THE TOWER OF K-THEORY OF TRUNCATED POLYNOMIAL ALGEBRAS

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Introduction

Suppose that A is a regular noetherian ring that is also an $\mathbb{F}_p$-algebra. Then it was proved by the author and Madsen [**6**, **8**, **4**] that there is a long-exact sequence

$$\cdots \longrightarrow \bigoplus_{i \geqslant 0} \mathbb{W}_{i+1}\Omega_A^{q-2i} \xrightarrow{V_m} \bigoplus_{i \geqslant 0} \mathbb{W}_{m(i+1)}\Omega_A^{q-2i} \xrightarrow{\varepsilon} K_{q+1}(A[x]/(x^m),(x)) \longrightarrow \cdots$$

which expresses the K-groups of the truncated polynomial algebra $A[x]/(x^m)$ relative to the ideal (x) in terms of the groups $\mathbb{W}_r\Omega_A^q$ of big de Rham-Witt forms; see also [**5**]. We remark that in the papers [**6**, **8**, **4**] the result was stated and proved under the more restrictive assumption that the ring A be a smooth $\mathbb{F}_p$-algebra. However, since Popescu [**14**] has proved that every ring A as above can be written as a filtered colimit of smooth $\mathbb{F}_p$-algebras, the general case immediately follows. In this paper, we consider the map of relative K-groups

$$f_* \colon K_{q+1}(A[x]/(x^m),(x)) \to K_{q+1}(A[x]/(x^n),(x))$$

induced by the canonical projection

$$f \colon A[x]/(x^m) \to A[x]/(x^n).$$

To state the result, we recall from [**8**] that the big de Rham-Witt groups $\mathbb{W}_r\Omega_A^q$ are modules over the ring $\mathbb{W}(A)$ of big Witt vectors in A. In particular, they are modules over $\mathbb{W}(\mathbb{F}_p)$. The ring $\mathbb{W}(\mathbb{F}_p)$ is canonically isomorphic to the product indexed by the set of positive integers not divisible by p of copies of the ring $\mathbb{Z}_p$ of p-adic integers. A unit α of the total quotient ring of $\mathbb{W}(\mathbb{F}_p)$ determines a divisor $\mathrm{div}(\alpha)$ on $\mathbb{W}(\mathbb{F}_p)$ and, conversely, the unit α is determined, up to multiplication by a unit of $\mathbb{W}(\mathbb{F}_p)$, by the divisor $\mathrm{div}(\alpha)$. The ring $\mathbb{W}_r(\mathbb{F}_p)$ of big Witt vectors of length r in $\mathbb{F}_p$ determines a divisor on $\mathbb{W}(\mathbb{F}_p)$ that we denote by $\mathrm{div}(\mathbb{W}_r(\mathbb{F}_p))$.

THEOREM A. *Let A be a regular noetherian ring and an $\mathbb{F}_p$-algebra. Then the canonical projection $f \colon A[x]/(x^m) \to A[x]/(x^n)$ induces a map of long-exact sequences*

$$\begin{array}{ccccccc}
\cdots \longrightarrow \bigoplus_{i \geqslant 0} \mathbb{W}_{i+1}\Omega_A^{q-2i} & \xrightarrow{V_m} & \bigoplus_{i \geqslant 0} \mathbb{W}_{m(i+1)}\Omega_A^{q-2i} & \xrightarrow{\varepsilon} & K_{q+1}(A[x]/(x^m),(x)) & \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow \bigoplus_{i \geqslant 0} \mathbb{W}_{i+1}\Omega_A^{q-2i} & \xrightarrow{V_n} & \bigoplus_{i \geqslant 0} \mathbb{W}_{n(i+1)}\Omega_A^{q-2i} & \xrightarrow{\varepsilon} & K_{q+1}(A[x]/(x^n),(x)) & \longrightarrow \cdots
\end{array}$$

where the right-hand vertical map is the map of relative K-groups induced by the canonical projection, where the middle vertical map takes the ith summand of the domain to the ith summand of the target by the composition

$$\mathbb{W}_{m(i+1)}\Omega_A^{q-2i} \xrightarrow{\mathrm{res}} \mathbb{W}_{n(i+1)}\Omega_A^{q-2i} \xrightarrow{m_\alpha} \mathbb{W}_{n(i+1)}\Omega_A^{q-2i}$$

of the restriction map and the multiplication by an element $\alpha = \alpha_p(m,n,i)$ *of* $\mathbb{W}(\mathbb{F}_p)$ *that is determined, up to a unit, by the effective divisor*

$$\mathrm{div}(\alpha) = \sum_{0 \leqslant h < i} \big(\mathrm{div}(\mathbb{W}_{m(h+1)}(\mathbb{F}_p)) - \mathrm{div}(\mathbb{W}_{n(h+1)}(\mathbb{F}_p)) \big),$$

and where the left-hand vertical map is zero.

The divisor $\mathrm{div}(\alpha_p(m,n,i))$ not only depends on the integers m, n, and i, but also on the prime number p. In more detail, recall that the length of a module M over a ring R is the length s of a longest ascending chain $M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_s$ of R-submodules of M. If a longest chain does not exists, the module M is said to have infinite length. A $\mathbb{Z}_p$-module M of finite order p^s has length s. Then the lengths as $\mathbb{Z}_p$-module of the domain and target of the map

$$f_* \colon K_{2i+1}(\mathbb{F}_p[x]/(x^m),(x)) \to K_{2i+1}(\mathbb{F}_p[x]/(x^n),(x))$$

are equal to $(m-1)(i+1)$ and $(n-1)(i+1)$, respectively, and hence, do not depend on the prime p. By contrast, the lengths as a $\mathbb{Z}_p$-module of the kernel $K_{2i+1}(\mathbb{F}_p[x]/(x^m),(x^n))$ and cokernel $K_{2i}(\mathbb{F}_p[x]/(x^m),(x^n))$ of this map are arithmetic functions of the prime number p.

We note that, if $\mathrm{div}(\alpha_p(m,n,i)) \geqslant \mathrm{div}(\mathbb{W}_{n(i+1)}(\mathbb{F}_p))$ then the multiplication by α map

$$m_\alpha \colon \mathbb{W}_{n(i+1)}\Omega_A^{q-2i} \to \mathbb{W}_{n(i+1)}\Omega_A^{q-2i}$$

is zero. We examine the divisor $\alpha_p(m,n,i)$ and prove the following result.

THEOREM B. *Let A be a regular noetherian ring and an $\mathbb{F}_p$-algebra and assume that A is finitely generated as an algebra over the subring $A^p \subset A$ of pth powers. Let m and n be positive integers with $m > n + 1$. Then there exists an integer $q_0 \geqslant 1$ such that the map*

$$f_* \colon K_{q+1}(A[x]/(x^m),(x)) \to K_{q+1}(A[x]/(x^n),(x))$$

induced by the canonical projection is zero, for all $q \geqslant q_0$.

The main purpose of the paper [**8**] was to evaluate the Nil-groups of the truncated polynomial rings $A[x]/(x^m)$. We recall that, for every ring R, the ring homomorphisms $\eta \colon R \to R[t]$ and $\epsilon \colon R[t] \to R$ defined by $\eta(a) = a$ and $\epsilon(f) = f(0)$ give rise to a direct sum decomposition

$$K_{q+1}(R[t]) = K_{q+1}(R) \oplus K_{q+1}(R[t],(t)).$$

The second summand on the right-hand side is the Nil-group

$$\mathrm{Nil}_q(R) = K_{q+1}(R[t],(t))$$

which measures the extent to which the functor K_{q+1} fails to be homotopy invariant. The group $\mathrm{Nil}_q(R)$ is zero, if the ring R is regular. This implies that, for $R = A[x]/(x^m)$ with A regular, the canonical map

$$\mathrm{Nil}_q(A[x]/(x^m),(x)) \to \mathrm{Nil}_q(A[x]/(x^m))$$

is an isomorphism. The ring homomorphisms $\eta \colon R \to R[t]$ and $\epsilon \colon R[t] \to R$ also give rise to a direct sum decomposition of big de Rham-Witt groups

$$\mathbb{W}_r\Omega_{R[t]}^q = \mathbb{W}_r\Omega_R^q \oplus \mathbb{W}_r\Omega_{(R[t],(t))}^q.$$

Hence, if A is a regular noetherian ring and an $\mathbb{F}_p$-algebra, the long-exact sequence at the beginning of the paper implies the long-exact sequence

$$\cdots \longrightarrow \bigoplus_{i \geqslant 0} \mathbb{W}_{i+1}\Omega_{(A[t],(t))}^{q-2i} \xrightarrow{V_m} \bigoplus_{i \geqslant 0} \mathbb{W}_{m(i+1)}\Omega_{(A[t],(t))}^{q-2i} \xrightarrow{\varepsilon} \mathrm{Nil}_q(A[x]/(x^m)) \longrightarrow \cdots$$

which expresses the Nil-groups of $A[x]/(x^m)$ in terms of the relative groups of big de Rham-Witt forms. The map V_m is injective, if p does not divide m, but is generally not injective, if p divides m. We remark that [**10**, Thm. B] gives an explicit description of the middle and left-hand terms of this sequence as functors of the big de Rham-Witt groups of the ring A. In particular, the group $\mathrm{Nil}_q(A[x]/(x^m))$ is non-zero, for all integers $q \geqslant 0$ and $m > 1$. We note that $\mathrm{Nil}_q(A[x]/(x^m))$ is zero, for $q < 0$ [**1**, Chap. XII, Prop. 10.1]. Thm. A above immediately implies the following result.

THEOREM C.   *Let A be a regular noetherian ring and an $\mathbb{F}_p$-algebra. Then the canonical projection $f \colon A[x]/(x^m) \to A[x]/(x^n)$ induces a map of long-exact sequences*

$$\cdots \longrightarrow \bigoplus_{i \geqslant 0} \mathbb{W}_{i+1}\Omega^{q-2i}_{(A[t],(t))} \xrightarrow{V_m} \bigoplus_{i \geqslant 0} \mathbb{W}_{m(i+1)}\Omega^{q-2i}_{(A[t],(t))} \xrightarrow{\varepsilon} \mathrm{Nil}_q(A[x]/(x^m)) \longrightarrow \cdots$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow \qquad\qquad\qquad\qquad \downarrow{\scriptstyle f_*}$$

$$\cdots \longrightarrow \bigoplus_{i \geqslant 0} \mathbb{W}_{i+1}\Omega^{q-2i}_{(A[t],(t))} \xrightarrow{V_n} \bigoplus_{i \geqslant 0} \mathbb{W}_{n(i+1)}\Omega^{q-2i}_{(A[t],(t))} \xrightarrow{\varepsilon} \mathrm{Nil}_q(A[x]/(x^n)) \longrightarrow \cdots$$

where the middle vertical map takes the ith summand of the domain to the ith summand of the target by the composition

$$\mathbb{W}_{m(i+1)}\Omega^{q-2i}_{(A[t],(t))} \xrightarrow{\mathrm{res}} \mathbb{W}_{n(i+1)}\Omega^{q-2i}_{(A[t],(t))} \xrightarrow{m_\alpha} \mathbb{W}_{n(i+1)}\Omega^{q-2i}_{(A[t],(t))}$$

of the restriction map and the multiplication by the element $\alpha = \alpha_p(m,n,i)$ of Thm. A, and where the left-hand vertical map is zero.

Similarly, Thm. B above immediately implies the following result.

THEOREM D.   *Let A be a regular noetherian ring and an $\mathbb{F}_p$-algebra and assume that A is finitely generated as an algebra over the subring $A^p \subset A$ of pth powers. Let m and n be positive integers with $m > n + 1$. Then there exists an integer $q_0 \geqslant 1$ such that the map*

$$f_* \colon \mathrm{Nil}_q(A[x]/(x^m)) \to \mathrm{Nil}_q(A[x]/(x^n))$$

induced by the canonical projection is zero, for all $q \geqslant q_0$.

We remark that, if A is an $\mathbb{F}_p$-algebra and m is a power of p, then $A[x]/(x^m)$ is equal to the group algebra $A[C_m]$ of the cyclic group of order m with generator $1 + x$.

All rings considered in this paper are assumed to be commutative and unital. We write $\mathbb{T}$ for the multiplicative group of complex numbers of modulus 1.

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1.   *p-typical decompositions*

The long-exact sequences that appear in Thm. A of the introduction have canonical p-typical product decompositions. We here recall the p-typical decomposition and state the equivalent Thm. 1.5.

We recall from [**8**, Cor. 1.2.6] that the big de Rham-Witt groups $\mathbb{W}_r\Omega^q_A$ decompose as a product of the p-typical de Rham-Witt groups $W_s\Omega^q_A$ of Bloch-Deligne-Illusie [**11**]. More generally, there is a big de Rham-Witt group $\mathbb{W}_S\Omega^q_A$ associated to every subset $S \subset \mathbb{N}$ of the set of positive integers stable under division. The group $\mathbb{W}_\emptyset\Omega^q_A$ is zero, the group $\mathbb{W}_r\Omega^q_A$ is the big de Rham-Witt group associated to the set of positive integers less than or equal to r, and

the p-typical de Rham-Witt group $W_s\Omega_A^q$ is the big de Rham-Witt group

$$W_s\Omega_A^q = \mathbb{W}_{\{1,p,\dots,p^{s-1}\}}\Omega_A^q.$$

For every pair of subset $T \subset S \subset \mathbb{N}$ stable under division, there is a map

$$\text{res}\colon \mathbb{W}_S\Omega_A^q \to \mathbb{W}_T\Omega_A^q \qquad \text{(restriction)}.$$

We mention that, if $S \subset \mathbb{N}$ is the union of a family of subsets $S_\alpha \subset \mathbb{N}$ each of which is stable under division, then the restriction maps induce an isomorphism

$$\mathbb{W}_S\Omega_A^q \xrightarrow{\sim} \lim_{\text{res}} \mathbb{W}_{S_\alpha}\Omega_A^q.$$

We also mention that every subset $S \subset \mathbb{N}$ stable under division is equal to the union of the subsets $\langle s \rangle \subset \mathbb{N}$, $s \in S$, where $\langle s \rangle$ is the set of divisors of s. Let $S \subset \mathbb{N}$ be a subset stable under division, let s be any positive integer, and let S/s be the set of positive integers t such that $st \in S$. Then there are maps

$$F_s\colon \mathbb{W}_S\Omega_A^q \to \mathbb{W}_{S/s}\Omega_A^q \qquad \text{(Frobenius)}$$
$$V_s\colon \mathbb{W}_{S/s}\Omega_A^q \to \mathbb{W}_S\Omega_A^q \qquad \text{(Verschiebung)}.$$

Now, let $S \subset \mathbb{N}$ be a finite subset stable under division, and let j be a positive integer that is not divisible by p. We define the non-negative integer $u = u_p(S,j)$ by

$$u_p(S,j) = \text{card}(S/j \cap \{1,p,p^2,\dots\}) = \text{card}(S \cap \{j,pj,p^2j,\dots\})$$

and note that u is non-zero if and only if $j \in S$. We then let

$$\eta_j\colon \mathbb{W}_S\Omega_A^q \to W_u\Omega_A^q$$

be the composite map

$$\mathbb{W}_S\Omega_A^q \xrightarrow{F_j} \mathbb{W}_{S/j}\Omega_A^q \xrightarrow{\text{res}} \mathbb{W}_{\{1,p,\dots,p^{u-1}\}}\Omega_A^q =\!\!=\!\!= W_u\Omega_A^q$$

and define

$$\eta\colon \mathbb{W}_S\Omega_A^q \to \prod_j W_u\Omega_A^q$$

to be the map given by the maps η_j as j ranges over the set I_p of all positive integers not divisible by p. The map η is an isomorphism, for every $\mathbb{Z}_{(p)}$-algebra, by [**8**, Cor. 1.2.6]. We remark that, on the right-hand side, the factors indexed by $j \notin S$ are zero. The maps res, F_s, and V_s are also expressed as products of their p-typical analogs

$$R = \text{res}\colon W_s\Omega_A^q \to W_{s-1}\Omega_A^q \qquad \text{(restriction)}$$
$$F = F_p\colon W_s\Omega_A^q \to W_{s-1}\Omega_A^q \qquad \text{(Frobenius)}$$
$$V = V_p\colon W_{s-1}\Omega_A^q \to W_s\Omega_A^q \qquad \text{(Verschiebung)}.$$

Let $T \subset S \subset \mathbb{N}$ be a pair of subsets stable under division. Then there is a commutative diagram

$$\begin{array}{ccc}
\mathbb{W}_S\Omega_A^q & \xrightarrow{\eta} & \prod_j W_u\Omega_A^q \\
\downarrow{\scriptstyle\text{res}} & & \downarrow{\scriptstyle\text{res}^\eta} \\
\mathbb{W}_T\Omega_A^q & \xrightarrow{\eta} & \prod_j W_{u'}\Omega_A^q,
\end{array}$$

where $u = u_p(S,j)$ and $u' = u_p(T,j)$, and where the map res^η takes the factor indexed by an integer $j \in T$ that is not divisible by p to the factor indexed by the same integer j by the map $R^{u-u'}$ and annihilates the remaining factors. Similarly, let $S \subset \mathbb{N}$ be a subset stable under division, and let s be a positive integer. We write $s = p^v s'$ with s' not divisible by p. Then

there are commutative diagrams

$$
\begin{array}{ccc}
\mathbb{W}_S\Omega_A^q & \xrightarrow{\ \eta\ } & \prod_j W_u\Omega_A^q \\
{\scriptstyle V_s}\Big\uparrow\Big\downarrow{\scriptstyle F_s} & & {\scriptstyle V_s^\eta}\Big\uparrow\Big\downarrow{\scriptstyle F_s^\eta} \\
\mathbb{W}_{S/s}\Omega_A^q & \xrightarrow{\ \eta\ } & \prod_j W_{u-v}\Omega_A^q,
\end{array}
$$

where $u = u_p(S, j)$, and where the maps F_s^η and V_s^η are defined as follows: The map F_s^η takes the factor indexed by an integer $j \in S$ that is not divisible by p, but is divisible by s', and satisfies $p^v j \in S$ to the factor indexed by the integer $j/s' \in S/s$ by the map F^v and annihilates the remaining factors. The map V_s^η takes the factor indexed by $j \in S/s$ not divisible by p to the factor indexed by $s'j \in S$ by the map $s'V^v$.

