The big de Rham-Witt complex

Lars Hesselholt

Introduction

The big de Rham-Witt complex was introduced by the author and Madsen in [15] with the purpose of giving an algebraic description of the equivariant homotopy groups in low degrees of Bökstedt's topological Hochschild spectrum of a commutative ring. This functorial algebraic description, in turn, is essential for understading algebraic K-theory by means of the cyclotomic trace map of Bökstedt-Hsiang-Madsen [4]; compare [16, 14, 10]. The original construction, which relied on the adjoint functor theorem, was very indirect and a direct construction has been lacking. In this paper, we give a new and explicit construction of the big de Rham-Witt complex and we also correct the 2-torsion which was not quite correct in the original construction.

The new construction is based on a theory, which is developed first, of modules and derivations over a λ-ring. The main result of this first part of the paper is that the universal derivation of a λ-ring is given by the universal derivation of the underlying ring together with an additional structure that depends directly on the λ-ring structure in question. In the case of the universal λ-ring, which is given by the ring of big Witt vectors, this additional structure consists in divided Frobenius operators on the module of Kähler differentials. It is the existence of these divided Frobenius operators that makes the new direct construction of the big de Rham-Witt complex possible. This is carried out in the second part of the paper, where we also show that the big de Rham-Witt complex behaves well with respect to étale morphisms. Finally, we explicitly evaluate the big de Rham-Witt complex of the ring of integers.

In more detail, let A be a ring, which we always assume to be commutative and unital. The ring $\mathbb{W}(A)$ of big Witt vectors in A is equipped with a natural action through ring homomorphisms by the multiplicative monoid $\mathbb{N}$ of positive integers,

Generous assistance from DNRF Niels Bohr Professorship, JSPS Grant-in-Aid 23340016, and CMI Senior Scholarship is gratefully acknowledged

Lars Hesselholt
Nagoya University, Nagoya, Japan and University of Copenhagen, Denmark
E-mail: larsh@math.nagoya-u.ac.jp and larsh@math.ku.dk

where the action by $n \in \mathbb{N}$ is given by the nth Frobenius map

$$\mathbb{W}(A) \xrightarrow{F_n} \mathbb{W}(A).$$

The Frobenius maps give rise to a natural ring homomorphism

$$\mathbb{W}(A) \xrightarrow{\Delta} \mathbb{W}(\mathbb{W}(A))$$

whose Witt components $\Delta_e \colon \mathbb{W}(A) \to \mathbb{W}(A)$ are characterized by the formula

$$F_n(a) = \sum_{e|n} e\Delta_e(a)^{n/e}.$$

The triple $(\mathbb{W}(-), \Delta, \varepsilon)$ with $\varepsilon \colon \mathbb{W}(A) \to A$ the first Witt component is a comonad on the category of rings and a λ-ring in the sense of Grothendieck [11] is precisely a coalgebra (A, λ_A) over this comonad.

Recently, Borger [5] has proposed that a λ-ring structure $\lambda_A \colon A \to \mathbb{W}(A)$ on a ring A be considered as descent data from $\mathbb{Z}$-algebras to algebras over a deeper base $\mathbb{F}_1$. This begs the question as to the natural notions of modules and derivations over λ-rings. We show here that the general approach of Beck [2] leads to the following answer. First, if (A, λ_A) is a λ-ring, then the ring A is equipped with an action by the multiplicative monoid $\mathbb{N}$ through ring homomorphisms, where the action by $n \in \mathbb{N}$ is given by the nth associated Adams operation

$$A \xrightarrow{\psi_{A,n}} A$$

defined by the formula

$$\psi_{A,n}(a) = \sum_{e|n} e\lambda_{A,e}(a)^{n/e}.$$

Here $\lambda_{A,e} \colon A \to A$ is the eth Witt component of $\lambda_A \colon A \to \mathbb{W}(A)$. Now, the category of (A, λ_A)-modules is identified with the category of left modules over the twisted monoid algebra $A^{\psi}[\mathbb{N}]$ with the product defined by the formula

$$n \cdot a = \psi_{A,n}(a) \cdot n.$$

Hence, an (A, λ_A)-module is a pair (M, λ_M) that consists of an A-module M and an $\mathbb{N}$-indexed family of maps $\lambda_{M,n} \colon M \to M$ such that $\lambda_{M,n}$ is $\psi_{A,n}$-linear, $\lambda_{M,1} = \mathrm{id}_M$, and $\lambda_{M,m}\lambda_{M,n} = \lambda_{M,mn}$. Moreover, we identify the derivations

$$(A, \lambda_A) \xrightarrow{D} (M, \lambda_M)$$

with the derivations $D \colon A \to M$ that satisfy the identities

$$\lambda_{M,n}(Da) = \sum_{e|n} \lambda_{A,e}(a)^{(n/e)-1} D\lambda_{A,e}(a).$$

It is now easy to show that there is a universal derivation

$$(A, \lambda_A) \xrightarrow{d} (\Omega^1_{(A,\lambda_A)}, \lambda_{\Omega^1_{(A,\lambda_A)}}).$$

We prove the following result.

Theorem A. *For every λ-ring (A, λ_A), the canonical map*

$$\Omega^1_A \longrightarrow \Omega^1_{(A,\lambda_A)}$$

is an isomorphism of A-modules.

It follows that for a λ-ring (A, λ_A), the A-module of differentials Ω^1_A carries the richer structure of an (A, λ_A)-module. In the case of $(\mathbb{W}(A), \Delta_A)$, this implies that there are natural F_n-linear maps $F_n \colon \Omega^1_{\mathbb{W}(A)} \to \Omega^1_{\mathbb{W}(A)}$ defined by

$$F_n(da) = \sum_{e|n} \Delta_e(a)^{(n/e)-1} d\Delta_e(a)$$

such that $F_1 = \mathrm{id}$, $F_m F_n = F_{mn}$, $dF_n(a) = nF_n(da)$, and $F_n(d[a]) = [a]^{n-1} d[a]$. The p-typical analog of F_p was also constructed by Borger and Wieland in [7, 12.8].

The construction of the de Rham-Witt complex begins with the following variant of the de Rham complex. The ring $\mathbb{W}(\mathbb{Z})$ contains exactly the four units $\pm[\pm 1]$, all of which are square roots of $[1]$, and the 2-torsion element

$$d\log[-1] = [-1]^{-1} d[-1] = [-1] d[-1] \in \Omega^1_{\mathbb{W}(A)}$$

plays a special rôle. We define the graded $\mathbb{W}(A)$-algebra

$$\hat{\Omega}^{\cdot}_{\mathbb{W}(A)} = T^{\cdot}_{\mathbb{W}(A)} \Omega^1_{\mathbb{W}(A)} / J^{\cdot}$$

to be the quotient of the tensor algebra of the $\mathbb{W}(A)$-module $\Omega^1_{\mathbb{W}(A)}$ by the graded ideal $J^{\cdot}$ generated by all elements of the form

$$da \otimes da - d\log[-1] \otimes F_2 da$$

with $a \in \mathbb{W}(A)$. It is an anticommutative graded ring which carries a unique graded derivation d that extends $d \colon \mathbb{W}(A) \to \Omega^1_{\mathbb{W}(A)}$ and satisfies

$$dd\omega = d\log[-1] \cdot d\omega.$$

Moreover, the maps $F_n \colon \mathbb{W}(A) \to \mathbb{W}(A)$ and $F_n \colon \Omega^1_{\mathbb{W}(A)} \to \Omega^1_{\mathbb{W}(A)}$ extend uniquely to a map of graded rings $F_n \colon \Omega^{\cdot}_{\mathbb{W}(A)} \to \Omega^{\cdot}_{\mathbb{W}(A)}$ which satisfies $dF_n = nF_n d$. Next, we show that the maps d and F_n both descend to the further quotient

$$\check{\Omega}^{\cdot}_{\mathbb{W}(A)} = \hat{\Omega}^{\cdot}_{\mathbb{W}(A)} / K^{\cdot}$$

by the graded ideal generated by all elements of the form

$$F_p dV_p(a) - da - (p-1) d\log[-1] \cdot a$$

with p a prime number and $a \in \mathbb{W}(A)$. We now recall the Verschiebung maps

$$V_n \colon \mathbb{W}(A) \to \mathbb{W}(A)$$

which are additive and satisfy the projection formula

$$aV_n(b) = V_n(F_n(a)b).$$

These maps, however, do not extend to $\hat{\Omega}^{\cdot}_{\mathbb{W}(A)}$ or $\check{\Omega}^{\cdot}_{\mathbb{W}(A)}$, and the de Rham-Witt complex, roughly speaking, is the largest quotient

$$\check{\Omega}^{\cdot}_{\mathbb{W}(A)} \xrightarrow{\eta} \mathbb{W}\Omega^{\cdot}_{A}$$

such that the Verschiebung maps extend to $\mathbb{W}\Omega^{\cdot}_{A}$ and such that the extended maps F_n and V_n satisfy the projection formula. The precise definition given in Section 4 below is by recursion with respect to the quotients $\mathbb{W}_S(A)$ of $\mathbb{W}(A)$ where S ranges over the finite subsets $S \subset \mathbb{N}$ that are stable under division. We further prove the following result to the effect that the de Rham-Witt complex may be characterized as the universal example of an algebraic structure called a Witt complex, the precise definition of which is given in Definition 4.1.

Theorem B. *There exists an initial Witt complex $S \mapsto \mathbb{W}_S\Omega^{\cdot}_{A}$ over the ring A. In addition, the canonical maps*

$$\check{\Omega}^{q}_{\mathbb{W}_S(A)} \xrightarrow{\eta_S} \mathbb{W}_S\Omega^{q}_{A}$$

are surjective, and the diagrams

$$\begin{array}{ccc} \check{\Omega}^{q}_{\mathbb{W}_S(A)} & \xrightarrow{\eta_S} & \mathbb{W}_S\Omega^{q}_{A} \\ \downarrow R^S_T & & \downarrow R^S_T \\ \check{\Omega}^{q}_{\mathbb{W}_T(A)} & \xrightarrow{\eta_T} & \mathbb{W}_T\Omega^{q}_{A} \end{array} \qquad \begin{array}{ccc} \check{\Omega}^{q}_{\mathbb{W}_S(A)} & \xrightarrow{\eta_S} & \mathbb{W}_S\Omega^{q}_{A} \\ \downarrow d & & \downarrow d \\ \check{\Omega}^{q+1}_{\mathbb{W}_S(A)} & \xrightarrow{\eta_S} & \mathbb{W}_S\Omega^{q+1}_{A} \end{array} \qquad \begin{array}{ccc} \check{\Omega}^{q}_{\mathbb{W}_S(A)} & \xrightarrow{\eta_S} & \mathbb{W}_S\Omega^{q}_{A} \\ \downarrow F_m & & \downarrow F_m \\ \check{\Omega}^{q}_{\mathbb{W}_{S/m}(A)} & \xrightarrow{\eta_{S/m}} & \mathbb{W}_{S/m}\Omega^{q}_{A} \end{array}$$

commute.

