

# The absolute and relative de Rham-Witt complexes

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## ABSTRACT

We compare the absolute and relative de Rham-Witt complexes considered by the author and Madsen [7, 6] and by Langer and Zink [13], which both generalize the classical de Rham-Witt complex of Bloch, Deligne, and Illusie [9] from  $\mathbb{F}_p$ -schemes to  $\mathbb{Z}_{(p)}$ -schemes. From this comparison, we derive a Gauss-Manin connection on the crystalline cohomology of  $X/W_n(S)$  for a smooth family  $X/S$ .

## Introduction

Let  $f: X \rightarrow S$  be a morphism of noetherian  $\mathbb{Z}_{(p)}$ -schemes and suppose that  $p$  is odd and nilpotent on  $S$ . There is a canonical surjective map

$$W_n\Omega_X \rightarrow W_n\Omega_{X/S}$$

from the absolute de Rham-Witt complex of  $X$  considered by the author and Madsen [7] to the relative de Rham-Witt complex of  $X/S$  considered by Langer and Zink [13]. In the classical case, where  $S$  is a perfect  $\mathbb{F}_p$ -scheme, the map is an isomorphism and the common complex coincides with the de Rham-Witt complex of Bloch, Deligne, and Illusie [9]. In general, the terms of the complexes are quasi-coherent  $W_n(\mathcal{O}_X)$ -modules on the small étale site of  $X$ , and the differential of the relative de Rham-Witt complex is  $f^{-1}W_n(\mathcal{O}_S)$ -linear. The kernel  $\mathcal{I}$  of the projection is equal to the differential graded ideal generated by the image of the canonical map  $f^{-1}W_n\Omega_S^1 \rightarrow W_n\Omega_X^1$ . The graded pieces for the  $\mathcal{I}$ -adic filtration are differential graded modules over the differential graded ring  $W_n\Omega_{X/S}$ , and hence, complexes of quasi-coherent  $f^{-1}W_n(\mathcal{O}_S)$ -modules on the small étale site of  $X$ . We show that the absolute and relative de Rham-Witt complexes are related as follows.

**THEOREM A.** *Let  $f: X \rightarrow S$  be a smooth morphism of noetherian  $\mathbb{Z}_{(p)}$ -schemes and suppose that  $p$  is odd and nilpotent on  $S$ . Then there is a canonical isomorphism*

$$f^{-1}W_n\Omega_S^s \otimes_{f^{-1}W_n(\mathcal{O}_S)}^{\mathbb{L}} W_n\Omega_{X/S}^{-s} \xrightarrow{\sim} \mathrm{gr}_{\mathcal{I}}^s W_n\Omega_X$$

in the derived category of quasi-coherent  $f^{-1}W_n(\mathcal{O}_S)$ -modules.

By Langer and Zink [13, Thm. 3.5], there is a canonical isomorphism

$$H_{\mathrm{crys}}^q(X/W_n(S)) \xrightarrow{\sim} H^q(X, W_n\Omega_{X/S}).$$

Suppose that  $S = \mathrm{Spec} R$  is affine and that the groups  $H_{\mathrm{crys}}^q(X/W_n(S))$  are flat  $W_n(R)$ -modules. This is true, for instance, if  $X$  is an abelian  $S$ -scheme [2]. Then the isomorphism of Thm. A gives rise to an integrable Gauss-Manin connection

$$\nabla: H_{\mathrm{crys}}^q(X/W_n(S)) \rightarrow W_n\Omega_R^1 \otimes_{W_n(R)} H_{\mathrm{crys}}^q(X/W_n(S))$$

on the crystalline cohomology groups. The structure of  $W_n\Omega_R^1$  is not well-understood in general,

but see [6, 4] for some partial results. However, we have

$$W_n \Omega_{\mathbb{Z}(p)}^1 = \prod_{1 \leq s < n} \mathbb{Z}/p^s \mathbb{Z} \cdot dV^s([1]_{n-s}).$$

The Gauss-Manin connection constructed here is related to the monodromy operator of Hyodo and Kato [8] on log-crystalline cohomology; see Rem. 3.5 below.

We remark that the proof of Thm. A given here also works in the relative situation, where  $f: X \rightarrow S$  and  $g: S \rightarrow T$  are morphisms of  $\mathbb{Z}(p)$ -schemes with  $f$  smooth and with  $p$  odd and nilpotent on  $T$ . If  $\mathcal{I}$  denotes the kernel of the projection

$$W_n \Omega_{X/T} \rightarrow W_n \Omega_{X/S},$$

then we obtain a canonical isomorphism

$$f^{-1} W_n \Omega_{S/T}^s \otimes_{f^{-1} W_n(\mathcal{O}_S)}^{\mathbb{L}} W_n \Omega_{X/S}^{-s} \xrightarrow{\sim} \mathrm{gr}_{\mathcal{I}}^s W_n \Omega_{X/T}$$

in the derived category of quasi-coherent  $f^{-1} W_n(\mathcal{O}_S)$ -modules. This slightly generalizes and gives a new proof of a result of Langer and Zink [12, Prop. 2].

The proof of Thm. A is based on a formula for the absolute de Rham-Witt complex of the polynomial algebra  $A[T_1, \dots, T_d]$  in terms of that of  $A$ . In fact, we prove a slightly stronger result. Let  $I$  be the kernel of the canonical projection

$$W_n \Omega_{A[T_1, \dots, T_d]} \rightarrow W_n \Omega_{A[T_1, \dots, T_d]/A}$$

from the absolute to the relative de Rham-Witt complex. We then give a formula for the graded pieces for the  $I$ -adic filtration of the left-hand side in terms of the absolute de Rham-Witt complex of  $A$ . The formula is similar to the formula for the right-hand side given by Langer and Zink [13, Thm. 2.8]. In particular, the following notation from *loc. cit.* will be used.

Let  $\mathbb{N}_0[\frac{1}{p}]$  denote the set of non-negative rational numbers with denominator a power of  $p$ . We define a *weight* of rank  $d$  to be a function

$$k: \{1, 2, \dots, d\} \rightarrow \mathbb{N}_0[\frac{1}{p}]$$

and the *filtration* of  $k$  to be the smallest non-negative integer  $u(k)$  with the property that  $p^{u(k)}k$  takes integer values. We choose, for every weight  $k$ , a total ordering

$$\mathrm{supp}(k) = \{i_1, \dots, i_s\}$$

of the support such that  $v_p(k_{i_1}) \leq \dots \leq v_p(k_{i_s})$ . We then say that a partition of the support into intervals

$$\mathcal{P}: \mathrm{supp}(k) = I_0 \sqcup I_1 \sqcup \dots \sqcup I_\ell$$

is *admissible*, if for all  $1 \leq s \leq \ell$ , the interval  $I_s$  is non-empty and the elements of  $I_s$  are greater than the elements of  $I_{s-1}$ . The non-negative integer  $\ell$  is called the *length* of the partition.

We inductively prove the following result starting from the case  $d = 1$ , which was proved by the author and Madsen in [7, Thm. B]. The statement for  $s = 0$  was proved by Langer and Zink [13, Thm. 2.8] by different methods.

**THEOREM B.** *Let  $A$  be a  $\mathbb{Z}(p)$ -algebra, where  $p$  is an odd prime. Then there is a canonical isomorphism of abelian groups*

$$\bigoplus_{(k, \mathcal{P})} W_{n-u(k)} \Omega_A^s \xrightarrow{\sim} \mathrm{gr}_{\mathcal{I}}^s W_n \Omega_{A[T_1, \dots, T_d]}^q$$

where, on the left-hand side, the sum runs over all weights  $k$  of rank  $d$  and filtration  $0 \leq u(k) < n$  and over all admissible partitions  $\mathcal{P}$  of  $\mathrm{supp}(k)$  of length  $\ell = q - s$ .

The paper consists of three paragraphs, where the first and second contains the proofs of theorems B and A, respectively. In the final paragraph three, we prove a version of theorems A and B for the de Rham-Witt complex with log-poles.

We denote by  $\mathbb{N}$  (resp. by  $\mathbb{N}_0$ , resp. by  $\mathbb{N}_0[\frac{1}{p}]$ ) the set of positive integers (resp. non-negative integers, resp. non-negative rational numbers whose denominator is a power of  $p$ ). By a pro-object of a category  $\mathcal{C}$  we mean a functor from  $\mathbb{N}$ , viewed as a category with one arrow from  $n+1$  to  $n$ , to  $\mathcal{C}$ , and by a strict map between pro-objects we mean a natural transformation.

The results of this paper are much inspired by the work of Langer and Zink [12]. I would like to thank Thomas Zink for explaining this work to me and for suggesting that similar results might hold in the situation considered here. This paper was written in part while the author was visiting the University of Tokyo. I would like to express my sincere gratitude to the university and to Takeshi Saito in particular for the kind hospitality and generous support that I received. Finally, I would like to thank the anonymous referee for a very careful reading of an earlier version of this paper.

## 1. Polynomial extensions

Let  $f: R \rightarrow A$  be a map of  $\mathbb{Z}_{(p)}$ -algebras, where  $p$  is an odd prime. We briefly recall the definition of the canonical projection

$$W_n \Omega_A^\bullet \rightarrow W_n \Omega_{A/R}^\bullet$$

from the absolute to the relative de Rham-Witt complex. The reader is referred to [7, 13] for more details.

We recall from [7] that a *Witt complex* over a  $\mathbb{Z}_{(p)}$ -algebra  $A$  is defined to be a quadruple  $(E, \lambda, F, V)$  where  $E = \{E_n\}_{n \in \mathbb{N}}$  is a pro-differential graded ring,  $\lambda$  is a strict map of pro-rings  $\lambda: W_n(A) \rightarrow E_n^0$  from the pro-ring of Witt vectors in  $A$ ,  $F$  is a strict map of pro-graded rings  $F: E_n^\bullet \rightarrow E_{n-1}^\bullet$  such that  $\lambda F = F \lambda$  and such that for all  $a \in A$ ,  $F d \lambda([a]_n) = \lambda([a]_{n-1})^{p-1} d \lambda([a]_{n-1})$ , and  $V$  is a strict map of graded  $E_n^\bullet$ -modules  $V: F_* E_{n-1}^\bullet \rightarrow E_n^\bullet$  such that  $\lambda V = V \lambda$ ,  $F d V = d$ , and  $F V = p$ . A map of Witt complexes over  $A$  is a strict map of pro-differential graded rings  $f: E^* \rightarrow E'^*$  such that  $\lambda' = f \lambda$ ,  $F' f = f F$  and  $V' f = f V$ . We also recall from [7, Lemma 1.2.1] that the relations  $dF = pF d$  and  $V d = p d V$  hold in every Witt complex.

The absolute de Rham-Witt complex  $W_n \Omega_A^\bullet$  considered by the author and Madsen [7] is defined to be the initial example of a Witt complex over  $A$ . It is proved in [7, Thm. A] that the initial object exists and that the canonical map

$$\lambda: \Omega_{W_n(A)}^\bullet \rightarrow W_n \Omega_A^\bullet$$

is surjective. Similarly, the relative de Rham-Witt complex  $W_n \Omega_{A/R}^\bullet$  considered by Langer and Zink [13] is defined to be the initial example of a Witt complex over  $A$  with  $W_n(R)$ -linear differentials. The proof of [7, Thm. 3] shows that the initial object exists and that the canonical map

$$\lambda: \Omega_{W_n(A)/W_n(R)}^\bullet \rightarrow W_n \Omega_{A/R}^\bullet$$

is surjective, but see also [13, Prop. 1.6] for a different and somewhat more direct construction.

