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Erik K. Pedersen: Masterclass on Manifolds, I

Guiding theorem

Thm (Bauer-Kitchloo-Notbohm-Pedersen): B topological space, B ≃ CW-comp, B based. If $H_*(\Omega B;\mathbb{Z})$ is fg as an abelian group, then $\Omega B \simeq M$ for a closed, compact, smooth, paralleli- manifold.

Question: Which groups act freely on a sphere?

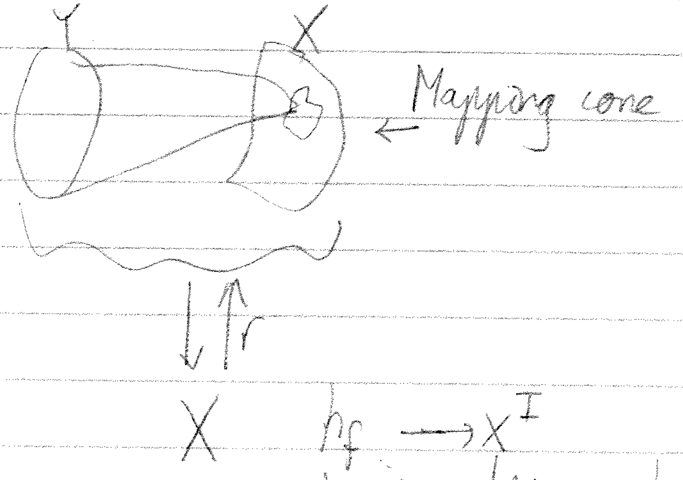
If π acts freely on S^n, then we get a manifold S^n/π.

Example:

$$\begin{tikzcd} S^1 \arrow[r] & S^\infty \arrow[r] & CP^\infty \\ \Omega CP^\infty \arrow[u, "\wr"] \arrow[r] & * \arrow[u] \arrow[ur] & \end{tikzcd}$$

homotopy fiber

Definition: Given a map $Y \xrightarrow{f} X$, construct



$Y \subseteq C_f$ cofibration

Dually

$$\begin{tikzcd} h_f \arrow[r] \arrow[d] & X^I \arrow[d, "\simeq"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

$h_f \to X$ fibration

Fact (Milnor): spaces equivalent to CW-complexes form a "convenient" category (closed under various constructions including Ω (loops))

Homology (old style)

Space X $X^{(0)} \subseteq X^{(1)} \subseteq \ldots \subseteq X$ skeleton

Free abelian groups generated by cells:

$$0 \leftarrow C_0 \leftarrow C_1 \leftarrow C_2 \leftarrow \ldots\ldots$$

Singular homology $S_i(X)$ = free abelian group on $\{\sigma : \Delta_i \to X\}$

Thm $C_*(X)$ is chain homotopy equivalent to $S_*(X)$.

One proof Use cellular approximation thm to approximate

$$|SX| \longrightarrow X \text{ by } \begin{tikzcd} {|SX|^{(n)}} \arrow[r] & X^{(n)} \arrow[l, bend left] \end{tikzcd} \qquad |SX| = \left(\amalg\, \sigma_i \times \Delta_i\right)/\!\sim$$

Def (Euler characteristic)

$$F_* \simeq C_*(X) \simeq S_*(X). \text{ Def } \chi = \sum_i (-1)^i \operatorname{rk}(C_i(X))$$

fg free ab gps

Def 1: X finitely dominated if there exists Y, finite complex with $X \underset{r}{\overset{i}{\rightleftarrows}} Y$, $r\circ i \simeq 1_X$

First obs Thm: $Z \underset{g}{\overset{f}{\rightleftarrows}} Y$, $T_{fg} \simeq T_{gf}$ (Mapping tori)
(Mather)

$$Y\times I/\sim \underset{f}{\overset{g}{\rightleftarrows}} Z\times I/\sim$$

So $S^1 \times X \simeq T_{ior}$. So X finitely dominated shows that $\pi_*(X)$ finitely presented.

Second obs $\tilde{X}$ universal cover, $\pi = \pi_1(X)$

$S_*(\tilde{X})$ is a chain complex of free $\mathbb{Z}\pi$-modules.

$$\begin{tikzcd} \Delta^i \arrow[r] \arrow[dr] & \tilde{X} \arrow[d] \\ & X \end{tikzcd}$$

$$S_*(X) = S_*(\tilde{X}) \otimes_{\mathbb{Z}\pi} \mathbb{Z}.$$

Def 2: X is finitely dominated if $\pi_1(X)$ is finitely presented and $S_*(\tilde{X}) \simeq P_*$, $\oplus P_i$ is a fg ~~free~~ projective $\mathbb{Z}\pi$-module.

