On a conjecture of Vorst

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Abstract Let k be an infinite perfect field of positive characteristic and assume that strong resolution of singularities holds over k. We prove that a localization of a d-dimensional commutative k-algebra R of finite type is K_{d+1}-regular if and only if it is regular. This partially affirms a conjecture of Vorst.

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Introduction

A ring R is defined to be K_n-regular, if the map $K_n(R) \to K_n(R[t_1,\ldots,t_r])$ induced by the canonical inclusion is an isomorphism for all $r \geqslant 0$ [1, Definition 2.2]. It was proved by Quillen [13, Corollary of Theorem 8] that a (left) regular noetherian ring is K_n-regular for all integers n. A conjecture of Vorst [14, Conjecture] predicts that, conversely, if R is a commutative ring of dimension d essentially of finite type over a field k, then K_{d+1}-regularity implies regularity. Recently, Cortiñas, Haesemeyer, and Weibel proved that the conjecture holds, if the field k has characteristic zero [2, Theorem 0.1]. In this paper, we prove, by somewhat different methods, that the following slightly weaker result holds, if k is an infinite perfect field of characteristic $p > 0$ and strong resolution of singularities holds over k in the sense of Section 1 below.

Theorem A. *Let k be an infinite perfect field of characteristic $p > 0$ such that strong resolution of singularities holds over k. Let R be a localization of a d-dimensional*

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commutative k-algebra of finite type and suppose that R is K_{d+1}-regular. Then R is a regular ring.

We also prove a number of results for more general fields of characteristic $p > 0$. For instance, we show in Theorem 3.2 that, if strong resolution of singularities holds over all infinite perfect fields of characteristic p, then for every field k that contains an infinite perfect subfield of characteristic p and every k-algebra R essentially of finite type, K_q-regularity for all q implies regularity.

We give a brief outline of the proof of Theorem A. Let $\mathfrak{m} \subset R$ be a maximal ideal, and let $d_\mathfrak{m} = \dim(R_\mathfrak{m})$ and $e_\mathfrak{m} = \dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$ be the dimension and embedding dimension, respectively. One always has $d_\mathfrak{m} \leqslant e_\mathfrak{m}$ and the ring R is said to be regular if $d_\mathfrak{m} = e_\mathfrak{m}$ for every maximal ideal $\mathfrak{m} \subset R$. Now, we show in Theorem 1.2 below that if R is K_{d+1}-regular, then the group $K_{d_\mathfrak{m}+1}(R_\mathfrak{m})/pK_{d_\mathfrak{m}+1}(R_\mathfrak{m})$ is zero for every maximal ideal $\mathfrak{m} \subset R$. We further show in Theorem 2.1 below that for every maximal ideal $\mathfrak{m} \subset R$, the group $K_q(R_\mathfrak{m})/pK_q(R_\mathfrak{m})$ is non-zero for all $0 \leqslant q \leqslant e_\mathfrak{m}$. Together the two theorems show that $d_\mathfrak{m} \geqslant e_\mathfrak{m}$ as desired. Theorem A follows.

1 *K*-theory

In this section, we prove Theorem 1.2 below. We say that strong resolution of singularities holds over the (necessarily perfect) field k if for every integral scheme X separated and of finite type over k, there exists a sequence of blow-ups

$$X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X$$

such that the reduced scheme X_r^{red} is smooth over k; the center Y_i of of the blow-up $X_{i+1} \to X_i$ is connected and smooth over k; the closed embedding of Y_i in X_i is normally flat; and Y_i is nowhere dense in X_i.