LEMMA 1.1. *Let  $f: R \rightarrow A$  be a map of  $\mathbb{Z}_{(p)}$ -algebras with  $p$  odd. Then there is a unique and surjective map of Witt complexes*

$$\pi: W_n \Omega_A^\bullet \rightarrow W_n \Omega_{A/R}^\bullet$$

and the kernel is equal to the differential graded ideal generated by the image of the canonical map  $W_n \Omega_R^1 \rightarrow W_n \Omega_A^1$ .

*Proof.* Since a Witt complex over  $A$  with  $W_n(R)$ -linear differentials is, in particular, a Witt complex over  $A$ , there exists a unique map of Witt complexes as in the statement. Similarly, there is a

commutative diagram of pro-differential graded rings

$$\begin{array}{ccc} \Omega_{W_n(A)} & \xrightarrow{\lambda} & W_n\Omega_A \\ \downarrow & & \downarrow \pi \\ \Omega_{W_n(A)/W_n(R)} & \xrightarrow{\lambda} & W_n\Omega_{A/R} \end{array}$$

where the left-hand vertical map is the canonical projection. Since the left-hand vertical map and the lower horizontal map are both surjective, so is the right-hand vertical map.

It remains to show that the kernel of the map of the statement is equal to the differential graded ideal  $I_n \subset W_n\Omega_A$  generated by the image of the canonical map  $W_n\Omega_R^1 \rightarrow W_n\Omega_A^1$ . The map of the statement induces a canonical surjective map

$$(W_n\Omega_A)/I_n \rightarrow W_n\Omega_{A/R}.$$

To produce an inverse map, we will show that the differential graded rings on the left-hand side form a Witt complex over  $A$  with  $W_n(R)$ -linear differentials. It suffices to show that  $F(I_n) \subset I_{n-1}$  and  $V(I_{n-1}) \subset I_n$ . We prove the first statement. It suffices to show that for all  $x \in W_n(R)$ , the element  $Fd\lambda(f(x)) \in W_n\Omega_A^1$  is contained in  $I_{n-1}$ . We can write  $x \in W_n(R)$  uniquely as

$$x = [x_0]_n + V[x_1]_{n-1} + \cdots + V^{n-1}[x_{n-1}]_1,$$

and hence, suppressing the map  $f$  from the notation, we have

$$\begin{aligned} Fd\lambda(x) &= Fd\lambda([x_0]_n) + FdV\lambda([x_1]_{n-1}) + \cdots + FdV^{n-1}\lambda([x_{n-1}]_1) \\ &= \lambda([x_0]_{n-1})^{p-1}d\lambda([x_0]_{n-1}) + d\lambda([x_1]_{n-1}) + \cdots + dV^{n-2}\lambda([x_{n-1}]_1). \end{aligned}$$

The lemma follows.  $\square$

Let  $f: A \rightarrow A'$  be a map of  $\mathbb{Z}_{(p)}$ -algebras, where  $p$  is an odd prime. It was proved in [7, Prop. 1.2.3] that the map  $f$  gives rise to an adjoint pair of functors  $(f^*, f_*)$  between the categories of Witt complexes over  $A$  and  $A'$ . Since a left adjoint functor preserves colimits, the canonical map

$$W_n\Omega_{A'} \rightarrow f^*W_n\Omega_A$$

is an isomorphism. In Thm. 1.2 below, we give a formula for the inverse image functor  $f^*$  associated with the map  $f: A \rightarrow A[T_1, \dots, T_d]$  that includes the constant polynomials. The proof is by induction starting from the case  $d = 1$  which was proved by the author and Madsen in [7, Thm. B].

Let  $A$  be a  $\mathbb{Z}_{(p)}$ -algebra with  $p$  odd, and let  $E$  be a Witt complex over  $A$ . Let

$$k: \{1, 2, \dots, d\} \rightarrow \mathbb{N}_0[\frac{1}{p}].$$

be a weight of rank  $d$  and filtration  $0 \leq u(k) < n$ , and let

$$I = \{i_{s+1}, \dots, i_{s+m}\} \subset \text{supp}(k)$$

be an interval with respect to the chosen total ordering. We write  $k_I$  for the weight with  $k_I(i) = k(i)$ , if  $i \in I$ , and  $k_I(i) = 0$ , otherwise, and let  $X^{k_I}$  be the image of

$$[T_{i_{s+1}}]_{n-u(k)}^{k_{i_{s+1}}} \cdots [T_{i_{s+m}}]_{n-u(k)}^{k_{i_{s+m}}} = [T_{i_{s+1}}]_{n-u(k)}^{k_{i_{s+1}}} \cdots [T_{i_{s+m}}]_{n-u(k)}^{k_{i_{s+m}}}$$

by the ring homomorphism

$$\lambda: W_{n-u(k)}(A[T_1, \dots, T_d]) \rightarrow (f^*E)_{n-u(k)}^0$$

which is part of the structure of a Witt complex over  $A[T_1, \dots, T_d]$ . Finally, let

$$\mathcal{P}: \text{supp}(k) = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_\ell$$

be an admissible partition of  $\text{supp}(k)$  of length  $0 \leq \ell < q$ . We associate to the pair  $(k, \mathcal{P})$  the map of abelian groups

$$e(k, \mathcal{P}): E_{n-u(k)}^{q-\ell} \rightarrow (f_* f^* E)_n^q = (f^* E)_n^q$$

that takes  $\xi$  to the basic Witt differential  $e = e(k, \mathcal{P})(\xi)$  of one of the following four types (A)–(D). Let  $t(I) = \max\{-v_p(k_i) \mid i \in I\}$ , and let  $\eta: E \rightarrow f_* f^* E$  is the unit of the adjunction  $(f^*, f_*)$ .

(A) If  $I_0 \neq \emptyset$  and if  $k$  is not integral, then

$$e = V^{u(k)}(\eta(\xi) X^{p^{u(k)} k_{I_0}} F^{u(k)-t(I_1)} dX^{p^{t(I_1)} k_{I_1}} \dots F^{u(k)-t(I_\ell)} dX^{p^{t(I_\ell)} k_{I_\ell}}).$$

(B) If  $I_0 \neq \emptyset$  and if  $k$  is integral, then

$$e = \eta(\xi) X^{k_{I_0}} F^{-t(I_1)} dX^{p^{t(I_1)} k_{I_1}} \dots F^{-t(I_\ell)} dX^{p^{t(I_\ell)} k_{I_\ell}}.$$

(C) If  $I_0 = \emptyset$  and if  $k$  is not integral, then

$$e = dV^{u(k)}(\eta(\xi) X^{p^{u(k)} k_{I_1}} F^{u(k)-t(I_2)} dX^{p^{t(I_2)} k_{I_2}} \dots F^{u(k)-t(I_\ell)} dX^{p^{t(I_\ell)} k_{I_\ell}}).$$

(D) If  $I_0 = \emptyset$  and if  $k$  is integral, then

$$e = \eta(\xi) F^{-t(I_1)} dX^{p^{t(I_1)} k_{I_1}} \dots F^{-t(I_\ell)} dX^{p^{t(I_\ell)} k_{I_\ell}}.$$

We note that the definition of the basic Witt differential given here is equivalent to the definition given in [13, (2.15)–(2.17)]. In the following, we shall often suppress the unit  $\eta$  in the notation. We prove the following result by induction on the number of variables  $d$  starting from the basic case  $d = 1$  which was proved in [7, Thm. B].

**THEOREM 1.2.** *Let  $E$  be a Witt complex over a  $\mathbb{Z}_{(p)}$ -algebra  $A$ , where  $p$  is an odd prime, and let  $f: A \rightarrow A[T_1, \dots, T_d]$  be the inclusion of the constant polynomials. Then the basic Witt differentials define an isomorphism of abelian groups*

$$e: \bigoplus_{(k, \mathcal{P})} E_{n-u(k)}^{q-\ell} \xrightarrow{\sim} (f^* E)_n^q.$$

Here the sum on the left-hand side runs over all weights  $k$  of rank  $d$  and filtration  $0 \leq u(k) < n$  and over all admissible partitions  $\mathcal{P}$  of  $\text{supp}(k)$ .

We spell out the statement of the theorem for  $d = 1$ . Let  $k$  be a weight of rank one. If  $k$  is zero, then  $\text{supp}(k) = \emptyset$ , the only admissible partition is  $\mathcal{P}: \text{supp}(k) = I_0$ , and  $e(k, \mathcal{P})(\xi) = \eta(\xi)$ . If  $k$  is non-zero, then  $\text{supp}(k) = \{1\}$ , there are two possible admissible partitions  $\mathcal{P}_0: \text{supp}(k) = I_0$  and  $\mathcal{P}_1: \text{supp}(k) = I_1$ ,  $e(k, \mathcal{P}_0)(\xi) = V^{u(k)}(\eta(\xi) X^{p^{u(k)} k})$ , and  $e(k, \mathcal{P}_1)(\xi)$  is either  $\eta(\xi) F^{-t(k)} dX^{p^{t(k)} k} = \eta(\xi) p^{t(k)} k X^{k-1} dX$  or  $dV^{u(k)}(\eta(\xi) X^{p^{u(k)} k})$  as  $k$  is integral or not.

As mentioned above, the proof of Thm. 1.2 is by induction on the rank  $d$ . In the induction step, we shall use the following combinatorial result.

**LEMMA 1.3.** *The following formula gives a bijection between the set of pairs of weights  $(k', k'')$  with domain  $\{0\}$  and  $\{1, 2, \dots, d\}$  and the set of weights  $k$  with domain  $\{0, 1, 2, \dots, d\}$ .*

$$k(i) = \begin{cases} p^{-u(k'')} k'(0) & \text{if } i = 0, \\ k''(i) & \text{if } i > 0. \end{cases}$$

For each weight  $k$  with domain  $\{0, 1, 2, \dots, d\}$ , let a total ordering  $\text{supp}(k) = \{i_1, \dots, i_r\}$  be chosen such that  $v_p(i_1) \leq \dots \leq v_p(i_r)$ , and give  $\text{supp}(k')$  and  $\text{supp}(k'')$  the induced total orderings. Then the following formulas establish a bijection between the set of pairs of admissible partitions  $(\mathcal{P}', \mathcal{P}'')$  of  $\text{supp}(k')$  and  $\text{supp}(k'')$  and the set of admissible partitions  $\mathcal{P}$  of  $\text{supp}(k)$ . We write (I–II–III) to indicate that  $(k', \mathcal{P}')$  is type I, that  $(k'', \mathcal{P}'')$  is type II, and that  $(k, \mathcal{P})$  is type III.

Case (A–A–A). Let  $I_0 = \{0\} \cup I''_0$  and  $I_s = I''_s$ , for  $0 < s \leq \ell''$ .

Case (B–A–A). If there exists  $0 \leq s < \ell''$  such that every element of  $I''_{s+1}$  is larger than 0, let  $m$  be the smallest such  $s$ ; otherwise, let  $m = \ell''$ . Then  $I_s = I''_s$ , if  $s \neq m$ , and  $I_m = I''_m \cup \{0\}$ .