If A is an $\mathbb{F}_p$-algebra and $S = \{1, p, \dots, p^{n-1}\}$, then $\mathbb{W}_S\Omega^{\cdot}_{A}$ agrees with the original p-typical de Rham-Witt complex $W_n\Omega^{\cdot}_{A}$ of Bloch-Deligne-Illusie [19]. More generally, if A is a $\mathbb{Z}_{(p)}$-algebra and $S = \{1, p, \dots, p^{n-1}\}$, then $\mathbb{W}_S\Omega^{\cdot}_{A}$ agrees with the p-typical de Rham-Witt complex $W_n\Omega^{\cdot}_{A}$ constructed by the author and Madsen [17] for p odd and by Costeanu [9] for $p = 2$. Finally, if 2 is either invertible or zero in A and S is arbitrary, then $\mathbb{W}_S\Omega^{\cdot}_{A}$ agrees with the big de Rham-Witt complex introduced by the author and Madsen [15]. We also note that if $f\colon R \to A$ is a map of $\mathbb{Z}_{(p)}$-algebras and $S = \{1, p, \dots, p^{n-1}\}$, then the relative p-typical de Rham-Witt complex $W_n\Omega^{\cdot}_{A/R}$ of Langer-Zink [22] agrees with the quotient of $\mathbb{W}_S\Omega^{\cdot}_{A}$ by the differential graded ideal generated by the image of $\mathbb{W}_S\Omega^1_R \to \mathbb{W}_S\Omega^1_A$.

We recall that van der Kallen [27, Theorem 2.4] and Borger [6, Theorem B] have proved independently that for every étale morphism $f\colon A \to B$ and every finite subset $S \subset \mathbb{N}$ stable under division, the induced morphism

$$\mathbb{W}_S(A) \xrightarrow{\mathbb{W}_S(f)} \mathbb{W}_S(B)$$

again is étale. Based on this theorem, we prove the following result.

Theorem C. *Let $f\colon A \to B$ be an étale map and let $S \subset \mathbb{N}$ be a finite subset stable under division. Then the induced map*

$$\mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega_A^q \longrightarrow \mathbb{W}_S\Omega_B^q$$

is an isomorphism, for all q.

To prove Theorem C, we verify that the left-hand terms form a Witt complex over the ring B and use Theorem B to obtain the inverse of the map in the statement. The verification of the Witt complex axioms, in turn, is significantly simplified by the existence of the divided Frobenius on $\Omega_{\mathbb{W}(A)}$ as follows from Theorem A.

Finally, we evaluate the de Rham-Witt complex of $\mathbb{Z}$. The result is that $\mathbb{W}\Omega_{\mathbb{Z}}^q$ is non-zero for $q \leqslant 1$ only. Moreover, we may consider $\mathbb{W}\Omega_{\mathbb{Z}}^{\cdot}$ as the quotient

$$\Omega_{\mathbb{W}(\mathbb{Z})}^{\cdot} \longrightarrow \mathbb{W}\Omega_{\mathbb{Z}}^{\cdot}$$

of the de Rham complex of $\mathbb{W}(\mathbb{Z})$ by a differential graded ideal generated by elements of degree 1. Hence, following Borger [5], we may interpret $\mathbb{W}\Omega_{\mathbb{Z}}^{\cdot}$ as the complex of differentials along the leaves of a codimension 1 foliation of $\operatorname{Spec}(\mathbb{Z})$ considered as an $\mathbb{F}_1$-space. We note that, by contrast, $\Omega_{\mathbb{W}(\mathbb{Z})}^q$ is non-zero for all q.

As mentioned earlier, the big de Rham-Witt complex was introduced in [15] with the purpose of giving an algebraic description of the equivariant homotopy groups

$$\mathrm{TR}_q^r(A) = [S^q \wedge (\mathbb{T}/C_r)_+, T(A)]_{\mathbb{T}}$$

of the topological Hochschild $\mathbb{T}$-spectrum $T(A)$ of the ring A. Here $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the circle group, $C_r \subset \mathbb{T}$ is the subgroup of order r, and $[-,-]_{\mathbb{T}}$ is the abelian group of maps in the homotopy category of orthogonal $\mathbb{T}$-spectra. We proved in [13, Section 1] that the groups $\mathrm{TR}_q^r(A)$ give rise to a Witt complex over the ring A in the sense of Definition 4.1 below. Therefore, by Theorem B, there is a unique map

$$\mathbb{W}_{\langle r\rangle}\Omega_A^q \longrightarrow \mathrm{TR}_q^r(A)$$

of Witt complexes over A, where $\langle r\rangle$ denotes the set of divisors of r. We will show elsewhere that this map is an isomorphism for all r and all $q \leqslant 1$.

1 Witt vectors

We begin with a review of Witt vectors and λ-rings. The material in this section is due to Cartier [8], Grothendieck [11], Teichmüller [26], and Witt [29] and accordingly we make no claim of originality. The reader is also referred to the very readable account by Bergman [24, Appendix] and to the more modern and general exposition by Borger [6]. We further mention the books by Hazewinkel [12] and Knutson [20], which focus more on the rôle of symmetric functions.

In the approach to Witt vectors taken here, all necessary congruences are isolated in following lemma, commonly attributed to Dwork [12, E.2.4]. Let $\mathbb{N}$ be the set of positive integers. We say that a subset $S \subset \mathbb{N}$, possibly empty, is a truncation set if whenever $n \in S$ and e is a divisor in n, then $e \in S$. The ring of big Witt vectors $\mathbb{W}_S(A)$ associated with the ring A and the truncation set S is defined to be the set A^S equipped with a ring structure such that the ghost map

$$\mathbb{W}_S(A) \xrightarrow{w} A^S$$

that takes the vector $a = (a_n \mid n \in S)$ to the sequence $w(a) = \langle w_n(a) \mid n \in S \rangle$ with

$$w_n(a) = \sum_{e \mid n} e a_e^{n/e}$$

is a natural transformation of functors from the category of rings to itself. Here the target A^S is considered a ring with componentwise addition and multiplication. The elements a_n and $w_n(a)$ are called the Witt components and the ghost components of the vector a, respectively. To prove that there exists a unique ring structure on $\mathbb{W}_S(A)$ characterized in this way, we first recall the following result, a different proof of which is given in [12, Lemma 17.6.1]. We write $v_p(n)$ for the p-adic valuation of n, normalized such that $v_p(p) = 1$.

Lemma 1.1. *Suppose that for every prime number $p \in S$, there exists a ring homomorphism $\phi_p \colon A \to A$ with the property that $\phi_p(a) \equiv a^p$ modulo pA. Then for every sequence $x = \langle x_n \mid n \in S \rangle$, the following* (i)–(ii) *are equivalent.*

(i) *The sequence x is in the image of the ghost map $w \colon \mathbb{W}_S(A) \to A^S$.*
(ii) *For every prime number $p \in S$ and every $n \in S$ with $v_p(n) \geqslant 1$,*

$$x_n \equiv \phi_p(x_{n/p}) \qquad \textit{modulo } p^{v_p(n)}A.$$

Proof. We first show that if $a \equiv b$ modulo pA, then $a^{p^{v-1}} \equiv b^{p^{v-1}}$ modulo $p^v A$. If we write $a = b + p\varepsilon$, then

$$a^{p^{v-1}} = b^{p^{v-1}} + \sum_{1 \leqslant i \leqslant p^{v-1}} \binom{p^{v-1}}{i} b^{p^{v-1}-i} p^i \varepsilon^i.$$

In general, the p-adic valuation of the binomial coefficient $\binom{m+n}{n}$ is equal to the number of carrys in the addition of m and n in base p. So in particular,

$$v_p\left(\binom{p^{v-1}}{i} p^i\right) = v - 1 + i - v_p(i) \geqslant v$$

which proves the claim. Now, suppose that $x = w(a)$ satisfies (i). Since $\phi_p \colon A \to A$ is a ring-homomorphism and lifts the Frobenius of A/pA, we have

$$\phi_p(w_{n/p}(a)) = \sum_{e \mid (n/p)} e \phi_p(a_e^{n/pe})$$

which is congruent to $\sum_{e \mid (n/p)} e a_e^{n/e}$ modulo $p^{v_p(n)}A$. But if e divides n but not n/p, then $v_p(e) = v_p(n)$ and hence, this sum is congruent to $\sum_{e \mid n} e a_e^{n/e} = w_n(a)$ modulo $p^{v_p(n)}A$. This shows that x satisfies (ii). Conversely, suppose that x satisfies (ii). We find a vector $a = (a_n \mid n \in S)$ with $w_n(a) = x_n$ as follows. We let $a_1 = x_1$ and assume, inductively, that a_e has been chosen, for all $e \neq n$ that divide n, such that $w_e(a) = x_e$. The calculation above shows that for every prime number p dividing n,

$$x_n - \sum_{e \mid n, e \neq n} e a_e^{n/e}$$

is congruent to zero modulo $p^{v_p(n)}A$. Therefore, this difference is divisible by n and hence is equal to na_n for some $a_n \in A$. This shows that x satisfies (i). □

Proposition 1.2. *There exists a unique ring structure on the domain of the ghost map*

$$\mathbb{W}_S(A) \xrightarrow{w} A^S$$

making it a natural transformation of functors from rings to rings.

Proof. Let A be the free ring generated by $\{a_n, b_n \mid n \in S\}$. The unique ring homomorphism $\phi_p \colon A \to A$ that maps a_n to a_n^p and b_n to b_n^p satisfies $\phi_p(f) = f^p$ modulo pA. Hence, if a and b are the vectors $(a_n \mid n \in S)$ and $(b_n \mid n \in S)$, respectively, then Lemma 1.1 shows that the sequences $w(a) + w(b)$, $w(a) \cdot w(b)$, and $-w(a)$ are in the image of the ghost map. It follows that there are sequences of polynomials $s = (s_n \mid n \in S)$, $p = (p_n \mid n \in S)$, and $i = (i_n \mid n \in S)$ such that $w(s) = w(a) + w(b)$, $w(p) = w(a) \cdot w(b)$, and $w(i) = -w(a)$. Moreover, since A is torsion free, the ghost map is injective, and accordingly, these polynomials are unique.

Let A' be any ring. If $a' = (a'_n \mid n \in S)$ and $b' = (b'_n \mid n \in S)$ are two vectors in $\mathbb{W}_S(A')$, then there is a unique ring homomorphism $f \colon A \to A'$ with the property that $\mathbb{W}_S(f)(a) = a'$ and $\mathbb{W}_S(f)(b) = b'$. We define $a' + b' = \mathbb{W}_S(f)(s)$, $a \cdot b = \mathbb{W}_S(f)(p)$, and $-a = \mathbb{W}_S(f)(i)$. To prove that the ring axioms are verified, suppose first that A' is torsion free. In this case, the ghost map is injective, and hence, the ring axioms hold since they do so in A^S. In general, we choose a surjective ring homomorphism $g \colon A'' \to A'$ from a torsion free ring A''. The induced map $\mathbb{W}_S(g) \colon \mathbb{W}_S(A'') \to \mathbb{W}_S(A')$ is again surjective, and since the ring axioms hold in the domain, they do so, too, in the target. □

If $T \subset S$ are two truncation sets, then the forgetful map

$$\mathbb{W}_S(A) \xrightarrow{R^S_T} \mathbb{W}_T(A)$$

is a natural ring homomorphism called the restriction from S to T. If $S \subset \mathbb{N}$ is a truncation set and $n \in \mathbb{N}$, then the set

$$S/n = \{e \in \mathbb{N} \mid ne \in S\}$$

again is a truncation set, possibly empty. For every $n \in \mathbb{N}$, there is a natural map

$$\mathbb{W}_{S/n}(A) \xrightarrow{V_n} \mathbb{W}_S(A)$$

that to the vector $a = (a_e \mid e \in S/n)$ assigns the vector $V_n(a) = (b_m \mid m \in S)$, where b_m is equal to a_e, if $m = ne$, and 0, otherwise. It is called the nth Verschiebung.