Proposition 1.1. *Let k be an infinite perfect field of characteristic $p > 0$ and assume that strong resolution of singularities holds over k. Let X be the limit of a cofiltered diagram $\{X_i\}$ with affine transition maps of d-dimensional schemes separated and of finite type over k. Then $KH_q(X, \mathbb{Z}/p\mathbb{Z})$ vanishes, for $q > d$.*

Proof. It follows from [5, Sect. IV.8.5] that for all integers q, the canonical map

$$\operatorname*{colim}_i K_q(X_i, \mathbb{Z}/p\mathbb{Z}) \to K_q(X, \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism. Therefore, using the natural spectral sequence

$$E^1_{s,t} = N_s K_t(U, \mathbb{Z}/p\mathbb{Z}) \Rightarrow KH_{s+t}(U, \mathbb{Z}/p\mathbb{Z}),$$

we conclude that for all integers q, the canonical map

$$\operatorname*{colim}_i KH_q(X_i, \mathbb{Z}/p\mathbb{Z}) \to KH_q(X, \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism. Hence, we may assume that X itself is a d-dimensional scheme separated and of finite type over k. In fact, we may even assume that X is integral. Indeed, it follows from [16, Theorem 2.3] that for all integers q, the canonical map

$$KH_q(X, \mathbb{Z}/p\mathbb{Z}) \to KH_q(X^{\mathrm{red}}, \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism, so we may assume that X is reduced. Moreover, if $X_1 \subset X$ is an irreducible component and $X_2 \subset X$ the closure of $X \smallsetminus X_1$, then $X_{12} = X_1 \cap X_2$ has smaller dimension than X and by [16, Corollary 4.10] there is a long exact sequence

$$\cdots \to KH_q(X,\mathbb{Z}/p\mathbb{Z}) \to KH_q(X_1,\mathbb{Z}/p\mathbb{Z}) \oplus KH_q(X_2,\mathbb{Z}/p\mathbb{Z}) \to KH_q(X_{12},\mathbb{Z}/p\mathbb{Z}) \to \cdots$$

Therefore, a downward induction on the number of irreducible components shows that we can assume X to be integral. So we let X be integral and proceed by induction on $d \geqslant 0$. In the case $d = 0$, X is a finite disjoint union of prime spectra of fields k_α with $[k_\alpha : k] < \infty$. It follows that the canonical maps

$$KH_q(X,\mathbb{Z}/p\mathbb{Z}) \leftarrow K_q(X,\mathbb{Z}/p\mathbb{Z}) \to \prod_\alpha K_q(k_\alpha,\mathbb{Z}/p\mathbb{Z})$$

are isomorphisms, and since the fields k_α again are perfect of characteristic $p > 0$, the right-hand group is zero, for $q > 0$ as desired [9]. So we let $d \geqslant 1$ and assume that the statement has been proved for smaller d. By the assumption that resolution of singularities holds over k, there exists a proper bi-rational morphism $X' \to X$ from a scheme X' smooth over k. We may further assume that X' is of dimension d. We choose a closed subscheme Y of X that has dimension at most $d-1$ and contains the singular set of X and consider the cartesian square

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X. \end{array}$$

Since the field k is assumed to be an infinite perfect field such that strong resolution of singularities holds over k, the proof of [6, Theorem 3.5] shows that the cartesian square above induces a long exact sequence

$$\cdots \to KH_q(X,\mathbb{Z}/p\mathbb{Z}) \to KH_q(X',\mathbb{Z}/p\mathbb{Z}) \oplus KH_q(Y,\mathbb{Z}/p\mathbb{Z}) \to KH_q(Y',\mathbb{Z}/p\mathbb{Z}) \to \cdots$$

Now, the schemes Y and Y' are of dimension at most $d-1$ and are separated and of finite type over k. Therefore, the groups $KH_q(Y,\mathbb{Z}/p\mathbb{Z})$ and $KH_q(Y',\mathbb{Z}/p\mathbb{Z})$ vanish, for $q > d-1$, by the inductive hypothesis. Finally, since the scheme X' is smooth over k, the canonical map defines an isomorphism

$$K_q(X',\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} KH_q(X',\mathbb{Z}/p\mathbb{Z}),$$

and by [4, Theorem 8.4] the common group vanishes for $q > d$. We conclude from the long exact sequence that $KH_q(X,\mathbb{Z}/p\mathbb{Z})$ vanishes, for $q > d$, as desired. □