Case (C–A–A). Let  $I_0 = \{0\}$  and  $I_s = I''_{s-1}$ , for  $0 < s \leq \ell$ .

Case (D–A–A). If there exists  $0 \leq s < \ell''$  such that every element of  $I''_{s+1}$  is larger than 0, let  $m$  be the smallest such  $s$ ; otherwise, let  $m = \ell''$ . Write  $I''_m = I''_{m,1} \sqcup I''_{m,2}$  with every element of  $I''_{m,1}$  smaller than 0 and every element of  $I''_{m,2}$  larger than 0. If  $m = 0$  and  $I''_{0,1} = \emptyset$ , let  $I_0 = \{0\}$ , and let  $I_s = I''_{s-1}$ , for  $0 < s \leq \ell$ . Otherwise, let  $I_s = I''_s$ , for  $0 \leq s < m$ , let  $I_m = I''_{m,1}$  and  $I_{m+1} = \{0\} \cup I''_{m,2}$ , and let  $I_s = I''_{s-1}$ , for  $m+1 < s \leq \ell$ .

Case (A–B–A). Let  $I_0 = \{0\} \cup I''_0$  and  $I_s = I''_s$ , for  $0 < s \leq \ell''$ .

Case (B–B–B). If there exists  $0 \leq s < \ell''$  such that every element of  $I''_{s+1}$  is larger than 0, let  $m$  be the smallest such  $s$ ; otherwise, let  $m = \ell''$ . Then  $I_s = I''_s$ , if  $s \neq m$ , and  $I_m = I''_m \cup \{0\}$ .

Case (C–B–C). Let  $I_1 = \{0\} \cup I''_0$ , and let  $I_s = I''_{s-1}$ , for  $1 < s \leq \ell$ .

Case (D–B–B). If there exists  $0 \leq s < \ell''$  such that every element of  $I''_{s+1}$  is larger than 0, let  $m$  be the smallest such  $s$ ; otherwise, let  $m = \ell''$ . Write  $I''_m = I''_{m,1} \sqcup I''_{m,2}$  with every element of  $I''_{m,1}$  smaller than 0 and every element of  $I''_{m,2}$  larger than 0. If  $m = 0$  and  $I''_{0,1} = \emptyset$ , let  $I_0 = \{0\}$ , and let  $I_s = I''_{s-1}$ , for  $0 < s \leq \ell$ . Otherwise, let  $I_s = I''_s$ , for  $0 \leq s < m$ , let  $I_m = I''_{m,1}$  and  $I_{m+1} = \{0\} \cup I''_{m,2}$ , and let  $I_s = I''_{s-1}$ , for  $m+1 < s \leq \ell$ .

Case (A–C–C). If there exists  $1 \leq s < \ell''$  such that every element of  $I''_{s+1}$  is larger than 0, let  $m$  be the smallest such  $s$ ; otherwise, let  $m = \ell''$ . Then  $I_s = I''_s$ , if  $s \neq m$ , and  $I_m = I''_m \cup \{0\}$ .

Case (B–C–C). If there exists  $1 \leq s < \ell''$  such that every element of  $I''_{s+1}$  is larger than 0, let  $m$  be the smallest such  $s$ ; otherwise, let  $m = \ell''$ . Then  $I_s = I''_s$ , if  $s \neq m$ , and  $I_m = I''_m \cup \{0\}$ .

Case (C–C–C). Let  $I_1 = \{0\}$ , and let  $I_s = I''_{s-1}$ , for  $1 < s \leq \ell$ .

Case (D–C–C). If there exists  $1 \leq s < \ell''$  such that every element of  $I''_{s+1}$  is larger than 0, let  $m$  be the smallest such  $s$ ; otherwise, let  $m = \ell''$ . Write  $I''_m = I''_{m,1} \sqcup I''_{m,2}$  with every element of  $I''_{m,1}$  smaller than 0 and every element of  $I''_{m,2}$  larger than 0. If  $m = 1$  and  $I''_{1,1} = \emptyset$ , let  $I_1 = \{0\}$ , and let  $I_s = I''_{s-1}$ , for  $1 < s \leq \ell$ . Otherwise, let  $I_s = I''_s$ , for  $1 \leq s < m$ , let  $I_m = I''_{m,1}$  and  $I_{m+1} = \{0\} \cup I''_{m,2}$ , and let  $I_s = I''_{s-1}$ , for  $m+1 < s \leq \ell$ .

Case (A–D–A). Let  $I_0 = \{0\}$ , and let  $I_s = I''_s$ , for  $0 < s \leq \ell$ .

Case (B–D–D). If there exists  $1 \leq s < \ell''$  such that every element of  $I''_{s+1}$  is larger than 0, let  $m$  be the smallest such  $s$ ; otherwise, let  $m = \ell''$ . Then  $I_s = I''_s$ , if  $s \neq m$ , and  $I_m = I''_m \cup \{0\}$ .

Case (C–D–C). Let  $I_1 = \{0\}$ , and let  $I_s = I''_{s-1}$ , for  $1 < s \leq \ell$ .

Case (D–D–D). If there exists  $1 \leq s < \ell''$  such that every element of  $I''_{s+1}$  is larger than 0, let  $m$  be the smallest such  $s$ ; otherwise, let  $m = \ell''$ . Write  $I''_m = I''_{m,1} \sqcup I''_{m,2}$  with every element of  $I''_{m,1}$  smaller than 0 and every element of  $I''_{m,2}$  larger than 0. If  $m = 1$  and  $I''_{1,1} = \emptyset$ , let  $I_1 = \{0\}$ , and let  $I_s = I''_{s-1}$ , if  $1 < s \leq \ell$ . Otherwise, let  $I_s = I''_s$ , for  $1 \leq s < m$ , let  $I_m = I''_{m,1}$  and  $I_{m+1} = \{0\} \cup I''_{m,2}$ , and let  $I_s = I''_{s-1}$ , for  $m+1 < s \leq \ell$ .

*Proof.* The assignment of the weight  $k$  to the pair of weights  $(k', k'')$  is a bijection. Indeed, the inverse assignment associates to a weight  $k$  with domain  $\{0, 1, 2, \dots, d\}$ , the pair of weights  $(k', k'')$  with domain  $\{0\}$  and  $\{1, 2, \dots, d\}$ , where  $k''$  is obtained from  $k$  by restriction, and where  $k'$  is given by  $k'(0) = p^{u(k'')}k(0)$ . We note that  $u(k) = u(k') + u(k'')$  and  $\ell = \ell' + \ell''$ .

To prove that the assignment of the partition  $\mathcal{P}$  to the pair of partitions  $(\mathcal{P}', \mathcal{P}'')$  is a bijection we explicitly give the inverse function. In all cases, the partition  $\mathcal{P}'$  is uniquely determined, so we only give the partition  $\mathcal{P}''$ . We consider the cases where  $k'$  and  $k''$  are integral and not integral separately.

Suppose first that  $k'$  and  $k''$  are both not integral.

*Case (A-A-A).* The partition  $\mathcal{P}$  ranges over all type A partitions of  $\text{supp}(k)$  such that  $I_0 \neq \{0\}$ . We let  $I''_0 = I_0 \setminus \{0\}$  and  $I''_s = I_s$ , for  $0 < s \leq \ell''$ .

*Case (C-A-A).* The partition  $\mathcal{P}$  ranges over all type A partitions of  $\text{supp}(k)$  such that  $I_0 = \{0\}$ . We define  $I''_s = I_{s+1}$ .

*Case (A-C-C).* The partition  $\mathcal{P}$  ranges over all type C partitions of  $\text{supp}(k)$  such that  $I_1 \neq \{0\}$ . We let  $I''_1 = I_1 \setminus \{0\}$  and  $I''_s = I_s$ , for  $1 < s \leq \ell''$ .

*Case (C-C-C).* The partition  $\mathcal{P}$  ranges over all type C partitions of  $\text{supp}(k)$  such that  $I_1 = \{0\}$ . We define  $I''_s = I_{s+1}$ .

Suppose next that  $k'$  is integral and that  $k''$  is not integral.

*Case (B-A-A).* The partition  $\mathcal{P}$  ranges over all type A partitions of  $\text{supp}(k)$  such that  $I_0 \neq \{0\}$  and such that  $\min I_s \neq 0$ , for all  $0 < s \leq \ell$ . We let  $I''_s = I_s$ , if  $0 \notin I_s$ , and  $I''_s = I_s \setminus \{0\}$ , if  $0 \in I_s$ .

*Case (D-A-A).* The partition  $\mathcal{P}$  ranges over all type A partitions of  $\text{supp}(k)$  such that  $I_0 = \{0\}$  or such that  $\min I_s = 0$ , for some  $0 < s \leq \ell$ . If  $I_0 = \{0\}$ , we let  $I''_s = I_{s+1}$ . If  $I_0 \neq \{0\}$ , then  $0 \in I_{m+1}$ , for some  $0 \leq m \leq \ell''$ , and we let  $I''_s = I_s$ , for  $0 \leq s < m$ ,  $I''_m = I_m \cup (I_{m+1} \setminus \{0\})$ , and  $I''_s = I_{s+1}$ , for  $m < s \leq \ell''$ .

*Case (B-C-C).* The partition  $\mathcal{P}$  ranges over all type C partitions of  $\text{supp}(k)$  such that  $I_1 \neq \{0\}$  and such that  $\min I_s \neq 0$ , for all  $1 < s \leq \ell$ . We let  $I''_s = I_s$ , if  $0 \notin I_s$ , and  $I''_s = I_s \setminus \{0\}$ , if  $0 \in I_s$ .

*Case (D-C-C).* The partition  $\mathcal{P}$  ranges over all type C partitions of  $\text{supp}(k)$  such that  $I_1 = \{0\}$  or such that  $\min I_s = 0$ , for some  $1 < s \leq \ell$ . If  $I_1 = \{0\}$ , we let  $I''_s = I_{s+1}$ . If  $I_1 \neq \{0\}$ , then  $0 \in I_{m+1}$ , for some  $1 \leq m \leq \ell''$ , and we let  $I''_s = I_s$ , for  $1 \leq s < m$ ,  $I''_m = I_m \cup (I_{m+1} \setminus \{0\})$ , and  $I''_s = I_{s+1}$ , for  $m < s \leq \ell''$ .

We next suppose that  $k'$  is not integral and that  $k''$  is integral.

*Case (A-B-A).* The partition  $\mathcal{P}$  ranges over all type A partitions of  $\text{supp}(k)$  such that  $I_0 \neq \{0\}$ . We let  $I''_0 = I_0 \setminus \{0\}$  and  $I''_s = I_s$ , for  $0 < s \leq \ell''$ .

*Case (C-B-C).* The partition  $\mathcal{P}$  ranges over all type C partitions of  $\text{supp}(k)$  such that  $I_1 \neq \{0\}$ . We let  $I''_0 = I_1 \setminus \{0\}$  and  $I''_s = I_{s+1}$ , for  $0 < s \leq \ell''$ .

*Case (A-D-A).* The partition  $\mathcal{P}$  ranges over all type A partitions of  $\text{supp}(k)$  such that  $I_0 = \{0\}$ . We let  $I''_s = I_s$ , for  $1 \leq s \leq \ell''$ .