Specializing to the case $S = \{1, 2, \ldots, m(i+1)\}$, we define

$$
s_p(m, i, j) = u_p(S, j) = \operatorname{card}(\{1, 2, \ldots, m(i+1)\} \cap \{j, pj, p^2 j, \ldots\}). \tag{1.1}
$$

The integer $s_p(m, i, j)$ is non-zero if and only if $1 \leqslant j \leqslant m(i+1)$, and in this case, is the unique integer s such that $p^{s-1}j \leqslant m(i+1) < p^s j$. Suppose now that A is a regular noetherian ring and an $\mathbb{F}_p$-algebra. It is then straightforward to verify that the p-typical decomposition of the top long-exact sequence in the diagram in Thm. A of the introduction takes the form

$$
\begin{aligned}
\cdots \to \bigoplus_{i \geqslant 0} \bigoplus_{j \in m'I_p} W_{s-v}\Omega_A^{q-2i} & \xrightarrow{m'V^v} \bigoplus_{i \geqslant 0} \bigoplus_{j \in I_p} W_s\Omega_A^{q-2i} \\
\xrightarrow{\varepsilon} \ K_{q+1}(A[x]/(x^m), (x)) & \xrightarrow{\partial} \bigoplus_{i \geqslant 0} \bigoplus_{j \in m'I_p} W_{s-v}\Omega_A^{q-1-2i} \to \cdots
\end{aligned} \tag{1.2}
$$

where $s = s_p(m, i, j)$, where $m = p^v m'$ with m' not divisible by p, and where I_p is the set of positive integers not divisible by p. Similarly, the bottom long-exact sequence in the diagram in the statement of Thm. A takes the form

$$
\begin{aligned}
\cdots \to \bigoplus_{i \geqslant 0} \bigoplus_{j \in n'I_p} W_{t-w}\Omega_A^{q-2i} & \xrightarrow{n'V^w} \bigoplus_{i \geqslant 0} \bigoplus_{j \in I_p} W_t\Omega_A^{q-2i} \\
\xrightarrow{\varepsilon} \ K_{q+1}(A[x]/(x^n), (x)) & \xrightarrow{\partial} \bigoplus_{i \geqslant 0} \bigoplus_{j \in n'I_p} W_{t-w}\Omega_A^{q-1-2i} \to \cdots
\end{aligned} \tag{1.3}
$$

where $t = s_p(n, i, j)$, and where $n = p^w n'$ with n' not divisible by p.

Finally, we explain the p-typical decomposition of the divisor $\operatorname{div}(\alpha_p(m, n, i))$ that appears in Thm. A. Let $\mathscr{S} \subset \mathbb{W}(\mathbb{F}_p)$ be the subset of non-zero-divisors. Then the map

$$
\operatorname{div}\colon (\mathscr{S}^{-1}\mathbb{W}(\mathbb{F}_p))^*/\mathbb{W}(\mathbb{F}_p)^* \to \operatorname{Div}(\mathbb{W}(\mathbb{F}_p))
$$

is injective. We refer to [**3**, §21] for the general theory of divisors. Let also $S \subset W(\mathbb{F}_p)$ be the subset of non-zero-divisors, and let

$$
\operatorname{ord}\colon (\mathscr{S}^{-1}\mathbb{W}(\mathbb{F}_p))^*/\mathbb{W}(\mathbb{F}_p)^* \xrightarrow{\sim} \prod_{j \in I_p} (S^{-1}W(\mathbb{F}_p))^*/W(\mathbb{F}_p)^* \xrightarrow{\sim} \prod_{j \in I_p} \mathbb{Z}
$$

be the composition of the isomorphism induced by the ring isomorphism η and the isomorphism given by the p-adic valuation. We recall the function $s_p(m, i, j)$ from (1.1) above. The p-typical decomposition of the ring $\mathbb{W}_{m(i+1)}(\mathbb{F}_p)$ immediately implies the following result.

LEMMA 1.4. *For every* $a \in (\mathscr{S}^{-1}\mathbb{W}(\mathbb{F}_p))^*$, *the following are equivalent:*
  (i) $\operatorname{div}(a) = \operatorname{div}(\mathbb{W}_{m(i+1)}(\mathbb{F}_p))$.
  (ii) $\operatorname{ord}(a) = (s_p(m, i, j) \mid j \in I_p)$.

The following statement is equivalent to Thm. A of the introduction. The proof is given in Sect. 5 below.

THEOREM 1.5. *Let A be a regular noetherian ring and an $\mathbb{F}_p$-algebra. Then the canonical projection $f\colon A[x]/(x^m) \to A[x]/(x^n)$ induces a map of long-exact sequences from the sequence (1.2) to the sequence (1.3) that is given, on the lower left-hand terms, by the map*

$$f_*\colon K_*(A[x]/(x^m),(x)) \to K_*(A[x]/(x^n),(x))$$

induced by the canonical projection, on the upper right-hand terms, by the map that takes the (i,j)th summand of the domain to the (i,j)th summand of the target by the composite map

$$W_s\Omega_A^{q-2i} \xrightarrow{R^{s-t}} W_t\Omega_A^{q-2i} \xrightarrow{m_\alpha} W_t\Omega_A^{q-2i},$$

where the right-hand map is multiplication by a p-adic integer $\alpha = \alpha_p(m,n,i,j)$ whose p-adic valuation is given by the sum

$$v_p(\alpha_p(m,n,i,j)) = \sum_{0 \leqslant h < i} \big(s_p(m,h,j) - s_p(n,h,j)\big),$$

and, on the upper left-hand terms, by the zero map. Here $s_p(m,i,j)$ is the integer (1.1).

REMARK 1.6. (i) The integer $s_p(m,h,j) - s_p(n,h,j)$ is equal to the number of positive integers r such that $n(h+1) < p^{r-1}j \leqslant m(h+1)$.

(ii) We recall from [**11**, Prop. I.3.4] that the map $m_\alpha\colon W_t\Omega_A^{q-2i} \to W_t\Omega_A^{q-2i}$ given by the multiplication by a p-adic integer α of p-adic valuation v factors canonically

$$W_t\Omega_A^{q-2i} \xrightarrow{R^v} W_{t-v}\Omega_A^{q-2i} \xrightarrow{m_\alpha} W_t\Omega_A^{q-2i}$$

as the composite of the surjective restriction map R^v and an injective map $\underline{m}_\alpha$. There is no analog of this factorization for the big de Rham-Witt groups. Indeed, the quotient of $\mathbb{W}(\mathbb{F}_p)$ by the ideal generated by the element $\alpha_p(m,n,i) \in \mathbb{W}(\mathbb{F}_p)$ that appears in Thm. A is generally not of the form $\mathbb{W}_S(\mathbb{F}_p)$ for some subset $S \subset \mathbb{N}$ that is stable under division.

2. *Topological Hochschild homology*

The proof of Thm. A of the introduction is based on the following result which we proved in [**6**, Prop. 4.2.3]: For every $\mathbb{F}_p$-algebra A, there is a long-exact sequence

$$\cdots \longrightarrow \lim_R \mathrm{TR}_{q-\lambda_d}^{r/m}(A) \xrightarrow{V_m} \lim_R \mathrm{TR}_{q-\lambda_d}^r(A) \xrightarrow{\varepsilon} K_{q+1}(A[x]/(x^m),(x)) \longrightarrow \cdots,$$

which expresses the groups $K_{q+1}(A[x]/(x^m),(x))$ in terms of the $RO(\mathbb{T})$-graded equivariant homotopy groups of the topological Hochschild $\mathbb{T}$-spectrum $T(A)$. (The corresponding long-exact sequence of homotopy groups with finite coefficients is valid for every ring A.) We briefly recall the terms in this sequence.

The topological Hochschild $\mathbb{T}$-spectrum $T(A)$ associated with the ring A is a cyclotomic spectrum in the sense of [**7**, Def. 2.2]. In particular, it is an object of the $\mathbb{T}$-stable homotopy category. Let λ be a finite dimensional orthogonal $\mathbb{T}$-representation, let S^λ be the one-point compactification, and let $C_r \subset \mathbb{T}$ be the subgroup of order r. Then one defines

$$\mathrm{TR}_{q-\lambda}^r(A) = [S^q \wedge (\mathbb{T}/C_r)_+, S^\lambda \wedge T(A)]_{\mathbb{T}}$$

to be the abelian group of maps in the $\mathbb{T}$-stable homotopy category between the indicated $\mathbb{T}$-spectra. Suppose that $r = st$. Then there are maps

$$F_s \colon \mathrm{TR}^r_{q-\lambda}(A) \to \mathrm{TR}^t_{q-\lambda}(A) \qquad \text{(Frobenius)}$$
$$V_s \colon \mathrm{TR}^t_{q-\lambda}(A) \to \mathrm{TR}^r_{q-\lambda}(A) \qquad \text{(Verschiebung)}$$

induced by maps $f_s \colon (\mathbb{T}/C_t)_+ \to (\mathbb{T}/C_r)_+$ and $v_s \colon (\mathbb{T}/C_r)_+ \to (\mathbb{T}/C_t)_+$ in the $\mathbb{T}$-stable homotopy category that are defined as follows. The map f_s is the map of suspension $\mathbb{T}$-spectra induced by the canonical projection $\mathrm{pr} \colon \mathbb{T}/C_t \to \mathbb{T}/C_r$. The map v_s is the corresponding transfer map. To define it, we choose an embedding $\iota \colon \mathbb{T}/C_t \hookrightarrow \lambda$ into a finite dimensional orthogonal $\mathbb{T}$-presentation. The product embedding $(\iota, \mathrm{pr}) \colon \mathbb{T}/C_t \to \lambda \times \mathbb{T}/C_r$ has trivial normal bundle, and the linear structure on λ determines a preferred trivialization. Hence, the Pontryagin-Thom construction gives a map of pointed $\mathbb{T}$-spaces

$$S^\lambda \wedge (\mathbb{T}/C_r)_+ \to S^\lambda \wedge (\mathbb{T}/C_t)_+$$

and v_s is the induced map of suspension $\mathbb{T}$-spectra.

The isomorphism $\rho_s \colon \mathbb{T} \to \mathbb{T}/C_s$ given by the sth root induces an equivalence of categories ρ_s^* from the $\mathbb{T}/C_s$-stable homotopy category to the $\mathbb{T}$-stable homotopy category. The additional cyclotomic structure of the $\mathbb{T}$-spectrum $T(A)$ gives rise to a map of $\mathbb{T}$-spectra

$$r_s \colon \rho_s^*(T(A)^{C_s}) \to T(A).$$

Suppose again that $r = st$. Then we get a map of $RO(\mathbb{T})$-graded equivariant homotopy groups

$$R_s \colon \mathrm{TR}^r_{q-\lambda}(A) \to \mathrm{TR}^t_{q-\lambda'}(A) \qquad \text{(restriction)},$$

where $\lambda' = \rho_s^*(\lambda^{C_s})$ is the $\mathbb{T}/C_s$-representation λ^{C_s} considered as a $\mathbb{T}$-representation via the isomorphism ρ_s. The map R_s is defined to be the composition

$$\mathrm{TR}^r_{q-\lambda}(A) = [S^q \wedge (\mathbb{T}/C_r)_+, S^\lambda \wedge T(A)]_{\mathbb{T}} \xleftarrow{\sim} [S^q \wedge (\mathbb{T}/C_r)_+, S^{\lambda^{C_s}} \wedge T(A)^{C_s}]_{\mathbb{T}/C_s}$$
$$\xrightarrow{\sim} [S^q \wedge (\mathbb{T}/C_t)_+, S^{\lambda'} \wedge \rho_s^*(T(A)^{C_s})]_{\mathbb{T}} \to [S^q \wedge (\mathbb{T}/C_t)_+, S^{\lambda'} \wedge T(A)]_{\mathbb{T}} = \mathrm{TR}^t_{q-\lambda'}(A)$$

of the canonical isomorphism, the isomorphism ρ_s^*, and the map induced by r_s. It is proved in [**7**, Addendum 3.3] that there is a canonical isomorphism of rings

$$\xi \colon \mathbb{W}_{\langle r \rangle}(A) \xrightarrow{\sim} \mathrm{TR}^r_0(A)$$

from the ring of big Witt vectors in A corresponding to the subset $\langle r \rangle \subset \mathbb{N}$ of divisors of r. The isomorphism ξ is compatible with the restriction, Frobenius, and Verschiebung operators.

We return to the long-exact sequence of [**6**, Prop. 4.2.3] displayed at the beginning of the section. In the middle and left-hand terms, the representation λ_d is defined to be the sum

$$\lambda_d = \mathbb{C}(d) \oplus \mathbb{C}(d-1) \oplus \cdots \oplus \mathbb{C}(1),$$

where $\mathbb{C}(i) = \mathbb{C}$ with $\mathbb{T}$ acting from the left by $z \cdot w = z^i w$, and where

$$d = d(m, r) = \left[\frac{r-1}{m} \right]$$

is the largest integer less than or equal to $(r-1)/m$. The limits range over the set of all positive integers and the set of positive integers divisible by m, respectively, and the structure map in both limit systems is the restriction map. We note that if $r = st$ then

$$\rho_s^*(\lambda^{C_s}_{d(m,r)}) = \lambda_{d(m,t)}$$

as required. To prove Thm. A of the introduction, we first prove the following general result which does not require the ring A to be regular or noetherian. The proof of this result is given in Sect. 3 below.