Theorem 1.2. *Let k be an infinite perfect field of positive characteristic p such that strong resolution of singularities holds over k. Let R be a localization of a d-dimensional k-algebra of finite type and assume that R is K_{d+1}-regular. Then the group $K_{d+1}(R)/pK_{d+1}(R)$ is zero.*

Proof. Since we assume R is K_{d+1}-regular, a theorem of Vorst [14, Corollary 2.1] shows that R is K_q-regular for all $q \leqslant d+1$, or equivalently, that the groups $N_sK_q(R)$ vanish for all $s > 0$ and $q \leqslant d+1$. The coefficient exact sequence

$$0 \to N_sK_q(R)/pN_sK_q(R) \to N_sK_q(R,\mathbb{Z}/p\mathbb{Z}) \to \mathrm{Tor}_1^{\mathbb{Z}}(N_sK_{q-1}(R),\mathbb{Z}/p\mathbb{Z}) \to 0$$

then shows that the groups $N_sK_q(R,\mathbb{Z}/p\mathbb{Z})$ vanish for $s > 0$ and $q \leqslant d+1$. Therefore, we conclude from the spectral sequence

$$E^1_{s,t} = N_sK_t(R,\mathbb{Z}/p\mathbb{Z}) \Rightarrow KH_{s+t}(R,\mathbb{Z}/p\mathbb{Z})$$

that the canonical map

$$K_q(R,\mathbb{Z}/p\mathbb{Z}) \to KH_q(R,\mathbb{Z}/p\mathbb{Z})$$

is an isomorphism for $q \leqslant d+1$. Now, for $q = d+1$, Proposition 1.1 shows that the common group is zero, and hence, the coefficient sequence

$$0 \to K_{d+1}(R)/pK_{d+1}(R) \to K_{d+1}(R,\mathbb{Z}/p\mathbb{Z}) \to \mathrm{Tor}(K_d(R),\mathbb{Z}/p\mathbb{Z}) \to 0$$

shows that the group $K_{d+1}(R)/pK_{d+1}(R)$ is zero as stated. □

2 Hochschild homology

In this section, we prove the following general result. The reader is referred to [11] for the general theory of cyclic objects in a category.

Theorem 2.1. *Let κ be a commutative ring, let r be a positive integer, and let A be the κ-algebra $A = \kappa[x_1,\dots,x_r]/(x_ix_j \mid 1 \leqslant i \leqslant j \leqslant r)$. Then, for all $1 \leqslant q \leqslant r$, the image of the symbol $\{1+x_1,\dots,1+x_q\}$ by the composition*

$$K_q(A) \to \mathrm{HH}_q(A) \to \mathrm{HH}_q(A/\kappa)$$

of the Dennis trace map and the canonical map from absolute Hochschild homology to Hochschild homology relative to the ground ring κ is non-trivial.

To prove Theorem 2.1, we first evaluate the groups $\mathrm{HH}_*(A/\kappa)$ that are target of the map of the statement. By definition, these are the homology groups of the chain complex associated with the cyclic κ-module $\mathrm{HH}(A/\kappa)[-]$ defined by

$$\mathrm{HH}(A)[n] = A \otimes_\kappa \cdots \otimes_\kappa A \qquad (n+1 \text{ factors})$$

with cyclic structure maps

$$\begin{aligned} d_i &\colon \mathrm{HH}(A/\kappa)[n] \to \mathrm{HH}(A/\kappa)[n-1] \qquad (0 \leqslant i \leqslant n)\\ s_i &\colon \mathrm{HH}(A/\kappa)[n] \to \mathrm{HH}(A/\kappa)[n+1] \qquad (0 \leqslant i \leqslant n)\\ t_n &\colon \mathrm{HH}(A/\kappa)[n] \to \mathrm{HH}(A/\kappa)[n] \end{aligned}$$

defined by

$$
\begin{aligned}
d_i(a_0 \otimes \cdots \otimes a_n) &= \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & (0 \leqslant i < n) \\ a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} & (i = n) \end{cases} \\
s_i(a_0 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n \\
t_n(a_0 \otimes \cdots \otimes a_n) &= a_n \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}.
\end{aligned}
$$