*Case (C-D-C).* The partition  $\mathcal{P}$  ranges over all type C partitions of  $\text{supp}(k)$  such that  $I_1 = \{0\}$ . We let  $I''_s = I_{s+1}$ , for  $1 \leq s \leq \ell''$ .

Finally, we suppose that  $k'$  and  $k''$  are both integral.

*Case (B-B-B).* The partition  $\mathcal{P}$  ranges over all type B partitions of  $\text{supp}(k)$  such that  $I_0 \neq \{0\}$  and such that  $\min I_s \neq 0$ , for all  $0 < s \leq \ell$ . We let  $I''_s = I_s$ , if  $0 \notin I_s$ , and  $I''_s = I_s \setminus \{0\}$ , if  $0 \in I_s$ .

*Case (D-B-B).* The partition  $\mathcal{P}$  ranges over all type B partitions of  $\text{supp}(k)$  such that  $I_0 = \{0\}$  or such that  $\min I_s = 0$ , for some  $0 < s \leq \ell$ . If  $I_0 = \{0\}$ , we let  $I''_s = I_{s+1}$ , for  $0 \leq s \leq \ell''$ . If  $I_0 \neq \{0\}$ , then  $0 \in I_{m+1}$ , for some  $0 \leq m \leq \ell''$ , and we let  $I''_s = I_s$ , for  $0 \leq s < m$ ,  $I''_m = I_m \cup (I_{m+1} \setminus \{0\})$ , and  $I''_s = I_{s+1}$ , for  $m < s \leq \ell''$ .

*Case (B-D-D).* The partition  $\mathcal{P}$  ranges over all type D partitions of  $\text{supp}(k)$  such that  $\min I_s \neq 0$ , for all  $1 < s \leq \ell$ . We let  $I''_s = I_s$ , if  $0 \notin I_s$ , and  $I''_s = I_s \setminus \{0\}$ , if  $0 \in I_s$ .

*Case (D-D-D).* The partition  $\mathcal{P}$  ranges over all type D partitions of  $\text{supp}(k)$  such that  $\min I_s = 0$ , for some  $1 < s \leq \ell$ . Then  $0 \in I_{m+1}$ , for some  $1 \leq m < \ell''$ , and we define  $I''_s = I_s$ , for  $1 \leq s < m$ ,  $I''_m = I_m \cup (I_{m+1} \setminus \{0\})$ , and  $I''_s = I_{s+1}$ , for  $m < s \leq \ell''$ .  $\square$

*Proof of Thm. 1.2.* The proof is by induction on the rank  $d$  starting from the case  $d = 1$  which was proved in [7, Thm. B]. So we assume that the statement of Thm. 1.2 is true for  $d$  and prove it is true for  $d + 1$ . Let  $f': A \rightarrow A[T_0]$ ,  $f'': A[T_0] \rightarrow A[T_0][T_1, \dots, T_d]$ , and  $f: A \rightarrow A[T_0, \dots, T_d]$

be the inclusions of the constant polynomials. Let  $k'$  and  $k''$  be two weights with domain  $\{0\}$  and  $\{1, 2, \dots, d\}$ , and let  $k$  be the corresponding weight with domain  $\{0, 1, 2, \dots, d\}$  given by the bijection of Lemma 1.3. We choose a total ordering

$$\text{supp}(k) = \{i_1, \dots, i_r\}$$

such that  $v_p(i_1) \leq \dots \leq v_p(i_r)$  and give  $\text{supp}(k')$  and  $\text{supp}(k'')$  the induced total orderings. We define a total ordering on the set of admissible partitions of  $\text{supp}(k)$  as follows. Two admissible partitions  $\mathcal{P}$  and  $\mathcal{P}'$  of  $\text{supp}(k)$  are either equal or there exists a smallest  $s$  such that the cardinality of  $I_s$  and  $I'_s$  are not equal. In the latter case, we define  $\mathcal{P}$  to be less than  $\mathcal{P}'$  if the cardinality of  $I_s$  is less than the cardinality of  $I'_s$  and vice versa. We note that the admissible partition with  $I_0$  the whole set is the largest and the admissible partition with  $I_0$  empty and the remaining  $I_s$  a one-element set is the smallest.

Let  $\mathcal{P}'$  be an admissible partition of  $\text{supp}(k')$  of length  $\ell'$ , let  $\mathcal{P}''$  be an admissible partition of  $\text{supp}(k'')$  of length  $\ell''$ , and let  $\ell = \ell' + \ell''$ . We claim that the composite

$$E_{n-u(k)}^{q-\ell} \xrightarrow{e(k', \mathcal{P}')} (f'^* E)_{n-u(k'')}^{q-\ell'} \xrightarrow{e(k'', \mathcal{P}'')} (f''^* f'^* E)_n^q = (f^* E)_n^q$$

is a linear combination of the maps

$$e(k, \mathcal{Q}): E_{n-u(k)}^{q-\ell} \rightarrow (f^* E)_n^q,$$

where  $\mathcal{Q}$  ranges over the admissible partition of  $\text{supp}(k)$  of length  $\ell$ . We further claim that the smallest partition  $\mathcal{Q}$  such that the coefficient  $e(k, \mathcal{Q})$  is not divisible by  $p$  is equal to the partition  $\mathcal{P}$  associated to the pair of partitions  $(\mathcal{P}', \mathcal{P}'')$  by the bijection of Lemma 1.3. It follows that the dotted map exists that makes the following diagram commute and that this map is an isomorphism.

$$\begin{array}{ccc} \bigoplus_{(k', \mathcal{P}')} \left( \bigoplus_{(k'', \mathcal{P}'')} E_{n-u(k'')-u(k')}^{q-\ell''-\ell'} \right) & \xrightarrow{\sim e} & \bigoplus_{(k', \mathcal{P}')} f'^* E_{n-u(k')}^{q-\ell'} \\ \downarrow \sim & & \sim \downarrow e \\ \bigoplus_{(k, \mathcal{Q})} E_{n-u(k)}^{q-\ell} & \xrightarrow{e} & (f^* E)_n^q. \end{array}$$

The top horizontal map and the right-hand vertical maps are isomorphisms by the inductive hypothesis. Hence, also the lower horizontal map is an isomorphism. This proves the induction step.

It remains to prove the claim. We will consider the several cases of Lemma 1.3 separately.

*Case (A–A–A).* We rewrite  $e(k'', \mathcal{P}'') \circ e(k', \mathcal{P}')$  as

$$V^{u(k)}(\xi X^{p^{u(k)}k_0} X^{p^{u(k)}k_{I_0''}} F^{u(k)-t(I_1'')} dX^{p^{t(I_1'')}k_{I_1''}} \dots F^{u(k)-t(I_{\ell''}''')} dX^{p^{t(I_{\ell''}''')}k_{I_{\ell''}''}})$$

which is equal to the basic Witt differential  $e(k, \mathcal{P})$ .

*Case (B–A–A).* We rewrite  $e(k'', \mathcal{P}'') \circ e(k', \mathcal{P}')$  by the same formula as in the case (A–A–A). If  $m = 0$ , this expression is equal to the basic Witt differential  $e(k, \mathcal{P})$ . If  $m > 0$ , this formula is not of the form of a basic Witt differential, and we further rewrite it as a linear combination of basic Witt differentials  $e(k, \mathcal{Q})$ . To this end we use that if  $t = t(k) \geq t' = t(k') \geq t'' = t(k'')$ , then

$$\begin{aligned} y^{k'} \cdot F^{-t} d(x^{p^t k} z^{p^t k''}) &= F^{-t} (y^{p^t k'} d(x^{p^t k} z^{p^t k''})) \\ &= F^{-t} d(x^{p^t k} y^{p^t k'} z^{p^t k''}) - x^k \cdot F^{-t} d(y^{p^t k'} z^{p^t k''}) + x^k y^{k'} \cdot F^{-t} d(z^{p^t k''}) \\ &= F^{-t} d(x^{p^t k} y^{p^t k'} z^{p^t k''}) - p^{t-t'} x^k \cdot F^{-t'} d(y^{p^t k'} z^{p^t k''}) \\ &\quad + p^{t-t''} x^k y^{k'} \cdot F^{-t''} d(z^{p^t k''}) \end{aligned}$$

The claim follows by repeated use of this relation.



Case (C-A-A). We rewrite  $e(k'', \mathcal{P}'') \circ e(k', \mathcal{P}')$  as

$$\begin{aligned} & V^{u(k'')} (dV^{u(k')} (\xi X^{p^{u(k')}} k'_0) X^{p^{u(I''_0)} k''_{I''_0}} \dots F^{u(k'')-t(I''_{\ell''})} dX^{p^{t(I''_{\ell''})} k''_{I''_{\ell''}}}) \\ &= p^{u(k'')} dV^{u(k)} (\xi X^{p^{u(k)}} k_0 X^{p^{u(k)} k_{I''_0}} \dots F^{u(k)-t(I''_{\ell''})} dX^{p^{t(I''_{\ell''})} k_{I''_{\ell''}}}) \\ &\quad - V^{u(k)} (\xi X^{p^{u(k)}} k_0 F^{u(k)-t(I''_0)} dX^{p^{t(I''_0)} k_{I''_0}} \dots F^{u(k)-t(I''_{\ell''})} dX^{p^{t(I''_{\ell''})} k_{I''_{\ell''}}}). \end{aligned}$$

This is a linear combination of two basic Witt differentials. The latter is  $e(k, \mathcal{P})$ , and the coefficient of the former is divisible by  $p$ , since  $u(k'') > 0$ .

Case (D-A-A). We rewrite  $e(k'', \mathcal{P}'') \circ e(k', \mathcal{P}')(\xi)$  as

$$V^{u(k)} (\xi F^{u(k)-t(k_0)} dX^{p^{t(k_0)} k_0} \cdot X^{p^{u(I''_0)} k_{I''_0}} \dots F^{u(k)-t(I''_{\ell''})} dX^{p^{t(I''_{\ell''})} k_{I''_{\ell''}}}).$$

This expression is not in the form of a basic Witt differential, but we can rewrite it as a linear combination of basic Witt differentials of the form  $e(k, \mathcal{Q})$  by repeated use of the following two relations. If  $t = t(k) \geq t' = t(k') \geq t'' = t(k'')$ , then

$$\begin{aligned} F^{u-t'} d(y^{p^{t'} k'}) F^{u-t} (x^{p^t k} z^{p^t k''}) &= F^{u-t} (x^{p^t k}) F^{u-t'} d(y^{p^{t'} k'} z^{p^t k''}) \\ &\quad - p^{t'-t''} F^{u-t} (x^{p^t k} y^{p^t k'}) F^{u-t''} d(z^{p^{t''} k''}) \end{aligned}$$

and

$$\begin{aligned} F^{u-t'} d(y^{p^{t'} k'}) F^{u-t} d(x^{p^t k} z^{p^t k''}) &= -F^{u-t} d(x^{p^t k}) F^{u-t'} d(y^{p^{t'} k'} z^{p^t k''}) \\ &\quad + p^{t'-t''} F^{u-t} d(x^{p^t k} y^{p^t k'}) F^{u-t''} d(z^{p^{t''} k''}). \end{aligned}$$

The relations are proved by expanding both sides of the equations. It follows that the composite map  $e(k'', \mathcal{P}'') \circ e(k', \mathcal{P}')$  is equal to a linear combination of maps  $e(k, \mathcal{Q})$  and that  $\mathcal{Q} = \mathcal{P}$  is the smallest admissible partition such that the coefficient of  $e(k, \mathcal{P})$  is not divisible by  $p$ .

