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Master Project in Mathematics

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Stable homotopy and the Adams Spectral Sequence

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Abstract

In the first part of the project, we define stable homotopy and construct the category of CW-spectra, which is the appropriate setting for studying it. CW-spectra are particular spectra with properties that resemble CW-complexes. We define the homotopy groups of CW-spectra, which are connected to stable homotopy groups of spaces, and we define homology and cohomology for CW-spectra.

Then we construct the Adams spectral sequence, in the setting of the category of CW-spectra. This spectral sequence can be used to get information about the stable homotopy groups of spaces.

The last part of the project is devoted to calculations. We show two examples of the use of the Adams sequence, the first one applied to the stable homotopy groups of spheres and the second one to the homotopy of the spectrum ko of the connective real K-theory.

Contents

Introduction

The stable homotopy groups of spaces are quite complicated to compute and this spectral sequence provides a tool that helps in this task. To get a clean construction of the spectral sequence, instead of working with space, we consider spectra, which are sequences of spaces (X_i) with structure maps $\Sigma X_i \to X_{i+1}$ and soon we restrict our attention to CW-spectra. So we construct a suitable category for these objects, in which many of the results that hold for CW-complexes are true. The construction of the spectral sequence is done following the approach of [Hat04].

Performing calculations of stable homotopy groups using of the Adams spectral sequence is not a mechanical process. One has to compute free resolutions over the Steenrod Algebra and the differentials of the spectral sequence to obtain the E_∞ page. Moreover, the E_∞ term gives only subquotients of the p-components of the stable homotopy groups for certain filtrations of them. So one has still to solve the extension problem to get the group itself. In Section 4, we show two examples of this process: the first one regards the stable homotopy groups of spheres, the second one the homotopy of the spectrum ko of the connective real K-theory. We work out the cited free resolutions explicitly, by actually computing the generators of each free module.

We assume that the reader is familiar with the material contained in [Hat02]. The proofs of some of the results here presented could not fit into this project: in such cases we sketch them and refer the reader to one of the references for a more detailed treatment.

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1 Stable homotopy

1.1 Generalities

In this section, we will always consider basepointed topological spaces with a CW-structure, i.e. pairs (X, x_0) where the space X is a CW-complex and the basepoint $x_0 \in X$ is a fixed 0-cell.

Definition 1.1. Given two basepointed spaces (X, x_0) and (Y, y_0), their *smash product* is defined as

$$X \wedge Y = (X \times Y)/(X \vee Y) = (X \times Y)/(X \times \{y_0\} \cup \{x_0\} \times Y).$$

This is again a basepointed space, with basepoint the subspace we quotient out.

For a basepointed space X, we consider (instead of the suspension) the *reduced suspension*, which is

$$\Sigma X := (SX)/(\{x_0\} \times I),$$

where SX is the suspension of X. The reduced suspension is again a basepointed space and it follows immediately for the definition that it can be expressed in terms of smash product with the sphere S^1, namely $\Sigma X = X \wedge S^1$.

Let us also note that, if X is a CW-complex, the reduced suspension is homotopy equivalent to the unreduced one. In fact ΣX is obtained from SX by collapsing the contractible subspace $\{x_0\} \times I$.

1.2 Stable homotopy groups

The term *stable* in Algebraic Topology is used to indicate phenomena that occur in any dimension, without depending essentially on it, given that the dimension is sufficiently large. [Ada74, Part III, Ch. 1]

Let us recall a classical result:

Theorem 1.2. *(Freudenthal Theorem) Let X be a $(n-1)$-connected CW-complex. Then the suspension map $\pi_i(X) \to \pi_{i+1}(\Sigma X)$ is an isomorphism for $i < 2n-1$ and a surjection for $i = 2n-1$.*

If X is n-connected, then ΣX is $(n+1)$-connected and so the Freudenthal theorem implies that, if X is a CW-complex of any connection and $i \in \mathbb{N}$, the morphisms in the sequence:

$$\pi_i(X) \to \pi_{i+1}(\Sigma X) \to \cdots \to \pi_{i+k}(\Sigma^k X) \to \dots$$

will become isomorphisms from a certain point. The group to which the sequence stabilized is called the i^{th} *stable homotopy group* of X, denoted $\pi_i^s(X)$. This is the same as saying that:

$$\pi_i^s(X) = \varinjlim \pi_{i+k}(\Sigma^k X),$$

where $\varinjlim$ denotes the direct limit.

In particular, taking $X = S^0$, $\pi_i^s(S^0)$ is called the i^{th} *stable homotopy groups of spheres* and is denoted by π_i^s. By the Freudenthal Theorem, we have:

$$\pi_i^s = \pi_i^s(S^0) \cong \pi_{n+i}(S^n),$$

for any $n > i + 1$.

The composition of maps $S^{i+j+k} \to S^{j+k} \to S^k$ induces a product structure on $\pi_*^s = \bigoplus_i \pi_i^s$. We have the following:

Proposition 1.3. *The composition product $\pi_i^s \times \pi_j^s \to \pi_{i+j}^s$ given by composition makes π_*^s into a graded ring in which we have the relation $\alpha\beta = (-1)^{ij}\beta\alpha$, for $\alpha \in \pi_i^s$, $\beta \in \pi_j^s$.*

Proof. See [Hat02, Proposition 4.56]. □

2 Spectra

2.1 Definition and homotopy groups

Definition 2.1. A *spectrum* E is a family $(E_n)_{n\in\mathbb{N}}$ of basepointed spaces X_n together with structure maps $\varepsilon_n \colon \Sigma E_n \to E_{n+1}$ that are basepoint-preserving.

Example 2.2. Let X be a pointed space. If we set $E_n = \Sigma^n(X)$ for all $n \in \mathbb{N}$ and ε_n the identity of Σ^{n+1} we get a spectrum, called *the suspension spectrum of* X. In particular, taking $X = S^0$, we get the *sphere spectrum*, denoted S^0. The n^{th} space of this spectrum is homeomorphic to S^n.

A second example shows the connection of spectra with cohomology: by the Brown Representation Theorem [Hat02, Thm 4E.1], for every reduced cohomology theory $\tilde{K}$ on the category of CW-complexes with basepoint, we have that the functor $\tilde{K}^n$ is natural equivalent to $[-, E_n]$ for a family of basepointed CW-complexes (E_n), where $[X, E_n]$ denotes the set of the homotopy classes of basepoint-preserving maps $X \to E_n$. By the suspension isomorphism for K, we have that: $\sigma\colon \tilde{K}^n(X) \to \tilde{K}^{n+1}(\Sigma X)$ is a natural isomorphism. Hence we get the sequence of natural isomorphisms:

$$[X, E_n] \cong \tilde{K}^n(X) \cong \tilde{K}^{n+1}(\Sigma X) \cong [\Sigma X, E_{n+1}] \cong [X, \Omega E_{n+1}],$$

where the last equivalence comes from the adjunction between reduced suspension and basepointed loops. By the (contravariant) Yoneda Lemma, the natural equivalence $[X, E_n] \cong [X, \Omega E_{n+1}]$ must be induced by a weak equivalence $\varepsilon'_n \colon E_n \to \Omega E_{n+1}$ which is the adjoint of a map $\varepsilon_n \colon \Sigma E_n \to E_{n+1}$ and so (E_n) forms a spectrum.

If we take the reduced cohomology with coefficients in an abelian group G, we get $E_n = K(G, n)$ and the spectrum is called the *Eilenberg-MacLane spectrum* for the group G and denoted HG.

This can be slightly generalized by shifting dimension of an integer m and taking $X_m = K(G, n+m)$ with the same maps $\Sigma K(G, n+m) \to K(G, n+m+1)$. We will denote this shifted Eilenberg-MacLane with $K(G, n)$.

It is possible to define homotopy groups for spectra in a quite natural way:

Definition 2.3. Let E be a spectrum. We define *the i^{th} homotopy group of* E, denoted $\pi_i(E)$, as the direct limit of the sequence:

$$\cdots \to \pi_{i+n}(E_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma E_n) \xrightarrow{\pi_{i+n+1}(\varepsilon_n)} \pi_{i+n+1}(E_{n+1}) \xrightarrow{\Sigma} \pi_{i+n+2}(\Sigma E_{n+1}) \to \dots.$$

In symbols: $\pi_i(E) = \varinjlim \pi_{i+n}(E_n)$.

Example 2.4. The homotopy groups of the suspension spectrum of a space are precisely the stable homotopy groups of that space, since the structure maps ε_i are all identities. For the Eilenberg-MacLane spectrum, we already know that $\Sigma E_n \to E_{n+1}$ is a weak equivalence for all n and the homomorphism $\pi_{r+n+i}(E_{n+1}) \to \pi_{r+n+i+1}(\Sigma E_{n+i})$ is an isomorphism in the range given by the Freudenthal suspension theorem, so we have that, as for the Eilenberg-MacLane spaces:

$$\pi_i(HG) = \begin{cases} G, & \text{if } i = 0, \\ 0, & \text{if } i \neq 0. \end{cases}$$

Note that, unlike in the definition of the stable homotopy group of a space, this direct limit is not always attained.

Let us give the definition of a subobject of a spectrum, which will be useful for the next section:

Definition 2.5. Given a spectrum E, a *subspectrum* A of E is a family of spaces $(A_n)_{n\in\mathbb{N}}$ such that $A_n \subseteq E_n$ and $\varepsilon_n(\Sigma A_n) \in A_{n+1}$ for all $n \in \mathbb{N}$. In particular, A is a spectrum if we take $\varepsilon_n|_{\Sigma A_n}$ as structure maps.

2.2 CW-Spectra

Our aim now is to build a category with spectra as objects and use this setting to study stable homotopy phenomena. To do this we will consider a particular class of spectra, called CW-spectra, which have a structure that resembles the one of CW-complexes.

Given a spectrum E in which the spaces E_n are a CW-complexes with a 0-cell as basepoint for all $n \in \mathbb{N}$, we can apply the procedure that we sketch here: first we make the structure maps cellular, via the cellular approximation theorem ([Hat02, Thm. 4.8]): since the structure maps of E map basepoints to basepoints and they are 0-cells in the CW-structures, we can take the cellular maps to be also basepoint-preserving. Then we proceed replacing each space E_n with union of the reduced mapping cylinders of the maps in the composition[1], which is again a CW-complex:

$$\Sigma^n(E_0) \xrightarrow{\Sigma^{n-1}\varepsilon_0} \Sigma^{n-1}(E_1) \to \cdots \to \Sigma(E_{n-1}) \xrightarrow{\varepsilon_{n-1}} E_n.$$

For example, we begin the construction by replacing E_1 with the homotopy-equivalent space $E_1' = M_{\varepsilon_0}$, so that we have an inclusion $\Sigma E_0 \hookrightarrow M_{\varepsilon_0}$. Then we consider

$$\Sigma^2(E_0) \xrightarrow{\Sigma\varepsilon_0} \Sigma(E_1) \xrightarrow{\varepsilon_1} E_2$$

and we have that $\Sigma E_1' = M_{\Sigma\varepsilon_0} \hookrightarrow M_{\Sigma\varepsilon_0} \cup M_{\varepsilon_1} \simeq M_{\varepsilon_1 \circ \Sigma\varepsilon_0} = E_2' \simeq E_2$.

Proceeding this way we get a new spectrum in which all spaces are CW-complexes with the basepoints as 0-cells and the structure maps are inclusions of subcomplexes. This construction gives motivates the following definition:

Definition 2.6. A *CW-spectrum* is a spectrum E in which the spaces E_n are CW-complexes with a 0-cell as basepoint for all $n \in \mathbb{N}$ and such that the structure maps are inclusions of subcomplexes.

A *CW-subspectrum* of a CW-spectrum E is a subspectrum A as defined earlier with the requirement that A_n is a subcomplex of E_n for all $n \in \mathbb{N}$.

This way, for each cell e_i of E_n, different from the basepoint, its suspension becomes a cell of dimension $i+1$ in E_{n+1}. We say that two cells e_α of E_n and e_β of E_m, with $n \leq m$ are equivalent if e_α becomes e_β by suspending (and including) a suitable number of times. We will say that a *stable cell* of E is an equivalence class under this relation. So, a stable cell consists of cells e^{n+k} of E_n for all $n \geq n_0$. The dimension of such a stable cells is defined to be k. This way, if we consider the suspension spectrum of a CW-complex X, the cells of X coincide with the stable cells of the spectrum, together with their dimensions.