THEOREM 2.1. *Let A be an $\mathbb{F}_p$-algebra, and let $f\colon A[x]/(x^m) \to A[x]/(x^n)$ be the canonical projection. Then there is a map of long-exact sequences*

$$\cdots \longrightarrow \lim_R \mathrm{TR}^{r/m}_{q-\lambda_d}(A) \xrightarrow{V_m} \lim_R \mathrm{TR}^r_{q-\lambda_d}(A) \xrightarrow{\varepsilon} K_{q+1}(A[x]/(x^m),(x)) \longrightarrow \cdots$$

$$\cdots \longrightarrow \lim_R \mathrm{TR}^{r/n}_{q-\lambda_e}(A) \xrightarrow{V_n} \lim_R \mathrm{TR}^r_{q-\lambda_e}(A) \xrightarrow{\varepsilon} K_{q+1}(A[x]/(x^n),(x)) \longrightarrow \cdots,$$

where $d = d(m,r) = [(r-1)/m]$ and $e = d(n,r) = [(r-1)/n]$, where the right-hand vertical map is the map of relative K-groups induced by the canonical projection, where the middle vertical map is the map of limits defined by the maps

$$\iota(m,n,r)_q\colon \mathrm{TR}^r_{q-\lambda_d}(A) \to \mathrm{TR}^r_{q-\lambda_e}(A)$$

induced by the canonical inclusions $\iota(m,n,r)\colon S^{\lambda_d} \to S^{\lambda_e}$, and where the left-hand vertical map is zero. The corresponding statement for the homotopy groups with finite coefficients is valid for every ring A.

We show in Lemma 2.6 below that the limits that appear in the statement of Thm. 2.1 stabilize. In particular, the corresponding derived limits vanish.

The long-exact sequences of Thm. 2.1 admit a p-typical decomposition analogous to the p-typical decomposition of the big de Rham-Witt groups which we discussed in Sect. 1 above. In the remainder of this section, we recall this decomposition and state the equivalent version Thm. 2.5 of Thm. 2.1.

It is proved in [**6**, Prop. 4.2.5] that the groups $\mathrm{TR}^r_{q-\lambda}(A)$, which appear in the statement of Thm. 2.1, decompose as a product of the p-typical groups

$$\mathrm{TR}^u_{q-\lambda}(A;p) = \mathrm{TR}^{p^{u-1}}_{q-\lambda}(A) = [S^q \wedge (\mathbb{T}/C_{p^{u-1}})_+, S^\lambda \wedge T(A)]_{\mathbb{T}}.$$

To state the result, we first write $r = p^{u-1}r'$, where r' is not divisible by p. Let j be a divisor of r', and let $\lambda' = \lambda'(j) = \rho^*_{r'/j}(\lambda^{C_{r'/j}})$. We let

$$\gamma_j\colon \mathrm{TR}^r_{q-\lambda}(A) \to \mathrm{TR}^u_{q-\lambda'}(A;p)$$

be the composite map

$$\mathrm{TR}^r_{q-\lambda}(A) \xrightarrow{F_j} \mathrm{TR}^{r/j}_{q-\lambda}(A) \xrightarrow{R_{r'/j}} \mathrm{TR}^{p^{u-1}}_{q-\lambda'}(A) =\!=\!= \mathrm{TR}^u_{q-\lambda'}(A;p)$$

and define

$$\gamma\colon \mathrm{TR}^r_{q-\lambda}(A) \to \prod_j \mathrm{TR}^u_{q-\lambda'}(A;p)$$

be the product of the maps γ_j as j ranges over the divisors of r'. The map γ is an isomorphism, for every $\mathbb{Z}_{(p)}$-algebra A, by [**6**, Prop. 4.2.5]. We remark that this decomposition is analogous to the p-typical decomposition of the big de Rham-Witt groups $\mathbb{W}_{\langle r \rangle}\Omega^q_A$ that we discussed in Sect. 1 above. Indeed, let j be a positive integer not divisible by p. Then the integer $u_p(\langle r \rangle, j)$ is equal to $u = v_p(r) + 1$, if j divides r', and is zero, otherwise. The maps R_s, F_s, and V_s are similarly expressed as products of their p-typical analogs

$$R = R_p\colon \mathrm{TR}^u_{q-\lambda}(A;p) \to \mathrm{TR}^{u-1}_{q-\lambda'}(A;p) \qquad \text{(restriction)}$$
$$F = F_p\colon \mathrm{TR}^u_{q-\lambda}(A;p) \to \mathrm{TR}^{u-1}_{q-\lambda}(A;p) \qquad \text{(Frobenius)}$$
$$V = V_p\colon \mathrm{TR}^{u-1}_{q-\lambda}(A;p) \to \mathrm{TR}^u_{q-\lambda}(A;p) \qquad \text{(Verschiebung)}.$$

Suppose that $r = st$ and write $s = p^v s'$ and $t = p^{u-v-1} t'$ with s' and t' not divisible by p. Then there are commutative diagrams

$$
\begin{array}{ccc}
\mathrm{TR}^r_{q-\lambda}(A) & \xrightarrow{\gamma} & \prod_j \mathrm{TR}^u_{q-\lambda'}(A;p) \\
\downarrow{\scriptstyle R_s} & & \downarrow{\scriptstyle R^\gamma_s} \\
\mathrm{TR}^t_{q-\lambda'}(A) & \xrightarrow{\gamma} & \prod_j \mathrm{TR}^{u-v}_{q-\lambda''}(A;p),
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{TR}^r_{q-\lambda}(A) & \xrightarrow{\gamma} & \prod_j \mathrm{TR}^u_{q-\lambda'}(A;p) \\
{\scriptstyle V_s}\uparrow\downarrow{\scriptstyle F_s} & & {\scriptstyle V^\gamma_s}\uparrow\downarrow{\scriptstyle F^\gamma_s} \\
\mathrm{TR}^t_{q-\lambda}(A) & \xrightarrow{\gamma} & \prod_j \mathrm{TR}^{u-v}_{q-\lambda'}(A;p)
\end{array}
$$

where the maps R^γ_s, F^γ_s, and V^γ_s are defined as follows: The map R^γ_s takes the factor indexed by a divisor j of t' to the factor indexed by the same divisor j of t' by the map R^v and annihilates the factors indexed by divisors j of r' that do not divide t'. The map F^γ_s takes the factor indexed by a divisor j of r' that is divisible by s' to the factor indexed by the divisor j/s' of t' by the map F^v and annihilates the remaining factors. Finally, the map V^γ_s takes the factor indexed by the divisor j of t' to the factor indexed by the divisor $s'j$ of r' by the map $s'V^v$.

It is now straightforward to check that the p-typical decomposition of the top long-exact sequence in the diagram in the statement of Thm. 2.1 takes the form

$$
\cdots \to \prod_{j \in m'I_p} \lim_R \mathrm{TR}^{u-v}_{q-\lambda_d}(A;p) \xrightarrow{m'V^v} \prod_{j \in I_p} \lim_R \mathrm{TR}^u_{q-\lambda_d}(A;p)
$$
$$
\xrightarrow{\varepsilon} K_{q+1}(A[x]/(x^m),(x)) \xrightarrow{\partial} \prod_{j \in m'I_p} \lim_R \mathrm{TR}^{u-v}_{q-1-\lambda_d}(A;p) \to \cdots
\tag{2.2}
$$

where $m = p^v m'$ with m' not divisible by p, and where $d = d_p(m,u,j)$ is defined by

$$
d_p(m,u,j) = \left\lceil \frac{p^{u-1}j - 1}{m} \right\rceil.
\tag{2.3}
$$

In the top line, the right-hand product ranges over the set I_p of positive integers not divisible by p, and the left-hand product ranges over the subset $m'I_p \subset I_p$. Similarly, the bottom long-exact sequence in the diagram in the statement of Thm. 2.1 takes the form

$$
\cdots \to \prod_{j \in n'I_p} \lim_R \mathrm{TR}^{u-w}_{q-\lambda_e}(A;p) \xrightarrow{n'V^w} \prod_{j \in I_p} \lim_R \mathrm{TR}^u_{q-\lambda_e}(A;p)
$$
$$
\xrightarrow{\varepsilon} K_{q+1}(A[x]/(x^n),(x)) \xrightarrow{\partial} \prod_{j \in n'I_p} \lim_R \mathrm{TR}^{u-w}_{q-1-\lambda_e}(A;p) \to \cdots
\tag{2.4}
$$

where $n = p^w n'$ with n' not divisible by p, and where $e = d_p(n,u,j)$. It follows that Thm. 2.1 above is equivalent to the following statement:

THEOREM 2.5. *Let A be an $\mathbb{F}_p$-algebra, and let $f\colon A[x]/(x^m) \to A[x]/(x^n)$ be the canonical projection. Then there is a map of long-exact sequence from the sequence (2.2) to the sequence (2.4) that is given, on the lower left-hand terms, by the map*

$$
f_*\colon K_*(A[x]/(x^m),(x)) \to K_*(A[x]/(x^n),(x))
$$

induced by the canonical projection, on the upper right-hand terms, by the map that takes the jth factor of the domain to the jth factor of the target by the map

$$
\iota_p(m,n,j)_q\colon \lim_R \mathrm{TR}^u_{q-\lambda_d}(A;p) \to \lim_R \mathrm{TR}^u_{q-\lambda_e}(A;p)
$$

induced from the canonical inclusions $\iota_p(m,n,j,u)\colon S^{\lambda_d} \to S^{\lambda_e}$, and, on the upper left-hand terms, by the zero map. The corresponding statement for the homotopy groups with $\mathbb{Z}/p^i\mathbb{Z}$-coefficients is valid for every ring A.

The following result shows that the limits that appear in the statements of Thms. 2.1 and 2.5 stabilize. In particular, the corresponding derived limits vanish. It also shows that, for every integer q, the products that appear in (2.2) and (2.4) are finite.

LEMMA 2.6. *Let $u, u' \geqslant 1$ and $m > n \geqslant 1$ be integers, and let j be an integer not divisible by p. Let $d = d_p(m, u, j)$ and $d' = d_p(m, u', j)$, and let $e = d_p(n, u, j)$ and $e' = d_p(n, u', j)$. Then, in the commutative diagram,*

$$\lim_R \mathrm{TR}^u_{q-\lambda_d}(A; p) \xrightarrow{\iota_p(m,n,j)_q} \lim_R \mathrm{TR}^u_{q-\lambda_e}(A; p)$$

$$\downarrow{\mathrm{pr}_{u'}} \qquad\qquad\qquad\qquad \downarrow{\mathrm{pr}_{u'}}$$

$$\mathrm{TR}^{u'}_{q-\lambda_{d'}}(A; p) \xrightarrow{\iota_p(m,n,j,u')_q} \mathrm{TR}^{u'}_{q-\lambda_{e'}}(k; p),$$

the left-hand vertical map is an isomorphism, if $q < 2d_p(m, u'+1, j)$, and right-hand vertical map is an isomorphism, if $q < 2d_p(n, u'+1, j)$.

Proof. Let r be a positive integer, let $\tilde{d} = d_p(m, r, j)$, and let $\tilde{d}' = d_p(m, r-1, j)$. It follows from [**7**, Thm. 2.2] that there is a long-exact sequence

$$\cdots \to \mathbb{H}_q(C_{p^{r-1}}, T(A) \wedge S^{\lambda_{\tilde{d}}}) \xrightarrow{N} \mathrm{TR}^r_{q-\lambda_{\tilde{d}}}(A; p) \xrightarrow{R} \mathrm{TR}^{r-1}_{q-\lambda_{\tilde{d}'}}(A; p) \to \cdots.$$

The Borel homology group that appear on the left-hand side is the abutment of a first quadrant homology type spectral sequence

$$E^2_{s,t} = H_s(C_{p^{r-1}}, \mathrm{TR}^1_{t-\lambda_{\tilde{d}}}(A; p)) \Rightarrow \mathbb{H}_{s+t}(C_{p^{r-1}}, T(A) \wedge S^{\lambda_{\tilde{d}}}),$$

and the groups $\mathrm{TR}^1_{q-\lambda_{\tilde{d}}}(A; p)$ are zero, for $q < 2\tilde{d}$. Hence, the map R in the long-exact sequence above is an isomorphism, for $q < 2\tilde{d}$. The lemma follows. $\qquad\square$

3. *The cyclic bar-construction*

In this section, we prove Thm. 2.1 above. We first recall the structure of the topological Hochschild $\mathbb{T}$-spectrum of the truncated polynomial algebra $A[x]/(x^m)$. The reader is referred to [**5**] for details on the topological Hochschild $\mathbb{T}$-spectrum and the cyclic bar-construction. We will write $X[-]$ to indicate a cyclic object with r-simplices $X[r]$.

We showed in [**7**, Thm. 7.1] (see also [**5**, Prop. 4]) that there is an $\mathscr{F}$-equivalence

$$\alpha \colon N^{\mathrm{cy}}(\Pi_m) \wedge T(A) \to T(A[x]/(x^m))$$

where $N^{\mathrm{cy}}(\Pi_m)$ is the geometric realization of the cyclic bar-construction $N^{\mathrm{cy}}(\Pi_m)[-]$ of the pointed monoid $\Pi_m = \{0, 1, x, x^2, \ldots, x^{m-1}\}$ with base point 0 and with $x^m = 0$. A map of $\mathbb{T}$-spectra is an $\mathscr{F}$-equivalence, if it induces an equivalence of C_r-fixed point spectra, for all finite subgroups $C_r \subset \mathbb{T}$, and hence, we have induced isomorphisms

$$[S^q \wedge (\mathbb{T}/C_r)_+, N^{\mathrm{cy}}(\Pi_m) \wedge T(A)]_{\mathbb{T}} \xrightarrow{\alpha_*} [S^q \wedge (\mathbb{T}/C_r)_+, T(A[x]/(x^m))]_{\mathbb{T}} = \mathrm{TR}^r_q(A[x]/(x^m)).$$

Moreover, the following diagram commutes

$$N^{\mathrm{cy}}(\Pi_m) \wedge T(A) \xrightarrow{\alpha} T(A[x]/(x^m))$$

$$\downarrow{f'_* \wedge \mathrm{id}} \qquad\qquad\qquad \downarrow{f_*}$$

$$N^{\mathrm{cy}}(\Pi_n) \wedge T(A) \xrightarrow{\alpha} T(A[x]/(x^m)),$$

where $f \colon A[x]/(x^m) \to A[x]/(x^n)$ and $f' \colon \Pi_m \to \Pi_n$ are the canonical projections.

The cyclic bar-construction of Π_m decomposes as the wedge sum of pointed cyclic sets

$$N^{\mathrm{cy}}(\Pi_m)[-] = \bigvee_{i \geqslant 0} N^{\mathrm{cy}}(\Pi_m, i)[-],$$

where $N^{\mathrm{cy}}(\Pi_m, i)[-] \subset N^{\mathrm{cy}}(\Pi_m)[-]$ is the sub-pointed cyclic set whose k-simplices are the $(k+1)$-tuples $(x^{i_0}, \ldots, x^{i_k})$, where $i_0 + \cdots + i_k = i$, and the base-point 0. The geometric realization decomposes accordingly as a wedge sum of pointed $\mathbb{T}$-spaces

$$N^{\mathrm{cy}}(\Pi_m) = \bigvee_{i \geqslant 0} N^{\mathrm{cy}}(\Pi_m, i).$$

The homotopy type of the pointed $\mathbb{T}$-space $N^{\mathrm{cy}}(\Pi_m, i)$ was determined in [**6**, Thm. B]. The result is that $N^{\mathrm{cy}}(\Pi_m, 0)$ is the discrete space $\{0, 1\}$ and that for positive i there is a canonical cofibration sequence of pointed $\mathbb{T}$-spaces

$$\mathbb{T}_+ \wedge_{C_{i/m}} S^{\lambda_d} \xrightarrow{\mathrm{pr}} \mathbb{T}_+ \wedge_{C_i} S^{\lambda_d} \xrightarrow{\varepsilon} N^{\mathrm{cy}}(\Pi_m, i) \xrightarrow{\partial} \Sigma \mathbb{T}_+ \wedge_{C_{i/m}} S^{\lambda_d},$$

where $d = d(m, i) = [(i-1)/m]$. The left-hand term is understood to be a point, if m does not divide i. More precisely, there is a canonical homotopy class of maps of pointed $\mathbb{T}$-spaces

$$\hat{\theta}_d \colon N^{\mathrm{cy}}(\Pi_m, i) \to \mathrm{cone}(\mathbb{T}_+ \wedge_{C_{i/m}} S^{\lambda_d} \xrightarrow{\mathrm{pr}} \mathbb{T}_+ \wedge_{C_i} S^{\lambda_d})$$

from $N^{\mathrm{cy}}(\Pi_m, i)$ to the mapping cone of the map pr and any map in this homotopy class is a weak equivalence of pointed $\mathbb{T}$-spaces. The canonical projection $f' \colon \Pi_m \to \Pi_n$ induces a map of cofibration sequences which we identify in Prop. 3.2 below. But first, we recall from [**6**, §3] how the homotopy class of maps $\hat{\theta}_d$ is defined.