The cyclic κ-module $\mathrm{HH}(A/\kappa)[-]$ admits a direct sum decomposition as follows. Recall that a word of length m with letters in a set S is defined to be a function

$$\omega \colon \{1, 2, \dots, m\} \to S.$$

The cyclic group C_m of order m acts on the set $\{1, 2, \dots, m\}$ by cyclic permutation of the elements. We define a cyclical word of length m with letters in S to be an orbit for the induced action on the set of words of length m with letters in S. We write $[\omega]$ for the orbit through ω and call the length of the orbit the period of $[\omega]$. In particular, the set that consists of the empty word is a cyclical word $[0]$ of length 0 and period 1. Then the cyclic κ-module $\mathrm{HH}(A/\kappa)[-]$ decomposes as the direct sum

$$\mathrm{HH}(A/\kappa)[-] = \bigoplus_{[\omega]} \mathrm{HH}(A/\kappa; [\omega])[-],$$

where the direct sum ranges over all cyclical words with letters in $\{x_1, \dots, x_r\}$, where the summand $\mathrm{HH}(A/\kappa; [0])[-]$ is the sub-cyclic κ-module generated by the 0-simplex 1, and where the summand $\mathrm{HH}(A/\kappa; [\omega])[-]$ with $\omega = (x_{i_1}, \dots, x_{i_m})$, $m \geqslant 1$, is the sub-cyclic κ-module generated by the $(m-1)$-simplex $x_{i_1} \otimes \cdots \otimes x_{i_m}$.

Lemma 2.2. *Let κ be a commutative ring, let r be a positive integer, and let A be the κ-algebra $A = \kappa[x_1, \dots, x_r]/(x_i x_j \mid 1 \leqslant i \leqslant j \leqslant r)$. Let $\omega = (x_{i_1}, \dots, x_{i_m})$ be a word with letters in $\{x_1, \dots, x_r\}$ of length $m \geqslant 0$ and period $\ell \geqslant 1$.*

(1) If $m = 0$, then $\mathrm{HH}_0(A/\kappa; [\omega])$ is the free κ-module of rank one generated by the class of the cycle 1 and the remaining homology groups are zero.

(2) If m is odd or ℓ is even, then $\mathrm{HH}_{m-1}(A/\kappa; [\omega])$ and $\mathrm{HH}_m(A/\kappa; [\omega])$ are free κ-modules of rank one generated by the classes of the cycles $x_{i_1} \otimes \cdots \otimes x_{i_m}$ and $\sum_{0 \leqslant u < \ell} (-1)^{(m-1)u} t_m s_{m-1} t_{m-1}^u (x_{i_1} \otimes \cdots \otimes x_{i_m})$, respectively, and the remaining homology groups are zero.

(3) If $m \geqslant 2$ is even and ℓ is odd, then $\mathrm{HH}_{m-1}(A/\kappa; [\omega])$ is isomorphic to $\kappa/2\kappa$ generated by the class of the cycle $x_{i_1} \otimes \cdots \otimes x_{i_m}$, there is an isomorphism of the 2-torsion sub-κ-module $\kappa[2] \subset \kappa$ onto $\mathrm{HH}_m(A/\kappa; [\omega])$ that takes $a \in \kappa[2]$ to the class of the cycle $a \cdot \sum_{0 \leqslant u < \ell} (-1)^{mu} t_m s_{m-1} t_{m-1}^u (x_{i_1} \otimes \cdots \otimes x_{i_m})$, and the remaining homology groups are zero.

Proof. Let D_* be the chain complex given by the quotient of the chain complex associated with the simplicial κ-module $\mathrm{HH}(A/\kappa; [\omega])[-]$ by the subcomplex of degenerate simplices. We recall that the canonical projection induces an isomorphism of $\mathrm{HH}_q(A/\kappa; [\omega])$ onto $H_q(D_*)$; see for example [17, Theorem 8.3.8]. We evaluate the chain complex D_* in the three cases (1)–(3).