Lemma 1.3. *For every $n \in \mathbb{N}$, the map $V_n \colon \mathbb{W}_{S/n}(A) \to \mathbb{W}_S(A)$ is additive.*

Proof. The following diagram, where V_n^w takes the sequence $\langle x_e \mid e \in S/n \rangle$ to the sequence whose mth component is nx_e, if $m = ne$, and 0, otherwise, commutes.

$$\begin{array}{ccc} \mathbb{W}_{S/n}(A) & \xrightarrow{V_n} & \mathbb{W}_S(A) \\ \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w} \\ A^{S/n} & \xrightarrow{V_n^w} & A^S \end{array}$$

Since V_n^w is additive, so is V_n. Indeed, if A is torsion free, the horizontal maps are both injective, and hence V_n is additive in this case. In general, we choose a surjective ring homomorphism $g \colon A' \to A$ and argue as in the proof of Proposition 1.2. □

Lemma 1.4. *For every $n \in \mathbb{N}$, there exists a unique natural ring homomorphism*

$$\mathbb{W}_S(A) \xrightarrow{F_n} \mathbb{W}_{S/n}(A)$$

called the nth Frobenius that makes the following diagram, where the map F_n^w takes the sequence $x = \langle x_m \mid m \in S \rangle$ to the sequence $F_n^w(x) = \langle x_{ne} \mid e \in S/n \rangle$, commute.

$$\begin{array}{ccc} \mathbb{W}_S(A) & \xrightarrow{F_n} & \mathbb{W}_{S/n}(A) \\ \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w} \\ A^S & \xrightarrow{F_n^w} & A^{S/n} \end{array}$$

Proof. The construction of the map F_n is similar to the proof of Proposition 1.2. We let A be the free ring generated by $\{a_m \mid m \in S\}$, and let a be the vector $(a_m \mid m \in S)$. By Lemma 1.1, the sequence $F_n^w(w(a)) \in A^{S/n}$ is the image by the ghost map of a vector

$$F_n(a) = (f_{n,e} \mid e \in S/n) \in \mathbb{W}_{S/n}(A),$$

this vector being unique since A is torsion free. If A' is any ring and $a' = (a'_m \mid m \in S)$ a vector in $\mathbb{W}_S(A')$, then we define $F_n(a') = \mathbb{W}_{S/n}(g)(F_n(a))$, where $g \colon A \to A'$ is the unique ring homomorphism that maps a to a'. Finally, since the map F_n^w is a ring homomorphism, an argument similar to the proof of Lemma 1.3 shows that also F_n is a ring homomorphism. □

The Teichmüller representative is the map

$$A \xrightarrow{[-]_S} \mathbb{W}_S(A)$$

whose mth component is a, if $m = 1$, and 0, otherwise. It is a multiplicative map. Indeed, the following diagram, where $[a]_S^w$ is the sequence with mth component a^m, commutes, and $[-]^w_S$ is a multiplicative map.

$$\begin{array}{ccc} A & \xrightarrow{[-]_S} & \mathbb{W}_S(A) \\ \| & & \downarrow w \\ A & \xrightarrow{[-]^w_S} & A^S \end{array}$$

In particular, the Teichmüller representative $[1]_S$ is the multiplicative identity element in the ring $\mathbb{W}_S(A)$.

Lemma 1.5. *Let $S \subset \mathbb{N}$ be a truncation set and let A be a ring.*

(i) *For all $a \in \mathbb{W}_S(A)$, there is a convergent sum*

$$a = \sum_{n \in S} V_n([a_n]_{S/n}).$$

(ii) *For all $m, n \in \mathbb{N}$ with greatest common divisor $c = (m,n)$,*

$$F_m V_n = c V_{n/c} F_{m/c} \colon \mathbb{W}_{S/c}(A) \to \mathbb{W}_{S/c}(A).$$

(iii) *For all $n \in \mathbb{N}$, $a \in \mathbb{W}_S(A)$, and $a' \in \mathbb{W}_{S/n}(A)$,*

$$a V_n(a') = V_n(F_n(a) a').$$

(iv) *For all $m, n \in \mathbb{N}$,*

$$F_m F_n = F_{mn} \colon \mathbb{W}_S(A) \to \mathbb{W}_{S/mn}(A),$$
$$V_n V_m = V_{mn} \colon \mathbb{W}_{S/mn}(A) \to \mathbb{W}_S(A).$$

(v) *For all $n \in \mathbb{N}$ and $a \in A$,*

$$F_n([a]_S) = [a]^n_{S/n}.$$

Proof. One readily verifies that the two sides of each equation have the same image by the ghost map. This shows that the relations hold, if A is torsion free, and hence, in general. In statement (i), the convergence, for S infinite, is with respect to the product topology on $\mathbb{W}_S(A)$ induced by the discrete topology on A. □

Proposition 1.6. *The ring of Witt vectors in $\mathbb{Z}$ is equal to the abelian group*

$$\mathbb{W}_S(\mathbb{Z}) = \prod_{n \in S} \mathbb{Z} \cdot V_n([1]_{S/n})$$

with the multiplication given by

$$V_m([1]_{S/m}) \cdot V_n([1]_{S/n}) = c \cdot V_e([1]_{S/e})$$

where $c = (m,n)$ and $e = [m,n]$ are the greatest common divisor and the least common multiple, respectively.

Proof. The formula for the product follows from Lemma 1.5 (ii)–(iv). For finite S, we prove the statement by induction beginning from the case $S = \emptyset$ which is trivial. So suppose that S is non-empty, let $m \in S$ be maximal, and let $T = S \smallsetminus \{m\}$. The sequence of abelian groups

$$0 \longrightarrow \mathbb{W}_{\{1\}}(\mathbb{Z}) \xrightarrow{V_m} \mathbb{W}_S(\mathbb{Z}) \xrightarrow{R^S_T} \mathbb{W}_T(\mathbb{Z}) \longrightarrow 0$$

is exact, and we wish to show that it is equal to the following exact sequence.

$$0 \longrightarrow \mathbb{Z}\cdot[1]_{\{1\}} \xrightarrow{V_m} \prod_{n\in S}\mathbb{Z}\cdot V_n([1]_{S/n}) \xrightarrow{R^S_T} \prod_{n\in T}\mathbb{Z}\cdot V_n([1]_{T/n}) \longrightarrow 0$$

The latter sequence maps to the former, and by induction, the right-hand terms of the two sequences are equal. Since also the left-hand terms are equal, so are the middle terms. This completes the proof for S finite. Finally, every truncation set S is the union of its finite sub-truncation sets $S_\alpha \subset S$ and $\mathbb{W}_S(\mathbb{Z}) = \lim_\alpha \mathbb{W}_{S_\alpha}(\mathbb{Z})$. □

The values of the restriction, Frobenius, and Verschiebung maps on the generators $V_n([1]_{S/n})$ are readily evaluated by using Lemma 1.5 (ii)–(iv). To give a formula for the Teichmüller representative, we recall the Möbius inversion formula. Let $g\colon \mathbb{N} \to \mathbb{Z}$ be a function and define the function $f\colon \mathbb{N} \to \mathbb{Z}$ by $f(n) = \sum_{e|n} g(e)$. Then the original function is given by $g(n) = \sum_{e|n}\mu(e)f(n/e)$, where $\mu\colon \mathbb{N} \to \{-1,0,1\}$ is the Möbius function defined by $\mu(e) = (-1)^r$, if e is a product of $r \geqslant 0$ distinct prime numbers, and $\mu(e) = 0$, otherwise.

Addendum 1.7. *If m is an integer and S a truncation set, then*

$$[m]_S = \sum_{n\in S}\frac{1}{n}\big(\sum_{e|n}\mu(e)m^{n/e}\big)V_n([1]_{S/n}).$$

In particular, the square root of unity $[-1]_S$ is equal to $-[1]_S + V_2([1]_{S/2})$.

Proof. It suffices to prove that the formula holds in $\mathbb{W}_S(\mathbb{Z})$. By Proposition 1.6, there are unique integers r_e, $e \in S$ such that

$$[m]_S = \sum_{e\in S} r_e V_e([1]_{S/e}).$$

Evaluating the nth ghost component of this equation, we find that

$$m^n = \sum_{e|n} e r_e$$

from which the stated formula follows by Möbius inversion. Finally, defining $g(n)$ to be -1, if $n = 1$; 2, if $n = 2$; and 0, otherwise, we get $f(n) = \sum_{e|n} g(e) = (-1)^n$, which proves the stated formula for $[-1]_S$. □

If $m = q$ is a prime power, then the coefficient of $V_n([1]_{S/n})$ in $[m]_S$ is equal to the number of monic irreducible polynomials of degree n over the finite field $\mathbb{F}_q$.

Lemma 1.8. *If A is an $\mathbb{F}_p$-algebra and S a truncation set, then*

$$F_p = R^S_{S/p} \circ \mathbb{W}_S(\varphi)\colon \mathbb{W}_S(A) \to \mathbb{W}_{S/p}(A),$$

where $\varphi\colon A \to A$ is the Frobenius endomorphism.

Proof. By definition $F_p(a) = (f_{p,e}(a) \mid e \in S/p)$ with the elements $f_{p,e}$ of the free ring on $\{a_n \mid n \in S\}$ characterized by the system of equations

$$\sum_{e \mid m} e f_{p,e}^{m/e} = \sum_{e \mid pm} e a_e^{pm/e}$$

indexed by $m \in S/p$. The lemma is equivalent to the statement that for all $m \in S/p$, $f_{p,m} \equiv a_m^p$ modulo p, which we proceed to prove by induction on $m \in S/p$. Since $f_{p,1} = a_1^p + pa_p$, the statement holds for $m = 1$. So we let $m > 1$ and assume that for all proper divisors e of m, $f_{p,e} \equiv a_e^p$ modulo p. This implies that $e f_{p,e}^{m/e} \equiv e a_e^{pm/e}$ modulo $p^{v_p(m)+1}$ by the argument at the beginning of the proof of Lemma 1.1. We now write

$$\sum_{e \mid m} e f_{p,e}^{m/e} = \sum_{e \mid m} e a_e^{pm/e} + \sum_{e \mid pm, e \nmid m} e a_e^{pm/e}$$

and note that if $e \mid pm$ and $e \nmid m$, then $v_p(e) = v_p(m) + 1$. Therefore, we may conclude that $m f_{p,m} \equiv m a_m^p$ modulo $p^{v_p(m)+1}A$. But the free ring on $\{a_n \mid n \in S\}$ is torsion free, so $f_{p,m} \equiv a_m^p$ modulo p as desired. This completes the proof. □

Lemma 1.9. *Let m be an integer, let A be a ring, and let S be a truncation set. If m is invertible in A, then m is invertible in $\mathbb{W}_S(A)$; and if m is a non-zero-divisor in A, then m is a non-zero-divisor in $\mathbb{W}_S(A)$.*

Proof. As in the proof of Proposition 1.6, we may assume that S is finite. We proceed by induction on S beginning from the trivial case $S = \emptyset$. So let S be non-empty and assume the statement for all proper sub-truncation sets of S. We let $n \in S$ be maximal, and let $T = S \smallsetminus \{n\}$. In this situation, we have exact sequence

$$0 \longrightarrow \mathbb{W}_{\{1\}}(A) \xrightarrow{V_n} \mathbb{W}_S(A) \xrightarrow{R^S_T} \mathbb{W}_T(A) \longrightarrow 0$$

from which the induction step readily follows, since $\mathbb{W}_{\{1\}}(A) = A$. □

Let p be prime number. We say that a sub-truncation set of the truncation set

$$P = \{1, p, p^2, \dots\} \subset \mathbb{N}$$

is a p-typical truncation set. For instance, if S is any truncation set, then $S \cap P$ is a p-typical truncation set. The p-typical truncation sets $T \subset P$ are $T = \emptyset$, $T = P$, and $T = \{1, p, \dots, p^{n-1}\}$, where n is a positive integer. The ring $\mathbb{W}_P(A)$ is called the ring of p-typical Witt vectors and the ring $\mathbb{W}_{\{1,p,\dots,p^{n-1}\}}(A)$ is called the ring of p-typical Witt vectors of length n in A.