We view the standard simplex Δ^{i-1} as the convex hull of the set of group elements C_i inside the regular representation $\mathbb{R}[C_i]$. Let $\zeta_i \in C_i \subset \mathbb{T}$ be the generator $\zeta_i = \exp(2\pi\sqrt{-1}/i)$, and let $\Delta^{i-m} \subset \Delta^{i-1}$ be the convex hull of the group elements $1, \zeta_i, \ldots, \zeta_i^{i-m}$. As a cyclic set, $N^{\mathrm{cy}}(\Pi_m, i)[-]$ is generated by the single $(m-1)$-simplex $(x, \ldots, x)$. Based on this observation, we showed in [**6**, Lemma 2.2.6] that there is a $\mathbb{T}$-equivariant homeomorphism

$$\varphi \colon \mathbb{T}_+ \wedge_{C_i} (\Delta^{i-1}/C_i \cdot \Delta^{i-m}) \xrightarrow{\sim} N^{\mathrm{cy}}(\Pi_m, i).$$

The regular representation $\mathbb{R}[C_i]$ has the canonical direct sum decomposition

$$\mathbb{R}[C_i] = \begin{cases} \mathbb{C}(s) \oplus \cdots \oplus \mathbb{C}(1) \oplus \mathbb{R}, & \text{if } i = 2s+1, \\ \mathbb{C}(s) \oplus \cdots \oplus \mathbb{C}(1) \oplus \mathbb{R} \oplus \mathbb{R}_-, & \text{if } i = 2s+2, \end{cases}$$

so if $d \leqslant s$, or equivalently, if $2d < i$, there is a canonical projection

$$\pi_d \colon \mathbb{R}[C_i] \to \mathbb{C}(d) \oplus \cdots \oplus \mathbb{C}(1) = \lambda_d.$$

Suppose first that $md < i < m(d+1)$. We showed in [**6**, Thm. 3.1.2] that, in this case, the image of $C_i \cdot \Delta^{i-m} \subset \mathbb{R}[C_i]$ does not contain the origin $0 \in \lambda_d$. Hence, we may compose π_d with the radial projection away from a small ball around $0 \in \lambda_d$ to get a C_i-equivariant map

$$\theta_d \colon \Delta^{i-1}/C_i \cdot \Delta^{i-m} \to D(\lambda_d)/S(\lambda_d) = S^{\lambda_d}$$

whose homotopy class is well-defined. The composite map of pointed $\mathbb{T}$-spaces

$$\hat{\theta}_d \colon N^{\mathrm{cy}}(\Pi_m, i) \xrightarrow{\varphi^{-1}} \mathbb{T}_+ \wedge_{C_i} (\Delta^{i-1}/C_i \cdot \Delta^{i-m}) \xrightarrow{\mathrm{id} \wedge \theta_d} \mathbb{T}_+ \wedge_{C_i} S^{\lambda_d}$$

is then the desired map $\hat{\theta}_d$. Its homotopy class is well-defined and [**6**, Prop. 3.2.6] shows that it is a weak equivalence.

Suppose next that $i = m(d+1)$. We let $\lambda_d' \subset \lambda_{d+1}$ be the image of the canonical inclusion $\iota \colon \lambda_d \to \lambda_{d+1}$ and let $\lambda_d^\perp \subset \lambda_{d+1}$ be the orthogonal complement of $\lambda_d' \subset \lambda_{d+1}$. The canonical projection from λ_{d+1} to $\mathbb{C}(d+1)$ restricts to an isomorphism of $\lambda_d^\perp$ onto $\mathbb{C}(d+1)$, and we

define $C'_m \subset \lambda_d^\perp$ to be the preimage by this isomorphism of $C_m \subset \mathbb{C}(d+1)$. It follows again from [**6**, Thm. 3.1.2] that the image $\pi_{d+1}(C_i \cdot \Delta^{i-m})$ does not contain the origin $0 \in \lambda_{d+1}$ and, in addition, [**6**, 3.3.5] shows that

$$\pi_{d+1}(C_i \cdot \Delta^{i-m}) \cap \lambda_d^\perp = C'_m.$$

We pick a small open ball $B \subset \lambda_{d+1} \smallsetminus C'_m$ around a point in $S(\lambda_d^\perp)$ and define

$$U = (C_i \cdot B) \cap S(\lambda_{d+1}) \subset S(\lambda_{d+1}).$$

If the ball B is small enough, then the composition of the projection π_{d+1} and the radial projection away from a (different) small ball around $0 \in \lambda_{d+1}$ defines a C_i-equivariant map

$$\theta'_d \colon \Delta^{i-1}/C_i \cdot \Delta^{i-m} \to D(\lambda_{d+1})/(S(\lambda_{d+1}) \smallsetminus U)$$

whose homotopy class is well-defined. Let $C(C_m) = \{0\} * C_m \subset D(\mathbb{C}(d+1))$ be the unreduced cone with base $C_m \subset \mathbb{C}(d+1)$ and with cone point $0 \in \mathbb{C}(d+1)$. Then the canonical homeomorphism of $D(\mathbb{C}(d+1)) \times D(\lambda_d)$ onto $D(\lambda_{d+1})$ induces an inclusion into the target of the map θ'_d of the pointed C_i-space

$$\frac{C(C_m) \times D(\lambda_d)}{C(C_m) \times S(\lambda_d) \cup C_m \times D(\lambda_d)} = (S^0 * C_m) \wedge S^{\lambda_d}.$$

One immediately verifies that this inclusion is a strong deformation retract of pointed C_i-spaces. Hence, the map θ'_d defines a homotopy class of maps of pointed C_i-spaces

$$\theta_d \colon \Delta^{i-1}/C_i \cdot \Delta^{i-m} \to (S^0 * C_m) \wedge S^{\lambda_d}.$$

The composite map of pointed $\mathbb{T}$-spaces

$$\hat{\theta}_d \colon N^{\mathrm{cy}}(\Pi_m, i) \xrightarrow{\varphi^{-1}} \mathbb{T}_+ \wedge_{C_i} (\Delta^{i-1}/C_i \cdot \Delta^{i-m}) \xrightarrow{\mathrm{id} \wedge \theta_d} \mathbb{T}_+ \wedge_{C_i} ((S^0 * C_m) \wedge S^{\lambda_d})$$

is the desired map $\hat{\theta}_d$. Its homotopy class is well-defined and [**6**, Prop. 3.3.9] shows that it is a weak equivalence. In preparation for the proof of Prop. 3.2 below, we first prove the following key result.

LEMMA 3.1. *Let m and n be positive integers with $m > n$.*
(i) If $md < i < m(d+1)$ and $ne < i < n(e+1)$, then $d \leqslant e$ and the diagram

$$
\begin{array}{ccc}
\Delta^{i-1}/C_i \cdot \Delta^{i-m} & \xrightarrow{\theta_d} & S^{\lambda_d} \\
{\scriptstyle \mathrm{pr}} \downarrow & & \downarrow {\scriptstyle \iota} \\
\Delta^{i-1}/C_i \cdot \Delta^{i-n} & \xrightarrow{\theta_e} & S^{\lambda_e}
\end{array}
$$

commutes up to C_i-equivariant homotopy. Here the right-hand vertical map ι is the canonical inclusion of S^{λ_d} in S^{λ_e}.
(ii) If $md < i < m(d+1)$ and $i = n(e+1)$, then $d \leqslant e$ and the diagram

$$
\begin{array}{ccc}
\Delta^{i-1}/C_i \cdot \Delta^{i-m} & \xrightarrow{\theta_d} & S^{\lambda_d} \\
{\scriptstyle \mathrm{pr}} \downarrow & & \downarrow {\scriptstyle \iota} \\
\Delta^{i-1}/C_i \cdot \Delta^{i-n} & \xrightarrow{\theta_e} & (S^0 * C_n) \wedge S^{\lambda_e}
\end{array}
$$

commutes up to C_i-equivariant homotopy. Here the right-hand vertical map ι is the canonical inclusion of S^{λ_d} in S^{λ_e} followed by the canonical inclusion of $S^{\lambda_e} = S^0 \wedge S^{\lambda_e}$ in $(S^0 * C_n) \wedge S^{\lambda_e}$.

(iii) If $i = m(d+1)$ and $ne < i < n(e+1)$, then $d < e$ and the diagram

$$
\begin{array}{ccc}
\Delta^{i-1}/C_i \cdot \Delta^{i-m} & \xrightarrow{\theta_d} & (S^0 * C_m) \wedge S^{\lambda_d} \\
\downarrow{\scriptstyle\mathrm{pr}} & & \downarrow{\scriptstyle\iota} \\
\Delta^{i-1}/C_i \cdot \Delta^{i-n} & \xrightarrow{\theta_e} & S^{\lambda_e}
\end{array}
$$

commutes up to C_i-equivariant homotopy. Here the right-hand vertical map ι is the canonical inclusion of $(S^0 * C_m) \wedge S^{\lambda_d}$ in $(S^0 * S(\mathbb{C}(d+1))) \wedge S^{\lambda_d} = S^{\lambda_{d+1}}$ followed by the canonical inclusion of $S^{\lambda_{d+1}}$ in S^{λ_e}.

(iv) If $i = m(d+1) = n(e+1)$, then $d < e$ and the diagram

$$
\begin{array}{ccc}
\Delta^{i-1}/C_i \cdot \Delta^{i-m} & \xrightarrow{\theta_d} & (S^0 * C_m) \wedge S^{\lambda_d} \\
\downarrow{\scriptstyle\mathrm{pr}} & & \downarrow{\scriptstyle\iota} \\
\Delta^{i-1}/C_i \cdot \Delta^{i-n} & \xrightarrow{\theta_e} & (S^0 * C_n) \wedge S^{\lambda_e}
\end{array}
$$

commutes up to C_i-equivariant homotopy. Here the right-hand vertical map ι is the canonical inclusion of $(S^0 * C_m) \wedge S^{\lambda_d}$ in $(S^0 * S(\mathbb{C}(d+1))) \wedge S^{\lambda_d} = S^{\lambda_{d+1}}$ followed by the canonical inclusions of $S^{\lambda_{d+1}}$ in S^{λ_e} and of $S^{\lambda_e} = S^0 \wedge S^{\lambda_e}$ in $(S^0 * C_n) \wedge S^{\lambda_e}$.

Proof. We first consider the case (i). Let $\lambda_e - \lambda_d \subset \lambda_e$ be the orthogonal complement of $\lambda_d \subset \lambda_e$. Then there is a canonical C_i-equivariant homeomorphism

$$
D(\lambda_e - \lambda_d) \times D(\lambda_d) \xrightarrow{\sim} D(\lambda_e)
$$

which restricts to a C_i-equivariant homeomorphism

$$
D(\lambda_e - \lambda_d) \times S(\lambda_d) \cup S(\lambda_e - \lambda_d) \times D(\lambda_d) \xrightarrow{\sim} S(\lambda_e).
$$

We note that the composition

$$
\Delta^{i-1}/C_i \cdot \Delta^{i-m} \xrightarrow{\mathrm{pr}} \Delta^{i-1}/C_i \cdot \Delta^{i-n} \xrightarrow{\theta_e} D(\lambda_e)/S(\lambda_e)
$$

of the left-hand vertical map and the lower horizontal map in the diagram of part (i) of the statement factors through the map

$$
\frac{D(\lambda_e - \lambda_d) \times D(\lambda_d)}{D(\lambda_e - \lambda_d) \times S(\lambda_d)} \to D(\lambda_e)/S(\lambda_e)
$$

induced by the canonical homeomorphism above. But the latter map is homotopic to the composite map

$$
\frac{D(\lambda_e - \lambda_d) \times D(\lambda_d)}{D(\lambda_e - \lambda_d) \times S(\lambda_d)} \xrightarrow{\mathrm{pr}_2} D(\lambda_d)/S(\lambda_d) \xrightarrow{\iota} D(\lambda_e)/S(\lambda_e)
$$

by the C_i-equivariant homotopy induced from the radial contraction

$$
h \colon D(\lambda_e - \lambda_d) \times [0,1] \to D(\lambda_e - \lambda_d)
$$

defined by $h(z,t) = tz$. This homotopy, in turn, induces a C_i-equivariant homotopy from the composite $\iota \circ \theta_d$ to the composite $\theta_e \circ \mathrm{pr}$ in the diagram in part (i) of the statement.

In the case (ii), we note similarly that the composition

$$
\Delta^{i-1}/C_i \cdot \Delta^{i-m} \xrightarrow{\mathrm{pr}} \Delta^{i-1}/C_i \cdot \Delta^{i-n} \xrightarrow{\theta_e} \frac{C(C_n) \times D(\lambda_e)}{C(C_n) \times S(\lambda_e) \cup C_n \times D(\lambda_e)}
$$

of the left-hand vertical map and the lower horizontal map in the diagram in part (ii) of the statement factors through the canonical projection

$$\frac{C(C_n) \times D(\lambda_e)}{C(C_n) \times S(\lambda_e)} \to \frac{C(C_n) \times D(\lambda_e)}{C(C_n) \times S(\lambda_e) \cup C_n \times D(\lambda_e)}$$

Again, the latter map is homotopic to the composition

$$\frac{C(C_n) \times D(\lambda_e)}{C(C_n) \times S(\lambda_e)} \xrightarrow{\mathrm{pr}_2} D(\lambda_e)/S(\lambda_e) \xrightarrow{\iota} \frac{C(C_n) \times D(\lambda_e)}{C(C_n) \times S(\lambda_e) \cup C_n \times D(\lambda_e)}$$

by the C_i-equivariant homotopy induced from the radial contraction

$$h \colon C(C_n) \times [0,1] \to C(C_n)$$

defined by $h(z,t) = tz$. This homotopy induces the desired homotopy from the composite map $\iota \circ \theta_d$ to the composite map $\theta_e \circ \mathrm{pr}$ in the diagram in part (ii) of the statement.

In the case (iii) of the statement, one proves as in the proof of part (i) of the statement that the composite map $\mathrm{pr} \circ \theta_e$ is homotopic to the composition

$$\Delta^{i-1}/C_i \cdot \Delta^{i-m} \xrightarrow{\theta_{d+1}} D(\lambda_{d+1})/S(\lambda_{d+1}) \xrightarrow{\iota} D(\lambda_e)/S(\lambda_e).$$

But the map θ_{d+1} is homotopic to the composition

$$\Delta^{i-1}/C_i \cdot \Delta^{i-m} \xrightarrow{\theta_d} \frac{C(C_m) \times D(\lambda_d)}{C(C_m) \times S(\lambda_d) \cup C_m \times D(\lambda_d)} \xrightarrow{\iota} D(\lambda_{d+1})/S(\lambda_{d+1}).$$

Indeed, this is a direct consequence of the construction of the map λ_d. This proves the case (iii) of the statement. Finally, the proof of the case (iv) is analogous to the proof of the case (i). This completes the proof. $\qquad\square$

PROPOSITION 3.2. *Let $m > n \geqslant 1$ be integers and $f' \colon \Pi_m \to \Pi_n$ the canonical projection. Let $i \geqslant 1$ be an integer, and set $d = [(i-1)/m]$ and $e = [(i-1)/n]$. Then there is a homotopy commutative diagram of pointed $\mathbb{T}$-spaces*

$$\begin{array}{ccccccc}
\mathbb{T}_+ \wedge_{C_{i/m}} S^{\lambda_d} & \xrightarrow{\mathrm{pr}} & \mathbb{T}_+ \wedge_{C_i} S^{\lambda_d} & \xrightarrow{\varepsilon} & N^{\mathrm{cy}}(\Pi_m, i) & \xrightarrow{\partial} & \Sigma\mathbb{T}_+ \wedge_{C_{i/m}} S^{\lambda_d} \\
\downarrow{*} & & \downarrow{\mathrm{id} \wedge \iota} & & \downarrow{f'_*} & & \downarrow{*} \\
\mathbb{T}_+ \wedge_{C_{i/n}} S^{\lambda_e} & \xrightarrow{\mathrm{pr}} & \mathbb{T}_+ \wedge_{C_i} S^{\lambda_e} & \xrightarrow{\varepsilon} & N^{\mathrm{cy}}(\Pi_n, i) & \xrightarrow{\partial} & \Sigma\mathbb{T}_+ \wedge_{C_{i/n}} S^{\lambda_e},
\end{array}$$

where the rows are cofibration sequences, the map $\mathrm{id} \wedge \iota$ is induced from the canonical inclusion of λ_d in λ_e, and the map $*$ is the null-map. The domain (resp. target) of the map $*$ is understood to be a point if m (resp. n) does not divide i.