These definitions are equivalent: Def 1 $\Rightarrow$ Def 2: Mahlers argument and let $\overline{Y}$ = covering of Y corresponding to $\pi_1(X)$

$S_*(\tilde{X}) \underset{r_*}{\overset{i_*}{\rightleftarrows}} S_*(\overline{Y}) \simeq C_*(\overline{Y})$. Now Ranicki gives a computational argum

$$S_*(\tilde{X}) \simeq \left(\cdots \to F_n \xrightarrow{p} F_n \xrightarrow{1-p} F_n \to \cdots \to F_{n-1} \to \cdots \to F_i \to \cdots \to F_0 \right)$$

where $p^2 = p$.

Now use that $F_n = \mathrm{Im}(p) \oplus \mathrm{Im}(1-p)$

$$\begin{tikzcd} \cdots \arrow[r] & F_n \arrow[r, "1-p"] \arrow[d, equal] & F_n \arrow[r, "p"] & F_n \\ & \mathrm{Im}(p) \arrow[r, "0"] & \mathrm{Im}(p) \arrow[r, "1"] & \mathrm{Im}(p) \\ & \oplus & \oplus & \oplus \\ & \mathrm{Im}(1-p) \arrow[r, "1"] & \mathrm{Im}(1-p) \arrow[r, "0"] & \mathrm{Im}(1-p) \end{tikzcd}$$

So $S_*(\tilde{X}) \simeq$ finite complex of fg projective $\mathbb{Z}\pi$-modules. Now use Eilenberg swindle to see definition 2.

Conversely, consider $K^{(2)} \longrightarrow X$. $C_*(\tilde{K}^{(2)}) \xrightarrow{\alpha} P_* \xrightarrow{\text{mapping cone}} C_\alpha$

In homology α induces iso in degs ≤ 1.

$$\begin{tikzcd} R_1 \oplus R_2 \arrow[d, "1"] & & C_1 \oplus P_2 \arrow[d] \arrow[dr] & \\ R_1 \oplus P_0 \arrow[d, "1"] & \simeq & C_0 \oplus P_1 \arrow[d] & 1,1 \\ P_0 & & P_0 & \end{tikzcd}$$

Crucial fact: R_2 is fg projective $\mathbb{Z}\pi$-module

<u>Ideally</u>:
$$\begin{tikzcd} K^{(2)} \arrow[r] & X \\ S^2 \arrow[u] \arrow[r, phantom, "\subseteq"] & D^3 \arrow[u] \end{tikzcd}$$

<u>Get</u> $K^{(2)} \vee (S^2)'s \longrightarrow X$

<u>Problem</u>: Hurewicz theorem for pairs works only in degree ≥ 3.

$$\begin{tikzcd} \pi_2(\tilde{X}) \arrow[d, equal, "\sim"] & & \\ H_2(\tilde{X}) \arrow[r, two heads] & H_2(\tilde{X}, \tilde{K}^{(2)}) \arrow[r, phantom, "\cong"] & R_2/\text{relations}. \end{tikzcd}$$

End up with $\tilde{K} \to \tilde{X}$ with $H_i(\tilde{X},\tilde{K}) = 0$ for $i \leq 2$
(slightly easier argument than Walls ~~orig~~ original)

$C_*(\tilde{K}) \to P_*$, $H_3(\tilde{X},\tilde{K})$ fg $\mathbb{Z}\pi$-module

$$\begin{tikzcd} H_3(\tilde{X},\tilde{K}) \arrow[d, equal] \\ \pi_3(\tilde{X},\tilde{K}) = \pi_3(X,K) \end{tikzcd}$$

$$\begin{tikzcd} K \arrow[r] & X \\ S^2 \arrow[u] \arrow[r] & D^3 \arrow[u] \end{tikzcd}$$

Now continue. Since $P_* = (0 \to 0 \to \cdots \to P_n \to \cdots \to P_0 \to 0)$ we end up with

$$H_*(\tilde{X},\tilde{K}) = \begin{cases} 0 & i \neq n \\ P_* & i = n \end{cases}, \; P \text{ projective}.$$

$$\begin{tikzcd} P_* \\ \oplus\mathbb{Z}\pi \arrow[u] \end{tikzcd}$$

A

Consequence 1: Continuing we end up with $X \simeq \overset{\text{infinite complex}}{K}$, K has finitely many cells in each dimension.