Case (A-B-A). This case is analogous to the case (A-A-A).

Case (B-B-B). This case is analogous to the case (B-A-A).

Case (C-B-C). We can rewrite  $e(k'', \mathcal{P}'') \circ e(k', \mathcal{P}')$  as

$$\begin{aligned} & dV^{u(k')} (\xi X^{p^{u(k')}} k'_0) \cdot X^{k''_{I''_0}} F^{-t(I''_1)} dX^{p^{t(I''_1)} k''_{I''_1}} \dots F^{-t(I''_{\ell''})} dX^{p^{t(I''_{\ell''})} k''_{I''_{\ell''}}} \\ &= dV^{u(k)} (\xi X^{p^{u(k)}} k_0 X^{p^{u(k)} k_{I''_0}} F^{u(k)-t(I''_1)} dX^{p^{t(I''_1)} k_{I''_1}} \dots F^{u(k)-t(I''_{\ell''})} dX^{p^{t(I''_{\ell''})} k_{I''_{\ell''}}}) \\ &\quad - V^{u(k)} (\xi X^{p^{u(k)}} k_0 F^{u(k)-t(I''_0)} dX^{p^{t(I''_0)} k_{I''_0}} \dots F^{u(k)-t(I''_{\ell''})} dX^{p^{t(I''_{\ell''})} k_{I''_{\ell''}}}). \end{aligned}$$

This is a linear combination of two basic Witt differentials  $e(k, \mathcal{Q})$ , which both have coefficient not divisible by  $p$ . The partition of the former is smaller than the partition of the latter in the total ordering of the set of admissible partitions of  $\text{supp}(k)$  which we defined at the beginning of the proof. This smaller partition is equal to the partition  $\mathcal{P}$  of Lemma 1.3.

Case (D-B-B). This is analogous to the case (D-A-A).

Case (A-C-C)—(D-C-C). These are completely parallel to the cases (A-A-A)—(D-A-A). For instance, in the case (A-C-C) we rewrite  $e(k'', \mathcal{P}'') \circ e(k', \mathcal{P}')$  as

$$dV^{u(k)} (\xi X^{p^{u(k)}} k_0 X^{p^{u(k)} k_{I''_1}} F^{u(k)-t(I''_2)} dX^{p^{t(I''_2)} k_{I''_2}} \dots F^{u(k)-t(I''_{\ell''})} dX^{p^{t(I''_{\ell''})} k_{I''_{\ell''}}})$$

which is equal to the basic Witt differential  $e(k, \mathcal{P})$ .

Case (A-D-A). In this case,  $e(k'', \mathcal{P}'') \circ e(k', \mathcal{P}') = e(k, \mathcal{P})$ .

*Case (B–D–D).* We claim that  $e(k'', \mathcal{P}'') \circ e(k', \mathcal{P}')$  can be expressed as a linear combination of basic Witt differentials  $e(k, \mathcal{Q})$  and that  $\mathcal{Q} = \mathcal{P}$  is the smallest partition which occurs with a non-zero coefficient. Moreover, this coefficient is equal to 1. Indeed, this follows by iterated use of the relation we considered in the case (A–A–A).

*Case (C–D–C).* In this case,  $e(k'', \mathcal{P}'') \circ e(k', \mathcal{P}') = e(k, \mathcal{P})$ .

*Case (D–D–D).* The composite  $e(k'', \mathcal{P}'') \circ e(k', \mathcal{P}')(\xi)$  is given by

$$\xi F^{-t(I_1')} dX^{p^{t(I_1')} k_0'} F^{-t(I_1'')} dX^{p^{t(I_1'')} k_{I_1''}''} \dots F^{-t(I_{\ell''}''')} dX^{p^{t(I_{\ell''}''')} k_{I_{\ell''}'''}''}.$$

We can use the same relations as in the case (D–A–A) above to write this as a linear combination of basic Witt differentials  $e(k, \mathcal{Q})$  with all partitions  $\mathcal{Q}$  of type D. The smallest partition  $\mathcal{Q}$  such that  $e(k, \mathcal{Q})$  occur with a non-zero coefficients is the partition  $\mathcal{P}$  of Lemma 1.3.  $\square$

*Proof of Thm. B.* Let  $f: A \rightarrow A[T_1, \dots, T_d]$  be the inclusion of the constant polynomials. Since  $f^*$  has a right adjoint, the unique map

$$W_n \Omega_{A[T_1, \dots, T_d]} \rightarrow f^* W_n \Omega_A$$

is an isomorphism. Hence Thm. 1.2 gives a canonical isomorphism

$$e: \bigoplus_{(k, \mathcal{P})} W_{n-u(k)} \Omega_A^{q-\ell} \xrightarrow{\sim} W_n \Omega_{A[T_1, \dots, T_d]}^q$$

where, on the left-hand side, the sum runs over all weights  $k$  of rank  $d$  and filtration  $0 \leq u(k) < n$  and over all admissible partitions  $\mathcal{P}$  of  $\text{supp}(k)$ . We must show that the length of the partition  $\mathcal{P}$  determines the  $I$ -adic filtration as stated.

The statement for  $s = 0$  was proved by Langer and Zink [13, Thm. 2.8] but is also an immediate consequence of Thm. 1.2 once we prove that

$$W_n \Omega_{A[T_1, \dots, T_d]/A} \xrightarrow{\sim} f^* W_n \Omega_{A/A}.$$

To see this, let  $E_n'$  be a Witt complex over  $A' = A[T_1, \dots, T_d]$  with  $W_n(A)$ -linear differentials. Then  $f_* E_n'$  is a Witt complex over  $A$  with  $W_n(A)$ -linear differentials, and hence there is a unique map  $W_n \Omega_{A/A} \rightarrow f_* E_n'$  of Witt complexes over  $A$ . It follows that there is a unique map

$$f^* W_n \Omega_{A/A} \rightarrow E_n'$$

of Witt complexes over  $A'$ , and hence  $f^* W_n \Omega_{A/A}$  is the initial Witt complex over  $A'$  with  $W_n(A)$ -linear differentials.

To prove the statement for  $s > 0$ , we define

$$\text{Fil}^s W_n \Omega_{A[T_1, \dots, T_d]}^q \subset W_n \Omega_{A[T_1, \dots, T_d]}^q$$

to be the image by the isomorphism  $e$  above of the summands  $(k, \mathcal{P})$  with  $\mathcal{P}$  of length  $\ell \leq q - s$ . We must show that

$$\text{Fil}_I^s W_n \Omega_{A[T_1, \dots, T_d]}^q = \text{Fil}^s W_n \Omega_{A[T_1, \dots, T_d]}^q.$$

This is true for  $0 \leq s \leq 1$ , since the theorem holds for  $s = 0$ . The  $I$ -adic filtration is multiplicative by definition, and it is straightforward to check that also the filtration

$$W_n \Omega_{A[T_1, \dots, T_d]} = \text{Fil}^0 W_n \Omega_{A[T_1, \dots, T_d]} \supset \text{Fil}^1 W_n \Omega_{A[T_1, \dots, T_d]} \supset \dots$$

is multiplicative. We then have

$$\text{Fil}_I^s W_n \Omega_{A[T_1, \dots, T_d]}^q \subset \text{Fil}^s W_n \Omega_{A[T_1, \dots, T_d]}^q$$

and must prove equality. Every element on the right-hand side is a sum of elements  $e(k, \mathcal{P})(\eta)$ , where  $k$  is a weight of rank  $d$  and filtration  $0 \leq u(k) < n$ ,  $\mathcal{P}$  an admissible partition of  $\text{supp}(k)$  of

length  $\ell \leq q - s$ , and  $\eta \in W_{n-u(k)}\Omega_A^{q-\ell}$ . So it suffices to show that these elements are contained in the left-hand side. Now if  $\xi \in W_n\Omega_A^\bullet$  and if  $\mathcal{P}: \text{supp}(k) = I_0 \sqcup \cdots \sqcup I_\ell$  is type A, B, or D, then

$$\xi \cdot e(k, \mathcal{P})(\eta) = e(k, \mathcal{P})(F^{u(k)}\xi \cdot \eta),$$

and if  $\mathcal{P}$  is of type C, then

$$\xi \cdot e(k, \mathcal{P})(\eta) = e(k, \mathcal{P})(F^{u(k)}\xi \cdot \eta) - e(k, \mathcal{P}')(F^{u(k)}d\xi \cdot \eta),$$

where  $\mathcal{P}': \text{supp}(k) = I'_0 \sqcup \cdots \sqcup I'_\ell$  is the type A partition given by  $I'_s = I_{s+1}$ . Hence, by Lemma 1.1, it suffices to show that the map

$$W_n\Omega_A^s \otimes W_{n-u(k)}\Omega_A^{q-\ell-s} \rightarrow W_{n-u(k)}\Omega_A^{q-\ell}$$

that to  $\xi \otimes \eta$  assigns  $F^{u(k)}\xi \cdot \eta$  is surjective. A general element on the right-hand side is a sum of elements of the form

$$V^{t_0}([a_0])dV^{t_1}([a_1]) \cdots dV^{t_{q-\ell}}([a_{q-\ell}]),$$

where  $a_0, \dots, a_{q-\ell} \in A$ . But this element is the image of

$$dV^{u(k)+t_1}([a_1]) \cdots dV^{u(k)+t_s}([a_s]) \otimes V^{t_0}([a_0])dV^{t_{s+1}}([a_{s+1}]) \cdots dV^{t_{q-\ell}}([a_{q-\ell}]).$$

The surjectivity and the theorem follows.  $\square$

## 2. The Gauss-Manin connection

Let  $f: X \rightarrow S$  be a map of noetherian  $\mathbb{Z}_{(p)}$ -schemes and assume that the prime  $p$  is odd and nilpotent on  $S$ . The induced map  $f^{-1}\mathcal{O}_S \rightarrow \mathcal{O}_X$  is a map of  $\mathbb{Z}_{(p)}$ -algebras in the topos of sheaves of sets on the small étale site of  $X$  and we define  $W_n\Omega_X^\bullet = W_n\Omega_{\mathcal{O}_X}^\bullet$  and  $W_n\Omega_{X/S}^\bullet = W_n\Omega_{\mathcal{O}_X/f^{-1}\mathcal{O}_S}^\bullet$ .