Proposition 1.10. *Let p be a prime number, let S be a truncation set, and let $I(S)$ be the set of $k \in S$ not divisible by p. If A is a ring in which every $k \in I(S)$ is invertible, then the ring homomorphism*

$$\mathbb{W}_S(A) \xrightarrow{g} \prod_{k \in I(S)} \mathbb{W}_{(S/k)\cap P}(A)$$

whose kth component is $g_k = R^{S/k}_{(S/k)\cap P} \circ F_k$ is an isomorphism.

Proof. We have a commutative diagram of ring homomorphisms

$$\begin{array}{ccc} \mathbb{W}_S(A) & \xrightarrow{g} & \prod \mathbb{W}_{(S/k)\cap P}(A) \\ \big\downarrow w & & \big\downarrow \prod w \\ A^S & \xrightarrow{g^w} & \prod A^{(S/k)\cap P} \end{array}$$

where the products on the right-hand side range over $k \in I(S)$ and where g^w is the map whose kth component g^w_k is given by $g^w_k(a)_{p^v} = a_{p^v k}$. The map g^w is a bijection since the sets $S \cap kP$ with $k \in I(S)$ partition S and since the maps $S \cap kP \to (S/k) \cap P$ that take $p^v k$ to p^v are bijections. Let h^w be the inverse of g^w. We claim that there exists a natural function $h\colon \prod \mathbb{W}_{(S/k)\cap P}(A) \to \mathbb{W}_S(A)$ such that $w \circ h = h^w \circ (\prod w)$. Granting this, the equalities $g^w \circ h^w = \mathrm{id}$ and $h^w \circ g^w = \mathrm{id}$ imply that $g \circ h = \mathrm{id}$ and $h \circ g = \mathrm{id}$, which proves the proposition.

To prove the claim, it suffices to show that, in the universal case, where A is the free $\mathbb{Z}[I(S)^{-1}]$-algebra generated by $\{a_{k,p^v} \mid k \in I(S), p^v \in (S/k) \cap P\}$ and $a = (a_k)$ with $a_k = (a_{k,p^v}) \in \mathbb{W}_{(S/k)\cap P}(A)$, the element $x = (h^w \circ (\prod w))(a)$ is in the image of $w\colon \mathbb{W}_S(A) \to A^S$. The unique $\mathbb{Z}[I(S)^{-1}]$-algebra homomorphism $\phi_p\colon A \to A$ that to a_{k,p^v} associates a^p_{k,p^v} is a lift of the Frobenius of A/pA. Moreover, all prime numbers $\ell \in S$ different from p are invertible in A. Therefore, we conclude from Lemma 1.1 that the sequence $x = \langle x_n \mid n \in S \rangle$ is in the image of the ghost map if and only if for all $n = p^v k \in S$ with $k \in I(S)$ and $v \geqslant 1$, $x_{p^v k} \equiv \phi_p(x_{p^{v-1}k})$ modulo $p^v A$. But $x_{p^v k} = w_{p^v}(a_k)$ and $\phi_p(x_{p^{v-1}k}) = \phi_p(w_{p^{v-1}}(a_k))$ which are congruent modulo $p^v A$ by Lemma 1.1. Hence, there exists a vector $h(a) \in \mathbb{W}_S(A)$ such that $x = w(h(a))$ and this vector is unique, as A is torsion free. The vector $h(a)$, in turn, uniquely determines the desired natural map h. This completes the proof. □

Example 1.11. If $S = \{1, 2, \dots, n\}$, then $(S/k) \cap P = \{1, p, \dots, p^{s-1}\}$ where $s = s(n,k)$ is the unique integer with $p^{s-1}k \leqslant n < p^s k$. Hence, if every integer $1 \leqslant k \leqslant n$ not divisible by p is invertible in A, then Proposition 1.10 gives an isomorphism

$$\mathbb{W}_{\{1,2,\dots,n\}}(A) \xrightarrow{\sim} \prod \mathbb{W}_{\{1,p,\dots,p^{s-1}\}}(A)$$

where the product ranges over integers $1 \leqslant k \leqslant n$ not divisible by p and $s = s(n,k)$.

Lemma 1.12. *If A is an $\mathbb{F}_p$-algebra, then for every truncation set S,*

$$V_p \circ F_p = p \cdot \mathrm{id}\colon \mathbb{W}_S(A) \to \mathbb{W}_S(A).$$

Proof. We first reduce to the case where S is a p-typical truncation set. It follows from Lemma 1.5 that the following diagram, where the products range over $k \in I(S)$ and where the vertical maps are the isomorphisms of Proposition 1.10, commutes,

$$\begin{array}{ccccc} \mathbb{W}_S(A) & \xrightarrow{F_p} & \mathbb{W}_{S/p}(A) & \xrightarrow{V_p} & \mathbb{W}_S(A) \\ \downarrow g & & \downarrow g & & \downarrow g \\ \prod \mathbb{W}_{(S/k)\cap P}(A) & \xrightarrow{\prod F_p} & \prod \mathbb{W}_{(S/pk)\cap P}(A) & \xrightarrow{\prod V_p} & \prod \mathbb{W}_{(S/k)\cap P}(A). \end{array}$$

Accordingly, it will suffice to prove the lemma for p-typical truncation sets S, and we may further assume that S is finite. It follows from Lemma 1.5 (iii) that

$$V_p \circ F_p = V_p([1]_{S/p}) \cdot \mathrm{id}\colon \mathbb{W}_S(A) \to \mathbb{W}_S(A)$$

and we proceed to prove that $V_p([1]_{S/p}) = p[1]_S$ by induction on the cardinality n of S. The case $n = 0$ holds trivially, so we let $S = \{1, p, \dots, p^{n-1}\}$ be the p-typical truncation set of cardinality $n > 0$ and assume that the identity in question has been proved for all proper sub-truncation sets $T \subset S$. The exact sequences

$$0 \longrightarrow \mathbb{W}_{\{1\}}(A) \xrightarrow{V_{p^{n-1}}} \mathbb{W}_S(A) \xrightarrow{R^S_{S/p}} \mathbb{W}_{S/p}(A) \longrightarrow 0$$

furnish an induction argument showing that $\mathbb{W}_S(A)$ is annihilated by p^n. In particular, $V_p([1]_{S/p})$ is annihilated by p^{n-1}. Moreover, it follows from Addendum 1.7 that

$$[p]_S = p[1]_S + \sum_{0<s<n} \frac{1}{p^s}(p^{p^s} - p^{p^{s-1}})V_{p^s}([1]_{S/p^s})$$

and the left-hand side vanishes, since A is an $\mathbb{F}_p$-algebra. The inductive hypothesis shows that $V_{p^s}([1]_{S/p^s}) = p^{s-1}V_p([1]_{S/p})$, so the formula above becomes

$$0 = p[1]_S + (p^{p^{n-1}-1} - 1)V_p([1]_{S/p}).$$

But $p^{n-1} - 1 \geqslant n-1$, so $V_p([1]_{S/p}) = p[1]_S$ which proves the induction step. □

Let A be a p-torsion free ring equipped with a ring homomorphism $\phi : A \to A$ such that $\phi(a) \equiv a^p$ modulo pA. By Lemma 1.1, there is a unique ring homomorphism

$$\lambda_\phi : A \to \mathbb{W}_P(A)$$

such that $w_{p^n} \circ \lambda_\phi = \phi^n$. We define $s_\phi : A \to \mathbb{W}_P(A/pA)$ to be the composition of λ_ϕ and the map induced by the canonical projection of A onto A/pA. We recall that A/pA is said to be perfect, if the Frobenius $\varphi\colon A/pA \to A/pA$ is an automorphism.

Proposition 1.13. *Let p be a prime number, let n be a non-negative integer, and let S be the finite p-typical truncation set of cardinality n. Let A be a p-torsion free ring equipped with a ring homomorphism $\phi : A \to A$ such that $\phi(a) \equiv a^p$ modulo pA and suppose that A/pA is perfect. In this situation, the map s_ϕ induces an isomorphism*

$$A/p^nA \xrightarrow{\bar{s}_\phi} \mathbb{W}_S(A/pA).$$

Proof. We claim that the map s_ϕ induces a map $\bar{s}_\phi$ as stated. Indeed, the restriction map $R^P_S\colon \mathbb{W}_P(A/pA) \to \mathbb{W}_S(A/pA)$ has kernel $V_{p^n}\mathbb{W}_P(A/pA)$, and

$$V_{p^n}\mathbb{W}_P(A/pA) = V_{p^n}\mathbb{W}_P(\varphi^n(A/pA)) = V_{p^n}F_{p^n}\mathbb{W}(A/pA) = p^n\mathbb{W}_P(A/pA),$$

where the left-hand equality follows from A/pA being perfect, the middle equality from Lemma 1.8, and the right-hand equality from Lemma 1.12. Now, the proof is completed by an induction argument based on the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A/pA & \xrightarrow{p^{n-1}} & A/p^nA & \xrightarrow{\mathrm{pr}} & A/p^{n-1}A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi^{n-1}} & & \downarrow{\scriptstyle \bar{s}_\phi} & & \downarrow{\scriptstyle \bar{s}_\phi} & & \\
0 & \longrightarrow & A/pA & \xrightarrow{V_{p^{n-1}}} & \mathbb{W}_S(A/pA) & \xrightarrow{R^S_{S/p}} & \mathbb{W}_{S/p}(A/pA) & \longrightarrow & 0,
\end{array}$$

where the top horizontal sequence is exact since A is p-torsion free, and where the left-hand vertical map is an isomorphism since A/pA is perfect. □

We return to the ring of big Witt vectors. We prove that the underlying additive group of the ring $\mathbb{W}(A)$ is naturally isomorphic to the multiplicative group

$$\Lambda(A) = (1 + tA[\![t]\!])^*$$

of power series with constant term 1. We also view the set $tA[\![t]\!]$ of power series with constant term 0 as an abelian group under coefficientwise addition. We recall the following result from [8, Section 1]; see also [12, Proposition 17.2.9].

Proposition 1.14. *The diagram of natural group homomorphisms*

$$\begin{array}{ccc}
\mathbb{W}(A) & \xrightarrow{\gamma} & \Lambda(A) \\
\downarrow{\scriptstyle w} & & \downarrow{\scriptstyle t\frac{d}{dt}\log} \\
A^{\mathbb{N}} & \xrightarrow{\gamma^w} & tA[\![t]\!],
\end{array}$$

where $\gamma(a_1, a_2, \dots) = \prod_{n\geqslant 1}(1 - a_nt^n)^{-1}$ and $\gamma^w\langle x_1, x_2, \dots\rangle = \sum_{n\geqslant 1} x_nt^n$, commutes, and the horizontal maps are isomorphisms.