The homology of a CW-spectrum E can be defined in terms of cellular chains. We consider the cellular chain complexes relative to the basepoint $C_*(E_n; G)$: the structure maps $\Sigma E_i \to E_{i+1}$ induce inclusions $C_*(E_n; G) \to C_*(E_{n+1}; G)$. If we take the union $C_*(E; G) = \cup_n C_*(E_n; G)$, it can be checked that we get a chain complex with one G summand for every stable cell of E, as for CW-complexes. Then we define $H_i(E; G)$ to be the i^{th} *homology group* of this chain complex.

A CW-spectrum with finitely many stable cells is called *finite*. If the number of cells in each dimension is finite, then the spectrum is of *finite type*.

[1]in the union, we identify, for each space in the composition, its copy in the mapping cylinders of the map from and to it.

Remark 2.7. Note that a CW-spectrum can also have cells in negative dimension: consider, for example, the CW-spectrum X defined by

$$X_n = S^1 \vee S^2 \vee \dots$$

for all $n \geq 0$, with structure map the obvious inclusion

$$\Sigma(X_n) = S^2 \vee S^3 \vee \dots \hookrightarrow X_{n+1}$$

and in which all the factors S^i are realized as a disc D^i with the boundary glued on one point. This spectrum has one cell for each dimension $k \in \mathbb{Z}$.

The Adams Spectral Sequence deals only with CW-spectra in which the dimension of the stable cells is bounded below. A CW-spectrum with this property is called *connective.*

The objects in the category we are going to build are of course CW-spectra. Instead, it is not that easy to find a suitable class of maps and it will require some work to get to a definition of morphism in our category. A reasonable goal would be to have that a morphism between two CW-spectra induces a homomorphism on homotopy.

The most obvious choice would be to take maps that commute with the structure maps of the spectra. Namely, we can give the following definition:

Definition 2.8. If E and F are CW-spectra and $r \in \mathbb{Z}$, a *strict map*[2] *of degree* r from E to F is a sequence of maps $f = (f_n)_{n\in\mathbb{Z}}$ with $f_n \colon E_n \to F_{n-r}$ a basepoint-preserving map for all $n \in \mathbb{Z}$ such that the following diagram commutes for all $n \in \mathbb{Z}$:

$$\begin{array}{ccc} \Sigma E_n & \longrightarrow & E_{n+1} \\ \Sigma f_n \downarrow & & \downarrow f_{n+1} \\ \Sigma F_{n-r} & \longrightarrow & F_{n-r+1}. \end{array}$$

We will just say *strict map* when the degree is 0.

Since we define graded maps, it is more convenient to index the spaces of the spectra over the integers rather than over $\mathbb{N}$. We will see soon that this does not cause any substantial difference.

Strict maps can be composed in the natural way and the identity of a CW-spectrum is a strict-map. Moreover, if we take a CW-subspectrum E' of a CW-spectrum E, the inclusion $E' \hookrightarrow E$ is a strict map.

We can however weaken the requirements of the definition and still get maps that induce homomorphisms on homotopy. In fact, if we had the maps f_n defined only for $n \geq n_0$ for some n_0, we would still get an induced map $\pi_i(E) \to \pi_{i-r}(F)$, since we can think of $\pi_i(X)$ as $\varinjlim \pi_{i+n}(E_n)$ and similarly for $\pi_i(F)$.

Actually, we can relax the condition some more. To do this, we first need a definition:

[2] strict maps are called *functions* in [Ada74].

Definition 2.9. A CW-subspectrum A of a CW-spectrum E is called *cofinal* if, for all $n \in \mathbb{N}$ and for every cell e^i of E_n, there exists a $k \in \mathbb{N}$ such that $\Sigma^k(e^i)$ is a cell of A_{n+k}.

Lemma 2.10. *If E', E'' are cofinal subspectra of a CW-spectrum E', then their intersection is also cofinal in E.*

Proof. Let $n \in \mathbb{N}$ and let e^i be a cell of E_n. By definition, there exist k_1 and k_2 such that $\Sigma^{k_1}(e^i)$ is a cell of E'_{n+k_1} and $\Sigma^{k_2}(e^i)$ is a cell of E''_{n+k_2}. Assume $k_1 \leq k_2$: if we suspend the cell $\Sigma^{k_1}(e^i)$ for $k_2 - k_1$ times, we get that $\Sigma^{k_2}(e^i)$ is a cell of E'_{n+k_2}. This shows that $E' \cap E''$ is cofinal. □

Let E be a CW-spectrum and F a spectrum. Let E', E'' be cofinal subspectra of E and $f\colon E' \to F$, $g\colon E'' \to G$ strict maps of degree r. We say that f and g are equivalent if there exists a cofinal subspectrum $\bar{E}$ of E contained in E' and E'' on which f and g coincide. We leave it as an exercise for the reader to check that this is an equivalence relation.

Definition 2.11. A *map* (of degree r) $E \to F$ of CW-spectra is an equivalence class of strict maps (of degree r) defined on cofinal subspectra.

Note that the previous lemma tells us that, if we take a map defined on a cofinal subspectrum E', we get the same map restricting f to another cofinal subspectrum $E'' \subseteq E'$.

The composition of maps of CW-spectra is defined by taking representatives, choosing them in a way such that their composition is defined. It is well defined by the following Proposition:

Proposition 2.12. *(a) Let $f\colon E \to F$ be a strict map of degree r of CW-spectra and $F' \subseteq F$ a cofinal subspectrum. Then, there exists a cofinal subspectrum $E' \subseteq E$ such that $f(E') \subseteq F'$.*

(b) If E' is a cofinal subspectrum of E and E'' a cofinal subspectrum of E', then E'' is a cofinal subspectrum of E.

Proof. (a) We set E' to be the subspectrum made by the cells of the spaces E_n which are mapped into F'_{n-r}. This is cofinal, in fact, if e^i is a cell in E_n, then it is mapped by f to a finite union of cells of F_{n-r} and so, for some k, their k-fold suspension lies in F'_{n-r+k}, since F' cofinal. f is a strict map and so it commutes with the structure maps. Then, the $\Sigma^k(e^i)$ is mapped in F' by f and so it is in E'.

(b) This is a straightforward consequence of the definition.

□

To see that also a map $f\colon E \to F$ of CW-spectra induces a morphism $f_*\colon \pi_i(E) \to \pi_{i-r}(F)$, assume that f is defined on the cells of a cofinal subspectrum E' of E. If we take a map $S^{i+n} \to X_n$, its image is compact and hence included in a finite subcomplex of X_n. If we suspend the cells of this subcomplex enough times, they will lie in E'_{n+k} (for some k) and f is defined on them. The same process can be done for a homotopy $S^{i+n} \times I \to X_n$.

We can now define homotopies between maps of CW-spectra, in a way all similar to what is done for spaces: for a CW-spectrum E, we define the *cylinder spectrum* $E \times I$ by

$$(E \times I)_n = \frac{(E_n \times I)}{e_0 \times I},$$

where e_0 denotes the basepoint. Clearly we have $\Sigma(E_n \times I) = \Sigma(E_n) \times I$. The structure maps of the cylinder spectrum are defined in the obvious way. We have two obvious inclusion maps:

$$i_0, i_1 \colon E \to E \times I,$$

that correspond to the two ends of the cylinder.

Definition 2.13. If $f, g \colon E \to F$ are maps of CW-spectra, we say that they are homotopic if there exists a map of CW-spectra $H \colon E \times I \to F$ such that $f = H \circ i_0$, $g = H \circ i_1$.

We leave to the reader to check that this is an equivalence relation and the composition of maps defined earlier induces the composition for homotopy classes, as in the familiar case of spaces.

The set of the homotopy classes of the maps $E \to F$ of degree r is denoted $[E, F]_r$

We have finally obtained a satisfactory definition for the morphisms in our category: the objects are CW-spectra and the *morphisms* of degree r are homotopy classes of maps of degree r. We will just say morphism to mean a morphism of degree 0. Note that a morphism f is an isomorphism in this category if it is a homotopy equivalence of spectra.

Remark 2.14. If E' is a cofinal subspectrum of a CW-spectrum E, we already know that the inclusion $E' \hookrightarrow E$ is a strict map and hence a map of CW-spectra. Moreover, if we take the equivalence class of the identity of E', it is a map of CW-spectra $E \to E'$, by definition, and one can easily check that the composition of these two maps in any order gives identity. This shows that every CW-spectrum is isomorphic to any of its cofinal subspectra.

Till now we have always considered spectra indexed over $\mathbb{N}$, but, if $E = (E_n)_{n \in \mathbb{Z}}$ is a CW-spectrum indexed over $\mathbb{Z}$, we can define E' by

$$E'_n = \begin{cases} E_n, & \text{if } n \geq 0, \\ \{*\}, & \text{if } n < 0. \end{cases}$$

E' is a cofinal subspectrum of E and so they are isomorphic: this shows that it does not really matter if we consider spectra indexed over $\mathbb{N}$ or over $\mathbb{Z}$.

Remark 2.15. We can always define a subspectrum E' of E by setting

$$E'_n = \Sigma E_{n-1} \subseteq E_n.$$

This is a cofinal subspectrum of E and so, by the previous remark, they are isomorphic.

More generally, we can define the suspension ΣF of a spectrum F by setting $(\Sigma F)_n = \Sigma(F_n)$. Then, the last assertion can be reformulated by saying that any CW-spectrum E is isomorphic to ΣF where F is the spectrum defined by $F_n = E_{n-1}$. This spectrum F can be denoted by $\Sigma^{-1}E$, so that $E = \Sigma(\Sigma^{-1}E)$. This argument shows that any CW-spectrum is isomorphic to the suspension of another CW-spectrum. In other words, the suspension is invertible in our category. We can iterate this construction: for $m \in \mathbb{N}$, we define a CW-spectrum G by:

$$G_n = E_{n-m}.$$

This way we get that $E \cong \Sigma^m G$.

Recall that the sphere spectrum S^0 is the suspension spectrum of the sphere S^0. The following Proposition is saying that our choice for morphisms is actually reasonable:

Proposition 2.16. *If E is the suspension spectrum for a finite CW-complex K and F is any spectrum, we have that:*

$$[E, F]_r = \varinjlim [E_{n+r}, F_n].$$

In particular, if $K = S^0$:

$$\pi_r(E) = [S^0, E]_r = [S^r, E]_0.$$

Proof. If we start with a map (of spaces) $f\colon \Sigma^{n+r}K \to F_n$, we can define a map of spectra $\bar{f}\colon E \to F$ of degree r: $\bar{f}_{n+r}\colon E_{n+r} \to F_n$ is taken to be f itself and, since we have $\Sigma E_{n+r} = E_{n+r+1}$, we are forced to define, for all $m > 0$:

$$E_{n+r+m} = \Sigma^m(E_{n+r}) \xrightarrow{\Sigma^m f} F_{n+m}.$$

This defines a strict map on the cofinal subspectrum:

$$E'_i = \begin{cases} E_i, & \text{if } i \geq n+m, \\ \{*\}, & \text{if } i < n+m. \end{cases}$$

If $f\colon \Sigma^{n+r}K \to F_n$, $g\colon \Sigma^{m+r}K \to F_m$ are two maps which are equivalent in the direct limit, it means that, for some p there exists a homotopy between the maps:

$$\Sigma^{p+r}K \xrightarrow{\Sigma^{p-n} f} \Sigma^{p-n}F_n \to F_p,$$

$$\Sigma^{p+r}K \xrightarrow{\Sigma^{p-m}g} \Sigma^{p-m}F_m \to F_p.$$

This homotopy induces a homotopy between the map of spectra corresponding to f and g and so we have a map:

$$\alpha\colon \varinjlim [E_{n+r}, F_n] \to [E, F]_r.$$

A map of CW-spectra $E \to F$ of degree r is a homotopy class of strict maps defined on a cofinal subspectrum of E. Any of these strict maps is equivalent to one defined on a subspectrum which has the form of E' defined above and so α is surjective. Moreover, if two maps of CW-spectra are homotopic, the homotopy is induced by a homotopy of maps of spaces, therefore α is injective. Therefore α is a bijection. □

Remark 2.17. One of the reasons to consider spectra instead of spaces is that, for E, F CW-spectra, the set $[E, F]$ is always an abelian group. This is because, as we have seen, a CW-spectrum is always isomorphic to the suspension of another CW-spectrum (and hence also to a double suspension), in other words it can always be de-suspended as many times as we want, allowing us to define an abelian sum operation using the extra coordinates given by the suspension, as one does for homotopy groups of spaces.

Proposition 2.18. *If E, F are CW-spectra, the suspension map*

$$[E, F] \to [\Sigma E, \Sigma F]$$

is a group isomorphism.