LEMMA 2.1. *If  $X' \rightarrow X$  is étale and  $X'$  is affine, then*

$$\Gamma(X', W_n\Omega_X^\bullet) = W_n\Omega_{\Gamma(X', \mathcal{O}_X)}^\bullet.$$

*Proof.* We recall that the small étale site of  $X$  consists of the category of étale morphisms  $X' \rightarrow X$  equipped with the étale topology. The inclusion of the full subcategory of étale morphisms  $X' \rightarrow X$  with  $X'$  affine induces an isomorphism of the associated topoi of sheaves of sets. We claim that the pre-sheaf  $E$  of Witt complexes on the latter category given by

$$\Gamma(X', E_n^\bullet) = W_n\Omega_{\Gamma(X', \mathcal{O}_X)}^\bullet$$

is a sheaf for the étale topology. In effect, we claim that the pre-sheaf  $E_n^q$  is a sheaf of quasi-coherent  $W_n(\mathcal{O}_X)$ -modules, for all  $q \geq 0$ . We recall from the proof of [14, III.1, Thm.-Def. 3] that this is equivalent to the statement that for all étale morphisms  $X'' \rightarrow X' \rightarrow X$  with  $X'$  and  $X''$  affine, the following canonical map is an isomorphism.

$$\Gamma(X'', W_n(\mathcal{O}_X)) \otimes_{\Gamma(X', W_n(\mathcal{O}_X))} \Gamma(X, E_n^q) \rightarrow \Gamma(X'', E_n^q).$$

To produce the inverse map, it suffices to show that the left-hand side constitutes a Witt complex over  $\Gamma(X'', \mathcal{O}_X)$ . In effect, by [7, Thm. D], it is enough to show that the left-hand side constitutes a  $V$ -pro-complex over  $\Gamma(X'', \mathcal{O}_X)$  in the sense of [9, I]. And this, given [13, Prop. A.8], follows from the proof of [9, Prop. I.1.14]. Hence  $E$  is a Witt complex over the  $\mathbb{Z}_{(p)}$ -algebra  $\mathcal{O}_X$  in the topos of sheaves of sets on the small étale site of  $X$ . Finally, it is clear from the definition that  $E$  is the initial Witt complex over  $\mathcal{O}_X$ .  $\square$

In a similar way, the following result follows from [13, Prop. 1.9].

LEMMA 2.2. *Suppose that there is a commutative diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

*with the upper and lower horizontal morphisms étale and unramified, respectively, and with  $X'$  and  $S'$  affine. Then*

$$\Gamma(X', W_n \Omega_{X'/S}) = W_n \Omega_{\Gamma(X', \mathcal{O}_X)/\Gamma(S', \mathcal{O}_S)}. \quad \square$$

*Proof of Thm. A.* We define the map of the statement. The canonical map

$$f^{-1} W_n \Omega_S^s \otimes_{f^{-1} W_n(\mathcal{O}_S)} W_n \Omega_X^{-s} \rightarrow \mathrm{gr}_{\mathcal{I}}^s W_n \Omega_X$$

factors through the projection

$$f^{-1} W_n \Omega_S^s \otimes_{f^{-1} W_n(\mathcal{O}_S)} W_n \Omega_X^{-s} \rightarrow f^{-1} W_n \Omega_S^s \otimes_{f^{-1} W_n(\mathcal{O}_S)} W_n \Omega_{X/S}^{-s}$$

and hence induces a map

$$f^{-1} W_n \Omega_S^s \otimes_{f^{-1} W_n(\mathcal{O}_S)} W_n \Omega_{X/S}^{-s} \rightarrow \mathrm{gr}_{\mathcal{I}}^s W_n \Omega_X.$$

The map of the statement is the composition of this map and the canonical map

$$f^{-1} W_n \Omega_S^s \otimes_{f^{-1} W_n(\mathcal{O}_S)}^{\mathbb{L}} W_n \Omega_{X/S}^{-s} \rightarrow f^{-1} W_n \Omega_S^s \otimes_{f^{-1} W_n(\mathcal{O}_S)} W_n \Omega_{X/S}^{-s}.$$

We wish to show that this composite map is an isomorphism in the derived category of quasi-coherent  $f^{-1} W_n(\mathcal{O}_S)$ -modules on the small étale site of  $X$ . The terms in the complexes are quasi-coherent  $W_n(\mathcal{O}_X)$ -modules, but the differential, in general, is not  $W_n(\mathcal{O}_X)$ -linear. However, since  $p$  is nilpotent on  $X$ , there exists  $m \geq n$  such that multiplication by  $p^{m-n}$  annihilates the terms of the complexes. We view the terms of the complexes as  $W_m(\mathcal{O}_X)$ -modules via  $F^{m-n}: W_m(\mathcal{O}_X) \rightarrow W_n(\mathcal{O}_X)$ . Then the terms of the complexes are quasi-coherent  $W_m(\mathcal{O}_X)$ -modules and the differential is  $W_m(\mathcal{O}_X)$ -linear, since  $dF^{m-n} = p^{m-n} F^{m-n} d$ .

We can thus assume that  $S = \mathrm{Spec} R$  and that  $X = \mathrm{Spec} R[T_1, \dots, T_d]$ . We recall from the proof of [13, Thm. 3.5] that the map of differential graded  $W_n(R)$ -algebras

$$\Omega_{W_n(R)[X_1, \dots, X_d]/W_n(R)} \rightarrow W_n \Omega_{R[T_1, \dots, T_d]/R}$$

that to  $X_i$  associates  $[T_i]$  is a quasi-isomorphism and the left-hand side is a complex of free  $W_n(R)$ -modules. Hence, we must show that the following map is a quasi-isomorphism of complexes of  $W_n(R)$ -modules.

$$W_n \Omega_R^s \otimes_{W_n(R)} \Omega_{W_n(R)[X_1, \dots, X_d]/W_n(R)}^{-s} \rightarrow \mathrm{gr}_{\mathcal{I}}^s W_n \Omega_{R[T_1, \dots, T_d]}.$$

We recall from Thm. B the isomorphism of abelian groups

$$e: \bigoplus_{(k, \mathcal{P})} W_{n-u(k)} \Omega_R^s \xrightarrow{\sim} \mathrm{gr}_{\mathcal{I}}^s W_n \Omega_{R[T_1, \dots, T_d]}^q$$

where  $k$  runs over all weights of rank  $d$  and filtration  $0 \leq u(k) < n$  and  $\mathcal{P}$  over all admissible partitions of  $\mathrm{supp}(k)$  of length  $\ell = q - s$ . Under this isomorphism, the map above is an isomorphism onto the summands  $(k, \mathcal{P})$  with  $k$  integral. Hence, the cokernel is identified with the summands  $(k, \mathcal{P})$  with  $k$  not integral. Suppose that  $k$  is not integral. Then there is a one-to-one correspondence between the type A partitions of  $\mathrm{supp}(k)$  of length  $\ell = q - s$  and the type C partitions of  $\mathrm{supp}(k)$  of length  $\ell' = q + 1 - s$  that to the type A partition  $\mathcal{P}: \mathrm{supp}(k) = I_0 \sqcup I_1 \sqcup \dots \sqcup I_\ell$  assigns the type C partition  $\mathcal{P}': \mathrm{supp}(k) = I'_1 \sqcup \dots \sqcup I'_{\ell'}$  with  $I'_r = I_{r-1}$ . Moreover, the differential maps the summand  $(k, \mathcal{P})$  isomorphically onto the summand  $(k, \mathcal{P}')$ . It follows that the cokernel is acyclic as desired.  $\square$

The isomorphism of Thm. A gives rise to a Gauss-Manin connection on the crystalline cohomology of  $X/W_n(S)$ . The construction is similar to the classical case [11] which we first recall.

Let  $X \xrightarrow{f} S \xrightarrow{g} T$  be two composable morphisms of schemes, and let  $\mathcal{I}$  be the kernel of the induced surjective map of de Rham complexes

$$\Omega_{X/T}^\bullet \rightarrow \Omega_{X/S}^\bullet.$$

The  $\mathcal{I}$ -adic filtration of  $\Omega_{X/T}^\bullet$  gives rise to a spectral sequence

$$E_1^{s,t} = \mathbb{R}^{s+t} f_* \text{gr}_{\mathcal{I}}^s \Omega_{X/T}^\bullet \Rightarrow \mathbb{R}^{s+t} f_* \Omega_{X/T}^\bullet$$

of sheaves of abelian groups on the small étale site of  $S$ . Suppose that  $f: X \rightarrow S$  is smooth. Then an argument similar to the proof of Thm. A gives a canonical isomorphism

$$f^{-1} \Omega_{S/T}^s \otimes_{f^{-1} \mathcal{O}_S}^{\mathbb{L}} \Omega_{X/S}^{-s} \xrightarrow{\sim} \text{gr}_{\mathcal{I}}^s \Omega_{X/T}^\bullet$$

in the derived category of quasi-coherent  $f^{-1} \mathcal{O}_S$ -modules, and hence, we obtain the following isomorphism in the derived category of quasi-coherent  $\mathcal{O}_S$ -modules.

$$\mathbb{R} f_* (f^{-1} \Omega_{S/T}^s \otimes_{f^{-1} \mathcal{O}_S}^{\mathbb{L}} \Omega_{X/S}^{-s}) \xrightarrow{\sim} \mathbb{R} f_* \text{gr}_{\mathcal{I}}^s \Omega_{X/T}^\bullet.$$

Suppose further that the schemes  $X$  and  $S$  are of finite Krull dimension. Then the projection formula [5, Prop. II.5.6] gives a canonical isomorphism

$$\Omega_{S/T}^s \otimes_{\mathcal{O}_S}^{\mathbb{L}} \mathbb{R} f_* \Omega_{X/S}^{-s} \xrightarrow{\sim} \mathbb{R} f_* (f^{-1} \Omega_{S/T}^s \otimes_{f^{-1} \mathcal{O}_S}^{\mathbb{L}} \Omega_{X/S}^{-s})$$

in the derived category of quasi-coherent  $\mathcal{O}_S$ -modules. Finally, suppose that for all  $i \geq 0$ , one or both of the quasi-coherent  $\mathcal{O}_S$ -modules  $\Omega_{S/T}^i$  or  $\mathbb{R}^i f_* \Omega_{X/S}^i$  are flat. Then we have

$$E_1^{s,t} = \Omega_{S/T}^s \otimes_{\mathcal{O}_S} \mathbb{R}^t f_* \Omega_{X/S}^i.$$

Since the  $\mathcal{I}$ -adic filtration is multiplicative, the spectral sequence is multiplicative in the sense that the  $E_r$ -terms are bi-graded rings and the  $d_r$ -differentials derivations [3, Chap. XV, Exer. 2]. In particular, the differential  $d_1: E_1^{0,t} \rightarrow E_1^{1,t}$  defines an integrable connection

$$\nabla: \mathbb{R}^t f_* \Omega_{X/S}^i \rightarrow \Omega_{S/T}^1 \otimes_{\mathcal{O}_S} \mathbb{R}^t f_* \Omega_{X/S}^i$$

and this is the Gauss-Manin connection. The differential  $d_1: E_1^{s,t} \rightarrow E_1^{s+1,t}$  is equal to the differential associated with the connection in the sense that

$$d_1(\eta \otimes \omega) = d\eta \otimes \omega + (-1)^s \eta \cdot \nabla(\omega).$$

Suppose that  $S = \text{Spec } A$  and  $T = \text{Spec } B$  are affine. Then the functor that takes a quasi-coherent  $\mathcal{O}_S$ -module  $M$  to the  $A$ -module of global sections  $\Gamma(S, M)$  is exact, and hence the Gauss-Manin connection induces the integrable connection

$$\nabla: H_{\text{dR}}^q(X/S) \rightarrow \Omega_{A/B}^1 \otimes_A H_{\text{dR}}^q(X/S)$$

on the de Rham cohomology of  $X/S$ . This completes our recollection of the classical Gauss-Manin connection.