Proof. It is clear that the maps in the diagram are natural transformations of functors from the category of rings to the category of sets. Moreover, the calculation

$$\begin{aligned}
t\frac{d}{dt}\log\Big(\prod_{e\geqslant 1}(1 - a_et^e)^{-1}\Big) &= -\sum_{e\geqslant 1} t\frac{d}{dt}\log(1 - a_et^e) = \sum_{e\geqslant 1}\frac{ea_et^e}{1 - a_et^e} \\
&= \sum_{e\geqslant 1}\sum_{q\geqslant 1} ea_e^qt^{qe} = \sum_{n\geqslant 1}\Big(\sum_{e\mid n} ea_e^{n/e}\Big)t^n
\end{aligned}$$

shows that the diagram commutes. It is also clear that the two vertical maps are group homomorphisms and that the map γ^w is an isomorphism of abelian groups. This implies that the map γ is a group homomorphism. Indeed, if A is torsion free, then the vertical maps both are injective, and in general, we choose a surjective ring homomorphism $A' \to A$ from a torsion free ring and use that $\mathbb{W}(-)$ and $\Lambda(-)$ both take surjective ring homomorphisms to surjective group homorphisms.

It remains to show that γ is a bijection. To this end, we write

$$\prod_{n \geqslant 1} (1 - a_n t^n)^{-1} = (1 + b_1 t + b_2 t^2 + \dots)^{-1}$$

where the coefficient b_n is given by the sum $b_n = \sum (-1)^r a_{i_1} \dots a_{i_r}$ that ranges over all $1 \leqslant i_1 < \dots < i_r \leqslant n$ such that $i_1 + 2i_2 + \dots + ri_r = n$. It follows that the Witt coordinates a_n are uniquely determined, recursively, by the coefficients b_n, and hence, that γ is a bijection as stated. □

Remark 1.15. We will always consider the set $\Lambda(A) = 1 + tA[[t]]$ as a ring with the unique ring structure that makes the map $\gamma\colon \mathbb{W}(A) \to \Lambda(A)$ a ring isomorphism. This ring structure is characterized by begin natural in A, by addition being given by power series multiplication, and by the product satisfying

$$(1 - at)^{-1} * (1 - bt)^{-1} = (1 - abt)^{-1}$$

for all $a, b \in A$; compare [11, Section 4]. We note that $(1-t)^{-1}$ is the multiplicative unit element in $\Lambda(A)$. The reader is warned, however, that there exists four different ring structures on the set $1 + tA[[t]]$ satisfying the first two of these requirements but with the last requirement replaced by the four possible choices of signs in the product formula $(1 \pm at)^{\pm 1} * (1 \pm bt)^{\pm 1} = (1 \pm abt)^{\pm 1}$. The choice $++$ is used in [11, 1, 3], while the choice $-+$ is used in [12, Section 17.2]. The four different rings $\Lambda(A)_{\pm\pm}$ are all naturally isomorphic, the natural isomorphism $u_{\pm\pm}\colon \Lambda(A) \to \Lambda(A)_{\pm\pm}$ given by $u_{\pm\pm}(f(t)) = (1 \pm t)^{\pm 1} * f(t)$, where the product is evaluated in $\Lambda(A)$. We also write $\gamma_{\pm\pm}\colon \mathbb{W}(A) \to \Lambda(A)_{\pm\pm}$ for the natural ring isomorphism $\gamma_{\pm\pm} = u_{\pm\pm} \circ \gamma$; in particular, $\gamma = \gamma_{--}$.

Addendum 1.16. *The map γ induces an isomorphism of abelian groups*

$$\mathbb{W}_S(A) \xrightarrow{\gamma_S} \Lambda_S(A)$$

where $\Lambda_S(A)$ is the quotient of the multiplicative group $\Lambda(A) = (1 + tA[[t]])^*$ by the subgroup $I_S(A)$ of all power series of the form $\prod_{n \in \mathbb{N} \smallsetminus S} (1 - a_n t^n)^{-1}$.

Proof. The kernel of the restriction map $R^{\mathbb{N}}_S\colon \mathbb{W}(A) \to \mathbb{W}_S(A)$ is equal to the subset of all vectors $a = (a_n \mid n \in \mathbb{N})$ such that $a_n = 0$, if $n \in S$. The image of this subset by the map γ is the subset $I_S(A) \subset \Lambda(A)$. □

Example 1.17. If $S = \{1, 2, \dots, n\}$, then $I_S(A) = (1 + t^{n+1} A[[t]])^*$. Hence, in this case, Addendum 1.16 gives an isomorphism of abelian groups

$$\gamma_S\colon \mathbb{W}_{\{1,2,\dots,n\}}(A) \xrightarrow{\sim} (1 + tA[[t]])^* / (1 + t^{n+1} A[[t]])^*.$$

For A a $\mathbb{Z}_{(p)}$-algebra, the structure of this group was examined in Example 1.11.

Lemma 1.18. *Let A be an arbitrary ring. For every prime number p, the natural ring homomorphism $F_p\colon \mathbb{W}(A) \to \mathbb{W}(A)$ satisfies that $F_p(a) \equiv a^p$ modulo $p\mathbb{W}(A)$.*

Proof. By naturality, it suffices to consider $A = \mathbb{Z}[a_1, a_2, \dots]$ and $a = (a_1, a_2, \dots)$ and show that there exists $b \in \mathbb{W}(A)$ with $F_p(a) - a^p = pb$. We have

$$w_n(F_p(a) - a^p) = \sum_{e \mid pn} e a_e^{pn/e} - \big(\sum_{e \mid n} e a_e^{n/e}\big)^p$$

which clearly is congruent to zero modulo pA. So we let $x = \langle x_n \mid n \in \mathbb{N} \rangle$ with

$$x_n = \frac{1}{p} w_n(F_p(a) - a^p)$$

and employ Lemma 1.1 to show that $x = w(b)$ with $b \in \mathbb{W}(A)$. To this end, we must show that for every prime number ℓ and every $n \in \ell\mathbb{N}$,

$$x_n \equiv \phi_\ell(x_{n/\ell})$$

modulo $\ell^{v_\ell(n)}A$, where $\phi_\ell \colon A \to A$ is the unique ring homomorphism that takes a_n to a_n^ℓ. The congruence in question is equivalent to the statement that

$$w_n(F_p(a) - a^p) \equiv \phi_\ell(w_{n/\ell}(F_p(a) - a^p))$$

modulo $\ell^{v_\ell(n)}A$, if $\ell \neq p$ and $n \in \ell\mathbb{N}$, and modulo $p^{v_p(n)+1}A$, if $\ell = p$ and $n \in p\mathbb{N}$. If $\ell \neq p$, the statement follows from Lemma 1.1, and if $\ell = p$ and $n \in p\mathbb{N}$, we find

$$\begin{aligned} & w_n(F_p(a) - a^p) - \phi_p(w_{n/p}(F_p(a) - a^p)) \\ & = \sum_{e \mid pn} e a_e^{pn/e} - \big(\sum_{e \mid n} e a_e^{n/e}\big)^p - \sum_{e \mid n} e a_e^{pn/e} + \big(\sum_{e \mid (n/p)} e a_e^{n/e}\big)^p. \end{aligned}$$

If $e \mid pn$ and $e \nmid n$, then $v_p(e) = v_p(n) + 1$, so

$$\sum_{e \mid pn} e a_e^{pn/e} \equiv \sum_{e \mid n} e a_e^{pn/e}$$

modulo $p^{v_p(n)+1}A$. Similarly, if $e \mid n$ and $e \nmid (n/p)$, then $v_p(e) = v_p(n)$, and hence,

$$\sum_{e \mid n} e a_e^{n/e} \equiv \sum_{e \mid (n/p)} e a_e^{n/e}$$

modulo $p^{v_p(n)}A$. But then

$$\big(\sum_{e \mid n} e a_e^{n/e}\big)^p \equiv \big(\sum_{e \mid (n/p)} e a_e^{n/e}\big)^p$$

modulo $p^{v_p(n)+1}A$ as required; compare the proof of Lemma 1.1. □

We next recall the following result of Cartier from [12, Theorem 17.6.17].

Proposition 1.19. *There exists a unique natural ring homomorphism*

$$\Delta = \Delta_A \colon \mathbb{W}(A) \to \mathbb{W}(\mathbb{W}(A))$$

such that for every positive integer n,

$$w_n \circ \Delta = F_n \colon \mathbb{W}(A) \to \mathbb{W}(A).$$

In addition, the following diagrams, where $\varepsilon_A = w_1 \colon \mathbb{W}(A) \to A$, *commute.*

$$\begin{array}{ccccc} \mathbb{W}(A) & \xleftarrow{\varepsilon_{\mathbb{W}(A)}} & \mathbb{W}(\mathbb{W}(A)) & \xrightarrow{\mathbb{W}(\varepsilon_A)} & \mathbb{W}(A) \\ & \diagdown\!\!\diagdown & \big\uparrow{\scriptstyle \Delta_A} & \diagup\!\!\diagup & \\ & & \mathbb{W}(A) & & \end{array} \qquad \begin{array}{ccc} \mathbb{W}(\mathbb{W}(\mathbb{W}(A))) & \xleftarrow{\Delta_{\mathbb{W}(A)}} & \mathbb{W}(\mathbb{W}(A)) \\ \big\uparrow{\scriptstyle \mathbb{W}(\Delta_A)} & & \big\uparrow{\scriptstyle \Delta_A} \\ \mathbb{W}(\mathbb{W}(A)) & \xleftarrow{\Delta_A} & \mathbb{W}(A) \end{array}$$

Proof. We first prove that a natural ring homomorphism as stated exists. It suffices to prove that in the universal case $A = \mathbb{Z}[a_1, a_2, \dots]$ and $a = (a_1, a_2, \dots)$, there exists an element $\Delta_A(a) \in \mathbb{W}(\mathbb{W}(A)$ whose image by the ghost map

$$w \colon \mathbb{W}(\mathbb{W}(A)) \to \mathbb{W}(A)^{\mathbb{N}}$$

is the sequence $\langle F_n(a) \mid n \in \mathbb{N} \rangle$. It follows from Lemma 1.9 that, in this case, the ghost map is injective, so the element $\Delta_A(a)$ necessarily is unique. Now Lemmas 1.1 and 1.18 show that the sequence $\langle F_n(a) \mid n \in \mathbb{N} \rangle$ is in the image of the ghost map if and only if for every prime number p and $n \in p\mathbb{N}$, the congruence

$$F_n(a) \equiv F_p(F_{n/p}(a)) \qquad \text{modulo } p^{v_p(n)}\mathbb{W}(A)$$

holds. But in fact equality holds by Lemma 1.5 (iv), so we conclude that the desired element $\Delta_A(a)$ with $w_n(\Delta_A(a)) = F_n(a)$ exists. Hence, there exists a unique natural ring homomorphism Δ such that $w_n \circ \Delta = F_n$ for every $n \in \mathbb{N}$. Finally, one readily verifies the commutativity of the two diagrams in the statement by evaluating the corresponding maps in ghost coordinates. □

Remark 1.20. The map $\Delta_n \colon \mathbb{W}(A) \to \mathbb{W}(A)$ given by the nth Witt component of the map Δ is generally not a ring homomorphism. For example, for a prime number p, the map Δ_p is the unique natural solution to the equation

$$F_p(a) = a^p + p\Delta_p(a).$$

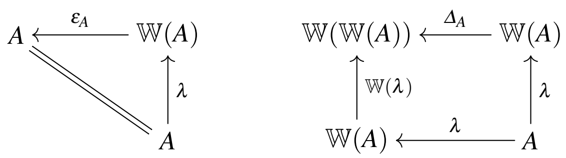
We also note that the map Δ has the property that for all $a \in A$,

$$\Delta([a]) = [[a]].$$

Indeed, we may assume that $A = \mathbb{Z}[a]$, in which case the ghost map is injective, and applying w_n on both sides, we get $F_n([a]) = [a]^n$ which holds by Lemma 1.5 (v).