Eilenberg swindle: $\underbrace{P \oplus Q}_{F} \oplus P \oplus \cdots$

Consequence 2: So ~~instead we~~ ~~we~~ can the wedge on infinitely many spheres to get a complex with cells in only finitely many dims and only infinitely many in two adjacent dimension

Consequence 3: $K \longrightarrow X$

$K\times S^1 \to X\times S^1$ in relative homology

$$\begin{tikzcd} P\otimes\mathbb{Z}[t,t^{-1}] \arrow[d, "1-t"] & \oplus & Q\otimes\mathbb{Z}[t,t^{-1}] \arrow[d, "1"] \\ P\otimes\mathbb{Z}[t,t^{-1}] & \oplus & Q\otimes\mathbb{Z}[t,t^{-1}] \end{tikzcd}$$

free

$X\times S^1 \simeq$ finite complex, So X is finitely dominated.

$C_*(\widetilde{X\times S^1}, \widetilde{K\times S^1})$

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X, $\pi=\pi_1(X)$ finitely presented

$S_*(\widetilde{X}) \simeq P_* = \{0 \to P_n \to \dots \to P_0 \to 0\}$ projective $\mathbb{Z}\pi$-modules

$K \to X$. Assume $H_i(\widetilde{X},\widetilde{K}) = \begin{cases} 0 & i<k \\ \neq 0 & i=k \end{cases}$

$H_k(\widetilde{X},\widetilde{K}) = \pi_k(\widetilde{X},\widetilde{K}) = \pi_k(X,K)$ represented by $(D^k, S^{k-1}) \to (X,K)$

$L = K \cup e^n$

$$0 \to H_{k+1}(\widetilde{X},\widetilde{K}) \to H_{k+1}(\widetilde{X},\widetilde{L}) \to (\mathbb{Z}\pi)^2 \to H_k(\widetilde{X},\widetilde{K}) \to H_k(\widetilde{X},\widetilde{L}) \to 0$$

↑ can make this surjective since $H_k(\widetilde{X},\widetilde{K})$ is a fg $\mathbb{Z}\pi$-module.

Argument: $C_*(\widetilde{K}) \to P_* \to D_*$

$$\begin{tikzcd} D_* : & D_2 \arrow[d] & & D_2 \arrow[d] & \\ & D_1 \arrow[d] & \cong & R_1 \oplus D_0 \arrow[d, "0"'] \arrow[r, phantom] & \\ & D_0 \arrow[d] & & D_0 \arrow[d] & \\ & 0 & & 0 & \end{tikzcd}$$

$R_1 \oplus D_0 \xrightarrow{\cong} D_0$

Eventually get to a situation where

$H_*(\widetilde{X},\widetilde{K}) = \begin{cases} P & *=n \quad (P \text{ fg } ~~\text{projective}~~) \\ 0 & * \neq n \end{cases}$

If P is free then $X \simeq$ finite complex. Otherwise we can keep on going or use Eilenberg swindle. (Note that for $\pi=1$ P is free).

$H_*(\widetilde{X \times S^1}, \widetilde{K \times S^1}) = H_*(\widetilde{X},\widetilde{K}) = P$

$0 \to \mathbb{Z}[t,t^{-1}] \xrightarrow{1-t} \mathbb{Z}[t,t^{-1}] \to \mathbb{Z} \to 0$

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$$\begin{tikzcd} 0 \arrow[r] & P\otimes\mathbb{Z}[t,t^{-1}] \arrow[r, "1-t"] & P\otimes\mathbb{Z}[t,t^{-1}] \arrow[r] & P \\ & \oplus & \oplus & \\ & Q\otimes\mathbb{Z}[t,t^{-1}] \arrow[r, "1"] & Q\otimes\mathbb{Z}[t,t^{-1}] & \end{tikzcd}$$

$\mathbb{Z}(\pi\times\mathbb{Z})$- modules.

If P is stably free (ie. $P\oplus(\mathbb{Z}\pi)^k$ is free) then also ok.

Given a ring R, consider the set of isomorphism classes of fg projective R-modules. Can define $[P]\oplus[Q]=[P\oplus Q]$. Grothendi construction gives Grothendieck group $K_0(R)$. Example : $K_0(\mathbb{Z})$

Consider $\sum(-1)^i[C_i(K)]\in K_0(\mathbb{Z})\cong\mathbb{Z}$.

