2(|\mathcal{S}_p(G)|)^P \otimes (\mathbb{Z}_p/\mathbb{Z}) \left[\frac{1}{p}\right] \to \pi_1(\mathrm{Map}(BP, |\mathcal{L}_S^c(G)|)_f) \to C_G(f(P))/\,\mathrm{Z}(G) \to 0$$

Proof. The map zig-zag of mod-p-equivalences from Corollary 15.14 induces a bijection on the set of components $[BP, |\mathcal{L}_S^c(G)|] \to \mathrm{Rep}(P, G)$.

We consider an $f \in \mathrm{Hom}(P, G)$ and aim to identify the corresponding component $\mathrm{Map}(BP, |\mathcal{L}_S^c(G)|)_f$ in terms of a Borel construction. Let $\pi \colon G \to G/\,\mathrm{Z}(G)$ be the quotient map. As $\mathrm{Z}(G)$ is a group of order prime to p, we have by Corollary 15.8 that π is fusion preserving. In particular the quotient map induces

$$\mathrm{Map}(BP, |\mathcal{L}^c_S(G)|)_f \simeq \mathrm{Map}(BP, |\mathcal{L}^c_{\pi(S)}(\pi(G))|)_{\pi\circ f}.$$

Recall that the component of $\mathrm{Map}(BP, |\mathcal{L}^c_{\pi(S)}(\pi(G))|)$ is defined using the Borel map $|\mathcal{S}_p(\pi(G))|_{h\pi(G)} \to B(\pi(G))$ using the the homotopy equivalences

$$|\mathcal{L}^c_{\pi(S)}(\pi(G))| \simeq |\mathcal{L}^{cr}_{\pi(S)}(\pi(G))| \simeq |\mathcal{S}_p(\pi(G))|_{h\pi(G)}.$$

In particular

$$\mathrm{Map}(BP, |\mathcal{L}^c_S(G)|)_f \simeq \mathrm{Map}(BP, |\mathcal{S}_p(\pi(G))|_{h\pi(G)})_{\pi\circ f}.$$

As described in [18, Theorem 2.6.6] we have that $\pi(G)$ is also a finite group of Lie type in characteristic p with the same Lie rank as G. The center of G is a subgroup of H by [18, Proposition 2.5.9]. In particular $\mathrm{Z}(G)$ is a subgroup of both the Borel group B and N in the fixed BN-pair for G. Thus, the split BN-pair of characteristic p for G, $(B, N, \mathcal{S}_W)$, descends to a split BN-pair for $\pi(G)$ of the form $(\pi(B), \pi(N), \mathcal{S}_W)$. This implies that the standard parabolic subgroups of $\pi(G)$ are of the form $\pi(P)$ for a standard parabolic subgroup P of G.

The homotopy group $\mathrm{Map}(BP, |\mathcal{L}^c_S(G)|)_f$ now follows from Theorem 9.2, Theorem 9.3 and Theorem 13.2, using the isomorphism on centralizers from Proposition 15.7 and the fact that the bijection $\mathcal{S}_p(G) \to \mathcal{S}_p(\pi(G))$ is an isomorphism of P-posets where the action is induced by f and $\pi \circ f$ respectively. □

References

[1] Peter Abramenko and Kenneth S Brown. Buildings and groups. *Buildings: Theory and Applications*, pages 1–79, 2008.

[2] Michael Aschbacher, Radha Kessar, and Robert Oliver. *Fusion systems in algebra and topology.* Number 391. Cambridge University Press, 2011.

[3] David Balavoine. Deformations of algebras over a quadratic operad. *Contemporary Mathematics*, 202:207–234, 1997.

[4] Donald Anthony Barkauskas. *Centralizers in fundamental groups of graphs of groups.* PhD thesis, UNIVERSITY of CALIFORNIA at BERKELEY, 2003.

[5] Serge Bouc. Homologie de certains ensembles ordonnés. *CR Acad. Sci. Paris Sér. I Math*, 299(2):49–52, 1984.

[6] Aldridge Knight Bousfield and Daniel Marinus Kan. *Homotopy limits, completions and localizations*, volume 304. Springer Science & Business Media, 1972.

[7] Carles Broto and Nitu Kitchloo. Classifying spaces of kač-moody groups. *Mathematische Zeitschrift*, 240(3):621–649, 2002.

[8] Carles Broto, Ran Levi, and Bob Oliver. Homotopy equivalences of p-completed classifying spaces of finite groups. *Inventiones mathematicae*, 151(3):611–664, 2003.

[9] Carles Broto, Ran Levi, and Bob Oliver. The homotopy theory of fusion systems. *Journal of the American Mathematical Society*, 16(4):779–856, 2003.

