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Algebraic K-theory and trace invariants

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(*Dedicated to Ib Madsen on his sixtieth birthday*)

1. Algebraic K-theory

The algebraic K-theory of Quillen [30], inherently, is a multiplicative theory. Trace invariants allow the study of this theory by embedding it in an additive theory. It is possible, by this approach, to evaluate the K-theory (with coefficients) of henselian discrete valuation fields of mixed characteristic. We first recall the expected value of the K-groups of a field k.

The groups $K_*(k)$ form a connected anti-commutative graded ring, there is a canonical isomorphism $\ell\colon k^* \xrightarrow{\sim} K_1(k)$, and $\ell(x) \cdot \ell(1-x) = 0$. One defines the Milnor K-groups $K_*^M(k)$ to be the universal example of this algebraic structure [29]. The canonical map $K_q^M(k) \to K_q(k)$ is an isomorphism, if $q \leq 2$. Let us now fix the attention on the K-groups with finite coefficients. (The rational K-groups, while of great interest, are of a rather different nature [11, 12].) The groups $K_*(k, \mathbb{Z}/m)$ form an anti-commutative graded $\mathbb{Z}/m$-algebra, at least if $v_2(m) \neq 1, 2$ and $v_3(m) \neq 1$. And if $\mu_m \subset k$, there is a canonical lifting

$$\begin{array}{ccc} & & K_2(k, \mathbb{Z}/m) \\ & \overset{b}{\nearrow} & \downarrow{\scriptstyle \beta_m} \\ \mu_m & \xrightarrow{\ \ell\ } & K_1(k), \end{array}$$

which to a primitive mth root of unity ζ associates the Bott element b_ζ. Hence, in this case, there is an additional map of graded rings $S_{\mathbb{Z}/m}(\mu_m) \to K_*(k, \mathbb{Z}/m)$. The Beilinson-Lichtenbaum conjectures predict that the combined map

$$K_*^M(k) \otimes_{\mathbb{Z}} S_{\mathbb{Z}/m}(\mu_m) \to K_*(k, \mathbb{Z}/m)$$

is an isomorphism of graded rings [1, 26]. The case $m = 2^v$ follows from the celebrated proof of the Milnor conjecture by Voevodsky [34]. We here consider the case of a henselian discrete valuation field of mixed characteristic $(0, p)$ with p odd and $m = p^v$ [20, 14]. The groups $K_*^M(k)/m$ typically are non-zero in only finitely many degrees. Hence, above this range, the groups $K_*(k, \mathbb{Z}/m)$ are two-periodic. All rings (resp. graded rings, resp. monoids) considered in this paper

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are assumed commutative (resp. anti-commutative, resp. commutative) and unital without further notice.

2. The de Rham-Witt complex

Let V be a henselian discrete valuation ring with quotient field K of characteristic zero and residue field k of odd characteristic p. (At this writing, we further require that V be of geometric type, i.e. that V be the henselian local ring at the generic point of the special fiber of a smooth scheme over a henselian discrete valuation ring $V_0 \subset V$ with *perfect* residue field.) A first example of a trace map is provided by the logarithmic derivative

$$K_*^M(K) \to \Omega^*_{(V,M)}$$

which to the symbol $\{a_1, \dots, a_q\}$ associates the form $d\log a_1 \dots d\log a_q$. The right hand side is the de Rham complex with log poles in the sense of Kato [25]: A log ring (A, M) is a ring A and a map of monoids $\alpha\colon M \to (A, \cdot)$; a log differential graded ring (E^*, M) is a differential graded ring E^* together with maps of monoids $\alpha\colon M \to (E^0, \cdot)$ and $d\log\colon M \to (E^1, +)$ such that $d \circ d\log = 0$ and such that $d\alpha(a) = \alpha(a) d\log a$ for all $a \in M$; the de Rham complex $\Omega^*_{(A,M)}$ is the universal log differential graded ring with underlying log ring (A, M). We will always consider the ring V with the *canonical* log structure

$$\alpha\colon M = V \cap K^* \hookrightarrow V.$$

(In this case, there are natural short-exact sequences

$$0 \to \Omega^q_V \to \Omega^q_{(V,M)} \to \Omega^{q-1}_k \to 0.)$$

The logarithmic derivative, however, is far from injective. It turns out that this can be rectified by incorborating the Witt vector construction which we now recall.