Observe that $[P]-[Q]=0$ in $K_0(R) \Longleftrightarrow \exists F$ free fg such th
$P\oplus F\cong Q\oplus F$

$R=\mathbb{Z}\pi$: Have homomorphisms $\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[rr, bend left] & \mathbb{Z}\pi \arrow[r] & \mathbb{Z} \end{tikzcd}$.

$R\to S$ induces

$K_0(R)\longrightarrow K_0(S)$

$P\longmapsto P\otimes_R S$

$R\longmapsto S$

$$\begin{tikzcd} K_0(\mathbb{Z}) \arrow[r] \arrow[d, phantom, "\parallel"] & K_0(\mathbb{Z}\pi) \arrow[r] \arrow[d, phantom, "\cong"] & K_0(\mathbb{Z}) \arrow[d, phantom, "\parallel"] \\ \mathbb{Z} & \widetilde{K_0}(\mathbb{Z}\pi)\oplus\mathbb{Z} & \mathbb{Z} \end{tikzcd}$$

Now, go back to original problem. X finitely dominated,

$X \underset{r}{\overset{i}{\rightleftarrows}} Y$. Then $X\times S^1\simeq$ finite complex

$S_*(\tilde{X})\simeq P_*$, Consider $\sum_i(-1)^i[P_i]\in K_0(\mathbb{Z}\pi)$

Since $S_*(\tilde{X}) \simeq P_*$ we have $S_*(X) \simeq S_*(\tilde{X}) \otimes_{\mathbb{Z}\pi} \mathbb{Z} \simeq P_* \otimes_{\mathbb{Z}\pi} \mathbb{Z}$

and under $K_0(\mathbb{Z}\pi) \to \mathbb{Z}$, $\sum (-1)^i [P_i]$ maps to χ (Euler char)

If $S_*(\tilde{X}) \simeq P_* \simeq Q_*$ then $\sum_i (-1)^i [P_i] = \sum_i (-1)^i [Q_i]$.

$\sum (-1)^i [P_i] = \chi(X) + \sigma(X)$ under iso $K_0(\mathbb{Z}\pi) \cong K_0(\mathbb{Z}) \oplus \tilde{K}_0(\mathbb{Z}$

<u>Thm</u>: $X \simeq$ finite complex $\Leftrightarrow \sigma(X) = 0$ in $\tilde{K}_0(\mathbb{Z}\pi)$.

$\Rightarrow S_*(\tilde{X}) \simeq S_*(\tilde{K}) \simeq C_*(\tilde{K})$ complex of fg free $\mathbb{Z}\pi$-modules.

$\Leftarrow$: Recall from proofs that we have $K \to X$ and
$C_*(\tilde{X}) \xrightarrow{f} P_* \longrightarrow D_*$ (mapping cone of f)

Clearly $\sigma(P_*) = \sigma(D_*)$

Eventually $D_* \simeq (\ldots 0 \to P \to 0 \ldots)$ so $\sigma(D_*) = [P]$. Hence

$0 = \sigma(X) = \sigma(P_*) = \sigma(D_*) = [P]$, so P is stably free and we are do

<u>Recall main thm</u>: If $H_*(\Omega B; \mathbb{Z})$ is fg abelian gp then $\Omega B \simeq$ ne con

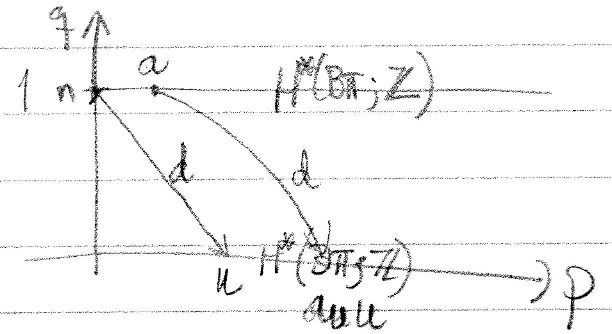
<u>First step</u>: ΩB finitely dominated (due to Mislin).
ΩB is a simple space. Mislin proves: X simple + $\bigoplus_i H_i(X; \mathbb{Z}$
is fg $\Rightarrow X$ finitely domin

Other motivating question: Which groups act freely (orientably
on a sphere?

