Morse theory, Graphs, and Strings

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In these lectures I will describe algebraic topological aspects of Morse theory, and applications both to the topology of finite dimensional manifolds, and to certain infinite dimensional manifolds such as loop spaces. The following is a summary of each of the four lectures.

1. *The "flow category" of a Morse function.*

   The objects of this category are the critical points of a Morse function, and the morphisms are the compactified moduli spaces of gradient flow lines. We will discuss an old (but unpublished!) theorem of Cohen, Jones, and Segal that describes the topology of a compact manifold in terms of this category. We discuss this with an eye toward applying it to infinite dimensions.

2. *The moduli space of graph flows and cohomology operations: "Morse Field Theory"*

   We will describe joint work with P. Norbury that uses "gradient graphs in a manifold" to describe classical cohomology operations such as Poincare duality, Steenrod operations, and characteristic classes. We will draw analogies with constructions in Gromov-Witten theory. In particular we study the moduli space of metrics in the graphs, and produce a virtual fundamental class, using algebraic topological, rather than algebraic geometric methods. We will discuss the field theoretic properties of these constructions.

3. *String topology - a Morse theoretic viewpoint.* By replacing ordinary graphs with ribbon graphs, we show how Morse Field Theory constructions lead to a Morse theoretic description of the "String Topology" operations of Chas and Sullivan. We use moduli spaces of ribbon graphs to give a description of the these operations in terms of graph flows of the Dirichlet energy functional on the loop space of a manifold.

4. *Floer theory and the cotangent bundle.* We will give a quick description of Floer homology of a symplectic manifold, and recall the underlying homotopy theory as described in old work of Cohen, Jones, and Segal. We then discuss recent work regarding the cotangent bundle of a manifold, with its canonical symplectic structure. We prove that the "Floer homotopy type" of T^*M, is stably homotopy equivalent to the loop space, LM. We use this to describe a relationship between the string topology of M and the Gromov-Witten theory of T^*M.