Proof. See [Ada74, Part III, Theorem 3.7] or [Hat04, p. 10]. □

Another advantage of working with CW-spectra is that many results which hold for CW-complexes also translate in this context. For instance we have cellular approximation of maps, as for CW-complexes. We have also a version of the Whitehead theorem:

Theorem 2.19. *A morphism of CW-spectra $f\colon E \to F$ that induces isomorphisms $f_*\colon \pi_i(E) \to \pi_i(F)$ for all i is an isomorphism and so a homotopy equivalence between e and F.*

Proof. See [Ada74, Part III, Corollary 3.5]. □

Another important result translate to the category of CW-spectra: the Hurewicz theorem. It must be noted that it holds only for connective spectra:

Theorem 2.20. *Let E be a connective CW-spectrum. It $\pi_i(E) = 0$ for all $i < n$, then there exists an isomorphism $h\colon \pi_n(E) \to H_n(E; \mathbb{Z})$.*

Proof. See [Mar83, Theorem 6.9]. □

2.3 Exact sequences for a cofibration of CW-spectra

In order to talk about cofibrations, we need to define mapping cylinders and mapping cones for a map of CW-spectra.

If $f\colon E \to F$ is a map of CW-spectra, we can pick as a representative for it a strict map $f'\colon E' \to F$ defined on a cofinal subspectra E' of E. Now we can deform f'_n to be cellular as a map of CW-complexes for each n and define the mapping cylinder of f to be the CW-spectrum M_f with:

$$(M_f)_n = M_{f'_n},$$

where $M_{f'_n}$ is a reduced mapping cylinder. This defines a CW-spectrum, since f' is a strict map and $\Sigma M_{f'_n} = M_{\Sigma f'_n}$. If we replace X' with a cofinal subspectrum, M_f is replaced with a subspectrum cofinal in it, hence isomorphic. Therefore the definition depends uniquely only on f, up to equivalence. The familiar deformation retraction of $M_{f'n}$ onto F_n, for each n, give a deformation retraction of the spectrum M_f onto F.

The *mapping cone*, is built up the same way, taking the reduced mapping cones of the maps f'_n instead of the cylinders. The resulting spectrum is denoted C_f. For an inclusion of a subspectrum $A \hookrightarrow E$, the mapping cone can be written as $X \cup CA$, as one does for CW-complexes, defining the cone cylinder CA similarly to what we did for the cylinder spectrum.

Definition 2.21. A subspectrum A of a CW-spectrum E is called *closed* if, when a cell e^i of E_n has an iterated suspension lying in A_{n+k}, then e^i is in A_n.

If A is a closed subspectrum of E, we can form the *quotient spectrum* E/A setting $(E/A)_n = E_n/A_n$. We have a map $E \cup CA \to E/A$, whose components are the homotopy equivalences of CW-complexes $E_n \cup CA_n \to E_n/A_n$. We can then apply Theorem 2.19 to get a homotopy equivalence $X \cup CA \simeq X/A$.

As for spaces, from the inclusion of a subspectrum, we can form a *cofibration sequence*:

$$A \hookrightarrow E \to E \cup CA,$$

which can we extended to the right by taking the mapping cone of the last inclusion:

$$A \hookrightarrow E \to E \cup CA \to (E \cup CA) \cup CE$$

and we have that $(E \cup CA) \cup CE \simeq (E \cup CA)/E \simeq \Sigma A$, so we can go on extending the sequence with $\Sigma A \to \Sigma E$, just as for spaces:

$$A \hookrightarrow E \to E \cup CA \to \Sigma A \to \Sigma E \to \cdots \tag{2.1}$$

And, as for spaces, we have an associated exact sequence:

$$[A, F] \leftarrow [E, F] \leftarrow [E/A, F] \leftarrow [\Sigma A, F] \leftarrow [\Sigma E, F] \leftarrow \cdots \tag{2.2}$$

But with CW-spectra we have also another exact sequence associated to a cofibration, which is not present for cofibrations of spaces. The reason is basically that we can always de-suspend spectra, which is is not true for spaces.

Proposition 2.22. *For a cofibration $A \hookrightarrow E \to E \cup CA$, the following sequence is exact:*

$$[F, A] \to [F, E] \to [F, E/A] \to [F, \Sigma A] \to [F, \Sigma E] \to \cdots \tag{2.3}$$

Proof. The sequence (2.1) is built including each space in the mapping cone of the map that is entering that space. So, it is enough to prove the exactness of the sequence:

$$[F, A] \to [F, E] \to [F, E \cup CA].$$

It is clear that the composition of the two maps is zero. Suppose now that we have $f\colon F \to E$ that becomes nullhomotopic when including E into $E \cup CA$. This homotopy $F \times I \to E \cup CA$ gives a map $H\colon CF \to E \cup CA$ (since CF can be seen as the quotient of $F \times I$ where we mod out one face of the cylinder) which forms a commutative square with f:

$$\begin{array}{ccc} F & \hookrightarrow & CF \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle H} \\ E & \hookrightarrow & E \cup CA. \end{array}$$

The two rows complete to cofibrations and we get the two next vertical maps at the right of the square to fill the following diagram, which commutes up to homotopy:

$$\begin{array}{ccccccccc} F & \xrightarrow{1} & F & \hookrightarrow & CF & \longrightarrow & \Sigma F & \xrightarrow{1} & \Sigma F \\ & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle \Sigma f} \\ A & \hookrightarrow & E & \hookrightarrow & E \cup CA. & \longrightarrow & \Sigma A & \xrightarrow{\Sigma i} & \Sigma E. \end{array}$$

By Proposition 2.18, $k = \Sigma l$, for some $l \in [F, A]$ and, by commutativity, $\Sigma f \simeq \Sigma(i)\Sigma(l) = \Sigma(il)$. Since the suspension is an isomorphism, this implies $f \simeq il$ and then f is in the image of $[F, A] \to [F, E]$. This completes the proof of the exactness. □

This exact sequence we built for a cofibration of CW-spectra is the analogous of the exact sequence $[Y, F] \to [Y, E] \to [Y, B]$ for a fibration of spaces $F \to E \to B$. In other words, in the category of CW-spectra, cofibrations are the same as fibrations.

Given two CW-spectra A and B we can form their *wedge sum* $A \vee B$ in the obvious way, setting $(A \vee B)_n = A_n \vee B_n$ for all n. The structure maps come from pasting the ones of A and B, since $\Sigma(A_n \vee B_n)$ is homeomorphic to $\Sigma A_n \vee \Sigma B_n$. Moreover the inclusions of the two factors, $A \to A \vee B \leftarrow B$ are strict maps of spectra.

Remark 2.23. For cofibration of the pair $(A \vee B, A)$, the exact sequence of Proposition 2.22:

$$\cdots \to [F, A] \to [F, A \vee B] \to [F, B] \to \cdots$$

has a splitting $[F, B] \to [F, A \vee B]$ given by inclusion and so we have that $[F, A \vee B] \cong [F, A] \oplus [F, B]$. This clearly generalizes to finite wedge sums.

2.4 Cohomology for CW-spectra

Definition 2.24. Given a CW-spectrum E, we define the n^{th} *E-cohomology* of a CW-spectrum X as:

$$E^n(X) = [X, E]_{-n}.$$

An element $f \in [X, E]_{-n}$ is a homotopy class of maps of spectra $X \to F$ of degree $-n$. It corresponds to a strict map f' defined on a subspectrum $X' \subseteq X$ with components $f'_i \colon X_{i-n} \to E_i$. As shown in Remark 2.15, the spectrum X is isomorphic to $\Sigma^n Y$, where Y is the spectrum defined by $Y_i = X_{i-n}$ for all i. Then we have:

$$[\Sigma^{-n} X, E] = [\Sigma^{-n}\Sigma^n Y, E] = [Y, E],$$

and a map $Y \to E$ of degree 0 corresponds to a strict map f' defined on a cofinal subspectrum $Y' \subseteq Y$ with components $f'_i \colon Y_n = X_{i-n} \to E_n$. Thus, $E^n(X)$ can be also written as $[\Sigma^{-n} X, E]$.

One check easily that, for all n, $E^n(X)$ is a contravariant functor from the category of CW-spectra to abelian groups in the variable X.

The reason for the name "E-cohomology" is that E^n satisfies a series of properties analogous to the axioms for a reduced cohomology theory on spaces (see, for instance, [May99, Ch. 19] for a reference on them):

Proposition 2.25. *If E is a CW-spectrum, the functor E^n satisfies the following properties:*

(a) For a cofibration $A \hookrightarrow X \to X/A$, the sequence:

$$E^n(X/A) \to E^n(X) \to E^n(A)$$

is exact.

(b) There is a natural isomorphism $E^n(X) \to E^{n+1}(\Sigma X)$.

(c) For a finite wedge sum of CW-spectra $A_1 \vee \cdots \vee A_k$, there is an isomorphism

$$E^n(A_1 \vee \cdots \vee A_k) \cong \prod_{i=1}^{k} E^n(A_i).$$

(d) A map of CW-spectra $f\colon X \to Y$ that is a weak equivalence induces a isomorphism:

$$f^*\colon E^n(Y) \to E^n(X).$$

Proof. (a) We can just use the exact sequence (2.2).

(b) Writing $E^n(X)$ as $[\Sigma^{-n}X, E]$, we have:

$$E^n(X) = [X, E]_{-n} \cong [\Sigma^{-n}(X), E] \cong [\Sigma^{-n-1}(\Sigma(X)), E] \cong [\Sigma(X), E]_{-n-1} = E^{n+1}(\Sigma X).$$

(c) By definition, $E^n(A_1 \vee \cdots \vee A_k) = [A_1 \vee \cdots \vee A_k, E]_{-n}$. To a map $f\colon A_1 \vee \cdots \vee A_k \to E$ we can associate, for $0 \leq i \leq k$, $f_i\colon A_i \to E$ by pre-composing with the inclusion $A_i \to A_1 \vee \cdots \vee A_k$ and, conversely, a sequence of maps $(f_i\colon A_i \to E)$ define a map from the wedge sum $A_1 \vee \cdots \vee A_k$ to E.

(d) If f is a weak equivalence, by Theorem 2.19 it is an isomorphism and so the induced map $E^n(Y) \to E^n(X)$ is also an isomorphism.

□

It is interesting to look at the E-cohomology functors for a CW-spectrum E that we know: given an integer n and an abelian group G, recall that we have the shifted Eilenberg-MacLane spectrum $K(G, m)$, with

$$K(G, m)_n = K(G, m + n).$$

Note that:[3]

$$HG_n = K(G, m)_{n-m}.$$

By Remark 2.15, HG is equivalent to $\Sigma^{-m}K(G, m)$. Hence, by Proposition 2.18, we have a natural isomorphism:

$$HG^m(X) = [\Sigma^{-m}X, HG] \cong [X, \Sigma^m HG] = [X, K(G, m)]. \qquad (2.4)$$

It is also possible to define a cohomology theory on CW-spectra in a more direct way, as we did for homology, building the cellular cochain complexes for stable cells and taking the cohomology of the complex, the same way one does for CW-complexes. For example, [Hat04] follows this approach. The resulting cohomology $H^*(X; G)$ is isomorphic to the HG-cohomology ([Hat04, Prop. 2.4]). Moreover, if we apply this functor to the suspension spectrum of a basepointed CW-complex X, it coincides with the ordinary reduced cohomology of X with coefficients in G.

If one defines the spectra cohomology with the cellular cochain complex, it is clear that, for a CW-spectrum of finite type, we get finitely generated

[3] HG is the Eilenberg-MacLane spectrum for the group G, defined in section 2.1.

cohomology groups. Then, the HG-cohomology groups for a spectrum of finite type are finitely generated.

Let us consider the natural transformations of functors:

$$HG^m(-) \to HG^s(-)$$

for an abelian group G and integers m, s. These natural transformation are cohomology operations and, by (2.4), if we can apply the (contravariant) Yoneda Lemma, we get that the set of them is in bijection with

$$HG^s(K(G,m)) \cong [K(G,m), K(G,s)] \cong [HG, K(G,s-m)] \cong HG^{s-m}(HG)$$

Namely, a natural transformation $HG^m(X) \to HG^s(X)$ corresponds to the composition:

$$X \to K(G,m) \to K(G,s). \tag{2.5}$$

So, every natural transformation $HG^*(-) \to HG^*(-)$ is identified with an element of $HG^*(HG)$.

We have that $HG^*(HG)$ is an abelian group, as in general for any $[X,Y]$, and it is also a ring, with multiplication given by composing maps. It can be easily checked that, for any CW-spectrum X of finite type, these facts give to $HG^*(X)$ a $HG^*(HG)$-module structure, with scalar multiplication given again by the composition.