The natural transformation Δ in Proposition 1.19 is called the universal λ-operation. Using it, we may restate Grothendieck's definition of a λ-ring from [11] as follows.

Definition 1.21. A λ-ring is a pair (A,λ) of a ring A and a ring homomorphism $\lambda\colon A \to \mathbb{W}(A)$ that makes the following diagrams commute.



A morphism of λ-rings $f\colon (A,\lambda_A) \to (B,\lambda_B)$ is a ring homomorphism $f\colon A \to B$ with the property that $\lambda_B \circ f = \mathbb{W}(f) \circ \lambda_A$.

If (A,λ) is a λ-ring (A,λ), then we write $\lambda_n\colon A \to A$ for the map that to a assigns the nth Witt component $\lambda_n(a)$ of the Witt vector $\lambda(a)$. The map λ_n is generally neither additive nor multiplicative.

Remark 1.22. We recall the translation between the above definition of a λ-ring and the original definition by Grothendieck as stated in [3, Definition V.2.4] (or in [11] and [1, Section 1], where a λ-ring is called a special λ-ring), emphasizing the choices of signs; see also [12, E.2.1]. The commutativity of the diagrams in Proposition 1.19 express that the triple $(\mathbb{W}(-),\Delta,\varepsilon)$ is a comonad on the category of commutative rings, and the commutativity of the diagrams in Definition 1.21 express that the pair (A,λ) is a coalgebra over this comonad. Similarly, in the original definition, a λ-ring is defined to be a coalgebra (A,λ_t) over the comonad $(\Lambda(-)_{++},\Delta_t,\varepsilon_t)$, where $\Lambda(-)_{++}$ is the functor from the category of commutative rings to itself defined in Remark 1.15; $\varepsilon_{t,A}\colon \Lambda(A)_{++} \to A$ is the natural ring homomorphism defined by $\varepsilon_{t,A}(1+a_1t+\dots) = a_1$; and $\Delta_{t,A}\colon \Lambda(A)_{++} \to \Lambda(\Lambda(A)_{++})_{++}$ is the unique natural ring homomorphism that is a section of $\varepsilon_{t,\Lambda(A)_{++}}$ and satisfies that for all $a \in A$,

$$\Delta_{t,A}(1+at) = 1+(1+at_2)t_1.$$

We claim that the natural ring isomorphism γ_{++} is an isomorphism of comonads from $(\mathbb{W}(-),\Delta,\varepsilon)$ to $(\Lambda(-)_{++},\Delta_t,\varepsilon_t)$ in the sense that if (A,λ) is a coalgebra over the former comonad, then $(A,\gamma_{++}\circ\lambda)$ is a coalgebra over the latter comonad. Indeed, this follows immediately from the above characterization of Δ_t and from the formula $\Delta_A([a]) = [[a]]$ from Remark 1.20. This shows that the two definitions of a λ-ring agree. Finally, we remark that if (A,λ) is a λ-ring and if we expand $\lambda_t = \gamma_{++}\circ\lambda$ as

$$\lambda_t(a) = 1+\lambda^1(a)t+\lambda^2(a)t^2+\cdots+\lambda^n(a)t^n+\cdots,$$

then $\lambda^n\colon A \to A$ is called the nth exterior operation associated with (A,λ); it should not be confused with $\lambda_n\colon A \to A$. Similarly, if we expand $\sigma_t = \gamma\circ\lambda$ as

$$\sigma_t(a) = 1+\sigma^1(a)t+\sigma^2(a)t^2+\cdots+\sigma^n(a)t^n+\cdots,$$

then $\sigma^n\colon A \to A$ is called the nth symmetric operation associated with (A,λ).

Definition 1.23. Let (A,λ) be a λ-ring. The associated nth Adams operation is the composite ring homomorphism $\psi_n = w_n\circ\lambda\colon A \to A$.

We note that, by Proposition 1.14, the series $\psi_t(a) = \sum_{n\geqslant 1} \psi_n(a)t^n$ is given by either one of the following formulas which are, perhaps, more familiar;

$$\psi_t(a) = t\frac{d}{dt}\log\sigma_t(a), \qquad \psi_{-t}(a) = -t\frac{d}{dt}\log\lambda_t(a).$$

We recall the following standard properties of the Adams operations, and mention Wilkerson's result [28, Proposition 1.2] that, if A is a ring flat over $\mathbb{Z}$ equipped with a family of ring endomorphisms ψ_n satisfying (i)–(iii) below, then there is a unique λ-ring structure on A for which the ψ_n are the associated Adams operators.

Lemma 1.24. *Let (A,λ) be a λ-ring. The associated Adams operations satisfy that*

(i) *the map ψ_1 is the identity map of A;*
(ii) *for all positive integers m and n, $\psi_m \circ \psi_n = \psi_{mn}$; and*
(iii) *for every prime number p and $a \in A$, $\psi_p(a) \equiv a^p$ modulo pA.*

Proof. The properties (i) and (iii) follow immediately from the definitions, and (ii) follows from the identities

$$\begin{aligned}\psi_m \circ \psi_n &= w_m \circ \lambda \circ w_n \circ \lambda = w_m \circ w_n \circ \mathbb{W}(\lambda) \circ \lambda \\ &= w_m \circ w_n \circ \Delta \circ \lambda = w_m \circ F_n \circ \lambda = w_{mn} \circ \lambda = \psi_{mn}.\end{aligned}$$

Here, the second identity follows from the naturality of w_n; the third identity from the definition of a λ-ring; the forth identity from the definition of the map Δ; and the fifth identity from the definition of the map F_n. □

Finally, we recall the following general theorem which was proved independently by Borger [6, Theorem B], [5, Corollary 15.4] and van der Kallen [27, Theorem 2.4].

Theorem 1.25. *Let $f\colon A \to B$ be an étale morphism, let S be a finite truncation set, and let n be a positive integer. Then the induced morphism*

$$\mathbb{W}_S(A) \xrightarrow{\mathbb{W}_S(f)} \mathbb{W}_S(B)$$

is étale and the square diagram

$$\begin{array}{ccc} \mathbb{W}_S(A) & \xrightarrow{\mathbb{W}_S(f)} & \mathbb{W}_S(B) \\ \downarrow F_n & & \downarrow F_n \\ \mathbb{W}_{S/n}(A) & \xrightarrow{\mathbb{W}_{S/n}(f)} & \mathbb{W}_{S/n}(B) \end{array}$$

is a cocartesian square of rings. □

We remark that in loc. cit., the Theorem 1.25 is stated only for the finite truncation sets $\langle n\rangle$ that consist of all divisors of a given positive integer n. However, as explained in [6, Section 9.5], the case of a general finite truncation set readily follows from the special case.

2 Modules and derivations over λ-rings

In general, if $\mathscr{C}$ is a category in which finite limits exist and if X is an object of $\mathscr{C}$, then Beck, in his thesis [2, Definition 5], defines the category of X-modules to be the category $(\mathscr{C}/X)_{\mathrm{ab}}$ of abelian group objects in the category over X. He also defines the derivations from X to the X-module $(Y/X, +_Y, 0_Y, -_Y)$ to be the set

$$\mathrm{Der}(X,(Y/X,+_Y,0_Y,-_Y)) = \mathrm{Hom}_{\mathscr{C}/X}(X/X, Y/X)$$

of morphisms in the category $\mathscr{C}/X$ equipped with the abelian group structure induced by the abelian group object structure on Y/X. In this section, we identify and study these notions in the case of the category $\mathscr{A}_\lambda$ of λ-rings.

We recall that, in general, an adjunction from a category $\mathscr{C}$ to a category $\mathscr{D}$ is a quadruple (F,G,ε,η) of functors $F\colon \mathscr{C}\to\mathscr{D}$ and $G\colon \mathscr{D}\to\mathscr{C}$ and natural transformations $\varepsilon\colon F\circ G\Rightarrow \mathrm{id}_{\mathscr{D}}$ and $\eta\colon \mathrm{id}_{\mathscr{C}}\Rightarrow G\circ F$ such that the following composite natural transformations are equal to the respective identity natural transformations,

$$F \xRightarrow{F\circ\eta} F\circ G\circ F \xRightarrow{\varepsilon\circ F} F, \qquad G \xRightarrow{\eta\circ G} G\circ F\circ G \xRightarrow{G\circ\varepsilon} G;$$

compare [23, Theorem IV.1.2]. We refer to this requirement by saying that the triangle identities hold. The natural transformations ε and η are called the counit and the unit of the adjunction, respectively, and the adjunction is said to be an adjoint equivalence if they both are isomorphisms. A functor $G\colon \mathscr{D}\to\mathscr{C}$ is said to admit a left adjoint, if there exists an adjunction (F,G,ε,η) with G as its second component, and in this case, the functor F is said to be a left adjoint of the functor G. If $(F',G,\varepsilon',\eta')$ is another such adjunction, then the composite

$$F \xRightarrow{F\circ\eta'} F\circ G\circ F' \xRightarrow{\varepsilon\circ F'} F'$$

is the unique natural transformation $\sigma\colon F\Rightarrow F'$ with the property that the diagrams

$$\begin{array}{ccc} F\circ G & \xRightarrow{\varepsilon} & \mathrm{id}_{\mathscr{D}} \\ \Big\Downarrow{\scriptstyle \sigma\circ G} & & \Big\| \\ F'\circ G & \xRightarrow{\varepsilon'} & \mathrm{id}_{\mathscr{D}} \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{id}_{\mathscr{C}} & \xRightarrow{\eta} & G\circ F \\ \Big\| & & \Big\Downarrow{\scriptstyle \sigma\circ G} \\ \mathrm{id}_{\mathscr{C}} & \xRightarrow{\eta'} & G\circ F' \end{array}$$

commute and is an isomorphism; see [23, Theorem IV.7.2]. In this sense, a left adjoint of a functor G, if it exits, is unique, up to unique isomorphism. Similar statements hold for right adjoint functors.

Let $\mathscr{A}$ be the category of rings. We always assume rings to be commutative and unital, unless otherwise stated. Given a ring A, we define an adjunction (F,G,ε,η) from the category $(\mathscr{A}/A)_{\mathrm{ab}}$ of abelian group objects in the over-category $\mathscr{A}/A$ to the category $\mathscr{M}(A)$ of A-modules in the usual sense, following Beck [2, Example 8]. So let $f\colon B\to A$ be an object of $\mathscr{A}/A$, and let

$$\begin{array}{ccc} B\times_A B & \xrightarrow{+_B} & B \\ \Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ A & = & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{0_B} & B \\ \Big\| & & \Big\downarrow{\scriptstyle f} \\ A & = & A \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{-_B} & B \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f} \\ A & = & A \end{array}$$

be abelian group object structure maps. The functor F associates to the abelian group object $(f, +_B, 0_B, -_B)$ the A-module M given by the kernel of f with the A-module structure $a \cdot x = 0_B(a)x$. Conversely, if M is an A-module, then we let $A \ltimes M$ be the ring given by the direct sum $A \oplus M$ equipped with the multiplication

$$(a,x) \cdot (a',x') = (aa', ax' + a'x)$$

and define $G(M)$ to be the abelian group object $(f, +, 0, -)$, where $f \colon A \ltimes M \to A$ is the canonical projection, and the abelian group object structure maps are given by $(a,x) + (a,x') = (a, x+x')$, $0(a) = (a,0)$, and $-(a,x) = (a,-x)$. Finally, we define $\varepsilon \colon G \circ F \Rightarrow \mathrm{id}$ by $\varepsilon(a,x) = 0_B(a) + x$ and $\eta \colon \mathrm{id} \Rightarrow F \circ G$ by $\eta(x) = (0,x)$. For later use, we include a proof of the following result of Beck [2, Example 8].