The ring of Witt vectors associated with a ring A is the set of "vectors"

$$W(A) = \{(a_0, a_1, \dots) \mid a_i \in A\}$$

with a new ring structure, see [19]. The ring operations are polynomial in the coordinates. The projection $W(A) \to A$, which to $(a_0, a_1, \dots)$ associates a_0, is a natural ring homomorphism with the unique natural multiplicative section

$$[\ ]\colon A \to W(A), \qquad [a] = (a, 0, 0, \dots).$$

If F is a perfect field of characteristic $p > 0$, then $W(F)$ is the unique (up to unique isomorphism) complete discrete valuation ring of mixed characteristic $(0, p)$ such that $W(F)/p \xrightarrow{\sim} F$. In general, the ring $W(A)$ is equal to the inverse limit of the rings $W_n(A)$ of Witt vectors of length n. But rather than forming the limit, we shall consider the limit system of rings $W.(A)$ as a pro-ring. There is a natural map of pro-rings $F\colon W.(A) \to W._{-1}(A)$, called the Frobenius, and a natural map of $W.(A)$-modules $V\colon F_* W._{-1}(A) \to W.(A)$, called the Verschiebung. The former is given, as the ring structure, by certain polynomials in the coordinates; the latter is given by $V(a_0, \dots, a_{n-2}) = (0, a_0, \dots, a_{n-2})$, and $FV = p$. Finally, we note that if (A, M) is a log ring, then the composite

$$M \xrightarrow{\alpha} A \xrightarrow{[\ ]} W.(A)$$

makes $(W.(A), M)$ a pro-log ring.

There is a natural way to combine differential forms and Witt vectors; the result is called the de Rham-Witt complex. It was considered first for $\mathbb{F}_p$-algebras by Bloch-Deligne-Illusie [3, 23] in connection with the crystalline cohomology of Berthelot-Grothendieck [2]. A generalization to log-$\mathbb{F}_p$-algebras was constructed by Hyodo-Kato [22]. The following extension to log-$\mathbb{Z}_{(p)}$-algebras was obtained in collaboration with Ib Madsen [19, 20]: Let (A, M) be a log ring such that A is a $\mathbb{Z}_{(p)}$-algebra with p odd. A *Witt complex* over (A, M) is:

(i) a pro-log differential graded ring $(E^*_{\cdot}, M_E)$ and a map of pro-log rings

$$\lambda\colon (W.(A), M) \to (E^0_{\cdot}, M_E);$$

(ii) a map of pro-log graded rings

$$F\colon E^*_{\cdot} \to E^*_{\cdot-1},$$

such that $\lambda F = F \lambda$ and such that

$$\begin{aligned} Fd\log_n a &= d\log_{n-1} a, && \text{for all } a \in M, \\ Fd\,\lambda[a]_n &= \lambda[a]_{n-1}^{p-1} d\,\lambda[a]_{n-1}, && \text{for all } a \in A; \end{aligned}$$

(iii) a map of pro-graded modules over the pro-graded ring $E^*_{\cdot}$,

$$V\colon F_* E^*_{\cdot-1} \to E^*_{\cdot},$$

such that $\lambda V = V \lambda$, $FV = p$ and $FdV = d$.

A map of Witt complexes over (A, M) is a map of pro-log differential graded rings which commutes with the maps λ, F and V. Standard category theory shows that there exists a universal Witt complex over (A, M). This, by defintion, is the de Rham-Witt complex $W.\,\Omega^*_{(A,M)}$. (The canonical maps $W.(A) \to W.\,\Omega^0_{(A,M)}$ and $\Omega^*_{(A,M)} \to W_1\,\Omega^*_{(A,M)}$ are isomorphisms, so the construction really does combine differential forms and Witt vectors.) We lift the logarithmic derivative to a map

$$K^M_q(K) \to W_n\,\Omega^q_{(V,M)}$$

which to the symbol $\{a_1, \dots, a_q\}$ associates $d\log_n a_1 \dots d\log_n a_q$. This trace map better captures the Milnor K-groups. Indeed, the following result was obtained in collaboration with Thomas Geisser [14]:

Theorem 2.1 *Suppose that $\mu_{p^v} \subset K$ and that k is separably closed. Then the trace map induces an isomorphism of pro-abelian groups*

$$K^M_q(K)/p^v \xrightarrow{\sim} \big(W.\,\Omega^q_{(V,M)}/p^v\big)^{F=1}.$$

To prove this, we first show that $W_n\,\Omega^q_{(V,M)}/p$ has a (non-canonical) k-vector space structure and find an explicit basis. The dimension is

$$\dim_k\left(W_n\,\Omega^q_{(V,M)}/p\right)=\binom{r+1}{q}\,e\sum_{s=0}^{n-1}p^{rs},$$

where $|k:k^p|=p^r$ and e the ramification index of K. It is not difficult to see that this is an upper bound for the dimension. The proof that it is also a lower bound is more involved and uses a formula for the de Rham-Witt complex of a polynomial extension by Madsen and the author [19]. We then evaluate the kernel of $1-F$ and compare with the calculation of $K^M_q(K)/p$ by Kato [24, 4]. The assumption that the residue field k be separably closed is not essential. In the general case, one instead has a short-exact sequence

$$0\to\left(W.\,\Omega^{q-1}_{(V,M)}\otimes\mu_{p^v}\right)_{F=1}\to K^M_q(K)/p^v\to\left(W.\,\Omega^q_{(V,M)}/p^v\right)^{F=1}\to 0,$$

where the superscript (resp. subscript) "$F=1$" indicates Frobenius invariants (resp. coinvariants).

We discuss a global version of theorem 2.1. Let V_0 be a henselian discrete valuation ring with quotient field K_0 of characteristic zero and *perfect* residue field k_0 of odd characteristic p. Let $\mathfrak{X}$ be a smooth V_0-scheme, and let i (resp. j) denote the inclusion of the special (resp. generic) fiber as in the cartesian diagram

$$\begin{array}{ccccc}
X & \overset{j}{\hookrightarrow} & \mathfrak{X} & \overset{i}{\hookleftarrow} & Y\\
\downarrow & & \downarrow{\scriptstyle f} & & \downarrow\\
\operatorname{Spec}K_0 & \hookrightarrow & \operatorname{Spec}V_0 & \hookleftarrow & \operatorname{Spec}k_0.
\end{array}$$

Suppose that $\mu_{p^v}\subset K_0$. Then the proof of theorem 2.1 shows that there is a short-exact sequence of sheaves of pro-abelian groups on Y for the étale topology

$$0\to i^*R^qj_*\mathbb{Z}/p^v(q)\to i^*(W.\,\Omega^q_{(\mathfrak{X},M)}/p^v)\xrightarrow{1-F}i^*(W.\,\Omega^q_{(\mathfrak{X},M)}/p^v)\to 0.$$

The left hand term is the sheaf of p-adic vanishing cycles.

3. The cyclotomic trace

We now turn to Quillen K-theory. The analog of the logarithmic derivative is the topological Dennis trace with values in topological Hochschild homology,

$$K_*(\mathcal{C})\to\mathrm{THH}_*(\mathcal{C}),$$

defined by Bökstedt [5]. It is a refinement of earlier trace maps by Dennis [9] and Waldhausen [35]. We will use a variant of the construction due to Dundas-McCarthy [10, 27] that can be applied to a category with cofibrations and weak equivalences in the sense of Waldhausen [36]. The category $\mathcal{C}$ we consider is the category of bounded chain complexes of finitely generated projective V-modules. The cofibrations are the degree-wise monomorphisms, and the weak equivalences are the chain maps $C \to C'$ such that $K \otimes_V C \to K \otimes_V C'$ is a quasi-isomorphism. The K-theory of this category is canonically isomorphic to Quillen's K-theory of the field K. We showed in [20] that the groups