Setting $G = \mathbb{Z}/p$ for a prime number p, we define the *modulo p Steenrod Algebra*:

$$\mathcal{A}_p = (H\mathbb{Z}/p)^*(H\mathbb{Z}/p),$$

the algebra of stable cohomology operations with $\mathbb{Z}/p$ coefficients. It is a $\mathbb{Z}/p$-algebra and it is generated by the Steenrod operations: the Steenrod squares Sq^i, if $p = 2$, the Steenrod powers P^k and the Bockstein map β_p, for $p > 2$ ([Ada74, Ch. III.12]).

In the construction of the Adams Spectral Sequence we will need an improved version of the isomorphism seen in Remark 2.23, that works for certain infinite wedges of CW-spectra. Note that here we get an isomorphism between $[X, \vee_i K_i]$ and the direct product $\prod_i [X, K_i]$, while for finite wedges we have a direct sum.

Proposition 2.26. *The natural map*

$$[X, \vee_i K(G,n_i)] \to \prod_i [X, K(G,n_i)]$$
$$f \mapsto (f_i),$$

where f_j is the composition of f with the projection $\vee_i K(G,n_i) \xrightarrow{p_j} K(G,n_j)$, is an isomorphism if X is a connective CW-spectrum of finite type and $n_i \to \infty$ as $i \to \infty$.

The proof of the Proposition is based on the two following lemmas. Recall that, if X is a connective CW-spectrum of finite type, the dimension of its stable cells is bounded below and X can be written as the increasing union of its k-skeleta X^k. Since X has finite type, each X^k is finite.

Lemma 2.27. *Let E^* be the E-cohomology for a fixed CW-spectrum E and X a connective CW-spectrum. Then there is an exact sequence:*

$$0 \to \varprojlim{}^1 E^{n-1}(X^k) \to E^n(X) \to \varprojlim E^n(X^k) \to 0,$$

where X^k denotes the k-skeleton of X.

Recall that $\varprojlim E^n(X^k)$ denotes the inverse limit of the groups $G_k = E^n(X^k)$, which can be seen as the kernel of the homomorphism:

$$\delta\colon \prod_i (G_k) \to \prod_i (G_k), \qquad \delta((g_i)) = (h_i),$$

with $h_j = g_j - i^*_{j+1}(g_{j+i})$, where i^*_k is the map induced on cohomology by the inclusion $X^{k+1} \to X^k$. Here $\varprojlim^1$ is the cokernel of δ.

A result analogous to this Lemma, but for CW-complexes instead that for CW-spectra, is proved in [Hat02, Thm. 3F.8]. and the argument can be easily translated to CW-spectra. The second ingredient for the proof of the proposition is the following algebraic lemma, whose proof is omitted from this project:

Lemma 2.28 (Mittag-Leffler criterion)**.** *If, for each k, the images of $G_{k+n} \to G_k$ are independent of n, for n big enough, then $\varprojlim^1 G_k = 0$.*

Proof (of Proposition 2.26). If X is a finite spectrum, every space X_j has a finite number of cells. Hence, by cellular approximation, all the factors $K(G, n_i)$ with n_i bigger than the maximum dimension of the cells in the spaces X_j can be omitted from the direct product and from the wedge sum and so we get back to the finite case seen in Remark 2.23. For the general case, we can see the spectrum X as the union of its skeleta X^k, that are finite. If we denote $h^n(X) = [\Sigma^{-n}X, \vee_i K(G, n_i)]$ is the cohomology associated to $\vee_i K(G, n_i)$, by the previous lemma we have the exact sequence:

$$0 \to \varprojlim{}^1 h^{n-1}(X^k) \to h^n(X) \to \varprojlim h^n(X^k) \to 0,$$

The third term of the spectral sequence $\varprojlim h^n(X^k)$ is the product we are interested in, $\prod_i [X, K(G, n_i)]$, because it is the inverse limit of the finite product from the previous case. Then it is enough to show that the term $\varprojlim^1 h^{n-1}(X^k)$ is zero to get the wanted isomorphism. We can apply the Mittag-Leffler criterion: in fact we have that the images of the maps

$$HG^i(X^{k+n}) \to HG^i(X^k)$$

are independent of n when $k + n > i$. Then $\varprojlim^1 h^{n-1}(X^k) = 0$ and the proof is complete. $\square$

3 The Adams Spectral Sequence

We begin the section doing the construction of an exact resolution of the cohomology of a CW-spectrum, before moving on to the construction of the Spectral Sequence. The spectral sequence deals with cohomology with coefficients in $\mathbb{Z}/p$, for a prime p. Hence in the following we will consider only this cohomology, with the notation $H^n(X) = (H\mathbb{Z}/p)^n(X) \cong [X, K(\mathbb{Z}, n)]$.

3.1 Exact resolutions

From now on we are restricting our attention to connective CW-spectra of finite type, so that the cohomology groups are finitely generated.

The results of the previous section allow us to use CW-spectra in a much similar way to CW-complexes, in the sense that we don't need anymore to look at the spaces composing a spectra. Instead, when looking at a sequence of spectra, we will use the notation X_n to refer to one of the spectra in the sequence.

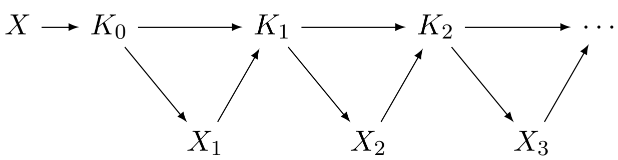
Let X be a connective CW-spectrum of finite type. Our goal is to build a free resolution of the cohomology $H^*(X)$, seen as a module over the modulo p Steenrod algebra $\mathcal{A}_p$. The first step is to take α_i generators for $H^*(X)$, with a finite number of them in each $H^n(X)$. By (2.4), each α_i can be seen as an element of $[X, K(\mathbb{Z}/p, n_i)]$ and so, by Proposition 2.26, they determine a map

$$X \to \vee_i K(\mathbb{Z}/p, n_i).$$

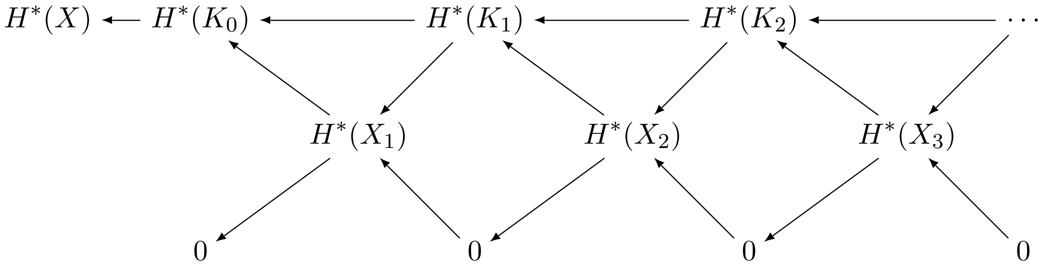
We set $K_0 = \vee_i K(\mathbb{Z}/p, n_i)$, that is a spectrum of finite type, and then, replacing this map with an inclusion via the mapping cylinder construction, we may form the quotient:

$$X_1 = K_0/X.$$

X_1 is again a connective spectrum of finite type, then we can repeat the construction with X_1 in place of X to build K_1 and $X_2 = K_1/X_1$. This yields the diagram:

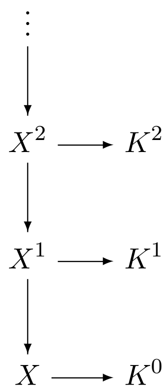


and, applying the cohomology functor, we get the diagram:



which gives a free resolution of $H^*(X)$ as a $\mathcal{A}_p$-module.

If we denote $X^n = \Sigma^{-n} X_n$ and $K^n = \Sigma^{-n} K_n$, we can rewrite the previous diagram in the form:



by using (2.1) for the cofibrations $X_s \to K_s \to X_{s+1}$ we built above and applying the suspension isomorphism of Proposition 2.18. The diagram is called the *Adams tower* over X.

3.2 Constructing the spectral sequence

In the construction of the exact resolution we defined cofibrations:

$$X_s \to K_s \to X_{s+1}.$$

Then, by Proposition 2.22, we get exact sequences:

$$[\Sigma^t Y, X_s] \to [\Sigma^t Y, K_s] \to [\Sigma^t Y, X_{s+1}] \to [\Sigma^t Y, \Sigma X_s] \to \cdots$$

Recall that we have the isomorphisms given by suspension (Proposition 2.18):

$$[\Sigma^t Y, \Sigma X_s] \cong [\Sigma^{t-1} Y, X_s],$$

and so the exact sequences above fit together to form a so-called *staircase diagram*:

$$
\begin{array}{ccccccccccccc}
\to & [\Sigma^{t+1}Y, X_s] & \longrightarrow & [\Sigma^{t+1}Y, K_s] & \longrightarrow & [\Sigma^{t+1}Y, X_{s+1}] & \longrightarrow & [\Sigma^{t+1}Y, \Sigma K_{s+1}] & \longrightarrow & [\Sigma^{t+1}Y, X_{s+2}] & \longrightarrow \\
& \downarrow & & & & \downarrow & & & & \downarrow & \\
\to & [\Sigma^{t}Y, X_{s-1}] & \longrightarrow & [\Sigma^{t}Y, K_{s-1}] & \longrightarrow & [\Sigma^{t}Y, X_{s}] & \longrightarrow & [\Sigma^{t}Y, \Sigma K_{s}] & \longrightarrow & [\Sigma^{t}Y, X_{s+1}] & \longrightarrow \\
& \downarrow & & & & \downarrow & & & & \downarrow & \\
\to & [\Sigma^{t-1}Y, X_{s-2}] & \longrightarrow & [\Sigma^{t-1}Y, K_{s-2}] & \longrightarrow & [\Sigma^{t-1}Y, X_{s-1}] & \longrightarrow & [\Sigma^{t-1}Y, \Sigma K_{s-1}] & \longrightarrow & [\Sigma^{t-1}Y, X_{s}] & \longrightarrow \\
& \downarrow & & & & \downarrow & & & & \downarrow &
\end{array}
$$

Such a diagram determines an exact couple, which is one of the standard ways to construct a spectral sequence (see [HS97, Chapter VIII] for a description of the construction in general).

Definition 3.1. The spectral sequence obtained from the previous staircase diagram is called the *modulo p Adams spectral sequence.*

To simplify the writing, we introduce the notation:

$$\pi_t^Y(X) = [\Sigma^t Y, X].$$

This way, taking Y equal to S^0, the sphere spectrum, we have: $\pi_t^{S^0}(X) = \pi_t(X)$.

To identify the terms in the first pages of the spectral sequence, we proceed as follows: by construction, each K_s is a wedge of Eilenberg-MacLane spaces and we know that its cohomology $H^*(K_s)$ is a free $\mathcal{A}_p$-module.

The elements of $\pi_t^Y(K_s) = [\Sigma^t Y, K_s]$ are tuples (α_i) of maps $\alpha_i \colon \Sigma^t Y \to K(\mathbb{Z}/p, n_i)$ or, in other words, $\alpha_i \in H^{n_i}(\Sigma^t Y)$. Since $H^*(K_s)$ is a free module, the natural map:

$$[\Sigma^t Y, K_s] \xrightarrow{\alpha^*} \operatorname{Hom}_{\mathcal{A}_p}(H^*(K_s), H^*(\Sigma^t Y))$$

is an isomorphism. If we introduce the notation Hom^t to indicate the homomorphism that lower degree by t, this can be rewritten as:

$$\pi_t^Y(K_s) \cong \operatorname{Hom}_{\mathcal{A}_p}(H^*(K_s), H^*(\Sigma^t Y)) \cong \operatorname{Hom}^t_{\mathcal{A}_p}(H^*(K_s), H^*(Y)).$$

Therefore, the terms of the first page of the spectral sequence have the form:

$$E_1^{s,t} = \operatorname{Hom}^t_{\mathcal{A}_p}(H^*(K_s), H^*(Y))$$

and the differential d_1 in this page is formed by the maps:

$$d_1^{s,t} \colon E_1^{s,t} \to E_1^{s+1,t},$$

induced by the map $K_s \to K_{s+1}$, that we defined when constructing the exact resolution of $H^*(X)$. In fact $E_1^{s,t} = \pi_t^Y(K_s)$ and $E_1^{s+1,t} = \pi_t^Y(K_{s+1})$, so we may view $d_1^{s,t}$ as a map $\pi_t^Y(K_s) \to \pi_t^Y(K_{s+1})$.