$$\begin{tikzcd} & S^n/\pi \arrow[r] & K(\pi,1)=B\pi \\ S^n \arrow[ur] & & \end{tikzcd}$$

Serre spectral sequence in cohomology has

$$E_2^{p,q} = H^p(B\pi; H^q(S^n;\mathbb{Z})) \Rightarrow H^{p+q}(S^n/\pi;\mathbb{Z})$$



So there exists $u \in H^n(B\pi;\mathbb{Z})$ such that $-\cup u$ is an isom
phism, ie π has periodic cohomology. This element
corresponds to sequence

$$0 \to \mathbb{Z} \to P_n \to \dots\dots \to P_2 \to P_1 \to P_0 \to \mathbb{Z}$$

By same procedure as before we get Swan complex X wi
$S_*(X) \simeq P_*$.

If π acts freely on S^n, then $\mathbb{Z}/p \times \mathbb{Z}/p \notin \pi$ since
$H^*(\mathbb{Z}/p \times \mathbb{Z}/p; \mathbb{F}_p)$ grows too quickly.

Milnor: D_{2p} does not act freely on a sphere. (elements of
order 2 hav
to centr

Thm (Hambleton+Pedersen) $G=\mathbb{Z}^k \rtimes_\alpha D_{2p}$, $\alpha : D_{2p} \to GL_k(\mathbb{Z})$ an inte
rep.

acts freely cocompactly on $S^n \times \mathbb{R}^l$ $\Longleftrightarrow$
$\nwarrow$ ie $(S^n \times \mathbb{R}^l)/G$ is compact

$\left\{ \begin{array}{l} n \equiv 3 \pmod{4},\ k = l \text{ ans} \\ \text{and } \alpha : D_{2p} \to GL_k(\mathbb{Z}) \to GL_k(\mathbb{R}) \text{ has at least } 2\ \mathbb{R}\text{- summa} \end{array} \right.$

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Thm If X is finitely dominated, then $X \simeq$ finite complex $\Longleftrightarrow \sigma(X) \in \widetilde{K}_0(\mathbb{Z}\pi_1 X)$ vanishes

Note: Any element in $\widetilde{K}_0(\mathbb{Z}\pi_1X)$ can be realized as $\sigma(X)$:
Moreover $\widetilde{K}_0(\mathbb{Z}\pi_1X)$ can be nonzero.

Take $[P] \in \widetilde{K}_0(\mathbb{Z}\pi$
π finitel
presen

$0 \to F \to F \to P \to 0$

What elements in $\widetilde{K}_0(\mathbb{Z}\pi_1X)$ can be realized:

Nilpotent spaces : some subgroup
Simple spaces : ? (Erik conjecture: image of Swan homomorphism)
H-spaces : ? (Conjecture: 0)
Ω-spaces : 0
compact manifolds : 0 (uses Kirby-Siebermann)
$K(\pi,1)$'s : ?? ($K(\pi,1)$ finitely dominated $\Rightarrow \pi$ torsionfree

Conj $\widetilde{K}_0(\mathbb{Z}\pi) = 0$ for π torsion free.

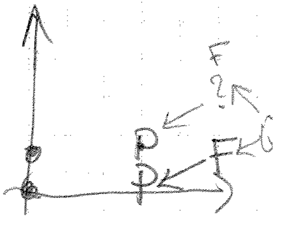
Loop space case: Say we have constructed $S^1 \to X \to Y$

finitely dominated (under X); finitely dominated (under Y)

$\pi_1(S^1) \xrightarrow{0} \pi_1(X)$

$$\begin{tikzcd} S^1 \arrow[r, equal] \arrow[d] & S^1 \arrow[d] \\ X \arrow[d] & L \arrow[l] \arrow[d] \\ Y & K \arrow[l] \arrow[u, dashed] \end{tikzcd} \qquad \begin{tikzcd} S^1 \arrow[d] \\ (\widetilde{X}, \widetilde{L}) \arrow[d] \\ (Y, K) \end{tikzcd}$$

Serre SS:



<u>Conclusion</u> : $Y \simeq$ finite complex $\Longrightarrow X \simeq$ finite complex

<u>Taylor & Pedersen</u> : Formula relating finiteness obstructions of E and B in fibration $S^1 \to E \to B$.

<u>Notbohm, Pedersen</u> : Construct S^1-fibration as above for any loop space

Conclusion $\sigma(X) = 0$ for X loop space.