Proof. As we recalled above, the statement that the rows in the diagram of the statement are cofibration sequences of pointed $\mathbb{T}$-spaces is equivalent to the statement that the maps of $\mathbb{T}$-spaces $\hat{\theta}_d$ are weak equivalences, and the latter statement is [**6**, Props. 3.2.6, 3.3.9]. The statement that the diagram commutes follows immediately from Lemma 3.1 upon applying the functor that to a pointed C_i-space X associates the pointed $\mathbb{T}$-space $\mathbb{T}_+ \wedge_{C_i} X$ with the exception of the statement that the left-hand vertical map is null-homotopic. Only the case (iii) needs proof. In this case, we have a homotopy commutative diagram of pointed C_i-spaces

$$\begin{array}{ccccccc}
C_{m+} \wedge S^{\lambda_d} & \xrightarrow{\mathrm{pr}_2} & S^{\lambda_d} & \longrightarrow & (S^0 * C_m) \wedge S^{\lambda_d} & \xrightarrow{\partial} & \Sigma C_{m+} \wedge S^{\lambda_d} \\
\downarrow & & \| & & \downarrow & & \downarrow \\
S(\mathbb{C}(d+1))_+ \wedge S^{\lambda_d} & \longrightarrow & S^{\lambda_d} & \xrightarrow{\iota} & S^{\lambda_{d+1}} & \xrightarrow{\partial} & \Sigma S(\mathbb{C}(d+1))_+ \wedge S^{\lambda_d}
\end{array}$$

which shows that the composite map $\iota \circ \mathrm{pr}_2$ is null-homotopic. It follows that the composite map of pointed $\mathbb{T}$-spaces

$$\mathbb{T}_+ \wedge_{C_{i/m}} S^{\lambda_d} \xrightarrow{\mathrm{pr}} \mathbb{T}_+ \wedge_{C_i} S^{\lambda_d} \xrightarrow{\mathrm{id} \wedge \iota} \mathbb{T}_+ \wedge_{C_i} S^{\lambda_{d+1}}$$

is null-homotopic as desired. □

Proof of Thm. 2.1. The proof that the horizontal sequences in the diagram in the statement are exact is given in [**7**, Prop. 4.2.3]. A direct, and perhaps more detailed, proof that the isomorphic sequences (2.2) and (2.4) are exact is given in [**5**, Prop. 11]. It is immediately clear from either proof that Prop. 3.2 implies that the diagram in the statement commutes. □

4. The map $\iota_* \colon \mathrm{TR}^u_{q-\lambda_{a-1}}(A;p) \to \mathrm{TR}^u_{q-\lambda_a}(A;p)$

The topological Hochschild $\mathbb{T}$-spectrum $T(A)$ gives rise to a generalized equivariant homology theory on the category of pointed $\mathbb{T}$-spaces defined by

$$T(A)^{C_r}_q(X) = \pi^{C_r}_q(X \wedge T(A)) = [S^q \wedge (\mathbb{T}/C_r)_+, X \wedge T(A)]_{\mathbb{T}},$$

and the map of the title is equal to the map of these homology groups induced by the canonical inclusion $\iota \colon S^{\lambda_{a-1}} \to S^{\lambda_a}$. We will evaluate this map in Prop. 4.2 below but, in preparation, we first prove a general result about $\mathbb{T}$-spectra. We define a $\mathbb{T}$-spectrum to be an orthogonal $\mathbb{T}$-spectrum in the sense of [**13**]. The category of $\mathbb{T}$-spectra is a model category and the associated homotopy category, by definition, is the $\mathbb{T}$-stable homotopy category. Moreover, the category of $\mathbb{T}$-spectra has a closed symmetric monoidal structure given by the smash product which induces a closed symmetric monoidal structure on the $\mathbb{T}$-stable homotopy category. We give the $\mathbb{T}$-stable homotopy category the triangulated structure defined in [**9**, §2].

Let T be a $\mathbb{T}$-spectrum, and let $C_r \subset \mathbb{T}$ be the subgroup of order r. We define

$$\pi^{C_r}_q(T) = [S^q \wedge (\mathbb{T}/C_r)_+, T]_{\mathbb{T}}$$

to be the set of maps in the $\mathbb{T}$-stable homotopy category between the indicated $\mathbb{T}$-spectra. In particular, $\mathrm{TR}^r_{q-\lambda}(A) = \pi^{C_r}_q(S^\lambda \wedge T(A))$. There are canonical isomorphisms

$$\pi_q(T^{C_r}) = [S^q, T^{C_r}] \xrightarrow{\sim} [S^q \wedge (\mathbb{T}/C_r)_+, T^{C_r}]_{\mathbb{T}/C_r} \xrightarrow{\sim} [S^q \wedge (\mathbb{T}/C_r)_+, T]_{\mathbb{T}} = \pi^{C_r}_q(T),$$

where, in the middle term, T^{C_r} denotes the C_r-fixed point $\mathbb{T}/C_r$-spectrum, and where, on the left-hand side, T^{C_r} denotes the underlying non-equivariant spectrum of this $\mathbb{T}/C_r$-spectrum. We consider the cofibration sequence of pointed $\mathbb{T}$-spaces

$$\mathbb{T}_+ \xrightarrow{\pi} S^0 \xrightarrow{\iota} S^{\mathbb{C}(1)} \xrightarrow{\partial} S^1 \wedge \mathbb{T}_+ = \Sigma \mathbb{T}_+,$$

where π collapses $\mathbb{T}$ onto the non-base point of S^0, and where ι is the canonical inclusion. We identify the underlying pointed spaces of S^2 and $S^{\mathbb{C}(1)}$ by the isomorphism

$$\varphi \colon S^2 \to S^{\mathbb{C}(1)}$$

that takes the class of (a, b) in $S^2 = S^1 \wedge S^1$ to the class of $a + b\sqrt{-1}$ in $S^{\mathbb{C}(1)}$. The composition of φ and ∂ defines a map in the non-equivariant stable homotopy category

$$\sigma \colon S^1 \to \mathbb{T}_+.$$

The cofibration sequence above induces a cofibration sequence in the non-equivariant stable homotopy category, and the latter sequence splits. We thus obtain a direct sum diagram in the non-equivariant stable homotopy category

$$S^1 \underset{\kappa}{\overset{\sigma}{\rightleftarrows}} \mathbb{T}_+ \underset{i}{\overset{\pi}{\rightleftarrows}} S^0$$

where i is the section of π that takes the the non-base point in S^0 to $1 \in \mathbb{T}$, and where the map κ is the corresponding retraction of the map σ. We define the map

$$d \colon \pi_q^{C_r}(T) \to \pi_{q+1}^{C_r}(T) \qquad \text{(Connes' operator)}$$

to be composition

$$\pi_q(T^{C_r}) \xrightarrow{\ \ell_\sigma\ } \pi_{q+1}(\mathbb{T}_+ \wedge T^{C_r}) \xrightarrow{\ \rho_{r*}\ } \pi_{q+1}((\mathbb{T}/C_r)_+ \wedge T^{C_r}) \xrightarrow{\ \mu_*\ } \pi_{q+1}(T^{C_r}),$$

where the left-hand map is left multiplication by σ, where the middle map is induced by the isomorphism $\rho_r \colon \mathbb{T} \to \mathbb{T}/C_r$ defined by the rth root, and where the right-hand map is induced from the left action by $\mathbb{T}$ on the underlying non-equivariant spectrum of T. It anti-commutes with the suspension isomorphism in the sense that the following diagram anti-commutes:

$$
\begin{array}{ccc}
\pi_q^{C_r}(T) & \xrightarrow{\ \ d\ \ } & \pi_{q+1}^{C_r}(T) \\
\Big\downarrow{\scriptstyle \text{susp}} \quad {\scriptstyle (-1)} & & \Big\downarrow{\scriptstyle \text{susp}} \\
\pi_{q+1}^{C_r}(\Sigma T) & \xrightarrow{\ \ d\ \ } & \pi_{q+2}^{C_r}(\Sigma T).
\end{array}
$$

Moreover, $dd(x) = \eta \cdot d(x)$ and $F_r dV_r(x) = d(x) + (r-1)\eta \cdot x$, where $\eta \in \pi_1(S^0)$ is the Hopf class, and, if T is a ring $\mathbb{T}$-spectrum, d is a derivation for the multiplication; see [**4**, §1]. If A is an $\mathbb{F}_p$-algebra, then multiplication by η is zero on $\pi_*^{C_r}(T(A))$. Hence, in this case, Connes' operator is a differential and satisfies $F_r dV_r = d$.

LEMMA 4.1. *Let T be a $\mathbb{T}$-spectrum, and let r and j be relative prime positive integers. Then, for every integer q, there is a commutative diagram*

$$
\begin{array}{ccc}
\pi_{q-1}(T) \oplus \pi_q(T) & \xrightarrow{\ \ \text{susp} \oplus \text{susp}\ \ } & \pi_q(S^1 \wedge T) \oplus \pi_{q+1}(S^1 \wedge T) \\
\Big\downarrow{\scriptstyle i_* \oplus i_*} & & \Big\uparrow{\scriptstyle \kappa_* \oplus \kappa_*} \\
\pi_{q-1}((\mathbb{T}/C_j)_+ \wedge T) \oplus \pi_q((\mathbb{T}/C_j)_+ \wedge T) & & \pi_q((\mathbb{T}/C_j)_+ \wedge T) \oplus \pi_{q+1}((\mathbb{T}/C_j)_+ \wedge T) \\
\Big\downarrow{\scriptstyle dV_r + V_r} & & \Big\uparrow{\scriptstyle (F_r, F_r d)} \\
\pi_q^{C_r}((\mathbb{T}/C_j)_+ \wedge T) & =\!=\!=\!=\!=\!=\!= & \pi_q^{C_r}((\mathbb{T}/C_j)_+ \wedge T),
\end{array}
$$

where the top horizontal map is the suspension isomorphism. Moreover, the compositions of the left-hand vertical maps and the right-hand vertical maps are both isomorphisms.

Proof. The general formulas that we recalled before the statement show that

$$\big((\kappa_* \oplus \kappa_*) \circ (F_r, F_r d) \circ (dV_r + V_r) \circ (i_* \oplus i_*)\big)(x, y) = \big((\kappa_* \circ d \circ i_*)(x), (\kappa_* \circ d \circ i_*)(y)\big),$$

and one verifies that $\kappa_* \circ d \circ i_*$ is equal to the suspension isomorphism. This shows that the diagram commutes. It was proved in [**10**, Prop. 3.4.1] that the composition of the left-hand vertical maps is an isomorphism, provided that $r = p^{v-1}$ is a power of a prime number p. The proof in the general case is completely similar. $\qquad\square$

Suppose now that r is a divisor in a. Then the isomorphism

$$\varphi \colon S^2 \to S^{\mathbb{C}(a)}$$

is an isomorphism of pointed C_r-spaces. We define

$$\hat{\varphi} \colon S^2 \wedge S^q \wedge (\mathbb{T}/C_r)_+ \to S^{\mathbb{C}(a)} \wedge S^q \wedge (\mathbb{T}/C_r)_+$$

be the isomorphism of pointed $\mathbb{T}$-spaces that takes the class of (x, y, zC_r) to the class of $(z\varphi(x), y, zC_r)$. It follows that, if r is a divisor in a, we have an isomorphism

$$\varphi^! \colon \mathrm{TR}^r_{q-\lambda_{a-1}}(A) \to \mathrm{TR}^r_{q+2-\lambda_a}(A)$$

defined by the composition

$$
\begin{aligned}
\mathrm{TR}^r_{q-\lambda_{a-1}}(A) &= [S^q \wedge (\mathbb{T}/C_r)_+, S^{\lambda_{a-1}} \wedge T(A)]_{\mathbb{T}} \\
&\xrightarrow{\mathrm{susp}} [S^{\mathbb{C}(a)} \wedge S^q \wedge (\mathbb{T}/C_r)_+, S^{\mathbb{C}(a)} \wedge S^{\lambda_{a-1}} \wedge T(A)]_{\mathbb{T}} \\
&\xrightarrow{\hat{\varphi}^*} [S^2 \wedge S^q \wedge (\mathbb{T}/C_r)_+, S^{\mathbb{C}(a)} \wedge S^{\lambda_{a-1}} \wedge T(A)]_{\mathbb{T}} = \mathrm{TR}^r_{q+2-\lambda_a}(A)
\end{aligned}
$$

We write $\varphi_!$ for the inverse of $\varphi^!$.

PROPOSITION 4.2. *Let A be a $\mathbb{Z}_{(p)}$-algebra, let $a, u \geqslant 1$ be integers, and let $v = v(a, u)$ be the minimum of u and $v_p(a) + 1$. Then there are natural long-exact sequences*

where the left-hand sequence is valid, for $v < u$, and where the right-hand sequence is valid, for $v = u$.

Proof. We consider the cofibration sequence of pointed $\mathbb{T}$-spaces

$$S(\mathbb{C}(a))_+ \xrightarrow{\pi} S^0 \xrightarrow{\iota} S^{\mathbb{C}(a)} \xrightarrow{\delta} S^1 \wedge S(\mathbb{C}(a))_+$$

and the induced cofibration sequence in the $\mathbb{T}$-stable category

$$S(\mathbb{C}(a))_+ \wedge T \xrightarrow{\pi'} T \xrightarrow{\iota'} S^{\mathbb{C}(a)} \wedge T \xrightarrow{\delta'} S^1 \wedge S(\mathbb{C}(a))_+ \wedge T$$

where $T = S^{\lambda_{a-1}} \wedge T(A)$. We also abbreviate $r = p^{u-1}$. The associated exact sequence of equivariant homotopy groups takes the form

$$\pi^{C_r}_q(S(\mathbb{C}(a))_+ \wedge T) \xrightarrow{\pi'_*} \pi^{C_r}_q(T) \xrightarrow{\iota'_*} \pi^{C_r}_q(S^{\mathbb{C}(a)} \wedge T) \xrightarrow{\partial} \pi^{C_r}_{q-1}(S(\mathbb{C}(a))_+ \wedge T),$$

where ∂ is the composition of ∂'_* and the inverse of the suspension isomorphism. We identify this sequence with the long-exact sequences of the statement. The terms $\pi^{C_r}_q(T)$ and $\pi^{C_r}_q(S^{\mathbb{C}(a)} \wedge T)$ are equal to $\mathrm{TR}^u_{q-\lambda_{a-1}}(A; p)$ and $\mathrm{TR}^u_{q-\lambda_a}(A; p)$, respectively, and the map ι'_* is equal to the map ι_*. It remains to identify the remaining term and the two maps π'_* and ∂_*.

Suppose first that $u = v$ such that $\mathbb{C}(a)$ is C_r-trivial. Then the map $i\colon S^0 \to S(\mathbb{C}(a))_+$ that takes the non-base point in S^0 to $1 \in S(\mathbb{C}(a))$ is C_r-equivariant and defines a section of the map π'_* in the exact sequence of equivariant homotopy groups above. This completes the proof in the case $u = v$.