This means that the E_1 page consists of the complexes

$$0 \to \mathrm{Hom}^t_{\mathcal{A}_p}(H^*(K_0), H^*(Y)) \xrightarrow{d_i^{0,t}} \mathrm{Hom}^t_{\mathcal{A}_p}(H^*(K_1), H^*(Y)) \xrightarrow{d_i^{1,t}} \mathrm{Hom}^t_{\mathcal{A}_p}(H^*(K_2), H^*(Y)) \to \dots \quad (3.1)$$

The terms of the next page E_2 are, by definition:

$$E_2^{s,t} = \ker(d_1^{s,t}) / \mathrm{Im}(d_1^{s-1,t}),$$

hence, computing them amounts to compute the homology of the complexes (3.1) and so, since the groups $H^*(K_s)$ are an exact resolution of $H^*(X)$, we get the Ext groups:

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(X), H^*(Y)).$$

3.3 The main theorem

Our main result deals with the convergence of the Adams Spectral Sequence.

Theorem 3.2. *Let X be a connective CW-spectrum of finite type. Then the Adams Spectral Sequence with E_2 terms of the form:*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(X), H^*(Y))$$

is convergent to $\pi_*^Y(X)$ modulo the subgroup given by torsion elements of order prime to p. In other words:

(a) The terms $E_\infty^{s,t}$ are isomorphic to the successive quotients $F^{s,t}/F^{s+1,t+1}$ for a filtration of $\pi_{t-s}^Y(X)$ in which $F^{s,t}$ is the image of the map $\pi_t(X_s) \to \pi_{t-s}(X)$.

(b) $\cap_n F^{s+n,t+n}$ is the subgroup of $\pi_{t-s}^Y(X)$ of the torsion elements of order prime to p.

Note that the filtration can very well be infinite: this happens if $\pi_{t-s}^Y(X)$ is infinite, since all the elements $E_r^{s,t}$ of the spectral sequence are finite dimensional $\mathbb{Z}/p$ vector spaces. In fact the terms on the E_1 page are finite dimensional, because they have the form $\mathrm{Hom}^t_{\mathcal{A}_p}(H^*(K_s), H^*(Y))$, and this implies that all the terms in the next pages are finite dimensional as well.

Let us introduce the following terminology: a sequence of spectra (L_i) with maps $Z \to L_0 \to L_1 \to \dots$ such that the composition of two consecutive maps is nullhomotopic is called a *complex* on Z. A complex such that every spectrum L_1 is a wedge of Eilenberg-MacLane spectra is named *Eilenberg-MacLane complex.* If the sequence induced in cohomology:

$$0 \leftarrow H^*(Z) \leftarrow H^*(L_0) \leftarrow H^*(L_1) \leftarrow \dots$$

is exact, then the complex is called *resolution.*

Let us prove a lemma that will be used in the proof of the theorem:

Lemma 3.3. *If we are given a resolution of Z:*

$$Z \to L_0 \to L_1 \to \dots,$$

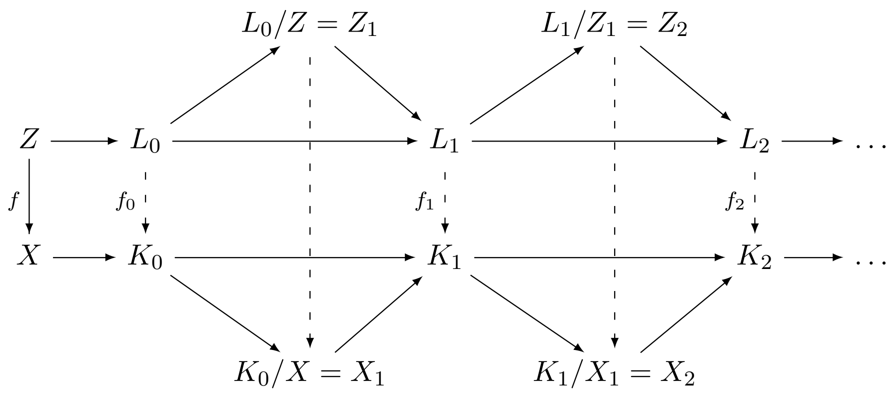
an Eilenberg-MacLane complex:

$$X \to K_0 \to K_1 \to \dots$$

and a map $f\colon Z \to X$, then we can build maps f_i forming the following diagram, commutative up to homotopy:

$$\begin{array}{ccccccccc} Z & \longrightarrow & L_0 & \longrightarrow & L_1 & \longrightarrow & L_2 & \longrightarrow & \dots \\ {\scriptstyle f}\downarrow & & {\scriptstyle f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & \\ X & \longrightarrow & K_0 & \longrightarrow & K_1 & \longrightarrow & K_2 & \longrightarrow & \dots \end{array}$$

Proof. Since the composition of two consecutive maps is nullhomotopic, we can factor the maps as in the diagram:



The map $X \to K_0 = \vee K(\mathbb{Z}/p, n_i)$ corresponds to a sequence of $\alpha_i\colon X \to K(G, n_i)$ and we can view each α_i as a class in $H^{n_i}(X)$, by (2.4). Since the map $H^*(L_0) \to H^*(Z)$ is surjective (because the first row is a resolution), there exist, for all i, some $\beta_i \in H^{n_i}(L_0)$ such that $f^*(\beta_i) = \alpha_i$. Each β_i gives a homotopy class in $[L_0, K(G, n_i)]$, again by (2.4), and so we get a map $f_0\colon L_0 \to K_0$, forming a homotopy-commutative square. By exactness, the map $H^*(L_1) \to H^*(Z_1)$ is surjective, so we can repeat the argument to build f_1 and further for all the other f_i's. $\square$

We can now prove out Theorem about the convergence of the Adams spectral sequence.

Proof (of Theorem 3.2). We prove the theorem in the case of the ordinary homotopy groups, in which Y is the sphere spectrum S^0: the proof for an arbitrary spectrum Y contains only minor changes. We start from (b) and for the proof we take Y to be the sphere spectrum, so that $H^*(Y) = \mathbb{Z}/p$: since the terms of the E_1 page are $\mathbb{Z}/p$ vector spaces, as observed, they have the form $E_1^{s,t} = \pi_t(K_s) = \mathrm{Hom}^t_{\mathcal{A}}(H^*(K_s), \mathbb{Z}/p)$. So, by exactness, the vertical maps in the staircase diagram are isomorphisms on the torsion of order prime to p, therefore we have that the torsion of order prime to p is contained in $\cap_n F^{s+n,t+n}$.

To prove the other inclusion, we do first a special case: suppose that $\pi_*(X)$ is entirely of p-torsion. We want to build a Eilenberg-MacLane complex over X: $X \to L_0 \to L_1 \to \dots$ as follows: take n such that $\pi_n(X)$ is the first non-trivial homotopy group. Then take generators $\alpha_i \colon X \to K(\mathbb{Z}/p, n)$ for $H^n(X)$ and let L_0 be a wedge of Eilenberg-MacLane spectra $K(\mathbb{Z}/p, n)$, one for each element α_i. This way the map $\alpha \colon X \to L_0$ induces isomorphism on cohomology and also on homology. By the Hurewicz Theorem 2.20, on homotopy it is the map $\pi_n(X) \to \pi_n(X) \otimes \mathbb{Z}_p$. Now, if we make the map $X \to L_0$ into an inclusion and look at $Z_1 = L_0/X$, we have $\pi_i(Z_1) = 0$ for $i \leq n$ and $\pi_{n+1}(Z_1) = \ker(\pi_n(X) \to \pi_n(L_0))$, so $\pi_{n+1}(Z_1)$ has a smaller order than $\pi_n(X)$. Then we can repeat the process with Z_1 in place of X and build a map $Z_1 \to L_1$ in the same way, with L_1 a wedge of Eilenberg-MacLane spectra. Turning this map into an inclusion, we form $Z_2 = L_1/Z_1$, whose first non-trivial homotopy group is $\pi_{n+1}(Z_2)$ and it has a smaller order than $\pi_{n+1}(Z_1)$. Continuing the process for a finite number of steps, we will get to Z_k, with $\pi_i(Z_k) = 0$ for $i \leq n + k$. At that point we consider $\pi_{n+k+1}(Z_k)$ and repeat the steps. This way we get the wanted complex $X \to L_0 \to L_1 \to \dots$. Now, if we denote $Z^i = \Sigma^{-i} Z_i$, we get the Adams tower associated to the complex, with maps $\cdots \to Z^2 \to Z^1 \to X$. By the previous construction, the map induced by $Z^1 \to X$ on homotopy is an isomorphism on all degrees except on π_n for which it is the inclusion of a proper subgroup. The same for $Z^2 \to Z^1$, until we reach the step k for which $\pi_n(Z^k) = 0$ and the inclusion of a proper subgroup is in degree $n + 1$. The tower continues: after another finite number of steps, $\pi_{n+1}(Z^j)$ becomes zero and so on: this implies that, for each i, $\pi_i(Z^j)$ is zero if j is big enough. Now we can apply Lemma 3.3, to get a map from the resolution of X that we used to build the spectral sequence and the complex $X \to L_0 \to L_1 \to \dots$. If we pass to the associated towers and apply homotopy, we get the diagram:

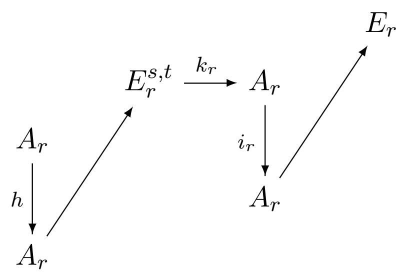
$$\begin{array}{ccccccc}
\dots \longrightarrow & \pi_i(X^2) & \longrightarrow & \pi_i(X^1) & \longrightarrow & \pi_i(X) \\
& \downarrow & & \downarrow & & \downarrow{=} \\
\dots \longrightarrow & \pi_i(Z^2) & \longrightarrow & \pi_i(Z^1) & \longrightarrow & \pi_i(X)
\end{array}$$

If there was an element in $\pi_i(X)$ pulling back in all the groups in the first row, it would do the same in the second row. But the Z^i's have been constructed so that $\pi_i(Z^j) = 0$ for j big enough, so this is impossible so the intersection the groups $F^{s,t} = \mathrm{Im}(\pi_t(X_s) \to \pi_{t-s}(X)) = \mathrm{Im}(\pi_{t-s}(X^s) \to \pi_{t-s}(X)$ have empty intersection, so (b) is proved in the case of $\pi_*(X)$ entirely of p-torsion. To prove this inclusion in general, let us consider an element $\alpha \in \pi_n(X)$ that is either of infinite order or of p-torsion. We shall prove that it is not contained in the intersection $\cap_n F^{s+n,t+n}$. Since the order of α is either infinite of p^t for some $t \geq 0$, we can pick a positive integer k such that α is not equal to p^k times another any other element of $\pi_n(X)$. If we consider the identity map in $[X, X]$ and add it to itself p^k times, we get a map $X \xrightarrow{p^k} X$ and we can form a cofibration $X \xrightarrow{p^k} X \to H$. By (2.1), we get a long exact sequence in homotopy:

$$\cdots \to \pi_i(X) \xrightarrow{p^k} \pi_i(X) \to \pi_i(H) \to \ldots$$

in which the map p^k is $x \mapsto x^{p^k}$. By exactness, we have that $\pi_i(H)$ is entirely of p-torsion for all i. We have a map $X \to H$ from the cofibration and so, by the Lemma, we get a map from the tower over X associated with the free resolution to the tower over H, as done in the previous case. By the way we have chosen α and k, we have that α is mapped to a non-zero element by the map $\pi_n(X) \to \pi_n(H)$, by looking at the previous exact sequence in degree n. If our element α pulled back in all the groups $\pi_i(X^j)$ in the tower over X, it would do the same in the tower over Z, but this is impossible by the case we proved earlier, because $\pi_*(Z)$ is entirely of p-torsion. Hence α is not contained in the intersection $\cap_n F^{s+n,t+n}$ and so (b) is proved.

For (a), we look at the r^{th} derived couple associated to the staircase diagram. We have the diagram:



We shall prove that, if r is big enough, then i_r is injective: injectivity on elements of infinite order or on torsion elements of order prime to p come from the exactness, since the in the E columns of the staircase diagram we have $\mathbb{Z}/p$ vector spaces. For elements of p-torsion, we know from (b) that $A_r^{s,t}$ does not contain such elements if r is sufficiently big, therefore the claim is proved.