$$\mathrm{THH}_*(V|K) = \mathrm{THH}_*(\mathcal{C})$$

form a log differential graded ring with underlying log ring (V, M), where the structure map $d\log$ is given by the composite

$$M = V \cap K^* \xrightarrow{\ell} K_1(K) \to \mathrm{THH}_1(V|K).$$

The canonical map from the de Rham complex

$$\Omega^q_{(V,M)} \to \mathrm{THH}_q(V|K)$$

is compatible with the trace maps and is an isomorphism, if $q \leq 2$. The topological Dennis trace, again, is far from injective. This can be rectified by a construction which, in retrospect, can be seen as incorporating Witt vectors. The result is the cyclotomic trace of Bökstedt-Hsiang-Madsen [6] which we now recall. The reader is referred to [20, 15, 10] for details.

The topological Dennis trace, we recall, is defined as the map of homotopy groups induced from a continuous map of spaces

$$K(\mathcal{C}) \to \mathrm{THH}(\mathcal{C}).$$

As a consequence of Connes' theory of cyclic sets, the right hand space is equipped with a continuous action by the circle group $\mathbb{T}$. Moreover, the image of the trace map is point-wise fixed by the $\mathbb{T}$-action. Let

$$\mathrm{TR}^n(\mathcal{C}; p) = \mathrm{THH}(\mathcal{C})^{C_{p^{n-1}}}$$

be the subset fixed by the subgroup $C_{p^{n-1}} \subset \mathbb{T}$ of the indicated order. It turns out that, as n and q varies, the homotopy groups

$$\mathrm{TR}^n_q(V|K; p) = \pi_q(\mathrm{TR}^n(\mathcal{C}; p))$$

form a Witt complex over (V, M); see [21, 18, 20]. The map F is induced from the obvious inclusion map, and the map V is the accompanying transfer map. The structure maps in the limit system and the map λ, however, are more difficult to define. The former was defined in [6] and the latter in [21]. The topological Dennis trace induces a map of pro-abelian groups

$$K_q(K) \to \mathrm{TR}^{\cdot}_q(V|K; p).$$

This is the cyclotomic trace. It takes values in the subset fixed by the Frobenius operator. The canonical map

$$W_n\Omega^q_{(V,M)} \to \mathrm{TR}^n_q(V|K; p)$$

is compatible with the trace maps from Milnor K-theory and Quillen K-theory, respectively, and is an isomorphism, if $q \leq 2$. The following is a combination of results obtained in collaboration with Thomas Geisser [15] and Ib Madsen [21, 20].

Theorem 3.1 *Suppose that k is separably closed. Then the cyclotomic trace induces an isomorphism of pro-abelian groups*

$$K_q(K, \mathbb{Z}/p^v) \xrightarrow{\sim} \mathrm{TR}_q^{\cdot}(V|K; p, \mathbb{Z}/p^v)^{F=1}.$$

We briefly outline the steps in the proof: We proved in [15] that the sequence

$$0 \to K_q(k, \mathbb{Z}/p^v) \to \mathrm{TR}_q^{\cdot}(k; p, \mathbb{Z}/p^v) \xrightarrow{1-F} \mathrm{TR}_q^{\cdot}(k; p, \mathbb{Z}/p^v) \to 0$$

is exact. This uses [4, 16, 18]. Given this, the theorem by McCarthy [28] that for nilpotent extensions, relative K-theory and relative topological cyclic homology agree, and the continuity results of Suslin [32] for K-theory and Madsen and the author [21] for TR show that also the sequence

$$0 \to K_q(V, \mathbb{Z}/p^v) \to \mathrm{TR}_q^{\cdot}(V; p, \mathbb{Z}/p^v) \xrightarrow{1-F} \mathrm{TR}_q^{\cdot}(V; p, \mathbb{Z}/p^v) \to 0$$

is exact. Theorem 3.1 follows by comparing the localization sequence of Quillen [30]

$$\cdots \to K_q(k, \mathbb{Z}/p^v) \xrightarrow{i^!} K_q(V, \mathbb{Z}/p^v) \xrightarrow{j_*} K_q(K, \mathbb{Z}/p^v) \to \cdots$$

to the corresponding sequence by Madsen and the author [20]

$$\cdots \to \mathrm{TR}_q^n(k; p, \mathbb{Z}/p^v) \xrightarrow{i^!} \mathrm{TR}_q^n(V; p, \mathbb{Z}/p^v) \xrightarrow{j_*} \mathrm{TR}_q(V|K; p, \mathbb{Z}/p^v) \to \cdots .$$

Again, the assumption in the statement of theorem 3.1 that the residue field k be separably closed is not essential. The general statement will be given below. It is also not necessary for theorem 3.1 to assume that V be of geometric type.