<u>Cartan (30's ?)</u> $H^*(G;\mathbb{Q})$ exterior for Lie groups G.

Hopf observed that the situation:

$$\begin{tikzcd}
G \times G \arrow[r, "m"] & G \\
G \times e \cup e \times G \arrow[u, hook] \arrow[d, equal] & \\
G \vee G \arrow[uur, "F"'] &
\end{tikzcd}$$

ie an H-space always has $H^*(X;\mathbb{Q})$ exterior. when $H^*(X;\mathbb{Q})$ is finite-dimensio

In particular finitedim H-spaces have Poincare duality.

Now let X be a loop space (or H-space for some of the arguments)

$$\begin{tikzcd}
X = \Omega B & \pi_1(X) = \pi_2(B) = \mathbb{Z} \oplus \rho \arrow[d, equal] & \\
& H_2(B) \arrow[r] & \mathbb{Z} \\
& & H_2(B) \arrow[u, phantom, "\cap"]
\end{tikzcd}$$

$$\begin{tikzcd}
B' \arrow[dr] & & \\
& B \arrow[r] & K(\mathbb{Z},2)
\end{tikzcd}$$

$S^1 \to \Omega B$, $\Omega B' \to \Omega B$ $S^1 \times \Omega B' \to \Omega B \times \Omega B \to \Omega B$ induces ISO on π_* so equivalen

So $\Omega B'$ finitely dominated if ΩB finitely dominated

This works for H-spaces as well.

Borel gives structure theorem for fg Hopf algebras / $\mathbb{F}_p$:

polynomial $\otimes$ truncated polynomial algebras $\otimes$ exterior algebras

<u>For $\pi_1(X)=0$</u>

Top dimensional classes have dimensions n_0 for $\mathbb{Q}$-coefficients
n_p for $\mathbb{F}_p$

So $n_p \geqslant n_0$ since $H^{n_0}(X;\mathbb{Z}) = \mathbb{Z} \oplus$ stuff

Since $H^1(X;\mathbb{F}_p)=0$ we have $H^{n_p-1}(X;\mathbb{F}_p)=0$. Thus $H^{n_p}(X;\mathbb{Z}) = \begin{cases} \mathbb{Z}/p^k \oplus \text{other} \\ \mathbb{Z}\oplus \cdots \end{cases}$

First possibility excluded since $H^{n_p-1}(X;\mathbb{F}_p)=0$. So $H^{n_p}(X;\mathbb{Z}) = \mathbb{Z}\oplus$ stuff

so $n_p \leqslant n_0$. Thus $n_0 = n_p$.

So lets set $n = n_0 = n_p$. Thus $H^n(X;\mathbb{Z}) = \mathbb{Z}$.

<u>Browder looked at</u> $\tilde{X} \searrow X$: using Hopf algebras techniques $n_p(\tilde{X}) \leqslant n_p(X)$

So any H-space has Poincare duality (also in modern terminology, taking into account π_1-structure). $H_n(X;\mathbb{Z}) \cong \mathbb{Z}$ generated by $[X]$ and

$[X] \cap — : H^i(X;\mathbb{Z}) \xrightarrow{\cong} H_{n-i}(X;\mathbb{Z})$

Assume now that we know that $o(X)=0$ ie $X \simeq$ finite complex

Then $X \hookrightarrow \mathbb{R}^n$ as subcomplex. Take second subdivision.

Take N = union of simplices touching X.

Then $X \subseteq N \subseteq \mathbb{R}^n$ and $X \xrightarrow[\simeq]{\subseteq} N$ plays role of normal bundle. Now $\pi_1(\partial N) \xrightarrow{\cong} \pi_1(N)$. ~~Spivak normal fibration~~ Map $\partial N \longrightarrow N \xrightarrow{\simeq} X$. ~~Fiber~~ Homotopy fiber is sphere.

Spivak normal fibration
$$\begin{tikzcd} S^k \arrow[r] & \partial N \arrow[d] \\ & X \end{tikzcd}$$
.