This implies that, for r big enough, the map preceding i_r in the exact couple, k_r, is zero, hence the differential d_r at $E_r^{s,t}$ is also zero. For the differential d_r mapping to $E_r^{s,t}$, we have that is is also zero if r is sufficiently big, because it originates at a zero group, since the terms in the E column of the staircase diagram are zero under some level. Hence $E_r^{s,t} = E_{r+1}^{s,t}$ from a certain value of r, since the differentials become zero. Since k_r is zero for r big enough, then $E_r^{s,t}$ is the cokernel of the map h in the diagram, which is the inclusion $F^{s+1,t+1} \to F^{s,t}$ and so $E_\infty^{s,t} \cong F^{s,t}/F^{s+1,t+1}$. □

We end this section with a result that will be useful for the calculations: if we want to use the spectral sequence with the spectrum Y equal to the sphere spectrum S^0, as in the case of the computation of stable homotopy groups, we have a way to simplify the calculation of $\mathrm{Ext}_{\mathcal{A}_p}^{s,t}(H^*(X), \mathbb{Z}_p)$: it is to build a minimal free resolution for $H^*(X)$. A free resolution

$$0 \leftarrow H^*(X) \leftarrow M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \dots$$

is called *minimal* if during the construction of the free modules, we take the minimal number of generators for each M_i in each degree.

The following proposition, which can be proved via an algebraic argument, shows why it is convenient to build such resolutions:

Proposition 3.4. *If a free resolution* $0 \leftarrow H^*(X) \leftarrow M_0 \leftarrow M_1 \leftarrow \dots$ *is minimal, then all the boundary maps in the complex:*

$$\dots \leftarrow \mathrm{Hom}_{\mathcal{A}}^t(M_2, \mathbb{Z}/p) \leftarrow \mathrm{Hom}_{\mathcal{A}}^t(M_1, \mathbb{Z}/p) \leftarrow \mathrm{Hom}_{\mathcal{A}}^t(M_0, \mathbb{Z}/p) \leftarrow 0$$

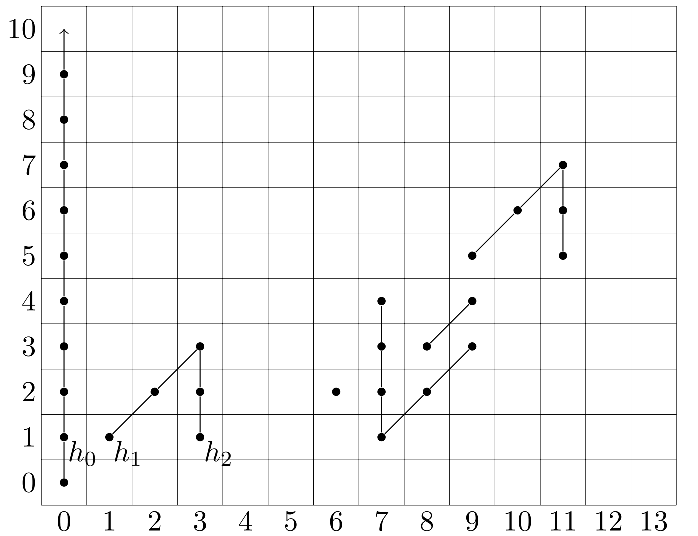
are zero and so $\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(X), \mathbb{Z}/p) \cong \mathrm{Hom}_{\mathcal{A}}^t(M_s, \mathbb{Z}/p)$.

Proof. See [Hat04, Ch.2, Lemma 2.8]. □

4 Calculations

4.1 Stable homotopy groups of spheres

One of the most classical applications of the Adams Spectral Sequence is the computation of the stable homotopy groups of spheres. If we take both X and Y equal to the sphere spectrum S^0 in Theorem 3.2, we obtain information about the *p-component* of the stable homotopy groups of spheres ${}_p\pi_*^s(S^0) = {}_p\pi_*^s$. In the calculation we will fix $p = 2$ and denote with $\mathcal{A}$ the modulo 2 Steenrod Algebra. The following diagram, that presents a portion of the page E_2, is an example of the result of these calculations.



The vertical axis of the diagram is the coordinate s and the horizontal one $t-s$, this way each column gives information about one homotopy group: namely, the column $t-s$ contains information about π_{t-s}^s. Every dot in the diagram represents one factor $\mathbb{Z}/2$ in E_2.

The differential $d_r^{s,t}$ in the page E_r goes from $E_r^{s,t}$ to $E_r^{s+r,t+r-1}$. Hence, in our diagram, d_r goes one cell to the left and r cells upward.

It can be shown that the vertical lines in the diagram represent multiplication by $p = 2$ in π_*^s, therefore, for instance, the infinite line of dots in the column $t - s = 0$ corresponds to the fact that $\pi_0^s = \mathbb{Z}$ and so one can keep multiplying by 2 without getting a zero. As a consequence of Theorem 3.2, each dot in this column is a quotient $2^n\mathbb{Z}/2^{n+1}\mathbb{Z}$ for the filtration $\cdots \subseteq 2^{n+1}\mathbb{Z} \subseteq 2^n\mathbb{Z} \subseteq 2^{n-1}\mathbb{Z} \subseteq \cdots$ of $\mathbb{Z}$.

All the differentials in the portion of E_2 drawn here are zero, hence all the dots shown here stay till the spectral sequence converges. To see this, one can use the fact that they are derivations and so satisfy the relation:

$$d_r(xy) = xd_r(y) + d_r(x)y.$$

This is not obvious, but it is true in our hypothesis as a consequence of [Rav03, Theorem 2.3.3]. The multiplication operation in every page of the spectral sequence corresponds to the graded multiplication that is present inside the graded stable homotopy group π_*^s:

$$\pi_i^s \times \pi_j^s \to \pi_{i+j}^s.$$

induced by the compositions $S^{i+j+k} \to S^{j+k} \to S^k$ (see Proposition 1.3). The vertical solid lines in the diagram represent multiplication by h_0.

Let us show how one can prove that the differential d_r (in the page E_r) originating at $(1,1)$ is zero. This differential maps to the cell $(0, r+1)$ and so, if we suppose that it is non-zero, it maps h_1 to the non-zero element at $(0, r+1)$: we would have $d_r(h_1) = h_0^{r+1}$. Then, by the derivation rule:

$$d_r(h_0h_1) = h_0d_r(h_1) + d_r(h_0)h_1 = h_0h_0^{r+1} = h_0^{r+2},$$

and h_0^{r+2} is non-zero, because multiplication by h_0 in the column $t-s=0$ is an isomorphism. But, looking at the diagram, we see that the product h_0h_1 is zero, and so the left hand side of the equality is zero, giving a contradiction. Therefore, we must have $d_r(h_1) = 0$.

To get the picture of the E_2 page that we have described, we need to calculate the $\mathrm{Ext}_{\mathcal{A}}(\mathbb{Z}/2, \mathbb{Z}/2)$ terms and for this we may work out a minimal free resolution

$$\cdots \to M_2 \to M_1 \to M_0 \to \mathbb{Z}/2 \to 0$$

of $\mathbb{Z}/2$ as a $\mathcal{A}$-module. The process begins by taking as M_0 a copy of $\mathcal{A}$, with a generator A_0 in degree zero that maps to the generator of $\mathbb{Z}/2$, as shown in the diagram of the next page. For the resolution to be exact, we need to choose M_1 so that it maps onto the kernel of the map $M_0 \to \mathbb{Z}/2$, that is composed by all the terms of the form $\mathrm{Sq}^I A_0$, where Sq^I, for $I = (i_1, i_2, \ldots, i_n)$, denotes the product of Steenrod squares $\mathrm{Sq}^{i_1} \ldots \mathrm{Sq}^{i_n}$. So we need a generator B_1 in M_1 that maps to $\mathrm{Sq}^1 A_0$. This implies that $\mathrm{Sq}^1 B_1$ maps to $\mathrm{Sq}^1 \mathrm{Sq}^1 A_0 = 0$, by the Adem relations. Hence we need a new generator B_2 in the position $s = 1$, $t - s = 1$, mapping to $\mathrm{Sq}^2 A_0$.

The process continues like this: note that in the diagram, in the column s, we have the elements of degree k in the row $t - s = k - s$. We use coordinates s and $t - s$ on the axes of the diagram to use the same notation that is used in the spectral sequence. For this reason, the maps of the resolution, which have degree 0, are sloping downward to the left. These maps are displayed as arrows in the chart and are omitted for elements that map to 0. An arrow with more than one head means that the element from which the arrow originates maps to the sum of the elements pointed.

All the resolution can be computed inductively, proceeding row by row and from left to right in each row: in every entry $(s, t - s)$ of the diagram we compute the boundary map on the elements that are already present

$t-s$ \ s	F_0	F_1	F_2	F_3	F_4	F_5
0	A_0	B_1	C_2	D_3	E_4	F_5
1	$\mathrm{Sq}^1 A_0$	$\mathrm{Sq}^1 B_1$ B_2	$\mathrm{Sq}^1 C_2$	$\mathrm{Sq}^1 D_3$	$\mathrm{Sq}^1 E_4$	$\mathrm{Sq}^1 F_5$
2	$\mathrm{Sq}^2 A_0$	$\mathrm{Sq}^2 B_1$ $\mathrm{Sq}^1 B_2$	$\mathrm{Sq}^2 C_2$ C_4	$\mathrm{Sq}^2 D_3$	$\mathrm{Sq}^2 E_4$	$\mathrm{Sq}^2 F_5$
3	$\mathrm{Sq}^{2,1} A_0$ $\mathrm{Sq}^3 A_0$	$\mathrm{Sq}^{2,1} B_1$ $\mathrm{Sq}^3 B_1$ $\mathrm{Sq}^2 B_2$ B_4	$\mathrm{Sq}^{2,1} C_2$ $\mathrm{Sq}^3 C_2$ $\mathrm{Sq}^1 C_4$ C_5	$\mathrm{Sq}^{2,1} D_3$ $\mathrm{Sq}^3 D_3$ D_6	$\mathrm{Sq}^{2,1} E_4$ $\mathrm{Sq}^3 E_4$	$\mathrm{Sq}^{2,1} F_5$ $\mathrm{Sq}^3 F_5$
4	$\mathrm{Sq}^{3,1} A_0$ $\mathrm{Sq}^4 A_0$	$\mathrm{Sq}^{3,1} B_1$ $\mathrm{Sq}^3 B_2$ $\mathrm{Sq}^4 B_1$ $\mathrm{Sq}^{2,1} B_2$ $\mathrm{Sq}^1 B_4$	$\mathrm{Sq}^{3,1} C_2$ $\mathrm{Sq}^4 C_2$ $\mathrm{Sq}^2 C_4$ $\mathrm{Sq}^1 C_5$	$\mathrm{Sq}^{3,1} D_3$ $\mathrm{Sq}^4 D_3$ $\mathrm{Sq}^1 D_6$	$\mathrm{Sq}^{3,1} E_4$ $\mathrm{Sq}^4 E_4$	$\mathrm{Sq}^{3,1} F_5$ $\mathrm{Sq}^4 F_5$
5	$\mathrm{Sq}^{4,1} A_0$ $\mathrm{Sq}^5 A_0$	$\mathrm{Sq}^{4,1} B_1$ $\mathrm{Sq}^4 B_2$ $\mathrm{Sq}^{3,1} B_2$ $\mathrm{Sq}^5 B_1$ $\mathrm{Sq}^2 B_4$	$\mathrm{Sq}^{4,1} C_2$ $\mathrm{Sq}^5 C_2$ $\mathrm{Sq}^3 C_4$ $\mathrm{Sq}^2 C_5$ $\mathrm{Sq}^{2,1} C_4$	$\mathrm{Sq}^{4,1} D_3$ $\mathrm{Sq}^5 D_5$ $\mathrm{Sq}^2 D_6$	$\mathrm{Sq}^{4,1} E_4$ $\mathrm{Sq}^5 E_4$	$\mathrm{Sq}^{4,1} F_5$ $\mathrm{Sq}^5 F_5$
6	$\mathrm{Sq}^{4,2} A_0$ $\mathrm{Sq}^{5,1} A_0$ $\mathrm{Sq}^6 A_0$	$\mathrm{Sq}^{4,2} B_1$ $\mathrm{Sq}^{5,1} B_1$ $\mathrm{Sq}^6 B_1$ $\mathrm{Sq}^{2,1} B_4$ $\mathrm{Sq}^{4,1} B_2$ $\mathrm{Sq}^5 B_2$ $\mathrm{Sq}^3 B_4$	$\mathrm{Sq}^{5,1} C_2$ $\mathrm{Sq}^{4,2} C_2$ $\mathrm{Sq}^6 C_2$ $\mathrm{Sq}^{2,1} C_5$ $\mathrm{Sq}^{3,1} C_4$ $\mathrm{Sq}^4 C_4$ $\mathrm{Sq}^3 C_5$ C_8	$\mathrm{Sq}^{5,1} D_3$ $\mathrm{Sq}^{4,2} D_3$ $\mathrm{Sq}^6 D_3$ $\mathrm{Sq}^{2,1} D_6$ $\mathrm{Sq}^3 D_6$	$\mathrm{Sq}^{5,1} E_4$ $\mathrm{Sq}^{4,2} E_4$ $\mathrm{Sq}^6 E_4$	$\mathrm{Sq}^{5,1} F_5$ $\mathrm{Sq}^{4,2} F_5$ $\mathrm{Sq}^6 F_5$ ⋮
7	$\mathrm{Sq}^{4,2,1} A_0$ $\mathrm{Sq}^{6,1} A_0$ $\mathrm{Sq}^{5,2} A_0$ $\mathrm{Sq}^7 A_0$	$\mathrm{Sq}^{4,2,1} B_1$ $\mathrm{Sq}^{6,1} B_1$ $\mathrm{Sq}^{5,1} B_2$ $\mathrm{Sq}^{5,2} B_1$ $\mathrm{Sq}^{4,2} B_2$ $\mathrm{Sq}^7 B_1$ $\mathrm{Sq}^{3,1} B_4$ $\mathrm{Sq}^6 B_2$ $\mathrm{Sq}^4 B_4$ B_8 ⋮	$\mathrm{Sq}^{4,2,1} C_2$ $\mathrm{Sq}^{6,1} C_2$ $\mathrm{Sq}^{4,1} C_4$ $\mathrm{Sq}^5 C_4$ $\mathrm{Sq}^7 C_2$ $\mathrm{Sq}^{3,1} C_5$ ⋮ C_9	$\mathrm{Sq}^{4,2,1} D_3$ $\mathrm{Sq}^{6,1} D_3$ ⋮ D_{10}	$\mathrm{Sq}^{4,2,1} E_4$ $\mathrm{Sq}^{6,1} E_4$ ⋮ E_{11}	⋮