We next assume that $v < u$ and consider first the case $v = 1$ such that a and r are relatively prime. The map $i\colon S^0 \to S(\mathbb{C}(a))_+$ is a non-equivariant section of the projection π. The following diagram commutes

$$
\begin{array}{ccc}
\pi_q^{C_r}(S(\mathbb{C}(a))_+ \wedge T) & \xrightarrow{\quad\pi'_*\quad} & \pi_q^{C_r}(T) \\
\Big\uparrow{\scriptstyle dV_r+V_r} & & \Big\uparrow{\scriptstyle dV_r+V_r} \\
\pi_{q-1}(S(\mathbb{C}(a))_+ \wedge T) \oplus \pi_q(S(\mathbb{C}(a))_+ \wedge T) & \xrightarrow{\pi'_*\oplus\pi'_*} & \pi_{q-1}(T) \oplus \pi_q(T) \\
\Big\uparrow{\scriptstyle i'_*\oplus i'_*} & & \\
\pi_{q-1}(T) \oplus \pi_q(T) & &
\end{array}
$$

and the composite map $(\pi'_* \oplus \pi'_*) \circ (i_* \oplus i_*)$ is the identity. Since $S(\mathbb{C}(a))_+$ is isomorphic to $(\mathbb{T}/C_a)_+$ as a pointed $\mathbb{T}$-space, Lemma 4.1 shows that the composition of the left-hand vertical maps is an isomorphism. We use this isomorphism to identify the left-hand term of the exact sequence of equivariant homotopy groups above. The composition of this isomorphism and the map π'_* is equal to the map

$$
dV_r + V_r\colon \mathrm{TR}^1_{q-1-\lambda_{a-1}}(A;p) \oplus \mathrm{TR}^1_{q-\lambda_{a-1}}(A;p) \to \mathrm{TR}^u_{q-\lambda_{a-1}}(A;p)
$$

as stated. Similarly, the following diagram commutes

$$
\begin{array}{ccc}
\pi_q^{C_r}(S^{\mathbb{C}(a)} \wedge T) & \xrightarrow{\quad\partial_*\quad} & \pi_{q-1}^{C_r}(S(\mathbb{C}(a))_+ \wedge T) \\
\Big\downarrow{\scriptstyle (F_r,-F_r d)} & & \Big\downarrow{\scriptstyle (F_r,F_r d)} \\
\pi_q(S^{\mathbb{C}(a)} \wedge T) \oplus \pi_{q+1}(S^{\mathbb{C}(a)} \wedge T) & \xrightarrow{\partial_*\oplus\partial_*} & \pi_{q-1}(S(\mathbb{C}(a))_+ \wedge T) \oplus \pi_q(S(\mathbb{C}(a))_+ \wedge T). \\
& & \Big\downarrow{\scriptstyle \mathrm{susp}^{-1}\circ\kappa'_*\oplus\mathrm{susp}^{-1}\circ\kappa'_*} \\
& & \pi_{q-2}(T) \oplus \pi_{q-1}(T)
\end{array}
$$

and the composition of the lower horizontal map and the lower right-hand vertical map is equal to the map $\varphi_! \oplus \varphi_!$. Finally, Lemma 4.1 shows that the composition of the right-hand vertical maps is equal to the inverse of the map $dV_r i_* \oplus V_r i_*$. This completes the proof in the case where $1 = v < u$.

Finally, we suppose that $1 < v < u$ and abbreviate $r = p^{u-1}$, $t = p^{v-1}$, and $s = p^{u-v}$. We recall that the root isomorphism $\rho_t\colon \mathbb{T} \to \mathbb{T}/C_t$ induces an equivalence of categories ρ_t^* from the $\mathbb{T}/C_t$-stable category to the $\mathbb{T}$-stable category. There is a commutative diagram

$$
\begin{array}{ccccccc}
\pi_q^{C_r}(S(\mathbb{C}(a))_+ \wedge T) & \xrightarrow{\pi'_*} & \pi_q^{C_r}(T) & \xrightarrow{\iota'_*} & \pi_q^{C_r}(S^{\mathbb{C}(a)} \wedge T) & \xrightarrow{\partial} & \pi_{q-1}^{C_r}(S(\mathbb{C}(a))_+ \wedge T) \\
\Big\uparrow & & \Big\uparrow & & \Big\uparrow & & \Big\uparrow \\
\pi_q^{C_s}(S(\mathbb{C}(j))_+ \wedge T') & \xrightarrow{\pi'_*} & \pi_q^{C_s}(T') & \xrightarrow{\iota'_*} & \pi_q^{C_s}(S^{\mathbb{C}(j)} \wedge T') & \xrightarrow{\partial} & \pi_{q-1}^{C_r}(S(\mathbb{C}(j))_+ \wedge T'),
\end{array}
$$

where $T = S^{\lambda_{a-1}} \wedge T(A)$ and $T' = \rho_t^*(T^{C_t})$, and where $j = a/t$. The vertical maps are the composition of the isomorphism $\rho_t^*\colon \pi_q^{C_r/C_t}(X^{C_t}) \to \pi_q^{C_s}(\rho_t^*(X^{C_t}))$ and the canonical isomorphism of $\pi_q^{C_r}(X)$ and $\pi_q^{C_r/C_t}(X^{C_t})$. The case of the lower sequence was treated above. This completes the proof. $\square$

REMARK 4.3.   Let A be a regular noetherian ring that is also an $\mathbb{F}_p$-algebra. There is a statement similar to Lemma 4.2 above for the p-typical de Rham-Witt complex of A. Suppose that $v < u$. Then there is a long-exact sequence sequence

$$\cdots \xrightarrow{\ \beta\ } W_v\Omega_A^{q-1} \oplus W_v\Omega_A^q \xrightarrow{\ \alpha\ } W_u\Omega_A^q \xrightarrow{\ p^{u-v}\ } W_u\Omega_A^q \xrightarrow{\ \beta\ } W_v\Omega_A^q \oplus W_v\Omega_A^{q+1} \xrightarrow{\ \alpha\ } \cdots,$$

where $\alpha = dV^{u-v} + V^{u-v}$ and $\beta = (F^{u-v}, -F^{u-v}d)$. This sequence is a more precise version of the statement [**12**, III (3.3.3.3)]. We outline the steps in the proof. First, the basic case $A = \mathbb{F}_p$ is clear. Next, one uses [**10**, Thm. B] to show that the exactness of the sequence for A implies the exactness of the sequence for $A[x]$. Third, one uses [**10**, Prop. 6.2.3] or [**11**, Prop. I.1.14] to conclude that the sequence is exact whenever A is a smooth $\mathbb{F}_p$-algebra. Finally, one uses the theorem of Popescu [**14**] that every regular noetherian $\mathbb{F}_p$-algebra is a filtered colimit of smooth $\mathbb{F}_p$-algebras and the fact that, being an initial object, the de Rham-Witt complex commutes with colimits.

5.   Proof of Thm. A

In this section we evaluate the maps $\iota_p(m,n,j)_q$ that appear in the statement of Thm. 2.5. This proves Thm. 1.5 and hence the equivalent Thm. A. We first recall the following result which was proved in [**7**, Prop. 9.1].

PROPOSITION 5.1.   *Let a and u be positive integers. Then for every non-negative integer i, there exists an isomorphism*

$$\sigma_p(a,u,i)\colon W_r(\mathbb{F}_p) \xrightarrow{\ \sim\ } \mathrm{TR}_{2i-\lambda_a}^u(\mathbb{F}_p; p),$$

where the length $r = r_p(a,u,i)$ is given by

$$r_p(a,u,i) = \begin{cases} u, & \text{if } a \leqslant i, \\ u-s, & \text{if } [a/p^s] \leqslant i < [a/p^{s-1}] \text{ and } 1 \leqslant s < u, \\ 0, & \text{if } i < [a/p^{u-1}]. \end{cases}$$

The group $\mathrm{TR}_{q-\lambda_a}^u(\mathbb{F}_p; p)$ is zero, if q is a negative or odd integer. In addition, the isomorphisms $\sigma_p(a,u,i)$ may be chosen in such a way that the square diagrams

$$
\begin{array}{ccc}
W_r(\mathbb{F}_p) & \xrightarrow{\ \sigma_p(a,u,i)\ } & \mathrm{TR}_{2i-\lambda_a}^u(\mathbb{F}_p; p) \\
{\scriptstyle V}\big\uparrow\big\downarrow{\scriptstyle F} & & {\scriptstyle V}\big\uparrow\big\downarrow{\scriptstyle F} \\
W_{r-1}(\mathbb{F}_p) & \xrightarrow{\ \sigma_p(a,u-1,i)\ } & \mathrm{TR}_{2i-\lambda_a}^{u-1}(\mathbb{F}_p; p),
\end{array}
$$

where $r = r_p(a,u,i)$, commute.

We recall the integer functions $s_p(m,i,j)$ and $d_p(m,u,j)$ defined in (1.1) and (2.3).

COROLLARY 5.2.   *Let $m, u, u' \geqslant 1$ and $i \geqslant 0$ be integers, let $j \geqslant 1$ be an integer that is not divisible by p, and let $d = d_p(m,u,j)$ and $d' = d_p(m,u',j)$. Then the canonical projection*

$$\mathrm{pr}_{u'}\colon \lim_R \mathrm{TR}_{2i-\lambda_d}^u(\mathbb{F}_p; p) \to \mathrm{TR}_{2i-\lambda_{d'}}^{u'}(\mathbb{F}_p; p)$$

is an isomorphism, if $p^{u'}j > m(i+1)$, and the common group is a cyclic $W(\mathbb{F}_p)$-module of length $s_p(m,i,j)$. The group $\lim_R \mathrm{TR}_{q-\lambda_d}^u(\mathbb{F}_p; p)$ is zero, if q is a negative or odd integer.

Proof. We proved in Lemma 2.6 that the map of the statement is an isomorphism, provided that $i < d_p(m, u'+1, j) = [(p^{u'}j - 1)/m]$. This constraint on the integer u' is equivalent to the inequality $i + 1 \leqslant [(p^{u'}j - 1)/m]$, which is equivalent to $i + 1 \leqslant (p^{u'}j - 1)/m$, which, in turn, is equivalent to the stated inequality $m(i+1) < p^{u'}j$. Finally, a similar calculation based on Prop. 5.1 shows that the length of the $W(\mathbb{F}_p)$-module $\lim_R \mathrm{TR}^u_{2i-\lambda_d}(\mathbb{F}_p; p)$ is equal to $s_p(m, i, j)$ as stated. $\square$

LEMMA 5.3. *Let $m > n \geqslant 1$, $u \geqslant 1$, and $i \geqslant 0$ be integers. Let $1 \leqslant j \leqslant m(i+1)$ be an integer that is not divisible by p, and let $d = d_p(m, u, j)$ and $e = d_p(n, u, j)$. Then the map*

$$\iota_p(m, n, j, u)_{2i} \colon \mathrm{TR}^u_{2i-\lambda_d}(\mathbb{F}_p; p) \to \mathrm{TR}^u_{2i-\lambda_e}(\mathbb{F}_p; p)$$

takes a generator of the domain to the product of a generator of the target and an element $\alpha' = \alpha'_p(m, n, i, j, u)$ of $W(\mathbb{F}_p)$ with p-adic valuation

$$v_p(\alpha') = \sum_{d < a \leqslant e} \mathrm{length}_{W(\mathbb{F}_p)} \mathrm{TR}^{v(u,a)}_{2i-\lambda_a}(\mathbb{F}_p; p),$$

where $v(u, a) = \min\{u, v_p(a) + 1\}$.

Proof. The canonical inclusion $\iota_p(m, n, j, u) \colon \lambda_d \to \lambda_e$ is equal to the composition of the canonical inclusions $\iota \colon \lambda_{a-1} \to \lambda_a$ for $d < a \leqslant e$. Since the groups $\mathrm{TR}^v_{q-\lambda_a}(\mathbb{F}_p; p)$ are zero, for q odd, Prop. 4.2 above identifies the cokernel of the map

$$\iota_* \colon \mathrm{TR}^u_{2i-\lambda_{a-1}}(\mathbb{F}_p; p) \to \mathrm{TR}^u_{2i-\lambda_a}(\mathbb{F}_p; p)$$

with the $W(\mathbb{F}_p)$-module $\mathrm{TR}^{v(u,a)}_{2i-\lambda_a}(\mathbb{F}_p; p)$. Equivalently, ι_* takes a generator of the domain to the product of a generator of the target and an element $\alpha'_a \in W(\mathbb{F}_p)$ of p-adic valuation

$$v_p(\alpha'_a) = \mathrm{length}_{W(\mathbb{F}_p)} \mathrm{TR}^{v(u,a)}_{2i-\lambda_a}(\mathbb{F}_p; p).$$

The lemma follows. $\square$

We proceed to manipulate the sum that appears in Lemma 5.3. To this end, we fix the positive integer u, and consider the bi-graded $\mathbb{F}_p$-vector space $E(u)$ defined by the associated graded for the p-adic filtration of the $W(\mathbb{F}_p)$-modules that appear in Lemma 5.3,

$$E(u)_{i,a} = \bigoplus_{r \geqslant 0} \mathrm{gr}^r_p \mathrm{TR}^{v(u,a)}_{2i-\lambda_a}(\mathbb{F}_p; p).$$

The following results identifies the structure of this bi-graded $\mathbb{F}_p$-vector space.

LEMMA 5.4. *The bi-graded $\mathbb{F}_p$-vector space $E(u)$ is isomorphic to the bi-graded $\mathbb{F}_p$-vector space defined by the sum of symmetric algebras*

$$A(u) = \bigoplus_{1 \leqslant r \leqslant u} S_{\mathbb{F}_p}\{x_r, \sigma_r\},$$

where $\deg x_r = (p^{r-1}, 1)$ and $\deg \sigma_r = (0, 1)$.

Proof. The statement is equivalent to the equality

$$\dim_{\mathbb{F}_p} A(u)_{a,i} = \mathrm{length}_{W(\mathbb{F}_p)} \mathrm{TR}^{v(u,a)}_{2i-\lambda_a}(\mathbb{F}_p; p),$$

which we verify by direct calculation. If $a \leqslant i$, then $A(u)_{a,i}$ has basis

$$x_1^a \sigma_1^{i-a}, x_2^{a/p} \sigma_2^{i-a/p}, \ldots, x_{v(u,a)}^{a/p^{v(u,a)-1}} \sigma_{v(u,a)}^{i-a/p^{v(u,a)-1}}$$

so $\dim_{\mathbb{F}_p} A(u)_{a,i} = v(u,a)$ as required. Similarly, if $a/p^s \leqslant i < a/p^{s-1}$ with $1 \leqslant s < v(u,a)$, then $A(u)_{a,i}$ has basis

$$x_{s+1}^{a/p^s} \sigma_{s+1}^{i-a/p^s}, x_{s+2}^{a/p^{s+1}} \sigma_{s+2}^{i-a/p^{s+1}}, \ldots, x_{v(u,a)}^{a/p^{v(u,a)-1}} \sigma_{v(u,a)}^{i-a/p^{v(u,a)-1}}$$

which shows that $\dim_{\mathbb{F}_p} A(u)_{a,i} = v(u,a) - s$ as desired. Finally, if $i < a/p^{v(u,a)-1}$, then $A(u)_{a,i} = 0$. This completes the proof. $\qquad\square$

PROPOSITION 5.5. *Let $m > n \geqslant 1$ and $i \geqslant 0$ be integers, and let $1 \leqslant j \leqslant m(i+1)$ be an integer that is not divisible by p. Then the map*

$$\iota_p(m,n,j)_{2i}\colon \lim_R \mathrm{TR}^u_{2i-\lambda_d}(\mathbb{F}_p;p) \to \lim_R \mathrm{TR}^u_{2i-\lambda_e}(\mathbb{F}_p;p),$$

where $d = d_p(m,u,j)$ and $e = d_p(n,u,j)$, takes a generator of the domain to the product of a generator of the target and an element $\alpha = \alpha_p(m,n,i,j)$ of $W(\mathbb{F}_p)$ with p-adic valuation

$$v_p(\alpha) = \sum_{0 \leqslant h < i} \big(s_p(m,h,j) - s_p(n,h,j)\big).$$

Proof. By Cor. 5.2 above, we may instead consider the map

$$\iota_p(m,n,j,u)_{2i}\colon \mathrm{TR}^u_{2i-\lambda_d}(\mathbb{F}_p;p) \to \mathrm{TR}^u_{2i-\lambda_e}(\mathbb{F}_p;p),$$

for a fixed positive integer u with $p^u j > m(i+1)$. We showed in Lemma 5.3 above that this map takes a generator of the domain to the product of a generator of the target and an element $\alpha' = \alpha'_p(m,n,i,j,u)$ of $W(\mathbb{F}_p)$ whose p-adic valuation is given by the sum

$$v_p(\alpha') = \sum_{d < a \leqslant e} \mathrm{length}_{W(\mathbb{F}_p)} \mathrm{TR}^{v(u,a)}_{2i-\lambda_a}(\mathbb{F}_p;p)$$

where $v(u,a) = \min\{u, v_p(a)+1\}$. We show that this sum is equal to the sum $v_p(\alpha)$ of the statement. It follows from Lemma 5.4 that

$$v_p(\alpha') = \dim_{\mathbb{F}_p} \Big(\bigoplus_{d < a \leqslant e} A(u)_{a,i} \Big).$$