Similarly, for  $f: X \rightarrow S$  as in the statement of Thm. A, the  $\mathcal{I}$ -adic filtration of the absolute de Rham-Witt complex  $W_n \Omega_X^\bullet$  gives rise to a spectral sequence

$$E_1^{s,t} = \mathbb{R}^{s+t} f_* \text{gr}_{\mathcal{I}}^s W_n \Omega_X^\bullet \Rightarrow \mathbb{R}^{s+t} f_* W_n \Omega_X^\bullet$$

of sheaves of abelian groups on the small étale site of  $S$ . The canonical isomorphism

$$f^{-1} W_n \Omega_S^s \otimes_{f^{-1} W_n(\mathcal{O}_S)}^{\mathbb{L}} W_n \Omega_{X/S}^{-s} \xrightarrow{\sim} \text{gr}_{\mathcal{I}}^s W_n \Omega_X^\bullet$$

of Thm. A induces a canonical isomorphism

$$\mathbb{R}f_*(f^{-1}W_n\Omega_S^s \otimes_{f^{-1}W_n(\mathcal{O}_S)}^{\mathbb{L}} W_n\Omega_{X/S}^{-s}) \xrightarrow{\sim} \mathbb{R}f_* \operatorname{gr}_{\mathcal{I}}^s W_n\Omega_X$$

in the derived category of quasi-coherent  $W_n(\mathcal{O}_S)$ -modules. Suppose further the schemes  $X$  and  $S$  are of finite Krull dimension. Then the projection formula [5, Prop. II.5.6] gives the following canonical isomorphism in the derived category of quasi-coherent  $W_n(\mathcal{O}_S)$ -modules.

$$W_n\Omega_S^s \otimes_{W_n(\mathcal{O}_S)}^{\mathbb{L}} \mathbb{R}f_*(W_n\Omega_{X/S}^{-s}) \xrightarrow{\sim} \mathbb{R}f_*(f^{-1}W_n\Omega_S^s \otimes_{f^{-1}W_n(\mathcal{O}_S)}^{\mathbb{L}} W_n\Omega_{X/S}^{-s}).$$

Suppose, finally, that the quasi-coherent  $W_n(\mathcal{O}_S)$ -modules  $\mathbb{R}^t f_* W_n\Omega_{X/S}^{-s}$  are flat, for all  $i \geq 0$ . This is true, for instance, if  $X$  is an abelian  $S$ -scheme [2]. Then the  $E_1$ -term becomes identified as

$$E_1^{s,t} = W_n\Omega_S^s \otimes_{W_n(\mathcal{O}_S)} \mathbb{R}^t f_* W_n\Omega_{X/S}^{-s}.$$

Since the  $\mathcal{I}$ -adic filtration is multiplicative, the spectral sequence is multiplicative in the sense that the  $E_r$ -terms are bi-graded rings and the  $d_r$ -differentials derivations [3, Chap. XV, Exer. 2]. In particular, the differential  $d_1: E_1^{0,t} \rightarrow E_1^{1,t}$  defines an integrable Witt connection

$$\nabla: \mathbb{R}^t f_* W_n\Omega_{X/S}^{-s} \rightarrow W_n\Omega_S^1 \otimes_{W_n(\mathcal{O}_S)} \mathbb{R}^t f_* W_n\Omega_{X/S}^{-s}, \quad (2.3)$$

and the differential  $d_1: E_1^{s,t} \rightarrow E_1^{s+1,t}$  is then given by

$$d_1(\eta \otimes \omega) = d\eta \otimes \omega + (-1)^s \eta \cdot \nabla(\omega).$$

We remark that Berthelot [1, IV Cor. 3.6.2] has defined a connection

$$\tilde{\nabla}: \mathbb{R}^t f_* W_n\Omega_{X/S}^{-s} \rightarrow \Omega_{W_n(S)}^1 \otimes_{W_n(\mathcal{O}_S)} \mathbb{R}^t f_* W_n\Omega_{X/S}^{-s} \quad (2.4)$$

which for  $n = 1$  agrees with the Gauss-Manin connection [1, Prop. 3.6.4]. We expect that the Witt connection (2.3) is equal to the composite of the connection (2.4) and the canonical projection

$$\Omega_{W_n(S)}^1 \otimes_{W_n(\mathcal{O}_S)} \mathbb{R}^t f_* W_n\Omega_{X/S}^{-s} \rightarrow W_n\Omega_S^1 \otimes_{W_n(\mathcal{O}_S)} \mathbb{R}^t f_* W_n\Omega_{X/S}^{-s}.$$

Suppose that  $S = \operatorname{Spec} R$  is affine. Then the functor that takes a quasi-coherent  $W_n(\mathcal{O}_S)$ -module  $M$  to the  $W_n(R)$ -module of global sections  $\Gamma(S, M)$  is exact. Hence the Witt connection (2.3) induces the integrable Witt connection

$$\nabla: H_{\text{crys}}^q(X/W_n(S)) \rightarrow W_n\Omega_R^1 \otimes_{W_n(R)} H_{\text{crys}}^q(X/W_n(S))$$

on the crystalline cohomology of  $X/W_n(S)$  that we mentioned in the introduction.

### 3. The de Rham-Witt complex with log-poles

Let  $f: (X, M) \rightarrow (S, N)$  be a map of log- $\mathbb{Z}_{(p)}$ -schemes, where  $p$  is an odd prime. We recall the definition of the canonical projection

$$W_n\Omega_{(X,M)} \rightarrow W_n\Omega_{(X,M)/(S,N)}$$

from the absolute to the relative de Rham-Witt complex. But first we recall some basic notions concerning log-schemes from Kato [10].

A pre-log structure on a ring  $A$  in a topos is a map of monoids  $\alpha: M \rightarrow A$  from a monoid  $M$  to the underlying multiplicative monoid of  $A$ . A log-structure is a pre-log structure  $\alpha: M \rightarrow A$  such that the map  $\alpha^{-1}(A^*) \rightarrow A^*$  induced by  $\alpha$  is an isomorphism, and a log-ring  $(A, M)$  is a ring  $A$  with a log-structure  $\alpha: M \rightarrow A$ . The forgetful functor from rings with a log-structure to rings with a pre-log structure has a right adjoint functor which to a ring  $A$  with a pre-log structure  $\alpha: M \rightarrow A$  associates the ring  $A$  with the log-structure  $\alpha^a: M^a \rightarrow A$  given by the following push-out square of

(commutative) monoids.

$$\begin{array}{ccc} \alpha^{-1}(A^*) & \xrightarrow{\alpha} & A^* \\ \downarrow & & \downarrow \\ M & \longrightarrow & M^a. \end{array}$$

If  $(A, M)$  is a log-ring, then we write  $(W_n(A), W_n(M))$  for the ring of Witt vectors of length  $n$  in  $A$  with the log-structure associated to the pre-log structure given by the composition of  $\alpha: M \rightarrow A$  and  $[-]_n: A \rightarrow W_n(A)$ .

A log-differential graded ring  $(E^\bullet, M)$  is a differential graded ring  $E^\bullet$  together with a log-structure  $\alpha: M \rightarrow E^0$  and a map of monoids  $d \log: M \rightarrow E^1$  from  $M$  to the underlying additive monoid of  $E^1$  such that  $d \circ d \log = 0$  and such that  $d\alpha(x) = \alpha(x)d \log x$ , for all  $x \in M$ . Maps of log-rings and log-differential graded rings are defined in the obvious way.

We recall from [6, Def. 3.2.1] that if  $(A, M)$  is a log- $\mathbb{Z}_{(p)}$ -algebra, where  $p$  is an odd prime, then a *Witt complex* over  $(A, M)$  consist of the following data.

(i) a pro-log differential graded ring  $(E, M_E) = \{(E_n^\bullet, M_{E,n})\}_{n \in \mathbb{N}}$  together with a strict map of pro-log rings

$$\lambda: (W_n(A), W_n(M)) \rightarrow (E_n^0, M_{E,n});$$

(ii) a strict map of pro-log graded rings

$$F: (E_n^\bullet, M_{E,n}) \rightarrow (E_{n-1}^\bullet, M_{E,n-1})$$

such that  $\lambda F = F\lambda$  and such that

$$\begin{aligned} Fd\lambda([a]_n) &= \lambda([a]_{n-1})^{p-1}d\lambda([a]_{n-1}), & \text{for all } a \in A, \\ Fd \log_n \lambda(x) &= d \log_{n-1} \lambda(x), & \text{for all } x \in M; \end{aligned}$$

(iii) a strict map of pro-graded  $E_n^\bullet$ -modules

$$V: F_*E_{n-1}^\bullet \rightarrow E_n^\bullet$$

such that  $\lambda V = V\lambda$ ,  $FdV = d$ , and  $FV = p$ .

A map of Witt complexes over  $(A, M)$  is a strict map  $f: (E_n^\bullet, M_{E,n}) \rightarrow (E_n'^\bullet, M_{E',n})$  of pro-log differential graded rings such that  $\lambda' = f\lambda$ ,  $F'f = fF$  and  $V'f = fV$ .

Let  $f: (R, N) \rightarrow (A, M)$  be a map of log- $\mathbb{Z}_{(p)}$ -algebras with  $p$  odd. The absolute de Rham-Witt complex of  $(A, M)$  is defined to be the initial example of a Witt complex over  $(A, M)$ . The relative de Rham-Witt complex of  $(A, M)/(R, N)$  is defined to be the initial example of a Witt complex over  $(A, M)$  with  $W_n(R)$ -linear differentials and with  $d \log_n \lambda(f(x)) = 0$ , for all  $x \in N$  and all  $n \geq 1$ . Standard category theory shows that the initial objects exist and that the canonical map

$$W_n\Omega_{(A,M)}^\bullet \rightarrow W_n\Omega_{(A,M)/(R,N)}^\bullet$$

is surjective. An argument similar to the proof of Lemma 1.1 shows that the kernel  $I$  of the canonical projection is equal to the differential graded ideal generated by the image of the canonical map  $W_n\Omega_{(R,N)}^1 \rightarrow W_n\Omega_{(A,M)}^1$ .

A map of log- $\mathbb{Z}_{(p)}$ -algebras  $f: (A, M) \rightarrow (A', M')$  gives rise to an adjoint pair  $(f^*, f_*)$  between the categories of Witt complexes over  $(A, M)$  and  $(A', M')$ . The direct image functor  $f_*$  is given by viewing a Witt complex over  $(A', M')$  as a Witt complex over  $(A, M)$  by replacing the map  $\lambda$  by the composite  $\lambda \circ W_n(f)$ , and one can prove that the inverse image functor  $f^*$  exists by an argument similar to [7, Prop. 1.2.3]. We describe the inverse image functor  $f^*$  in the case where the log-structure  $M'$  and the inverse image log-structure induced from  $M$  agree.