4. The Tate spectral sequence

If G is a finite group and X a G-space, it is usually not possible to evaluate the groups $\pi_*(X^G)$ from knowledge of the G-modules $\pi_*(X)$. At first glance, this is the problem that one faces in evaluating the groups

$$\mathrm{TR}_q^n(\mathcal{C}; p) = \pi_q\big(\mathrm{THH}(\mathcal{C})^{C_{p^{n-1}}}\big).$$

However, the mapping fiber of the structure map $\mathrm{TR}^n(\mathcal{C}; p) \to \mathrm{TR}^{n-1}(\mathcal{C}; p)$, it turns out, is given by the Borel construction $\mathbb{H}.(C_{p^{n-1}}, \mathrm{THH}(\mathcal{C}))$ whose homotopy groups are the abutment of a (first quadrant) spectral sequence

$$E^2_{s,t} = H_s(C_{p^{n-1}}, \mathrm{THH}_t(\mathcal{C})) \Rightarrow \pi_{s+t}\mathbb{H}.(C_{p^{n-1}}, \mathrm{THH}(\mathcal{C})).$$

This suggests that the groups $\mathrm{TR}_q^n(\mathcal{C}; p)$ can be evaluated inductively starting from the case $n = 1$. However, it is generally difficult to carry out the induction step.

In addition, the absence of a multiplicative structure makes the spectral sequence above difficult to solve. The main vehicle to overcome these problems, first employed by Bökstedt-Madsen in [7], is the following diagram of fiber sequences

$$\begin{array}{ccccc} \mathbb{H}_{\cdot}(C_{p^{n-1}}, \mathrm{THH}(\mathcal{C})) & \longrightarrow & \mathrm{TR}^n(\mathcal{C};p) & \longrightarrow & \mathrm{TR}^{n-1}(\mathcal{C};p) \\ \| & & \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \hat{\Gamma}} \\ \mathbb{H}_{\cdot}(C_{p^{n-1}}, \mathrm{THH}(\mathcal{C})) & \longrightarrow & \mathbb{H}^{\cdot}(C_{p^{n-1}}, \mathrm{THH}(\mathcal{C})) & \longrightarrow & \hat{\mathbb{H}}(C_{p^{n-1}}, \mathrm{THH}(\mathcal{C})) \end{array}$$

together with a multiplicative (upper half-plane) spectral sequence

$$E^2_{s,t} = \hat{H}^{-s}(C_{p^{n-1}}, \mathrm{THH}_t(\mathcal{C})) \Rightarrow \pi_{s+t}\hat{\mathbb{H}}(C_{p^{n-1}}, \mathrm{THH}(\mathcal{C}))$$

starting from the Tate cohomology of the (trivial) $C_{p^{n-1}}$-module $\mathrm{THH}_t(\mathcal{C})$. The lower fiber sequence is the Tate sequence; see Greenlees and May [17] or [20]. In favorable cases, the maps Γ and $\hat{\Gamma}$ induce isomorphisms of homotopy groups in non-negative degrees. Indeed, this is true in the case at hand (if k is perfect). The differential structure of the spectral sequence

$$E^2_{s,t} = \hat{H}^{-s}(C_{p^{n-1}}, \mathrm{THH}_t(V|K, \mathbb{Z}/p)) \Rightarrow \pi_{s+t}(\hat{\mathbb{H}}(C_{p^{n-1}}, \mathrm{THH}(V|K)), \mathbb{Z}/p)$$

was determined in collaboration with Ib Madsen [20] in the case where the residue field k is perfect. This is the main calculational result of the work reported here. The following result, for perfect k, is a rather immediate consequence. The extension to non-perfect k is given in [19].