The $\mathbb{F}_p$-vector space on the right-hand side has a basis given by the elements $x_r^k \sigma_r^l$, where $1 \leqslant r \leqslant u$, and where k and l are non-negative integers such that $k + l = i$ and $d < p^{r-1}k \leqslant e$. Therefore, we have

$$v_p(\alpha') = \sum_{1 \leqslant r \leqslant u} \mathrm{card}\{0 \leqslant k \leqslant i \mid d < p^{r-1}k \leqslant e\} = \sum_{1 \leqslant r \leqslant u} \mathrm{card}\{1 \leqslant k \leqslant i \mid d < p^{r-1}k \leqslant e\}$$

$$= \sum_{1 \leqslant k \leqslant i} \mathrm{card}\{1 \leqslant r \leqslant u \mid d < p^{r-1}k \leqslant e\} = \sum_{0 \leqslant h < i} \mathrm{card}\{1 \leqslant r \leqslant u \mid d < p^{r-1}(h+1) \leqslant e\}$$

$$= \sum_{0 \leqslant h < i} \big(\mathrm{card}\{1 \leqslant r \leqslant u \mid d < p^{r-1}(h+1)\} - \mathrm{card}\{1 \leqslant r \leqslant u \mid e < p^{r-1}(h+1)\}\big),$$

where, we recall, $d = d_p(m,u,j)$ and $e = d_p(n,u,j)$. The inequality $d < p^{r-1}(h+1)$ is equivalent to the inequality $(p^{u-1}j-1)/m < p^{r-1}(h+1)$ which, in turn, is equivalent to the inequality

$$p^{u-r}j < m(h+1).$$

Suppose that $s = s_p(m,h,j)$ satisfies $1 \leqslant s \leqslant u$. Then $p^{s-1}j \leqslant m(h+1) < p^s j$. Therefore, the inequality $p^{u-1}j < m(h+1)$ is equivalent to the inequality $u - r \leqslant s - 1$. Hence,

$$\mathrm{card}\{1 \leqslant r \leqslant u \mid d < p^{r-1}(h+1)\} = \mathrm{card}\{u - (s-1) \leqslant r \leqslant u\} = s = s_p(m,h,j).$$

Finally, if $m(h+1) < j$, we also find that

$$\mathrm{card}\{1 \leqslant r \leqslant u \mid d < p^{r-1}(h+1)\} = 0 = s_p(m,h,j).$$

The proof that $\operatorname{card}\{1 \leqslant r \leqslant u \mid e < p^{r-1}(h+1)\} = s_p(n, h, j)$ is similar. $\qquad\square$

Proof of Thm. A. We prove the equivalent Thm. 1.5. Suppose that $A = \mathbb{F}_p$. We first identify the sequences (2.2) and (1.2) and the sequences (2.4) and (1.3). Let $m \geqslant 1$ and $i \geqslant 0$ be integers, and let $j \geqslant 1$ be an integer not divisible by p. We know from Cor. 5.2 that, if $u' \geqslant 1$ is an integer such that $p^{u'} j > m(i+1)$, then the canonical projection

$$\operatorname{pr}_{u'} \colon \varprojlim_R \operatorname{TR}^u_{2i-\lambda_d}(\mathbb{F}_p; p) \to \operatorname{TR}^{u'}_{2i-\lambda_{d'}}(\mathbb{F}_p; p),$$

where $d = d_p(m, u, j)$ and $d' = d_p(m, u', j)$, is an isomorphism. Hence, the composition of the isomorphism $\sigma_p(d', u', i)$ of Prop. 5.1 and the inverse of the isomorphism $\operatorname{pr}_{u'}$ defines an isomorphism

$$\tau_p(m, i, j, u') \colon W_s(\mathbb{F}_p) \xrightarrow{\sim} \varprojlim_R \operatorname{TR}^u_{2i-\lambda_d}(\mathbb{F}_p; p),$$

where $s = s_p(m, i, j)$. Then, if $p^{u'} > m(i+1)$, we define

$$\tau_p(m, i, u') \colon \bigoplus_{j \in I_p} W_s(\mathbb{F}_p) \to \prod_{j \in I_p} \varprojlim_R \operatorname{TR}^u_{2i-\lambda_d}(\mathbb{F}_p; p)$$

to be the isomorphism that takes the jth summand on the left-hand side to the jth factor on the right-hand side by the map $\tau_p(m, i, j, u')$. Similarly, we define

$$\tau'_p(m, i, u') \colon \bigoplus_{j \in m'I_p} W_{s-v}(\mathbb{F}_p) \to \prod_{j \in m'I_p} \varprojlim_R \operatorname{TR}^{u-v}_{2i-\lambda_d}(\mathbb{F}_p; p),$$

where $m = p^v m'$ with m' not divisible by p, where $s = s_p(m, i, j)$, and where $d = d_p(m, u, j)$, to be the isomorphism that takes the jth summand on the left-hand side to the jth factor on the right-hand side by the isomorphism given by the composition of $\sigma_p(d', u' - v, i)$ and the inverse of the isomorphism

$$\operatorname{pr}_{u'} \colon \varprojlim_R \operatorname{TR}^{u-v}_{2i-\lambda_d}(\mathbb{F}_p; p) \to \operatorname{TR}^{u'-v}_{2i-\lambda_{d'}}(\mathbb{F}_p; p).$$

Then, for every u' such that $p^{u'} > m(i+1)$, the diagram

$$
\begin{array}{ccc}
\displaystyle\bigoplus_{j \in m'I_p} W_{s-v}(\mathbb{F}_p) & \xrightarrow{\ m'V^v\ } & \displaystyle\bigoplus_{j \in I_p} W_s(\mathbb{F}_p) \\[2ex]
\Big\downarrow{\scriptstyle \tau'_p(m,i,u')} & & \Big\downarrow{\scriptstyle \tau_p(m,i,u')} \\[2ex]
\displaystyle\prod_{j \in m'I_p} \varprojlim_R \operatorname{TR}^{u-v}_{2i-\lambda_d}(\mathbb{F}_p; p) & \xrightarrow{\ m'V^v\ } & \displaystyle\prod_{j \in I_p} \varprojlim_R \operatorname{TR}^u_{2i-\lambda_d}(\mathbb{F}_p; p)
\end{array}
$$

commutes. This identifies the sequences (2.2) and (1.2). However, we do not know that the family of isomorphisms $\sigma_p(a, u, i)$ in Prop. 5.1 can be chosen with the additional property that, if both $p^{u'} > m(i+1)$ and $p^{u''} > m(i+1)$, the isomorphisms $\tau_p(m, i, u')$ and $\tau_p(m, i, u'')$ are equal. Therefore, we choose $u' = u'(m, i)$ to be the unique integer that satisfies

$$p^{u'-1} \leqslant m(i+1) < p^{u'}$$

and use the isomorphisms $\tau_p(m, i, u')$ and $\tau'_p(m, i, u')$ to identify (2.2) and (1.2). We use the same $u' = u'(m, i)$ and the isomorphisms $\tau_p(n, i, u')$ and $\tau'_p(n, i, u')$ to identify the sequences (2.4) and (1.3). In particular, the sequences (1.2) and (1.3) are exact as proved in [**6**, Thm. 4.2.10].

Finally, Prop. 5.5 shows that there is a commutative diagram

$$
\begin{array}{ccc}
W_s(\mathbb{F}_p) \xrightarrow{\ R^{s-t}\ } W_t(\mathbb{F}_p) \xrightarrow{\ m_\alpha\ } W_t(\mathbb{F}_p) \\
\Big\downarrow{\scriptstyle \tau_p(m,i,j,u')} \qquad\qquad\qquad \Big\downarrow{\scriptstyle \tau_p(n,i,j,u')} \\
\lim_R \mathrm{TR}^u_{2i-\lambda_d}(\mathbb{F}_p;p) \xrightarrow{\ \iota_p(m,n,i,j)\ } \lim_R \mathrm{TR}^u_{2i-\lambda_e}(\mathbb{F}_p;p),
\end{array}
$$

where the map m_α is given by multiplication by an element $\alpha = \alpha_p(m,n,i,j) \in W(\mathbb{F}_p)$ with p-adic valution

$$
v_p(\alpha) = \sum_{0 \leqslant h < i} \big(s_p(m,h,j) - s_p(n,h,j)\big).
$$

This completes the proof of Thm. 1.5 for $A = \mathbb{F}_p$.

Suppose next that A is any regular noetherian ring that is also an $\mathbb{F}_p$-algebra. The groups $\lim_R \mathrm{TR}^u_{q-\lambda_d}(A;p)$ were evaluated in [**8**, Thm. 2.2.2], but see also [**5**, Thm. 16]. By the universal property of the de Rham-Witt complex, there is a canonical map of graded rings

$$
\xi_u \colon W_u\Omega_A^* \to \mathrm{TR}^u_*(A;p)
$$

that commutes with R, F, V, and d. Let $m \geqslant 1$ and $i \geqslant 0$ be integers, and let $u' = u'(m,i)$. Let j be an integer not divisible by p, and let $s = s_p(m,i,j)$. We consider the map

$$
\tilde{\omega}_p(m,q,i,j,u')\colon \lim_R W_u\Omega_A^{q-2i} \otimes_{W(\mathbb{F}_p)} W_s(\mathbb{F}_p) \to \lim_R \mathrm{TR}^u_{q-\lambda_d}(A;p)
$$

defined by the composition

$$
\lim_R W_u\Omega_A^{q-2i} \otimes_{W(\mathbb{F}_p)} W_s(\mathbb{F}_p) \to \lim_R \mathrm{TR}^u_{q-2i}(A;p) \otimes_{W(\mathbb{F}_p)} \lim_R \mathrm{TR}^u_{2i-\lambda_d}(\mathbb{F}_p;p)
$$
$$
\to \lim_R \mathrm{TR}^u_{q-2i}(A;p) \otimes_{W(\mathbb{F}_p)} \lim_R \mathrm{TR}^u_{2i-\lambda_d}(A;p) \to \lim_R \mathrm{TR}^u_{q-\lambda_d}(A;p),
$$

where the first map is induced by the maps $\xi_u \otimes \tau_p(m,i,j,u')$, where the second map is induced from the unit map $\eta\colon \mathbb{F}_p \to A$, and where the last map is induced by the $T(A)$-module spectrum structure on $S^{\lambda_d} \wedge T(A)$. It follows from [**5**, Thm. 16] that the maps $\tilde{\omega}_p(m,q,i,j,u')$ factor through the canonical projections

$$
\mathrm{pr}_s \otimes \mathrm{id}\colon \lim_R W_u\Omega_A^{q-2i} \otimes_{W(\mathbb{F}_p)} W_s(\mathbb{F}_p) \to W_s\Omega_A^{q-2i}
$$

and that the induced maps $\omega_p(m,q,i,j,u')$ define an isomorphism

$$
\omega_p(m,q,j) = \bigoplus \omega_p(m,q,i,j,u')\colon \bigoplus_{i \geqslant 0} W_s\Omega_A^{q-2i} \xrightarrow{\sim} \lim_R \mathrm{TR}^u_{q-\lambda_d}(A;p),
$$

where $u' = u'(m,i)$ and $s = s_p(m,i,j)$. We define

$$
\omega_p(m,q)\colon \bigoplus_{i \geqslant 0} \bigoplus_{j \in I_p} W_s\Omega_A^{q-2i} \to \prod_{j \in I_p} \lim_R \mathrm{TR}^u_{q-\lambda_d}(A;p)
$$

to be the isomorphism that takes the j summand on the left-hand side to the jth factor on the right-hand side by the isomorphism $\omega_p(m,q,j)$. We define the isomorphism

$$
\omega'_p(m,q)\colon \bigoplus_{i \geqslant 0} \bigoplus_{j \in m'I_p} W_{s-v}\Omega_A^{q-2i} \to \prod_{j \in m'I_p} \lim_R \mathrm{TR}^{u-v}_{q-\lambda_d}(A;p)
$$

in a completely similar manner substituting the isomorphisms $\tau'_p(m,i,j,u')$ for the isomorphisms $\tau_p(m,i,j,u')$ in the definition of $\omega_p(m,q)$. Then the diagram

$$
\begin{array}{ccc}
\displaystyle\bigoplus_{i\geqslant 0}\bigoplus_{j\in m'I_p} W_{s-v}\Omega_A^{q-2i} & \xrightarrow{\ m'V^v\ } & \displaystyle\bigoplus_{i\geqslant 0}\bigoplus_{j\in I_p} W_s\Omega_A^{q-2i} \\[2ex]
\Big\downarrow{\scriptstyle\omega'_p(m,q)} & & \Big\downarrow{\scriptstyle\omega_p(m,q)} \\[2ex]
\displaystyle\prod_{j\in m'I_p}\varprojlim_R \mathrm{TR}_{q-\lambda_d}^{u-v}(A;p) & \xrightarrow{\ m'V^v\ } & \displaystyle\prod_{j\in I_p}\varprojlim_R \mathrm{TR}_{q-\lambda_d}^u(A;p)
\end{array}
$$

commutes. Hence, the isomorphisms $\omega_p(m,q)$ and $\omega'_p(m,q)$ identify the sequences (2.2) and (1.2). In particular, the sequence (1.2) is exact. Similarly, the isomorphisms $\omega_p(n,q)$ and $\omega'_p(n,q)$ identify the sequences (2.4) and (1.3). Finally, we have a commutative diagram

$$
\begin{array}{ccccc}
W_s\Omega_A^{q-2i} & \xrightarrow{\ R^{s-t}\ } & W_t\Omega_A^{q-2i} & \xrightarrow{\ m_\alpha\ } & W_t\Omega_A^{q-2i} \\[2ex]
\Big\downarrow{\scriptstyle\omega_p(m,q,i,j,u')} & & & & \Big\downarrow{\scriptstyle\omega_p(n,q,i,j,u')} \\[2ex]
\varprojlim_R \mathrm{TR}_{q-\lambda_d}^u(A;p) & & \xrightarrow{\ \iota_p(m,n,i,j)\ } & & \varprojlim_R \mathrm{TR}_{q-\lambda_e}^u(A;p),
\end{array}
$$

where the top horizontal map m_α is multiplication by the *same* element $\alpha = \alpha_p(m,n,i,j)$ of $W(\mathbb{F}_p)$ as in the basic case $A = \mathbb{F}_p$. This completes the proof of Thm. 1.5. $\qquad\square$

6. The divisor $\mathrm{div}(\alpha_p(m,n,i))$

In this section we examine the divisor $\mathrm{div}(\alpha_p(m,n,i))$ which appears in the statement of Thm. A and prove the precise Thm. 6.3 below. We then use this result to derive Thm. B of the introduction and Thm. 6.5 below. We recall the integer function $s_p(m,i,j)$ defined in (1.1).

LEMMA 6.1. *Suppose that the integer $t = s_p(n,i,j)$ satisfies*

$$\frac{m-n}{mn}\cdot\frac{p^t-1}{p-1}\cdot j \geqslant 2t.$$

Then $v_p(\alpha_p(m,n,i,j)) \geqslant s_p(n,i,j)$.