More precisely, we write  $f = (g, h): (A, M) \rightarrow (A', M')$  and assume that the map of monoids  $h: g^*M \rightarrow M'$  is an isomorphism. Let  $((E, M_E), \lambda, F, V)$  be a Witt complex over  $(A, M)$ . We define a Witt complex  $((E', M_{E'}), \lambda', F', V')$  over  $(A', M')$  and a map of Witt complexes over  $(A, M)$

$$((E, M_E), \lambda, F, V) \rightarrow f_*((E', M_{E'}), \lambda', F', V') \quad (3.1)$$

in the following way. The Witt complex  $((E, M_E), \lambda, F, V)$  over  $(A, M)$  determines, by neglect of structure, a Witt complex  $(E, \varphi, F, V)$  over  $A$ , and we define

$$(E', \varphi', F', V') = g^*(E, \varphi, F, V)$$

to be the inverse image Witt complex over  $A'$ . The composition of the map  $\alpha_{E,n}: M_{E,n} \rightarrow E_n^0$  and the unit  $\eta: E_n^0 \rightarrow (g_*g^*E_n^0) = E_n^0$  of the adjunction  $(g^*, g_*)$  gives rise to a pre-log structure on  $E_n^0$ , and we define

$$\alpha_{E',n}: M_{E',n} \rightarrow E_n^0$$

to be the associated log-structure. The map  $M_{E,n} \rightarrow E_n^1$  which is given as the composition of  $d \log_n: M_{E,n} \rightarrow E_n^1$  and the unit  $\eta: E_n^1 \rightarrow E_n^1$  and the map  $(E_n^0)^* \rightarrow E_n^1$  that takes  $a$  to  $a^{-1}da$  determines a unique strict map of pro-monoids

$$d \log'_n: M_{E',n} \rightarrow E_n^1.$$

The maps  $\lambda: (W_n(A), W_n(M)) \rightarrow (E_n^0, M_{E,n})$  and  $\varphi': W_n(A') \rightarrow E_n^0$  give rise to a strict map of pro-log rings

$$\lambda' = (\varphi', \psi'): (W_n(A'), W_n(M')) \rightarrow (E_n^0, M_{E',n}).$$

Finally, the unit maps  $\eta_E: E \rightarrow E'$  and  $\eta_M: M_E \rightarrow M_{E'}$  define the map (3.1).

LEMMA 3.2. *Let  $f = (g, h): (A, M) \rightarrow (A', M')$  be a map of  $\log\text{-}\mathbb{Z}_{(p)}$ -algebras, where  $p$  is an odd prime, and assume that the map  $h$  induces an isomorphism of monoids  $g^*M \xrightarrow{\sim} M'$ . Then the adjoint of the map (3.1)*

$$f^*((E, M_E), \lambda, F, V) \rightarrow ((E', M_{E'}), \lambda', F', V')$$

is an isomorphism of Witt complexes over  $(A', M')$ .

*Proof.* One readily verifies that  $((E', M_{E'}), \lambda', F', V')$  is indeed a Witt complex over  $(A', M')$  and that the map (3.1)

$$\eta = (\eta_E, \eta_M): ((E, M_E), \lambda, F, V) \rightarrow f_*((E', M_{E'}), \lambda', F', V')$$

is a map of Witt complexes over  $(A, M)$ . Suppose that  $((E'', M_{E''}), \lambda'', F'', V'')$  is a Witt complex over  $(A', M')$  and that

$$\xi = (\xi_E, \xi_M): ((E, M_E), \lambda, F, V) \rightarrow f_*((E'', M_{E''}), \lambda'', F'', V'')$$

is a map of Witt complexes over  $(A, M)$ . We must show that there is a unique map

$$\gamma = (\gamma_E, \gamma_M): ((E', M_{E'}), \lambda', F', V') \rightarrow ((E'', M_{E''}), \lambda'', F'', V'')$$

of Witt complexes over  $(A', M')$  such that  $\xi = f_*(\gamma) \circ \eta$ . Since the map  $\eta_E$  is the unit for the adjunction  $(g^*, g_*)$ , there is a unique map  $\gamma_E$  of Witt complexes over  $A'$  such that  $\xi_E = g_*(\gamma_E) \circ \eta_E$ . The maps  $\xi_M: M_E \rightarrow M''_E$  and  $\gamma_E: (E^0)^* \rightarrow (E''^0)^*$  determine a unique map of pro-monoids  $\gamma_M: M'_E \rightarrow M''_E$  and the pair of maps  $\gamma = (\gamma_E, \gamma_M)$  is the desired map of Witt complexes over  $(A', M')$ .  $\square$

Let  $(A, M)$  be a  $\log\text{-}\mathbb{Z}_{(p)}$ -algebra, let  $g: A \rightarrow A[T_1, \dots, T_d]$  be the inclusion of the constant polynomials, and let  $M[T_1, \dots, T_d] = g^*M$  be the inverse image log-structure on the polynomial algebra. We let  $I$  be the kernel of the projection

$$W_n\Omega_{(A[T_1, \dots, T_d], M[T_1, \dots, T_d])} \rightarrow W_n\Omega_{(A[T_1, \dots, T_d], M[T_1, \dots, T_d])/(A, M)}$$



and consider the  $I$ -adic filtration of the absolute de Rham-Witt complex on the left-hand side. We have the following extension of Thm. B of the introduction.

**THEOREM 3.3.** *Let  $(A, M)$  be a  $\log\text{-}\mathbb{Z}_{(p)}$ -algebra, where  $p$  is an odd prime. Then there is a canonical isomorphism of abelian groups*

$$\bigoplus_{(k, \mathcal{P})} W_{n-u(k)} \Omega_{(A, M)}^s \xrightarrow{\sim} \text{gr}_I^s W_n \Omega_{(A[T_1, \dots, T_d], M[T_1, \dots, T_d])}^q$$

where, on the left-hand side, the sum runs over all weights  $k$  of rank  $d$  and filtration  $0 \leq u(k) < n$  and over all admissible partitions  $\mathcal{P}$  of  $\text{supp}(k)$  of length  $\ell = q - s$ .

*Proof.* We apply Lemma 3.2 to the map

$$f = (g, h): (A, M) \rightarrow (A[T_1, \dots, T_d], M[T_1, \dots, T_d]) = (A', M'),$$

where  $h: M \rightarrow g_* g^* M$  is the unit of the adjunction  $(g^*, g_*)$ . Since the inverse image functor  $f^*$  has a right adjoint, the unique map is an isomorphism

$$W_n \Omega_{(A[T_1, \dots, T_d], M[T_1, \dots, T_d])} \xrightarrow{\sim} f^* W_n \Omega_{(A, M)}.$$

Hence Lemma 3.2 and Thm. 1.2 gives a canonical isomorphism

$$e: \bigoplus_{(k, \mathcal{P})} W_{n-u(k)} \Omega_{(A, M)}^{q-\ell} \xrightarrow{\sim} W_n \Omega_{(A[T_1, \dots, T_d], M[T_1, \dots, T_d])}^q$$

where, on the left-hand side, the sum runs over all weights  $k$  of rank  $d$  and filtration  $0 \leq u(k) < n$  and over all admissible partitions  $\mathcal{P}$  of  $\text{supp}(k)$ . The proof that the length of the partition  $\mathcal{P}$  determines the  $I$ -adic filtration as stated is completely analogous to the argument given in the proof of Thm. B.  $\square$

Let  $f: (X, M) \rightarrow (S, L)$  be a morphism of  $\log\text{-}\mathbb{Z}_{(p)}$ -schemes and assume that the schemes  $X$  and  $S$  are noetherian and that  $p$  is odd and nilpotent on  $S$ . We consider the canonical projection

$$W_n \Omega_{(X, M)} \rightarrow W_n \Omega_{(X, M)/(S, L)}$$

from the absolute to the relative de Rham-Witt complex. The terms of the complexes are quasi-coherent  $W_n(\mathcal{O}_X)$ -modules on the small étale site of  $X$ , and the differential of the relative complex is  $f^{-1} W_n(\mathcal{O}_S)$ -linear. The kernel  $\mathcal{I}$  of the projection is equal to the differential graded ideal generated by the image of the canonical map  $f^{-1} W_n \Omega_{(S, L)}^1 \rightarrow W_n \Omega_{(X, M)}^1$ . The graded pieces for the  $\mathcal{I}$ -adic filtration are complexes of quasi-coherent  $f^{-1} W_n(\mathcal{O}_S)$ -modules on the small étale site of  $X$ . We prove the following extension of Thm. A of the introduction.

**THEOREM 3.4.** *Let  $(S, L)$  be a  $\log\text{-}\mathbb{Z}_{(p)}$ -scheme with  $S$  noetherian and  $p$  odd and nilpotent on  $S$ . Let  $f: X \rightarrow S$  be a smooth morphism and let  $M = f^* L$  be the inverse image log-structure on  $X$ . Then there is a canonical isomorphism*

$$f^{-1} W_n \Omega_{(S, L)}^s \otimes_{f^{-1} W_n(\mathcal{O}_S)}^{\mathbb{L}} W_n \Omega_{(X, M)/(S, L)}^{-s} \xrightarrow{\sim} \text{gr}_{\mathcal{I}}^s W_n \Omega_{(X, M)}$$

in the derived category of quasi-coherent  $f^{-1} W_n(\mathcal{O}_S)$ -modules.

*Proof.* The proof is completely analogous to the proof of Thm. A with the exception that Thm. 3.3 is used in place of Thm. B.  $\square$

*Remark 3.5.* Let  $k$  be a perfect field of odd characteristic  $p$ , and let  $(S, L)$  be  $S = \text{Spec } k$  with the log-structure associated to the pre-log structure given by the map  $\alpha: \mathbb{N}_0 \rightarrow k$  that takes 1 to 0. Let  $t \in L$  be the image of  $1 \in \mathbb{N}_0$ . Then the absolute de Rham-Witt complex is the exterior algebra

$$W_n \Omega_{(S, L)} = \Lambda_{W_n(k)} \{d \log_n t\}$$

with zero differential. Let  $f: Y \rightarrow S$  be a smooth morphism and let  $N = f^*L$  be the inverse image log-structure on  $Y$ . Then theorem 3.4 gives an exact triangle

$$W_n\Omega_{(Y,N)/(S,L)}^{-1} \rightarrow W_n\Omega_{(Y,N)} \rightarrow W_n\Omega_{(Y,N)/(S,L)} \rightarrow W_n\Omega_{(Y,N)/(S,L)}.$$

It was proved by Hyodo and Kato [8, Prop. 1.5] that such an exact triangle exists, more generally, if  $f: Y \rightarrow S$  is the special fiber of a scheme  $X$  with semi-stable reduction over a complete discrete valuation ring of mixed characteristic  $(0, p)$  with residue field  $k$  and if  $\alpha: N \rightarrow Y$  is the inverse image log-structure of the log-structure on  $X$  defined by  $Y$ . We recall from *loc. cit.* that, by definition, the map of cohomology groups induced by the boundary map of the triangle above is the monodromy operator on log-crystalline cohomology

$$\mathcal{N}: H_{\text{crys}}^*((Y, M)/(W_n(S), W_n(N))) \rightarrow H_{\text{crys}}^*((Y, M)/(W_n(S), W_n(N))).$$

This map is zero, if  $f: Y \rightarrow S = \text{Spec } k$  is smooth, since the composite

$$W_n\Omega_Y \rightarrow W_n\Omega_{(Y,N)} \rightarrow W_n\Omega_{(Y,N)/(S,L)}$$

is an isomorphism. It is an interesting problem to also generalize theorem 3.4 to the semi-stable reduction case.

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