First, in the case (1), D_0 is the free κ-module generated by 1 and D_q is zero, for $q > 0$. This proves statement (1).

Next, in the case (2), let C_ℓ be the cyclic group of order ℓ, and let τ be a generator. We define D'_* to be the chain complex with $D'_q = \kappa[C_\ell]$, if $q = m-1$ or $q = m$, and zero, otherwise, and with differential $d' \colon D'_m \to D'_{m-1}$ given by multiplication by $1-\tau$. Then the map $\alpha \colon D'_* \to D_*$ defined by

$$\begin{aligned} \alpha_{m-1}(\tau^u) &= (-1)^{(m-1)u} t_{m-1}^u (x_{i_0} \otimes \cdots \otimes x_{i_m}) \\ \alpha_m(\tau^u) &= (-1)^{(m-1)u} t_m s_{m-1} t_{m-1}^u (x_{i_0} \otimes \cdots \otimes x_{i_m}) \end{aligned}$$

is an isomorphism of chain complexes, since $(m-1)\ell$ is even. Now, the homology groups $H_{m-1}(D'_*)$ and $H_m(D'_*)$ are free κ-modules of rank 1 generated by the class of 1 and the norm element $N = 1 + \tau + \cdots + \tau^{\ell-1}$, respectively. This proves the statement (2).

Finally, in the case (3), let C_ℓ be the cyclic group of order ℓ, and let τ be a generator. We define D''_* to be the chain complex with $D''_q = \kappa[C_\ell]$, if $q = m-1$ or $q = m$, and zero, otherwise, and with differential $d'' \colon D''_m \to D''_{m-1}$ given by multiplication by $1+\tau$. Then the map $\beta \colon D''_* \to D_*$ defined by

$$\begin{aligned} \beta_{m-1}(\tau^u) &= (-1)^{mu} t_{m-1}^u (x_{i_0} \otimes \cdots \otimes x_{i_m}) \\ \beta_m(\tau^u) &= (-1)^{mu} t_m s_{m-1} t_{m-1}^u (x_{i_0} \otimes \cdots \otimes x_{i_m}) \end{aligned}$$

is an isomorphism of chain complexes, since m is even. Hence, to prove statement (3), it suffices to show that the following sequence of κ-modules is exact.

$$0 \to \kappa[2] \xrightarrow{N} \kappa[C_\ell] \xrightarrow{1+\tau} \kappa[C_\ell] \xrightarrow{\bar{\varepsilon}} \kappa/2\kappa \to 0.$$

To this end, we consider the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I[C_\ell] & \longrightarrow & \kappa[C_\ell] & \xrightarrow{\varepsilon} & \kappa & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1+\tau} & & \downarrow{\scriptstyle 1+\tau} & & \downarrow{\scriptstyle 2} & & \\ 0 & \longrightarrow & I[C_\ell] & \longrightarrow & \kappa[C_\ell] & \xrightarrow{\varepsilon} & \kappa & \longrightarrow & 0 \end{array}$$

The augmentation ideal $I[C_\ell]$ is equal to the sub-$k[C_\ell]$-module generated by $1-\tau$. Since ℓ is odd, τ^2 is a generator of C_ℓ, and hence, $1-\tau^2 = (1+\tau)(1-\tau)$ is a generator of $I[C_\ell]$. This shows that the left-hand vertical map $1+\tau$ is an isomorphism. Finally, the following diagram commutes.

$$\begin{array}{ccc} \kappa[2] & = & \kappa[2] \\ \downarrow{\scriptstyle N} & & \downarrow \\ \kappa[C_\ell] & \xrightarrow{\varepsilon} & \kappa \end{array}$$

Indeed, $\varepsilon \circ N$ is equal to multiplication by ℓ which is congruent to 1 modulo 2. This shows that the sequence in question is exact. Statement (3) follows. $\square$

Remark 2.3. For κ a field of characteristic zero, the Hochschild homology of the κ-algebra A in Lemma 2.2 was first evaluated by Lindenstrauss [10, Theorem 3.1] who also determined the product structure of the graded κ-algebra $\mathrm{HH}_*(A/\kappa)$.