Lemma 2.1. *If A is a ring, then the quadruple $(F, G, \eta, \varepsilon)$ defined above is an adjoint equivalence of categories from $(\mathscr{A}/A)_{\mathrm{ab}}$ to $\mathscr{M}(A)$.*

Proof. It is clear that η is well-defined and a natural isomorphism, and it is also clear that ε is a natural isomorphism of underlying additive groups. We must show that ε is a multiplicative map and a map of abelian group objects; we first consider the latter statement. So we fix an object $(f \colon B \to A, +_B, 0_B, -_B)$ of $(\mathscr{A}/A)_{\mathrm{ab}}$ and let $M = \ker(f)$ with the A-module structure defined above. By definition, we have $\varepsilon(a,0) = 0_B(a)$ which shows that ε preserves zero maps. To see that ε preserves addition maps, we first note that, since $+_B$ is a ring homomorphism,

$$(u+v) +_B (u'+v') = (u +_B u') + (v +_B v')$$

for all $(u,u'), (v,v') \in B \times_A B$. In particular, if $x, y \in M$, then

$$x +_B y = (x+0) +_B (0+y) = (x +_B 0) + (0 +_B y) = x + y,$$

where we also use that $0 = 0_B(0)$ is a common zero element for the two compositions $+$ and $+_B$ on M. We therefore conclude that for all $a \in A$ and $x, x' \in M$,

$$\begin{aligned} &\varepsilon(a,x) +_B \varepsilon(a,x') = (0(a)+x) +_B (0(a)+x') \\ &= (0(a) +_B 0(a)) + (x +_B x') = 0(a) + (x+x') = \varepsilon(a, x+x'), \end{aligned}$$

as desired. We have showed that ε is compatible with the zero and addition maps; but then it is also compatible with negation maps.

It remains to prove that the map ε is multiplicative, or equivalently, that $M \subset B$ is a square-zero ideal. Since $+_B \colon B \times_A B \to B$ is a ring homomorphism, we have that $uv +_B u'v' = (u +_B u')(v +_B v')$, for all $(u,u'), (v,v') \in B \times_A B$. In particular,

$$xy + x'y' = (x+y)(x'+y')$$

for all $x, x', y, y' \in M$, since $+_B = +$ on M. Taking $y = x' = 0$, we find that $xy' = 0$ for all $x, y' \in M$ as desired. This completes the proof. ∎

We will prove the analogous statement for λ-rings in Proposition 2.10 below, but first we examine the Witt vectors of $A \ltimes M$. The polynomials s_n, p_n, and i_n that define the sum, product, and opposite in the ring of Witt vectors all have constant term zero. Therefore, the ring of Witt vectors is defined also for non-unital rings. Moreover, modulo terms of higher degree, these polynomials are congruent to $a_n + b_n$, $a_n b_n$, and $-a_n$, respectively, as one readily proves by induction. Therefore, if M is an abelian group considered as a non-unital ring with zero multiplication, then the non-unital ring $\mathbb{W}_S(M)$ has zero multiplication and its underlying additive group is equal to M^S with componentwise addition. In the same way, one shows that the polynomials $f_{n,e}$ and $d_{m,e}$ that define the nth Frobenius and the universal λ-operation all have constant term zero and that they are congruent to na_{ne} and a_{me}, respectively, modulo terms of higher degree. Therefore, for M as above, the map $F_n \colon \mathbb{W}_S(M) \to \mathbb{W}_{S/n}(M)$ takes $(x_m \mid m \in S)$ to $(nx_{ne} \mid e \in S/n)$ and the map $\Delta_M \colon \mathbb{W}(M) \to \mathbb{W}(\mathbb{W}(M))$ takes $(x_m \mid m \in \mathbb{N})$ to $((x_{me} \mid e \in \mathbb{N}) \mid m \in \mathbb{N})$.

Lemma 2.2. *Let S be truncation set, let A be a ring, and let M be an A-module. The canonical inclusions induce a ring isomorphism*

$$\mathrm{in}_{1*} + \mathrm{in}_{2*} \colon \mathbb{W}_S(A) \ltimes \mathbb{W}_S(M) \to \mathbb{W}_S(A \ltimes M),$$

provided that $\mathbb{W}_S(M)$ is given the $\mathbb{W}_S(A)$-module structure, where for $a \in \mathbb{W}_S(A)$ and $x \in \mathbb{W}_S(M)$, $ax \in \mathbb{W}_S(M)$ has Witt components $(ax)_n = w_n(a)x_n$.

Proof. We consider the following diagram of rings and ring homomorphisms, whose underlying diagram of additive groups is split-exact.

$$0 \longrightarrow M \xrightarrow{\mathrm{in}_2} A \ltimes M \underset{\mathrm{pr}_1}{\overset{\mathrm{in}_1}{\leftrightarrows}} A \longrightarrow 0$$

It induces the following diagram of rings and ring homomorphisms, whose underlying diagram of additive groups again is split-exact.

$$0 \longrightarrow \mathbb{W}_S(M) \xrightarrow{\mathrm{in}_{2*}} \mathbb{W}_S(A \ltimes M) \underset{\mathrm{pr}_{1*}}{\overset{\mathrm{in}_{1*}}{\leftrightarrows}} \mathbb{W}_S(A) \longrightarrow 0$$

It follows that the map of the statement is a ring isomorphism, if $\mathbb{W}_S(M)$ is given the $\mathbb{W}_S(A)$-module structure such that $\mathrm{in}_{2*}(ax) = \mathrm{in}_{1*}(a)\,\mathrm{in}_{2*}(x)$, for all $a \in \mathbb{W}_S(A)$ and $x \in \mathbb{W}_S(M)$. It remains to prove that ax is equal to the Witt vector y with nth component $w_n(a)x_n$. Since every ring admits a surjective ring homomorphism from a torsion free ring, we may assume that A and M are both torsion free. Moreover, since the ghost map is injective in this case, it will suffice to show that $w_n(ax) = w_n(y)$, or equivalently, that $\mathrm{in}_2(w_n(ax)) = \mathrm{in}_2(w_n(y))$, for all $n \geqslant 1$. Now, since w_n is a natural ring homomorphism, we find that for all $n \geqslant 1$,

$$\begin{aligned}\mathrm{in}_2(w_n(ax)) &= w_n(\mathrm{in}_{2*}(ax)) = w_n(\mathrm{in}_{1*}(a)\,\mathrm{in}_{2*}(x)) = w_n(\mathrm{in}_{1*}(a))w_n(\mathrm{in}_{2*}(x)) \\ &= \mathrm{in}_1(w_n(a))\,\mathrm{in}_2(w_n(x)) = \mathrm{in}_2(w_n(a)w_n(x)) = \mathrm{in}_2(nw_n(a)x_n) \\ &= \mathrm{in}_2(ny_n) = \mathrm{in}_2(w_n(y))\end{aligned}$$

as desired. Here the fifth equality follows from the definition of the multiplication on the ring $A \ltimes M$. ∎

Addendum 2.3. *Let S be a truncation set, let A be a ring, and let M be an A-module. If $a \in \mathbb{W}_S(A)$ and $x \in \mathbb{W}_S(M)$, then the components $b_n = (a_n, y_n)$ of the Witt vector $b = \mathrm{in}_{1*}(a) + \mathrm{in}_{2*}(x) \in \mathbb{W}_S(A \ltimes M)$ satisfy that for all $n \in S$,*

$$\sum_{e \mid n} a_e^{(n/e)-1} y_e = x_n.$$

Proof. We may assume that A and M are torsion free and proceed to calculate $w_n(b)$ in two different ways. First, since w_n is a natural ring homomorphism, we have

$$\begin{aligned} w_n(b) &= w_n(\mathrm{in}_{1*}(a)) + w_n(\mathrm{in}_{2*}(x)) = \mathrm{in}_1(w_n(a)) + \mathrm{in}_2(w_n(x)) \\ &= (w_n(a), w_n(x)) = (\sum_{e \mid n} e a_e^{n/e}, n x_n). \end{aligned}$$

Second, by the definition of the multiplication in $A \ltimes M$, we have

$$w_n(b) = \sum_{e \mid n} e b_e^{n/e} = \sum_{e \mid n} e(a_e, y_e)^{n/e} = (\sum_{e \mid n} e a_e^{n/e}, \sum_{e \mid n} n a_e^{(n/e)-1} y_e).$$

The stated formula follows as M was assumed to be torsion free. □

Example 2.4. Let p be a prime number. Then $y_p = x_p - a_1^{p-1} x_1$.

In general, if $f\colon A \to B$ is a ring homomorphism and if M and N are modules over A and B, respectively, then we define an f-linear map $h\colon M \to N$ to be an additive map such that $h(ax) = f(a)h(x)$, for all $a \in A$ and $x \in M$. In the following, given an A-module M and a truncation set $S \subset \mathbb{N}$, we write $\mathbb{W}_S(M)$ for the $\mathbb{W}_S(A)$-module given by the set M^S with componentwise addition and with the scalar multiplication of $a \in \mathbb{W}_S(A)$ and $x \in \mathbb{W}_S(M)$ defined by to be the element $ax \in \mathbb{W}_S(M)$ with

$$(ax)_n = \psi_{A,n}(a) x_n$$

for all $n \in S$; compare Definition 1.23 and Lemma 2.2. We remark that if M is the ring A considered a module over itself via multiplication, then the $\mathbb{W}_S(A)$-module $\mathbb{W}_S(M)$ defined above usually is not the same as the ring $\mathbb{W}_S(A)$ considered as a module over itself via multiplication. To avoid confusion, we will use $\mathbb{W}_S(A)$ to indicate the ring of Witt vectors only and will not use it indicate either module over this ring.

Definition 2.5. Let (A, λ_A) be a λ-ring. An (A, λ_A)-module is a pair (M, λ_M) of an A-module M and a λ_A-linear map

$$\lambda_M\colon M \to \mathbb{W}(M)$$

with the property that the diagrams

$$\begin{array}{ccc} M & \xleftarrow{\ \varepsilon_M\ } & \mathbb{W}(M) \\ & \diagdown\!\!\diagdown & \uparrow{\scriptstyle \lambda_M} \\ & & M \end{array} \qquad \begin{array}{ccc} \mathbb{W}(\mathbb{W}(M)) & \xleftarrow{\ \Delta_M\ } & \mathbb{W}(M) \\ \uparrow{\scriptstyle \mathbb{W}(\lambda)} & & \uparrow{\scriptstyle \lambda_M} \\ \mathbb{W}(M) & \xleftarrow{\ \lambda_M\ } & M \end{array}$$

commute. A morphism $h\colon (M, \lambda_M) \to (N, \lambda_N)$ of (A, λ)-modules is an A-linear map $h\colon M \to N$ such that $\lambda_N \circ h = \mathbb{W}(h) \circ \lambda_M$.