$S^n = \mathbb{R}^n \cup \{*\} \longrightarrow N/\partial N$. Then $N/\partial N \simeq \bar{E}/E$

Suppose
$$\begin{tikzcd} & F \arrow[r] & S^m \arrow[d] & & S^m \arrow[d] & D^{m+1} \arrow[d] \\ N \arrow[drr] & \supseteq & \partial N \arrow[d] & \simeq & E \arrow[dll] \arrow[r, hook] & \bar{E} \arrow[dlll] \\ & & X \end{tikzcd}$$

In our case we can tak $E = X \times S^m$, $\bar{E} = X \times D^m{}^{+1}$. So

$$\begin{tikzcd} {} \arrow[r] \arrow[d, equal] & N/\partial N \simeq X\times D^{m+1} / X \times S^m \\ {} \arrow[r, "\cancel{M}"] & X \times \mathring{D}^{m+1} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r] & \mathring{D}^{m+1} \end{tikzcd}$$
, U smooth manifold. Can approximate lower map by smooth map. We will actually aproximate upper map. Take regular value $p \in \mathring{D}^{m+1}$. Take premages

$$\begin{tikzcd} U \arrow[r] & X \times \mathring{D}^{m+1} \arrow[r] & \mathring{D}^{m+1} \\ M \arrow[u, hook] \arrow[r] & X \times p \arrow[u, hook] \arrow[r] & p \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

Thom isomorphism shows that $[M] \longmapsto [X]$.

Get $M \longrightarrow X$ with

$$\begin{tikzcd}
\nu_M \arrow[r] \arrow[d] & X \times \mathbb{R}^{m+1} \arrow[d] \\
M \arrow[r] & X
\end{tikzcd}$$

$$\begin{tikzcd}
{[M]} \arrow[d, phantom, ""] \arrow[rr, bend right, mapsto] & \in H_k(M^k; \mathbb{Z}) \arrow[r] & H_k(X; \mathbb{Z}) \arrow[d, phantom, "\Psi"] \\
& & {[X]}
\end{tikzcd}$$

ie $M \to X$ is an iso in top dimension.

$M \subseteq \mathbb{R}^n$. $D(\nu_M)/S(\nu_M) = TM$ (Thom space)

$K \subseteq S^n \xrightarrow{\text{dual}} S^n - K$

$M_+ \xrightarrow{\text{dual}} TM$

$D(X_+) = TX = N/\partial N$

Can build model for X with only one top cell $X \simeq L^{(k-1)} \cup e^k$

$$\begin{tikzcd}
X \simeq L^{(k-1)} \cup e^k \arrow[d] \\
S^k
\end{tikzcd}$$

$X_+ \wedge X_+ \longrightarrow X \times X \longrightarrow X \simeq L^{(k-1)} \cup e^k \longrightarrow S^k$

Stable homotopy theory shows that this is a dualizing map.

So the spherical fibration $\begin{tikzcd} S^m \arrow[r] & \partial N \arrow[d] \\ & X \end{tikzcd}$ is trivial: $D(X_+) \simeq X_+$

$\underset{N/\partial N}{\wr\wr}$. So $N/\partial N \simeq \ell(X_+)$

$X \times B^{\ell} \subseteq \Sigma$

$\wr$

$N/\partial N$

$\uparrow$

S

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Setup $X^k \simeq \Omega B$ $H_*(X;\mathbb{Z})$ fg abelian group

or H-space with $\sigma(X)=0$.

X^k is a finite complex, $X \subseteq \mathbb{R}^n$ $X \subseteq N \subseteq \mathbb{R}^n$

$D(X_+) = N/\partial N$ $X_+ \wedge X_+ = X \times X_+ \longrightarrow X \times X \to X \to S^k$

$^n \to N/\partial N \simeq \Sigma^d X_+$ $d=n-k$

$\|$ ie Spivak normal bundle is trivial

S^n

$\|$

$(X \times \mathbb{R}^d)_+$

$U \longrightarrow X \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$

$M^k \longrightarrow X \times p \longrightarrow p$

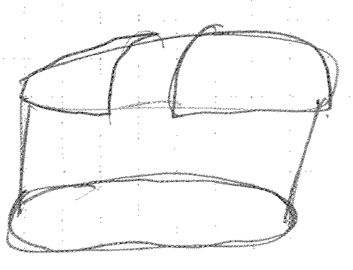
$[M] \longrightarrow [X]$

Now we do surgery. $S^i \times D^{k-i} \subset M^k$

$$M' = (M - S^i \times D^{k-i}) \cup (D^{i+1} \times S^{k-i-1})$$

Another way to think of this is as follows

$$M' \simeq (M \times I) \cup_{S^i \times D^{k-i}} (D^{i+1} \times D^{k-i})$$



<u>Thm (Smale)</u> Any bordism is a sequence of these.
Proof: Using Morse theory

$M \longrightarrow X$ $\pi_*(X, M)$ measures the deviation from $M \to X$ being a homotopy equivalence.