Proof of Theorem 2.1. We let ω be the word $(x_1,\dots,x_q)$ and consider the following composition of the map of the statement and the projection onto the summand $[\omega]$.

$$K_q(A) \to \mathrm{HH}_q(A) \to \mathrm{HH}_q(A/\kappa) \xrightarrow{\mathrm{pr}_{[\omega]}} \mathrm{HH}_q(A/\kappa,[\omega])$$

The Dennis trace map is a map of graded rings and takes the symbol $\{1+x_i\}$ to the Hochschild homology class $d\log(1+x_i)$ represented by the cycle $1\otimes x_i - x_i\otimes x_i$; see for example [3, Corollary 6.4.1], [8, Proposition 2.3.1], and [7, Proposition 1.4.5]. Hence, $\{1+x_1,\dots,1+x_q\}$ is mapped to $d\log(1+x_1)\dots d\log(1+x_q)$. The product on Hochschild homology is given by the shuffle product $*$, and moreover,

$$\mathrm{pr}_{[\omega]}(d\log(1+x_1)*\cdots*d\log(1+x_q)) = \mathrm{pr}_{[\omega]}((1\otimes x_1)*\cdots*(1\otimes x_q))$$

since summands that include a factor $x_i\otimes x_i$ are annihilated by $\mathrm{pr}_{[\omega]}$. Now,

$$(1\otimes x_1)*\cdots*(1\otimes x_q) = \sum_{\sigma}\mathrm{sgn}(\sigma) 1\otimes x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(q)},$$

where the sum ranges over all permutations of $\{1,2,\dots,q\}$, and hence,

$$\mathrm{pr}_{[\omega]}((1\otimes x_1)*\cdots*(1\otimes x_q)) = \sum_{\tau}\mathrm{sgn}(\tau) 1\otimes x_{\tau(1)}\otimes\cdots\otimes x_{\tau(q)},$$

where the sum range over all cyclic permutations of $\{1,2,\dots,q\}$. By Lemma 2.2 (2), this class is the generator of $\mathrm{HH}_q(A/\kappa;[\omega])$. The theorem follows. □

3 Proof of Theorem A

In this section, we prove Theorem A of the introduction and a number of generalizations of this result.

Proof of Theorem A. It suffices to show that for every maximal ideal $\mathfrak{m}\subset R$, the local ring $R_{\mathfrak{m}}$ is regular. The assumption that R is K_{d+1}-regular implies by [15, Theorem 2.1] and [14, Corollary 2.1] that the local ring $R_{\mathfrak{m}}$ is K_q-regular for all $q\leqslant d+1$. The local ring $R_{\mathfrak{m}}$ has dimension $d_{\mathfrak{m}}\leqslant d$. We first argue that we may assume that $d_{\mathfrak{m}}=d$. Let $I\subset R$ be the intersection of the minimal prime ideals $\mathfrak{p}_1,\dots,\mathfrak{p}_n\subset R$ that are not contained in $\mathfrak{m}$. We claim that $\mathfrak{m}+I=R$. For if not, the ideal $\mathfrak{m}+I$ would be contained in a maximal ideal of R which necessarily would be $\mathfrak{m}$. Now, for each $1\leqslant i\leqslant n$, we choose $y_i\in\mathfrak{p}_i$ with $y_i\notin\mathfrak{m}$. Then $y=y_1\dots y_n$ is in I, but not in $\mathfrak{m}$. The claim follows. Now, by the Chinese remainder theorem, there exists $r\in R$ such that $r\equiv 1 \mod \mathfrak{m}$ and $r\equiv 0 \mod I$. We define $R'=R[1/r]$ and $\mathfrak{m}'=\mathfrak{m}R'$. Then $\mathfrak{m}'\subset R'$ is a maximal ideal, since $R'/\mathfrak{m}'=(R/\mathfrak{m})[1/r]=R/\mathfrak{m}$, and the canonical map $R_{\mathfrak{m}}\to R'_{\mathfrak{m}'}$ is an isomorphism. Moreover, the k-algebra R' is of finite type, and since every minimal prime ideal of R' is contained in $\mathfrak{m}'$, we have $\dim R' = \dim R'_{\mathfrak{m}'} = d_{\mathfrak{m}}$. Therefore, we may assume that $d = d_{\mathfrak{m}}$. Hence, Theorem 1.2 shows that

$$K_{d_{\mathfrak{m}}+1}(R_{\mathfrak{m}})/pK_{d_{\mathfrak{m}}+1}(R_{\mathfrak{m}}) = 0.$$