Theorem 4.1 *Suppose that $\mu_{p^v} \subset K$. Then the canonical map is an isomorphism of pro-abelian groups*

$$W_{\cdot}\,\Omega^*_{(V,M)} \otimes_{\mathbb{Z}} S_{\mathbb{Z}/p^v}(\mu_{p^v}) \xrightarrow{\sim} \mathrm{TR}^{\cdot}_*(V|K;p,\mathbb{Z}/p^v)$$

We can now state the general version of theorem 3.1 which does not require that the residue field k be separably closed. The second tensor factor on the left hand side in the statement of theorem 4.1 is the symmetric algebra on the $\mathbb{Z}/p^v$-module μ_{p^v}, which is free of rank one. Spelling out the statement for the group in degree q, we get an isomorphism of pro-abelian groups

$$\bigoplus_{s\geq 0} W_{\cdot}\,\Omega^{q-2s}_{(V,M)} \otimes \mu_{p^v}^{\otimes s} \xrightarrow{\sim} \mathrm{TR}^{\cdot}_q(V|K;p,\mathbb{Z}/p^v).$$

In the case of a separably closed residue field, theorem 3.1 identifies the Frobenius fixed set of the common pro-abelian group with $K_q(K,\mathbb{Z}/p^v)$. In the general case, one has instead a short-exact sequence

$$0 \to \bigoplus_{s\geq 1} \left(W_{\cdot}\,\Omega^{q+1-2s}_{(V,M)} \otimes \mu_{p^v}^{\otimes s}\right)_{F=1} \to K_q(K,\mathbb{Z}/p^v) \to \bigoplus_{s\geq 0} \left(W_{\cdot}\,\Omega^{q-2s}_{(V,M)} \otimes \mu_{p^v}^{\otimes s}\right)^{F=1} \to 0,$$

valid for all integers q. (There is a similar sequence for the topological cyclic homology group $\mathrm{TC}^{\cdot}_q(V|K;p,\mathbb{Z}/p^v)$ [20] which includes the summand "$s=0$" on the left.) Comparing with the general version of theorem 2.1, we obtain the following result promised earlier [20, 14].

Theorem 4.2 *Suppose that $\mu_{p^v} \subset K$. Then the canonical map*

$$K_*^M(K) \otimes_{\mathbb{Z}} S_{\mathbb{Z}/p^v}(\mu_{p^v}) \xrightarrow{\sim} K_*(K, \mathbb{Z}/p^v)$$

is an isomorphism.

5. Galois descent

We now assume that the residue field k be perfect. In homotopy theoretic terms, theorem 4.1 states that the pro-spectrum $\mathrm{TR}^{\cdot}(V|K;p)$ is equivalent to the (-1)-connected cover of its localization with respect to complex periodic K-theory, see [8]. This suggests the possibility of completely understanding the homotopy type of this pro-spectrum. We expect that this, in turn, is closely related to the following question. Let $\bar{K}$ be an algebraic closure of K with Galois group G_K, and let $\bar{V}$ be the integral closure of V in $\bar{K}$. (The ring $\bar{V}$ is a valuation ring with value group the additive group of rational numbers.)

Conjecture 5.1 *If k is perfect then for all $q > 0$, the canonical map*

$$\mathrm{TR}_q^{\cdot}(V|K;p,\mathbb{Q}_p/\mathbb{Z}_p) \to \mathrm{TR}_q^{\cdot}(\bar{V}|\bar{K};p,\mathbb{Q}_p/\mathbb{Z}_p)^{G_K}$$

is an isomorphism of pro-abelian groups and the higher continuous cohomology groups $H^i_{\mathrm{cont}}(G_K, \mathrm{TR}_q^n(\bar{V}|\bar{K};p,\mathbb{Q}_p/\mathbb{Z}_p))$ vanish.