Proof. We recall that

$$v_p(\alpha_p(m,n,i,j)) = \sum_{0\leqslant h<i}\big(s_p(m,h,j) - s_p(n,h,j)\big).$$

In this sum, every summand is non-negative, and the summand indexed by h is positive if and only if there exists an integer $r \geqslant 1$ with

$$n(h+1) < p^{r-1}j \leqslant m(h+1).$$

We estimate the number of indices $0 \leqslant h < i$ that satisfy this inequality, for some $r \geqslant 1$. For a given $r \geqslant 1$, the number of integers h that satisfy the inequality

$$n(h+1) < p^{r-1}j \leqslant m(h+1)$$

is at least

$$\frac{(m-n)p^{r-1}j}{mn} - 1.$$

If $t = s_p(n, i, j) = 0$, then the statement of the lemma is trivially true. So assume that $t > 0$. Then t is the unique integer that satisfies $p^{t-1}j \leqslant n(i+1) < p^t j$. Now, if $1 \leqslant r \leqslant t$, then

$$n(h+1) < p^{r-1}j \leqslant p^{t-1}j \leqslant n(i+1),$$

implies that $h < i$. Therefore, we find

$$v_p(\alpha_p(m, n, i, j)) \geqslant \sum_{1 \leqslant r \leqslant t} \Big(\frac{(m-n)p^{r-1}j}{mn} - 1 \Big) = \frac{m-n}{mn} \cdot \frac{p^t - 1}{p - 1} \cdot j - t.$$

Hence, if the latter integer is greater than or equal to t, or equivalently, if

$$\frac{m-n}{mn} \cdot \frac{p^t - 1}{p - 1} \cdot j \geqslant 2t,$$

then $v_p(\alpha_p(m, n, i, j)) \geqslant s_p(n, i, j)$. $\qquad \square$

LEMMA 6.2. *Suppose that $i \geqslant n/(m-n)$ and $v_p(\alpha_p(m, n, i-1, j)) \geqslant s_p(n, i-1, j)$. Then also $v_p(\alpha_p(m, n, i, j)) \geqslant s_p(n, i, j)$.*

Proof. The assumption that $i \geqslant n/(m-n)$ implies that $mi \geqslant n(i+1)$ which, in turn, implies that $s_p(m, i-1, j) \geqslant s_p(n, i, j)$. Suppose $v_p(\alpha_p(m, n, i-1, j)) \geqslant s_p(n, i-1, j)$. Then

$$v_p(\alpha_p(m, n, i, j)) = v_p(\alpha_p(s, n, i-1, j)) + s_p(m, i-1, j) - s_p(n, i-1, j) \geqslant s_p(m, i-1, j).$$

The statement follows. $\qquad \square$

THEOREM 6.3. *The divisor $\mathrm{div}(\alpha_p(m, n, i))$ satisfies the following:*
(i) For every pair of integers $m > n > 1$ and for every prime number p, there exists an integer $i_0 = i_0(m, n, p)$ such that, for all integers $i \geqslant i_0$,

$$\mathrm{div}(\alpha_p(m, n, i)) \geqslant \mathrm{div}(\mathbb{W}_{n(i+1)}(\mathbb{F}_p)).$$

(ii) For every pair of integers $m > n > 1$, the integer $i_0(m, n, p)$ tends to infinity as the prime number p tends to infinity.
(iii) For every integer $n > 1$ and every prime number p, there exists an integer $m_0 = m_0(n, p)$ such that, for all $m \geqslant m_0(n, p)$ and for all integers $i > 0$,

$$\mathrm{div}(\alpha_p(m, n, i)) \geqslant \mathrm{div}(\mathbb{W}_{n(i+1)}(\mathbb{F}_p)).$$

Proof. We first prove the statement (i), which, by Lemma 1.4 is equivalent to the statement that there exists an integer $i_0 = i_0(m, n, p)$ such that, for all positive integers j not divisible by p, $v_p(\alpha_p(m, n, i, j)) \geqslant s_p(n, i, j)$. Suppose first that $j \geqslant 2mn/(m-n)$. Then

$$\frac{m-n}{mn} \cdot \frac{p^t - 1}{p - 1} \cdot j \geqslant 2 \cdot \frac{p^t - 1}{p - 1}$$

and, for every prime number p and every integer $t \geqslant 0$, the right-hand side is greater than or equal to $2t$. Therefore, Lemma 6.1 shows that $v_p(\alpha_p(m, n, i, j)) \geqslant s_p(n, i, j)$, for all $i \geqslant 0$. Suppose next that $1 \leqslant j < 2mn/(m-n)$. Then the inequality of the statement of Lemma 6.1 is satisfied, if t is large enough. Since $t = s_p(n, i, j)$ tends to infinity as i tends to infinity, it follows that there exists an integer $i_0(m, n, p, j)$ such that $v_p(\alpha_p(m, n, i, j)) \geqslant s_p(n, i, j)$, for all $i \geqslant i_0(m, n, p, j)$. This proves that part (i) of the statement holds with $i_0(m, n, p)$ equal to the maximum of the integers $i_0(m, n, p, j)$, where $1 \leqslant j < 2mn/(m-n)$.

We next prove (ii). Suppose that $p > m(i+1)$. Then, for every integer $0 \leqslant h \leqslant i$, and for every integer $1 \leqslant j \leqslant n$ not divisible by p, $s_p(i, j) = s_p(n, i, j) = 1$, and hence, $v_p(\alpha_p(m, n, i, j)) < s_p(n, i, j)$. Therefore, if $p > m(i+1)$, then $i_0(m, n, p) > i$. It follows that $i_0(m, n, p) \geqslant [(p-1)/m]$ which tends to infinity as p tends to infinity.

Finally, we prove (iii). For fixed i and j, the integer $s_p(m, i, j)$ tends to infinity as m tends to infinity. Hence, there exists $m_0 = m_0(n, p, i, j)$ such that $v_p(\alpha_p(m, n, i, j)) \geqslant s_p(n, i, j)$, for $m \geqslant m_0(n, p, i, j)$. Assume that $m_0(n, p, i, j)$ is chosen minimal with this property. Then an induction argument based on Lemma 6.2 shows that $m_0(n, p, i, j) \leqslant m_0(n, p, i - 1, j)$, if $i \geqslant n/(m - n)$. Moreover, if $j > n(i + 1)$, then $s_p(n, i, j) = 0$, and therefore, $m_0(n, p, i, j) = n$. This shows that (iii) holds with $m_0(n, p)$ equal to the maximum of the finitely many integers $m_0(n, p, i, j)$, where $0 \leqslant i \leqslant n/(m - n)$ and $1 \leqslant j \leqslant n(i + 1)$. $\qquad\square$

LEMMA 6.4. *Let A be an $\mathbb{F}_p$-algebra and suppose that A is generated by N elements as an algebra over the subring A^p of pth powers. Then, for every subset $S \subset \mathbb{N}$ stable under division, the big de Rham-Witt group $\mathbb{W}_S \Omega_A^q$ is zero, if $q > N + 1$.*

Proof. It suffices to show that, for all $u \geqslant 1$, the p-typical de Rham-Witt group $W_u \Omega_A^q$ is zero, if $q > N + 1$; compare Sect. 1 above. We first show that Ω_A^q is zero, if $q > N$. By assumption, there exists a surjective ring homomorphism $f \colon A^p[x_1, \dots, x_N] \to A$ from a polynomial algebra in finitely many variables over A^p. The induces map

$$f_* \colon \Omega_{A^p[x_1, \dots, x_N]}^q \to \Omega_A^q$$

is again surjective. Moreover, every element of of the domain is a sum of elements of the form

$$\omega = \eta \, dx_{i_1} \dots dx_{i_s}$$

where $0 \leqslant s \leqslant N$, where $1 \leqslant i_1 < \cdots < i_s \leqslant N$, and where $\eta \in \Omega_{A^p}^{q-s}$. Now, we claim that $f_*(\eta) = 0$ unless $s = q$. Indeed, the element $\eta \in \Omega_{A^p}^{q-s}$ can be written as a sum of elements of the form $b_0 db_1 \dots db_{q-s}$, where $b_0, \dots, b_{q-s} \in A^p$, and

$$f_*(b_0 db_1 \dots db_{q-s}) = f(b_0) df(b_1) \dots df(b_{q-s}).$$

But $f(b_i) = a_i^p$, for some $a_i \in A$, and hence, $df(b_i) = d(a_i^p) = p a_i^{p-1} da_i = 0$. It follows that the image of f_* is zero, if $q > N$. This shows that Ω_A^q is zero, if $q > N$, as stated.

Finally, we show by induction on $u \geqslant 1$ that $W_u \Omega_A^q$ is zero, if $q > N + 1$. The case $u = 1$ holds, since the canonical map $\Omega_A^q \to W_1 \Omega_A^q$ is an isomorphism. Finally, the induction step follows from the exact sequence

$$\Omega_A^q \oplus \Omega_A^{q-1} \xrightarrow{\ V^{u-1} + dV^{u-1}\ } W_u \Omega_A^q \xrightarrow{\ R\ } W_{u-1} \Omega_A^q \longrightarrow 0$$

which is proved in [**9**, Prop. 3.2.6]. $\qquad\square$

Proof of Thm. B. Since $m > n + 1$, we can choose $n < k < m$ and write f as the composition of the canonical projections $g \colon A[x]/(x^m) \to A[x]/(x^k)$ and $h \colon A[x]/(x^k) \to A[x]/(x^n)$. We consider the following maps of long-exact sequences from Thm. A.

$$
\begin{array}{ccccccc}
\cdots \to \bigoplus_{i \geqslant 0} \mathbb{W}_{m(i+1)} \Omega_A^{q-2i} & \xrightarrow{\ \varepsilon\ } & K_{q+1}(A[x]/(x^m), (x)) & \xrightarrow{\ \partial\ } & \bigoplus_{i \geqslant 0} \mathbb{W}_{i+1} \Omega_A^{q-1-2i} & \to & \cdots \\
\downarrow{\scriptstyle g_*'} & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle 0} & & \\
\cdots \to \bigoplus_{i \geqslant 0} \mathbb{W}_{k(i+1)} \Omega_A^{q-2i} & \xrightarrow{\ \varepsilon\ } & K_{q+1}(A[x]/(x^k), (x)) & \xrightarrow{\ \partial\ } & \bigoplus_{i \geqslant 0} \mathbb{W}_{i+1} \Omega_A^{q-1-2i} & \to & \cdots \\
\downarrow{\scriptstyle h_*'} & & \downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle 0} & & \\
\cdots \to \bigoplus_{i \geqslant 0} \mathbb{W}_{n(i+1)} \Omega_A^{q-2i} & \xrightarrow{\ \varepsilon\ } & K_{q+1}(A[x]/(x^n), (x)) & \xrightarrow{\ \partial\ } & \bigoplus_{i \geqslant 0} \mathbb{W}_{i+1} \Omega_A^{q-1-2i} & \to & \cdots
\end{array}
$$

It follows from Lemma 6.4 and Thm. 6.3(i) that there exists an integer q_0 such that both the maps g_*' and h_*' are zero, for $q \geqslant q_0$. Finally, a diagram chase based on the diagram above shows that the composite map $f_* = h_* \circ g_*$ is zero, for $q \geqslant q_0$. $\qquad\square$

Suppose that A is a regular noetherian ring and an $\mathbb{F}_p$-algebra. We let

$$\varepsilon_i \colon \mathbb{W}_{n(i+1)}\Omega_A^{q-2i} \to K_{q+1}(A[x]/(x^n),(x))$$

be the ith summand of the map ε in the long-exact sequence

$$\cdots \longrightarrow \bigoplus_{i\geqslant 0} \mathbb{W}_{i+1}\Omega_A^{q-2i} \xrightarrow{V_n} \bigoplus_{i\geqslant 0} \mathbb{W}_{n(i+1)}\Omega_A^{q-2i} \xrightarrow{\varepsilon} K_{q+1}(A[x]/(x^n),(x)) \longrightarrow \cdots.$$

We show that the image of the map ε_0 has the following interpretation.

THEOREM 6.5. *Let A be a regular noetherian ring that is also an $\mathbb{F}_p$-algebra, and let n be a positive integer. Then*

$$\bigcap_{m>n} \operatorname{im}\left(f_* \colon K_q(A[x]/(x^m),(x)) \to K_q(A[x]/(x^n),(x))\right)$$
$$= \operatorname{im}\left(\varepsilon_0 \colon \mathbb{W}_n\Omega_A^{q-1} \to K_q(A[x]/(x^n),(x))\right).$$

Proof. Let $g \colon A[x]/(x^m) \to A[x]/(x^k)$ and $h \colon A[x]/(x^k) \to A[x]/(x^n)$ be the canonical projections, with $m > k > n$. A diagram based on the diagram from the proof of Thm. B above shows that, for every integer $q \geqslant 0$, we have inclusions

$$\operatorname{im}\left(\bigoplus_{i\geqslant 0} \mathbb{W}_{m(i+1)}\Omega_A^{q-2i} \xrightarrow{h'_* g'_*} \bigoplus_{i\geqslant 0} \mathbb{W}_{n(i+1)}\Omega_A^{q-2i} \xrightarrow{\varepsilon} K_{q+1}(A[x]/(x^n),(x))\right)$$
$$\subset \operatorname{im}\left(K_{q+1}(A[x]/(x^m),(x)) \xrightarrow{f_*} K_{q+1}(A[x]/(x^n),(x))\right)$$
$$\subset \operatorname{im}\left(\bigoplus_{i\geqslant 0} \mathbb{W}_{k(i+1)}\Omega_A^{q-2i} \xrightarrow{h'_*} \bigoplus_{i\geqslant 0} \mathbb{W}_{n(i+1)}\Omega_A^{q-2i} \xrightarrow{\varepsilon} K_{q+1}(A[x]/(x^n),(x))\right).$$

The maps $h'_* g'_*$ and h'_* map the summand indexed by $i = 0$ to the summand indexed by $i = 0$ by the appropriate restriction map, which is surjective. Thm. 6.3 shows that, on summands indexed by $i > 1$, the maps $h'_* g'_*$ and h'_* are zero, if m is sufficiently large. The statement follows. $\square$

REMARK 6.6. Let k be a perfect field of positive characteristic p, and let n be a positive integer. It follows from Thms. A and 6.3 that, for $m \gg n$, the composite map

$$\mathbb{W}_{n(i+1)}(k) \xrightarrow{\varepsilon} K_{2i+1}(k[x]/(x^n),(x)) \xrightarrow{\delta} K_{2i}(k[x]/(x^m),(x^n))$$

induces an isomorphism

$$\Delta'_n \colon \mathbb{W}_{n(i+1)}(k)/V_n\mathbb{W}_{i+1}(k) \xrightarrow{\sim} K_{2i}(k[x]/(x^m),(x^n)).$$

We expect that, for $i = 1$, the map Δ'_n is equal to the map Δ_n of Stienstra [**16**, Cor. 3.6].

REMARK 6.7. One generally expects that, for every scheme X, there exists an Atiyah-Hirzebruch type spectral sequence

$$E_{s,t}^2 = H^{t-s}(X;\mathbb{Z}(t)) \Rightarrow K_{s+t}(X)$$

from the motivic cohomology of X to the algebraic K-theory of X. Such a sequence has been constructed in the case where X is a smooth scheme over a field [**17**]. However, in general, there presently does not exist a definition of motivic cohomology that could serve as the E^2-term of a spectral sequence of this form. However, see [**2**, **15**]. The long-exact sequence relating $K(A[x]/(x^n),(x))$ and the big de Rham-Witt groups and Thm. 6.5 suggests that, for $X = \operatorname{Spec}(A[x]/(x^n))$, where A is a regular noetherian $\mathbb{F}_p$-algebra, the spectral sequence takes the form

$$E_{s,t}^1 = \mathbb{W}_s\Omega_A^{t-(s-1)} \oplus \mathbb{W}_{(s+1)n}\Omega_A^{t-s} \Rightarrow K_{s+t}(A[x]/(x^n),(x))$$

with the d^1-differential induced by the Verschiebung operator

$$V_n \colon \mathbb{W}_s\Omega_A^{t-(s-1)} \to \mathbb{W}_{sn}\Omega_A^{t-(s-1)}$$

and with all higher differentials zero. The E^2-term is of this hypothetical spectral sequence then is our candidate for the value of the motivic cohomology groups of $A[x]/(x^n)$ relative to the ideal (x). We note that this value of the motivic cohomology groups is in agreement with the Beilinson-Soulé vanishing conjecture that $H^q(A[x]/(x^n), (x); \mathbb{Z}(t))$ is zero, for $q \leqslant 0$.

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