[10] Kenneth S Brown. Euler characteristics of groups: Thep-fractional part. *Inventiones mathematicae*, 29(1):1–5, 1975.

[11] Kenneth S Brown. High dimensional cohomology of discrete groups. *Proceedings of the National Academy of Sciences*, 73(6):1795–1797, 1976.

[12] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics.* Springer-Verlag, New York-Berlin, 1982.

[13] Kenneth S Brown. *Buildings.* Springer, 1989.

[14] Wojciech Chachólski and Jérôme Scherer. *Homotopy theory of diagrams.* Number 736. American Mathematical Soc., 2002.

[15] Andrew Chermak. Fusion systems and localities. *Acta mathematica*, 211(1):47–139, 2013.

[16] W Dwyer and Alexander Zabrodsky. Maps between classifying spaces. *Algebraic Topology Barcelona 1986*, pages 106–119, 1987.

[17] W. G. Dwyer. Homology decompositions for classifying spaces of finite groups. *Topology*, 36(4):783–804, 1997.

[18] Daniel Gorenstein, Richard Lyons, and Ronald Solomon. *The Classification of the Finite Simple Groups. Number 3. Part I. Almost Simple K-groups*, volume 40 of *Math. Surveys and Monog.* Amer. Math. Soc., Providence, RI, 1998.

[19] Jesper Grodal. Higher limits via subgroup complexes. *Annals of mathematics*, pages 405–457, 2002.

[20] Jesper Grodal. Endotrivial modules for finite groups via homotopy theory. *ArXiv e-prints*, August 2016.

[21] Peter John Hilton. On the homotopy groups of the union of spheres. *J. London Math. Soc.*, 30:154–172, 1955.

[22] Irving Kaplansky. *Infinite abelian groups.* Number 2. University of Michigan Press Ann Arbor, 1954.

[23] Chao Ku. A new proof of the solomon-tits theorem. *Proceedings of the American Mathematical Society*, 126(7):1941–1944, 1998.

[24] Jean Lannes. Sur les espaces fonctionnels dont la source est le classifiant d'un p-groupe abélien élémentaire. *Publications mathématiques de l'IHÉS*, 75(1):135–244, 1992.

[25] Assaf Libman and Antonio Viruel. On the homotopy type of the non-completed classifying space of a p-local finite group. In *Forum Mathematicum*, volume 21, pages 723–757, 2009.

[26] Gunter Malle and Donna Testerman. *Linear algebraic groups and finite groups of Lie type*, volume 133. Cambridge University Press, 2011.

[27] John Martino and Stewart Priddy. Stable homotopy classification of $bg_p^\wedge$. *Topology*, 34(3):633–649, 1995.

[28] Haynes Miller. The sullivan conjecture and homotopical representation theory. In *Proceedings of the International Congress of Mathematicians*, volume 1, page 2, 1986.

[29] Guido Mislin. On group homomorphisms inducing mod-p cohomology isomorphisms. *Commentarii Mathematici Helvetici*, 65(1):454–461, 1990.

[30] Dietrich Notbohm. Maps between classifying spaces. *Mathematische Zeitschrift*, 207(1):153–168, 1991.

[31] Silvia Onofrei. Saturated fusion systems with parabolic families. *arXiv:1008.3815*, 2010.

[32] Lluis Puig. Frobenius categories. *Journal of Algebra*, 303(1):309–357, 2006.

[33] Daniel Quillen. Homotopy properties of the poset of nontrivial p-subgroups of a group. *Advances in Mathematics*, 28(2):101 – 128, 1978.

[34] Christophe Reutenauer. Free lie algebras, volume 7 of london mathematical society monographs. new series, 1993.

[35] Hans Samelson. A connection between the whitehead and the pontryagin product. *American Journal of Mathematics*, 75(4):744–752, 1953.

[36] Jean Pierre Serre. *Trees Translated from the French by John Stillwell.* Springer, 1980.

[37] Louis Solomon. The steinberg character of a finite group with bn-pair. In *Theory of Finite Groups (Symposium, Harvard Univ., Cambridge, Mass., 1968)*, pages 213–221, 1969.

[38] Richard P Stanley. On the number of reduced decompositions of elements of coxeter groups. *European Journal of Combinatorics*, 5(4):359–372, 1984.

[39] Dennis Parnell Sullivan and Andrew Ranicki. *Geometric Topology: Localization, Periodicity and Galois Symmetry: the 1970 MIT Notes*, volume 8. Springer, 2005.

[40] Jacques Tits. On buildings and their applications. In *Proceedings of the International Congress of Mathematicians (Vancouver, BC, 1974)*, volume 1, pages 209–220, 1974.