It follows from Tate [33] that the groups $H^i_{\mathrm{cont}}(G_K, \mathrm{TR}_q^n(\bar{V}|\bar{K};p,\mathbb{Q}_p))$ vanish for $i \geq 0$ and $q > 0$. One may hope that these methods will help shed some light on the structure of the groups $H^i_{\mathrm{cont}}(G_K, \mathrm{TR}_q^n(\bar{V}|\bar{K};p,\mathbb{Q}_p/\mathbb{Z}_p))$. We now describe the structure of these G_K-modules; proofs will appear elsewhere.

The group $\mathrm{TR}_q^n(\bar{V}|\bar{K};p)$ is divisible, if $q > 0$, and uniquely divisible, if $q > 0$ and even. The Tate module $T_p\,\mathrm{TR}_1^n(\bar{V}|\bar{K};p)$ is a free module of rank one over $\mathrm{TR}_0^n(\bar{V}|\bar{K};p,\mathbb{Z}_p)$, and the canonical map an isomorphism:

$$S_{\mathrm{TR}_0^n(\bar{V}|\bar{K};p,\mathbb{Z}_p)}(T_p\,\mathrm{TR}_1^n(\bar{V}|\bar{K};p)) \xrightarrow{\sim} \mathrm{TR}_*^n(\bar{V}|\bar{K};p,\mathbb{Z}_p)$$

(note that $\mathrm{TR}_q^n(\bar{V}|\bar{K};p,\mathbb{Q}_p/\mathbb{Z}_p) \xleftarrow{\sim} \mathrm{TR}_q^n(\bar{V}|\bar{K};p,\mathbb{Z}_p) \otimes \mathbb{Q}_p/\mathbb{Z}_p)$. We note the formal analogy with the results on $K_*(\bar{K})$ by Suslin [31, 32].

The structure of the ring $\mathrm{TR}_0^n(\bar{V}|\bar{K};p,\mathbb{Z}_p) = W_n(\bar{V})^{\wedge}$ is well-understood (unlike that of $W_n(V)$): Following Fontaine [13], we let $R_{\bar{V}}$ be the inverse limit of the diagram $\bar{V}/p \leftarrow \bar{V}/p \leftarrow \cdots$ with the Frobenius as structure map. This is a perfect $\mathbb{F}_p$-algebra and an integrally closed domain whose quotient field is algebraically closed. There is a surjective ring homomorphism $\theta_n \colon W(R_{\bar{V}}) \twoheadrightarrow W_n(\bar{V})^{\wedge}$ whose kernel is a principal ideal. If $\epsilon = \{\epsilon^{(v)}\}_{v \geq 1}$ is a compatible sequence of primitive p^{v-1}st roots of unity considered as an element of $R_{\bar{V}}$, and if ϵ_n is the unique p^nth root of ϵ, then $([\epsilon]-1)/([\epsilon_n]-1)$ is a generator. Moreover, as n varies, the maps θ_n constitute a map of pro-rings compatible with the Frobenius maps.

The Bott element $b_{\epsilon,n} \in T_p\,\mathrm{TR}_1^n(\bar{V}|\bar{K};p)$ determined by the sequence ϵ is not a generator (so the statement of theorem 4.1 is not valid for $\bar{K}$). Instead there is a generator $\alpha_{\epsilon,n}$ such that $b_{\epsilon,n} = ([\epsilon_n]-1)\cdot\alpha_{\epsilon,n}$. The structure maps of the pro-abelian group $T_p\,\mathrm{TR}_1^{\boldsymbol{\cdot}}(\bar{V}|\bar{K};p)$ (resp. the Frobenius) take $\alpha_{\epsilon,n}$ to $([\epsilon_{n-1}]-1)/([\epsilon_n]-1)\cdot\alpha_{\epsilon,n-1}$ (resp. to $\alpha_{\epsilon,n-1}$), and the action of the Galois group is given by

$$\alpha_{\epsilon,n}^{\sigma} = \chi(\sigma)\frac{[\epsilon_n]-1}{[\epsilon_n^{\sigma}]-1}\cdot\alpha_{\epsilon,n},$$

where $\chi\colon G_K \to \mathrm{Aut}(\mu_{p^\infty}) = \mathbb{Z}_p^*$ is the cyclotomic character.

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