$X$ is a finite complex

$X$ satisfies Poincaré duality.

$x \in \mathbb{R}^n$  \quad x \in \mathbb{C} \mathbb{R}^n$

Regular $M^{n+1}$

Dual of $X$ : $D(x) = \frac{1}{N}$

If $X$ is a space, then $x = (x \times x)$, $x \times x \to x$.

$\Rightarrow x = D(x)$

$\Rightarrow \sum x = \frac{1}{N}$

(for $n$ big enough)

($d = n - k$)

$S^n \to \frac{1}{N}$

$U \to x \times \mathbb{R}^d \to \mathbb{R}^d$

$U$ can assume this.

Map is smooth

$X \times p \to \mathbb{P}$ regular value

Manifold

$[M] \to [X]$ in $H_k$
\[ S^i \times D^{2-i} \subset M^2 \]

**Subject:**

\[ M = (S^i \times D^{2-i}) \cup (D^{i+1} \times S^{2-i}) \]

want to use this process to change \( M \) to get a homotopy equivalence to \( X \).

\[ M \xrightarrow{\sim} \text{cobordism} \]

**Attaching a Handle:**

\[ M \times I \cup \left( S^i \times D^{2-i} \right) \cup \left( D^{i+1} \times D \right) \]

**In homotopy view point:** We are just attaching a cell: \( M \times I \cong M \), \( S^i \times D^{2-i} \cong S^i \).

so that we change \( M \) by a cobordism.

(Every cobordism is a sequence of surgeries — via Morse theory)

\[ M \longrightarrow X \]

Construction to hom equiv is \( \mathbb{H}_*(X,M) \).

If \( i \leq \frac{d-1}{2} \), can assume \( S^i \subset M \) embedding.
\[ V^m \to X \times \mathbb{R}^n \quad \text{BDL map} \]
\[ M \to X \]
Gives a trivialization of normal BDL of \( S^i \subset M \)

\[ \to S^i \times D^{2-i} \subset M \]

(We are using immersion theorem. Immersions are determined by bundle information)

If \( S^i \times D^{2-i} \to M \) is covered by a good BDL map, we can change the map to an immersion.

**Ex.**

![Diagram](image)

Obvious surgery

There is an obstruction in dim 2:

\[ V^m \to V^2 \]

BDL data gives the immersion of \( S^i \)

\[ \to \text{might not be able to do surgeries} \]

Would need to "retrace the pb" - change the BDL - to be able to do surgery
\[ \text{Invariance of } \pi_1 \]  
\[ \downarrow \] 
\[ J^M \rightarrow J^N \]  
\[ M^k \rightarrow N^k \]  

**Exact Map Covered by a BDL Map is Homotopic to an Imbedding.**

\[ M^k \rightarrow X^k \]

**Suppose**

\[ \frac{\pi_0}{\mathbb{Z}} \text{ connected} \]

By PD if \& if even \exists one non-trivial relative \( \pi_0 \)-GP (on universal cover).

If \( k \) is odd \& \( 2 \) non-trivial left.

\[ \Rightarrow \text{ surgery obstruction} \]

(Attaching Cells Rather Than Handles Makes Us Stay in Manifolds)


\[ \text{Men's \( \pi_1 \) Theorem: } (M, \partial M) \rightarrow (X, \partial X) \]

AND

PD pair

If \( \partial X \cong \pi X \) then can always do surgery:

\[ \text{cobordant to } (M, \partial M) \]

\[ (\text{note, } \partial X \neq \emptyset) \]
For loop space, Timm's thesis discusses the dual of spacelike. Can use the TIT theorem for self-dual spaces.

(TIT)-Theorem: Can get rid of self-interpenetrations in middle dimension, might have some double points. We want to get rid of.

Example of surgery with boundaries:

Need dim > 5 because need to embed a 2-disc.

Ideal world:

\[ \mathbb{S}^1 \text{ loop space} \quad \mathbb{Y} \text{ finite CP}^X \quad \mathbb{X}, \mathbb{X} \cong \mathbb{Y} \]

\[ \mathbb{S}^1 \to \mathbb{X} \]

\[ \downarrow \]

\[ \mathbb{Y} \]

\[ G_2 = E(2) \to \mathbb{S}^1 \text{-fiberation is } \mathbb{S}^1 \text{-bundle.} \]
\[ S^1 \rightarrow D^2 \]
\[ L \]
\[ \chi \leq E \]
\[ \therefore \]
\[ \pi_1(X) \cong \pi_1(E) \]

\[ \rightarrow \text{APPLY } (\pi_1, \pi_1) \rightarrow \text{THM} \]
\[ \text{TO } (E, X) \]

**Thom's Thesis:** \( Q \) is 

- **Set-up:** A surgery problem on \( Q : M \to Q \)
- **Pull it back to** \( (E, X) \)

\[ W \to B\mathcal{R} Y \]  
\[ W \to X \times \mathbb{R}^n \]
\[ M' \simeq X \]

\[ \rightarrow \text{NORMAL BOL SPLIT PARAMELIZABLE} \]

**Reality:** Don't know \( Q \) is finite, but know is \( \text{finitely dominated} \).

**Use a generalization of** \((\pi_1, \pi_1)\) **- Theorem**

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**Surgery below the middle dimension**

\[ V_M \to Z \]
\[ \to X \]

\[ M \to X \]

Make this into an inclusion by replacing \( X \) by the mapping cylinder.
\[ m \times i \]

**RING OF**

\[ x^{(0)} \text{ NOT IN M} \]

\[ \text{So can miss MUX}^{(0)} \]

\[ M \rightarrow \text{MUX}^{(0)} \rightarrow \text{MUX}^{(1)} \rightarrow \text{MUX}^{(2)} \rightarrow \]

\[ N^{(0)} \rightarrow N^{(1)} \]

\[ M \enspace \sim \enspace M' \enspace \sim \enspace M'' \]

\[ N \sim M' \cup \text{CEUS ABOVE MID-DIMENSION} \]

\[ \text{IS} \]

\[ \text{MUX}^{(2)} \rightarrow X \]

\[ X = M' \cup \text{CEUS ABOVE MID-DIMENSION} \]

\[ \rightarrow M' \rightarrow X \text{ HIGHLY CONNECTED} \]