(and have the form Sq^I times a generator introduced in a previous row, for all the admissible monomials Sq^I of the appropriate degree). Then, to decide if we need a new generator in the cell, we look at the map going from $(s-1, t-s+1)$ to $(s-2, t-s+2)$: if it is not injective, then we have to introduce a new generator. It does not matter if in the next row we will introduce new generators in $(s-1, t-s+1)$, because generators map injectively, so they are never in a kernel. This process makes use of the Adem relations, to write the image of the product $\mathrm{Sq}^I X$ as admissible monomials.

This example gives the feeling of how one uses this spectral sequence for calculations: first there is the algebraic problem of building an exact resolution for $H^*(X)$ and computing the homology of the Hom complex of the resolution, to get the terms $\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(X), H^*(Y))$. Then one has to understand what differentials d_r are non-trivial and to compute them to understand the convergence of the spectral sequence. As shown, E_∞ only contains the successive quotients in filtrations of the homotopy groups, so one has still to solve the group extension problem to recover the homotopy groups.

4.2 The homotopy of the spectrum ko

In this section we will show a computation of the 2-component ${}_2\pi_*(ko)$ of the homotopy of the spectrum ko, the spectrum of real connective K-theory. In this specific case, the computation of an exact resolution of $H^*(ko)$ is easier, because, via a certain change-of-ring isomorphism, we can avoid working with the (mod 2) Steenrod algebra $\mathcal{A}$ and use instead the finite-dimensional subalgebra $\mathcal{A}(1)$, which is defined as the subalgebra generated by Sq^1 and Sq^2. This is a module (or, in other words, a vector space) over $\mathbb{Z}/2$. By an explicit computation of the products of elements of $\mathcal{A}(1)$, applying the Adem relations, one sees that the following elements give an explicit basis for it, so $\mathcal{A}(1)$ is 8-dimensional over $\mathbb{Z}/2$:

$$B = \{\mathrm{Sq}^0, \mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^3, \mathrm{Sq}^2\,\mathrm{Sq}^1, \mathrm{Sq}^3\,\mathrm{Sq}^1, \mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2, \mathrm{Sq}^5\,\mathrm{Sq}^1\}.$$

We recall a shortened notation that is used to denote products of Steenrod squares: $\mathrm{Sq}^{i_1,\dots,{}_n} = \mathrm{Sq}^{i_1} \dots \mathrm{Sq}^{i_n}$. In the following computations we will use repeatedly the Adem relations, so we list here the ones that we are going to need:

$$\begin{aligned} &\mathrm{Sq}^0 = 1 && \mathrm{Sq}^{1,1} = 0 \\ &\mathrm{Sq}^{1,2} = \mathrm{Sq}^3 && \mathrm{Sq}^{1,3} = 0 \\ &\mathrm{Sq}^{2,2} = \mathrm{Sq}^{1,2,1} = \mathrm{Sq}^{3,1} && \mathrm{Sq}^{2,1,2} = \mathrm{Sq}^{2,3} \\ &\mathrm{Sq}^{2,2,2} = \mathrm{Sq}^{3,1,2} = \mathrm{Sq}^{5,1} && \mathrm{Sq}^{2,1,2,1} = \mathrm{Sq}^{5,1} \\ &\mathrm{Sq}^{1,2,2} = \mathrm{Sq}^{3,2} = 0 && \mathrm{Sq}^{2,2,1} = 0. \end{aligned}$$

All elements in our basis for $\mathcal{A}(1)$ have degree lower or equal than 6, hence every product of squares Sq^I where $I = (i_0, \dots, i_n)$ with $i_0 + \dots + i_n > 6$ is automatically zero. As said, our aim is to compute the Adams Spectral Sequence for $p = 2$ to compute ${}_2\pi_*(ko)$. We will sketch here the reason for which we can switch to $\mathcal{A}(1)$ in this case. For a more detailed explanation of this, we refer the reader to [Rav03, Ch. 3.1].

By Theorem 3.2, the terms $E_2^{s,t}$ of the spectral sequence converging to the 2-component of $\pi_*(ko)$ are:

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(ko), \mathbb{Z}/2).$$

Now, we have that $H^*(ko)$ is a module over $\mathcal{A}(1)$ and, by [Ada74, Part III, Prop. 16.6]:

$$H^*(ko) = \mathcal{A}//\mathcal{A}(1) = \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{Z}/2.$$

So we can substitute in the equation for the terms of E_2 and apply a change-of-ring isomorphism, obtaining:

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{Z}/2, \mathbb{Z}/2) = \mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(\mathbb{Z}/2, \mathbb{Z}/2).$$

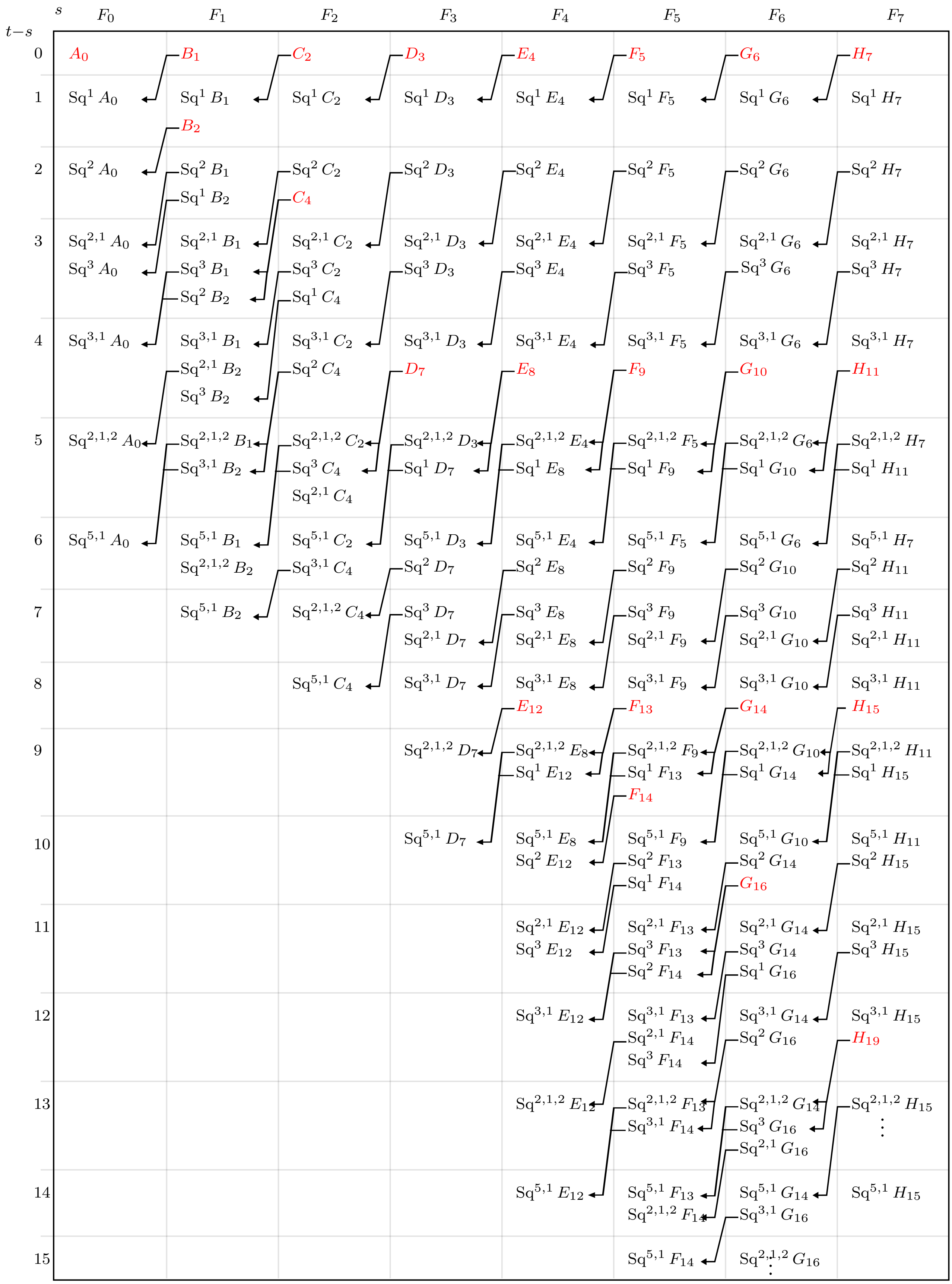
Hence we can compute a free resolution of $\mathbb{Z}/2$ as a $\mathcal{A}(1)$-module to compute this $\mathrm{Ext}_{\mathcal{A}(1)}$. To do this we proceed as we did in the previous section for computing a minimal free resolution of $\mathbb{Z}/2$ over $\mathcal{A}$, but in this case the computation is easier, since $\mathcal{A}(1)$ is finite dimensional.

We want to construct free $\mathcal{A}(1)$-modules M_i such that

$$\cdots \to M_2 \to M_1 \to M_0 \to \mathbb{Z}/2 \to 0$$

is exact.

Our computation is displayed in the chart in the following page. Note that the chart is indexed in the same way of the spectral sequence, with s on horizontal axis and $t-s$ on the vertical one. This choice is convenient to translate the information to the spectral sequence, but we have to be careful because this introduces a vertical shifting in the columns: the module M_i, which is in the column i, has its part of degree k in the row $k-s$. For example, the entry $(3, 1)$ contains the 4-degree part of M_3. Another consequence is