We choose a set of generators $x_1,\dots,x_r$ of the maximal ideal of the local ring $R_{\mathfrak{m}}$. Then $r \geqslant d_{\mathfrak{m}}$ with equality if and only if $R_{\mathfrak{m}}$ is regular. By [12, Theorem 28.3], we may choose a k-algebra section of the canonical projection $R_{\mathfrak{m}}/\mathfrak{m}^2R_{\mathfrak{m}} \to R/\mathfrak{m} = \kappa$. These choices give rise to a k-algebra isomorphism

$$A = \kappa[x_1,\dots,x_r]/(x_ix_j \mid 1 \leqslant i \leqslant j \leqslant r) \xrightarrow{\sim} R_{\mathfrak{m}}/\mathfrak{m}^2R_{\mathfrak{m}}.$$

Hence, Theorem 2.1 shows that for all $1 \leqslant q \leqslant r$, the symbol

$$\{1+x_1,\dots,1+x_q\} \in K_q(R_{\mathfrak{m}})/pK_q(R_{\mathfrak{m}})$$

has non-trivial image in $K_q(A)/pK_q(A)$, and therefore, is non-zero. Since the group $K_{d_{\mathfrak{m}}+1}(R_{\mathfrak{m}})/pK_{d_{\mathfrak{m}}+1}(R_{\mathfrak{m}})$ is zero, we conclude that $r \leqslant d_{\mathfrak{m}}$ which shows that $R_{\mathfrak{m}}$ is a regular local ring. This completes the proof. □

Theorem 3.1. *Let k be a field of positive characteristic p that is finitely generated over an infinite perfect subfield k', and assume that strong resolution of singularities holds over k'. Let R be a localization of a d-dimensional commutative k-algebra of finite type and suppose that R is K_{d+r+1}-regular where r is the transcendence degree of k over k'. Then R is a regular ring.*

Proof. We can write R as the localization $f\colon R' \to S^{-1}R' = R$ of a $(d+r)$-dimensional commutative k'-algebra R' of finite type with respect to a multiplicative subset $S \subset R'$. Let $\mathfrak{p} \subset R$ be a prime ideal. Then, by [15, Theorem 2.1], the local ring $R_{\mathfrak{p}}$ again is K_{d+r+1}-regular. Now, let $\mathfrak{p}' = f^{-1}(\mathfrak{p}) \subset R'$. Then the map f induces an isomorphism of $R'_{\mathfrak{p}'}$ onto $R_{\mathfrak{p}}$. Therefore, we conclude from Theorem A that $R_{\mathfrak{p}}$ is a regular ring. This proves that R is a regular ring as stated. □

Theorem 3.2. *Let p be a prime number and assume that strong resolution of singularities holds over all infinite perfect fields of characteristic p. Let k be any field that contains an infinite perfect subfield of characteristic p, let R be a commutative k-algebra essentially of finite type, and assume that R is K_q-regular for all q. Then R is a regular ring.*

Proof. We can write R as a localization of $R' \otimes_{k'} k$ where k' is a finitely generated field that contains an infinite perfect subfield and where R' is a commutative k'-algebra of finite type. Then we can write R as the filtered colimit

$$R = \operatorname*{colim}_{\alpha} R' \otimes_{k'} k_\alpha$$

where k_α runs through the finitely generated extensions of k' contained in k. It follows from Theorem 3.1 that the rings $R' \otimes_{k'} k_\alpha$ are all regular. Therefore the ring R is regular by [5, Prop. IV.5.13.7]. □

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