Homotopy theoretically

$$\begin{tikzcd} M \arrow[r] & X \\ S^i \arrow[u] \arrow[r, hook] & D^{i+1} \arrow[u] \end{tikzcd}$$

By insisting that

$$\begin{tikzcd} \nu_M \arrow[r] \arrow[d] & \xi \arrow[d] \\ M \arrow[r] & X \end{tikzcd}$$

commutes for some bundles ν_M, ξ the normal bundle is trivialized.

Thm (using ~~immersion theory~~ Walls idea): There is a unique regular homotopy class preserving the bundles..... Want the bordism $M \sim M'$ covered by bundle map.

Thm $M^k \longrightarrow N^k$
open
no compact components

$$\begin{tikzcd} \left\{ \text{Space of immersions } M^k \xrightarrow{f} N^k \right\} \arrow[d, "\simeq"] \\ \left\{ \text{Space of maps } \left\{ \begin{matrix} \xi_M \xrightarrow{\cong} \xi_N \\ \downarrow \quad\quad \downarrow \\ M^k \xrightarrow{f} N^k \end{matrix} \right\} \right\} \end{tikzcd}$$

homotopy equivalence

As a consequence (also using Poincare duality) we can make surgery to get $M' \longrightarrow X^k$ $\left[\frac{k}{2}\right]$-connected.

$$\begin{tikzcd} M' \arrow[r] & X^k \\ M \arrow[u, squiggly] \arrow[ur] & \\ \nu_M \arrow[r] & \xi \arrow[uu, squiggly] \end{tikzcd}$$

The ~~[illegible]~~ ~~[illegible]~~ dimension $\simeq k/2$ leads to an obstruction living in quadratic forms over ~~[illegible]~~ $(\mathbb{Z}/k)$.

<u>Trick (Walls π-π theorem)</u>: Assume $\pi_1(\partial X) \cong \pi_1(X)$.

$$\begin{tikzcd} \nu_M \arrow[rr] \arrow[d] & & \xi \arrow[d] \\ (M, \partial M) \arrow[rr] & & (X, \partial X) \end{tikzcd}$$

Then there is no obstruction and we get a homotopy equivalence of pairs after surgery.

Tilmans thesis is used to get us into this situation for loop spaces.

So for H-spaces there are two problems: finiteness obstruction
surgery obstruction.

<u>Loop spaces</u>: How do we get to the π-π situation?

$S^1 \to X \to Y$ $\quad$ Y finite complex, $\pi_1(X) \cong \pi_1(Y)$

<u>Thm (Quinn)</u>: Y will automatically have Poincare duality.

$$\begin{tikzcd} S^1 \arrow[r, phantom, "\subset"] \arrow[d] & D^2 \arrow[d] \\ X \arrow[r, hook] \arrow[dr] & E \arrow[d] \\ & Y \end{tikzcd}$$

If we can find a degree 1 normal map to Y then we can pull back and get into the π-π situation for the pair (E, X).

p-compact group theory + Tilmans ~~[illegible]~~ selfduality results constructs Y.

This shows that X is stably parallellizable. Using result of Dupont we get parallellizability. Obstruction $= 0$ by result on structure of cohomology

Actually the above is an oversimplification.
The Y constructed is not known to be a finite complex.

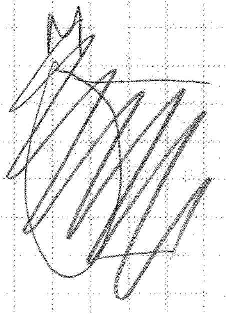
Actually need ~~[illegible]~~ double cover $Y \to Z$ with $\pi_1(Y) \to \pi_1(Z)$ split and Pedersen-Ranicki generalization of π-π theorem.

So now we can manipulate E to get a finite complex, and the above goes through.

Surgery below middle dimension.

$$\begin{tikzcd} \nu_M \arrow[r] \arrow[d] & \xi \arrow[d] \\ M \arrow[r, hook] & X \end{tikzcd}$$

inclusion (replace X by mapping cylinder)



Filter $M \hookrightarrow X$ as

$M \hookrightarrow M \cup X^{(0)} \hookrightarrow M \cup X^{(1)} \hookrightarrow \dots$

Use general position and surgery until this breaks down on middle dimension.