s
F_0
F_1
F_2
F_3
F_4
F_5
F_6
F_7
$t-s$
0
A_0
B_1
C_2
D_3
E_4
F_5
G_6
H_7
1
$\mathrm{Sq}^1 A_0$
$\mathrm{Sq}^1 B_1$
$\mathrm{Sq}^1 C_2$
$\mathrm{Sq}^1 D_3$
$\mathrm{Sq}^1 E_4$
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$\mathrm{Sq}^{3,1} C_2$
$\mathrm{Sq}^{3,1} D_3$
$\mathrm{Sq}^{3,1} E_4$
$\mathrm{Sq}^{3,1} F_5$
$\mathrm{Sq}^{3,1} G_6$
$\mathrm{Sq}^{3,1} H_7$
$\mathrm{Sq}^{2,1} B_2$
$\mathrm{Sq}^2 C_4$
D_7
E_8
F_9
G_{10}
H_{11}
$\mathrm{Sq}^3 B_2$
5
$\mathrm{Sq}^{2,1,2} A_0$
$\mathrm{Sq}^{2,1,2} B_1$
$\mathrm{Sq}^{2,1,2} C_2$
$\mathrm{Sq}^{2,1,2} D_3$
$\mathrm{Sq}^{2,1,2} E_4$
$\mathrm{Sq}^{2,1,2} F_5$
$\mathrm{Sq}^{2,1,2} G_6$
$\mathrm{Sq}^{2,1,2} H_7$
$\mathrm{Sq}^{3,1} B_2$
$\mathrm{Sq}^3 C_4$
$\mathrm{Sq}^1 D_7$
$\mathrm{Sq}^1 E_8$
$\mathrm{Sq}^1 F_9$
$\mathrm{Sq}^1 G_{10}$
$\mathrm{Sq}^1 H_{11}$
$\mathrm{Sq}^{2,1} C_4$
6
$\mathrm{Sq}^{5,1} A_0$
$\mathrm{Sq}^{5,1} B_1$
$\mathrm{Sq}^{5,1} C_2$
$\mathrm{Sq}^{5,1} D_3$
$\mathrm{Sq}^{5,1} E_4$
$\mathrm{Sq}^{5,1} F_5$
$\mathrm{Sq}^{5,1} G_6$
$\mathrm{Sq}^{5,1} H_7$
$\mathrm{Sq}^{2,1,2} B_2$
$\mathrm{Sq}^{3,1} C_4$
$\mathrm{Sq}^2 D_7$
$\mathrm{Sq}^2 E_8$
$\mathrm{Sq}^2 F_9$
$\mathrm{Sq}^2 G_{10}$
$\mathrm{Sq}^2 H_{11}$
7
$\mathrm{Sq}^{5,1} B_2$
$\mathrm{Sq}^{2,1,2} C_4$
$\mathrm{Sq}^3 D_7$
$\mathrm{Sq}^3 E_8$
$\mathrm{Sq}^3 F_9$
$\mathrm{Sq}^3 G_{10}$
$\mathrm{Sq}^3 H_{11}$
$\mathrm{Sq}^{2,1} D_7$
$\mathrm{Sq}^{2,1} E_8$
$\mathrm{Sq}^{2,1} F_9$
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$\mathrm{Sq}^{2,1} H_{11}$
8
$\mathrm{Sq}^{5,1} C_4$
$\mathrm{Sq}^{3,1} D_7$
$\mathrm{Sq}^{3,1} E_8$
$\mathrm{Sq}^{3,1} F_9$
$\mathrm{Sq}^{3,1} G_{10}$
$\mathrm{Sq}^{3,1} H_{11}$
E_{12}
F_{13}
G_{14}
H_{15}
9
$\mathrm{Sq}^{2,1,2} D_7$
$\mathrm{Sq}^{2,1,2} E_8$
$\mathrm{Sq}^{2,1,2} F_9$
$\mathrm{Sq}^{2,1,2} G_{10}$
$\mathrm{Sq}^{2,1,2} H_{11}$
$\mathrm{Sq}^1 E_{12}$
$\mathrm{Sq}^1 F_{13}$
$\mathrm{Sq}^1 G_{14}$
$\mathrm{Sq}^1 H_{15}$
F_{14}
10
$\mathrm{Sq}^{5,1} D_7$
$\mathrm{Sq}^{5,1} E_8$
$\mathrm{Sq}^{5,1} F_9$
$\mathrm{Sq}^{5,1} G_{10}$
$\mathrm{Sq}^{5,1} H_{11}$
$\mathrm{Sq}^2 E_{12}$
$\mathrm{Sq}^2 F_{13}$
$\mathrm{Sq}^2 G_{14}$
$\mathrm{Sq}^2 H_{15}$
$\mathrm{Sq}^1 F_{14}$
G_{16}
11
$\mathrm{Sq}^{2,1} E_{12}$
$\mathrm{Sq}^{2,1} F_{13}$
$\mathrm{Sq}^{2,1} G_{14}$
$\mathrm{Sq}^{2,1} H_{15}$
$\mathrm{Sq}^3 E_{12}$
$\mathrm{Sq}^3 F_{13}$
$\mathrm{Sq}^3 G_{14}$
$\mathrm{Sq}^3 H_{15}$
$\mathrm{Sq}^2 F_{14}$
$\mathrm{Sq}^1 G_{16}$
12
$\mathrm{Sq}^{3,1} E_{12}$
$\mathrm{Sq}^{3,1} F_{13}$
$\mathrm{Sq}^{3,1} G_{14}$
$\mathrm{Sq}^{3,1} H_{15}$
$\mathrm{Sq}^{2,1} F_{14}$
$\mathrm{Sq}^2 G_{16}$
H_{19}
$\mathrm{Sq}^3 F_{14}$
13
$\mathrm{Sq}^{2,1,2} E_{12}$
$\mathrm{Sq}^{2,1,2} F_{13}$
$\mathrm{Sq}^{2,1,2} G_{14}$
$\mathrm{Sq}^{2,1,2} H_{15}$
$\mathrm{Sq}^{3,1} F_{14}$
$\mathrm{Sq}^3 G_{16}$
⋮
$\mathrm{Sq}^{2,1} G_{16}$
14
$\mathrm{Sq}^{5,1} E_{12}$
$\mathrm{Sq}^{5,1} F_{13}$
$\mathrm{Sq}^{5,1} G_{14}$
$\mathrm{Sq}^{5,1} H_{15}$
$\mathrm{Sq}^{2,1,2} F_{14}$
$\mathrm{Sq}^{3,1} G_{16}$
15
$\mathrm{Sq}^{5,1} F_{14}$
$\mathrm{Sq}^{2,1,2} G_{16}$
⋮

that the maps of the resolution, which have degree 0, are sloping downward to the left. These maps are displayed as arrows in the chart and are omitted for elements that map to 0. An arrow with two heads means that the element from which the arrow originates maps to the sum of the two elements.

The first step of the resolution is of course to take as M_0 a copy of $\mathcal{A}(1)$ with a generator A_0 of degree 0 which is mapped to the generator $\bar{1} \in \mathbb{Z}/2$. The generator A_0 originates the 8 elements of the first column, given by products with the elements of the basis B of $\mathcal{A}(1)$. In general, every time that we introduce a generator in the cell $(s, t-s)$, we get 8 elements that reside in the entries from $(s, t-s)$ to $(s, t-s+6)$. In the chart, we write generators in red.

The kernel of the map $M_0 \to \mathbb{Z}/2$ consists of the products $\mathrm{Sq}^I A_0$ for $\mathrm{Sq}^I \in B$ except for $\mathrm{Sq}^0 = 1$. Then, we have to construct M_1 so that it maps on this kernel: to start with, we need a generator B_1 of degree 1 mapping to $\mathrm{Sq}^1 A_0$ in the first column. This defines the map on all the elements $\mathrm{Sq}^i B_1$: for instance, we have that $\mathrm{Sq}^1 B_1$ maps to $\mathrm{Sq}^1 \mathrm{Sq}^1 A_0 = 0$ and $\mathrm{Sq}^2 B_1$ maps to $\mathrm{Sq}^{2,1} A_0$. Now one notes that no element is mapping to $\mathrm{Sq}^2 A_0$, therefore we need a generator of M_1 mapping to it. Since $\mathrm{Sq}^2 A_0$ has degree 2, so does the generator, which we denote with B_2. In general we respect the convention that the subscript of a generator indicates its degree. Now we can compute the map on the elements of M_1 originating from B_2 and we see that we do not need any other generator in this column, because we are already hitting the entire kernel of $M_0 \to \mathbb{Z}/2$.

Then we can continue the process to compute every column of the chart: since we want to obtain a minimal resolution, we introduce a new generator only when it is needed to map on the kernel of the previous map.

After few columns, some patterns emerge: namely, we have a generator in every entry of the row $t-s=0$, while there is only the generator B_2 for $s=1$ and only C_4 mapping to $\mathrm{Sq}^3 B_1 + \mathrm{Sq}^2 B_2$.

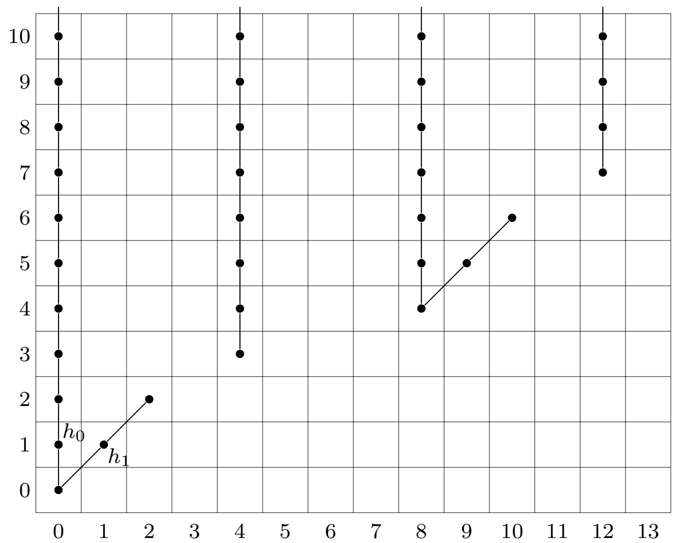
One can proceed row by row in the calculation, going from left to right in the row. In every cell $(s, t-s)$, to decide if a new generator is needed, we have to look at the map from $(s-1, t-s+1)$ to $(s-2, t-s+2)$: if it is not injective, then we have to introduce a new generator. As said, it does not matter if in the next row we have to introduce new generators in $(s-1, t-s+1)$, because generators map injectively, so they are never in a kernel.

We see that we need a generator D_7 in $s=3$, $t-s=4$, mapping to the sum $\mathrm{Sq}^{2,1,2} C_2 + \mathrm{Sq}^3 C_4$. This causes the existence of the elements $\mathrm{Sq}^{2,1,2} D_3$ and $\mathrm{Sq}^1 D_7$ in the cell below, that both map to $\mathrm{Sq}^{5,1} C_2$. Therefore we have to introduce a new generator in $s=4$, $t-s=4$, called E_8, to kill these two elements and so on for all the row $t-s=4$: hence this row presents an infinite tower beginning in $s=3$. We can see inductively for the following three rows that we do not need any new generator. When we come to $s=4$, $t-s=8$ we have to introduce a new generator E_{12} and, for a similar argument as before, we get a whole new tower on this row.

Now, when we come to the row below, $t-s=9$, we see that we are in the same situation that is present in $s=1$, $t-s=1$: the generators of the row $t-s=8$ play the role of there was of the ones in the row $t-s=0$: F_{14} maps to Sq^2 times the first generator in the tower. Hence the situation repeats and we get the same pattern of the rows $t-s=0,1,2$ in the rows $t-s=8,9,10$, shifted of 4 columns and 8 rows. Similarly, in $s=7$, $t-s=12$, we have the generator H_{19} that corresponds to D_7, shifted by the same number of rows and columns and so again the situation repeats.

Hence the resolution is composed by this pattern that repeats periodically, every time shifted of 4 columns to the right and 8 rows up.

By Proposition 3.4, since our resolution is minimal, we have the following pattern on the E_2 page of the (mod 2) spectral sequence for ${}_2\pi_*(ko)$:



Also in this diagram, the vertical lines indicate multiplication by h_0. The only differential that could be non-trivial is the one going from the column $8k+1$ to the column $8k$, for $k \in \mathbb{N}$. It is possible to see that it is zero, using again the fact that, in our hypothesis, differentials satisfy the derivation relation $d_r(xy) = xd_r(y) + d_r(x)y$ ([Rav03, Theorem 2.3.3]), as we have done for the case of the homotopy groups of spheres. To see that $d_r(h_1) = 0$, we can apply the derivation rule, which gives, since $h_0h_1 = 0$:

$$0 = d_r(h_0h_1) = h_0d_r(h_1) + d_r(h_0)h_1 = h_0d_r(h_1),$$

and so $d_r(h_1) = 0$, because multiplication by h_0 is an isomorphism between consecutive groups in the column $t-s=0$.

Hence, the diagram is also a picture of the E_∞ page.

As in the case of spheres, the vertical lines in the diagram represent multiplication by 2. In the E_∞ page, each dot in the i^{th} column represents a successive quotient isomorphic to $\mathbb{Z}/2$ in a filtration of ${}_2\pi_i(ko)$. The infinite towers of connected dots in the columns $t-s=4k$ correspond to the fact that one can keep multiplying by 2 without getting zero: since the homotopy groups of a spectrum of finite type are finitely generated, one can prove that these homotopy groups are isomorphic to $\mathbb{Z}$. For $t-s=8k+1$ or $t-s=8k+2$, we have only one quotient $\mathbb{Z}/2$ in the filtration of the homotopy group, which is then isomorphic to $\mathbb{Z}/2$. The absence of dots in the other columns implies that the corresponding homotopy groups are trivial. In conclusion, we get, for $n \geq 0$:

$$ {}_2\pi_n(ko) = \begin{cases} \mathbb{Z}, & \text{if } n \equiv 0 \pmod 4, \\ \mathbb{Z}/2, & \text{if } n \equiv 1 \text{ or } 2 \pmod 8, \\ 0, & \text{otherwise.} \end{cases} $$

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