Remark 2.6. Let (A,λ_A) be a λ-ring, M an A-module, and $\lambda_M\colon M\to \mathbb{W}(M)$ a map. Then (M,λ_M) is an (A,λ_A)-module if and only if the components $\lambda_{M,n}\colon M\to M$ are $\psi_{A,n}$-linear and satisfy $\lambda_{M,1}=\mathrm{id}_M$ and $\lambda_{M,m}\circ\lambda_{M,n}=\lambda_{M,mn}$, for all $m,n\in\mathbb{N}$. Hence, we may identify the category $\mathscr{M}(A,\lambda_A)$ with the category $\mathscr{M}(A^{\psi}[\mathbb{N}])$ of left modules over the twisted monoid algebra $A^{\psi}[\mathbb{N}]$ by associating to the (A,λ_A)-module (M,λ_M) the left $A^{\psi}[\mathbb{N}]$-module given by the A-module M and with $n\in\mathbb{N}$ acting through the map $\lambda_{M,n}\colon M\to M$. In particular, the category $\mathscr{M}(A,\lambda_A)$ is abelian.

Example 2.7. Let (A,λ_A) be a λ-ring. The functor that to an (A,λ_A)-module (M,λ_M) assigns the underlying set of M has a left adjoint functor that to the set S assigns the free (A,λ_A)-module $(F(S),\lambda_{F(S)})$ defined as follows. The A-module $F(S)$ is defined to be the free A-module generated by the symbols $\lambda_{F(S),n}(s)$, where $s\in S$ and $n\in\mathbb{N}$, and $\lambda_{F(S)}\colon F(S)\to\mathbb{W}(F(S))$ is defined to be the map with mth component

$$\lambda_{F(S),m}(\sum a_{s,n}\lambda_{F(S),n}(s))=\sum\psi_{A,m}(a_{s,n})\lambda_{F(S),mn}(s).$$

It follows from Remark 2.6 that the pair $(F(S),\lambda_{F(S)})$ is an (A,λ_A)-module. The unit of the adjunction maps $s\in S$ to $\lambda_{F(S),1}(s)\in F(S)$, and the counit of the adjunction maps $\sum a_{x,n}\lambda_{F(M),n}(x)\in F(M)$ to $\sum a_{x,n}\lambda_{M,n}(x)\in M$. It is straightforward to verify that the triangle identities hold.

Example 2.8. If (A,λ_A) is a λ-ring, then there is an (A,λ_A)-module (M,λ_M) defined by setting $M=A$ and $\lambda_{M,n}=\psi_{A,n}$. This (A,λ_A)-module is not a free (A,λ_A)-module in the sense of Example 2.7, except in trivial cases. We warn the reader that the pair (A,λ_A) is typically not an (A,λ_A)-module, let alone a free (A,λ_A)-module; compare the discussion preceding Definition 2.5.

Example 2.9. Let A be a ring, unital and commutative, and let $K_*(A)$ be the graded ring given by the Quillen K-groups. The ring $K_0(A)$ has a canonical λ-ring structure defined by Grothendieck [11], and for all $q\geqslant 1$, the group $K_q(A)$ has a canonical structure of a module over this λ-ring defined by Kratzer [21] and Quillen [18]. The $(K_0(A),\lambda_{K_0(A)})$-module structure maps are given by

$$\lambda_{K_q(A),n}=(-1)^{n-1}\lambda^n_{K_q(A)}\colon K_q(A)\to K_q(A)$$

with $\lambda^n_{K_q(A)}$ defined in [21, Théorème 5.1].

Let $U\colon\mathscr{A}_\lambda\to\mathscr{A}$ be the forgetful functor from the category of λ-rings to the category of rings that to a λ-ring (A,λ) assigns the underlying ring A. It admits the right adjoint functor $R\colon\mathscr{A}\to\mathscr{A}_\lambda$ defined by $R(A)=(\mathbb{W}(A),\Delta_A)$ with the counit and unit maps defined by $\lambda: (A,\lambda)\to(\mathbb{W}(A),\Delta_A)$ and $\varepsilon_A: \mathbb{W}(A)\to A$, respectively. The forgetful functor U also admits a left adjoint, but this will not be relevant for the discussion below. Since $\mathbb{W}(-)$ preserves limits, the forgetful functor U creates limits. Indeed, if $\{(A_i,\lambda_i)\}$ is a diagram of λ-rings and if $\{p_i\colon A\to A_i\}$ is a limit in $\mathscr{A}$ of the diagram $\{A_i\}$, then $\{\mathbb{W}(p_i)\colon\mathbb{W}(A)\to\mathbb{W}(A_i)\}$ is a limit in $\mathscr{A}$ of the diagram $\{\mathbb{W}(A_i)\}$. Therefore, we conclude that the pair (A,λ), where $\lambda: A\to\mathbb{W}(A)$ is defined to be the unique map with ith component $\lambda_i\circ p_i\colon A\to\mathbb{W}(A_i)$, is a λ-ring and that the family $\{p_i\colon (A,\lambda)\to(A_i,\lambda_i)\}$ is a limit in $\mathscr{A}_\lambda$ of the given diagram. It follows that that for (A,λ_A) a λ-ring, we obtain an adjunction

$$\mathscr{A}_\lambda/(A,\lambda_A) \underset{R_{(A,\lambda_A)}}{\overset{U_{(A,\lambda_A)}}{\rightleftarrows}} \mathscr{A}/A,$$

where the left adjoint functor $U_{(A,\lambda_A)}$ takes $f\colon (B,\lambda_B)\to(A,\lambda_A)$ to $f\colon B\to A$, the right adjoint functor $R_{(A,\lambda_A)}$ takes $f\colon B\to A$ to $p_2\colon (C,\lambda_C)\to(A,\lambda_A)$, for

$$\begin{array}{ccc} (C,\lambda_C) & \xrightarrow{p_1} & (\mathbb{W}(B),\Delta_B) \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \mathbb{W}(f)} \\ (A,\lambda_A) & \xrightarrow{\lambda_A} & (\mathbb{W}(A),\Delta_A) \end{array}$$

a choice of a pullback, the counit is $\varepsilon_B\circ p_1$, and the unit is the unique map with components λ_B and f. Since the functors $U_{(A,\lambda_A)}$ and $R_{(A,\lambda_A)}$ both preserve limits, this adjunction, in turn, induces an adjunction

$$(\mathscr{A}_\lambda/(A,\lambda_A))_{\mathrm{ab}} \underset{R_{(A,\lambda_A)}}{\overset{U_{(A,\lambda_A)}}{\rightleftarrows}} (\mathscr{A}/A)_{\mathrm{ab}}$$

between the associated categories of abelian group objects. Corresponding to this, we have the following adjunction

$$\mathscr{M}(A,\lambda_A) \underset{R'}{\overset{U'}{\rightleftarrows}} \mathscr{M}(A)$$

where the left adjoint functor U' takes (M,λ_M) to M, the right adjoint functor R' takes N to $(\lambda_{A*}(\mathbb{W}(N)),\Delta_N)$, and the counit and unit maps are defined to be the maps $\varepsilon_N\colon \lambda_{A*}(\mathbb{W}(N))\to N$ and $\lambda_M\colon (M,\lambda_M)\to(\lambda_{A*}(\mathbb{W}(M)),\Delta_M)$, respectively. Here we write $\lambda_{A*}(\mathbb{W}(N))$ for the $\mathbb{W}(A)$-module $\mathbb{W}(N)$ considered as an A-module via λ_A.

Proposition 2.10. *Let (A,λ_A) be a λ-ring. There exists, up to unique isomorphism, a unique adjunction $(F^\lambda,G^\lambda,\varepsilon^\lambda,\eta^\lambda)$ from $(\mathscr{A}_\lambda/(A,\lambda_A))_{\mathrm{ab}}$ to $\mathscr{M}(A,\lambda_A)$ such that, in the following diagram, the square of left adjoint functors commutes,*

$$\begin{array}{ccc} (\mathscr{A}/A)_{\mathrm{ab}} & \underset{G}{\overset{F}{\rightleftarrows}} & \mathscr{M}(A) \\ {\scriptstyle U_{(A,\lambda_A)}}\uparrow\downarrow{\scriptstyle R_{(A,\lambda_A)}} & & {\scriptstyle U'}\uparrow\downarrow{\scriptstyle R'} \\ (\mathscr{A}_\lambda/(A,\lambda_A))_{\mathrm{ab}} & \underset{G^\lambda}{\overset{F^\lambda}{\rightleftarrows}} & \mathscr{M}(A,\lambda_A). \end{array}$$

Moreover, the adjunction $(F^\lambda,G^\lambda,\varepsilon^\lambda,\eta^\lambda)$ is an adjoint equivalence of categories.

We remark that, by the uniqueness statement for adjoints, which we recalled at the beginning of the section, the commutativity of the square of left adjoint functors in the diagram in Proposition 2.10 implies that the corresponding square of right adjoint functors commutes, up to unique natural isomorphism.

Proof. If $(f,+,0,-)$ is an object of $(\mathscr{A}_\lambda/(A,\lambda_A))_{\mathrm{ab}}$ with underlying map of λ-rings $f\colon (B,\lambda_B)\to(A,\lambda_A)$, then we define $F^\lambda(f,+,-,0)$ to be the pair (M,λ_M) of the kernel $M=F(f)$ of f and the induced map $\lambda_M\colon M\to\mathbb{W}(M)$ of kernels of the vertical maps in the following diagram. We note that $U'\circ F^\lambda = F\circ U_{(A,\lambda_A)}$ as stated.

$$
\begin{array}{ccc}
B & \xrightarrow{\lambda_B} & \mathbb{W}(B) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathbb{W}(f)} \\
A & \xrightarrow{\lambda_A} & \mathbb{W}(A)
\end{array}
$$

Conversely, if (M,λ_M) is an (A,λ_A)-module, then we define $G^\lambda(M,\lambda_M)$ to be the abelian group object $G(M)$ in $\mathscr{A}/A$ with the underlying ring $B=A\ltimes M$ equipped with the λ-ring structure $\lambda_B\colon B\to\mathbb{W}(B)$ given by the composite map

$$
A\ltimes M \xrightarrow{\lambda_A\oplus\lambda_M} \mathbb{W}(A)\ltimes\mathbb{W}(M) \xrightarrow{\mathrm{in}_{1*}+\mathrm{in}_{2*}} \mathbb{W}(A\ltimes M);
$$

compare Lemma 2.2. To prove that $G^\lambda(M,\lambda_M)$ is well-defined, we must show (a) that (B,λ_B) is a λ-ring; (b) that the canonical projection $f\colon(B,\lambda_B)\to(A,\lambda_A)$ is a λ-ring homomorphism; and (c) that the abelian group object structure maps $+_B$, 0_B, and $-_B$ on $f\colon B\to A$ are λ-ring homomorphisms. First, the map $\lambda_A\oplus\lambda_M$ is a ring homomorphism, since λ_M is a λ_A-linear map. Moreover, Lemma 2.2 shows that also $\mathrm{in}_{1*}+\mathrm{in}_{2*}$ is a ring homomorphism, so λ_B is a ring homomorphism. To prove (a), it remains to show that the diagrams in Definition 1.21 commute. The left-hand diagram commutes, since $\varepsilon_A\circ\lambda_A=\mathrm{id}_A$ and $\varepsilon_M\circ\lambda_M=\mathrm{id}_M$ and since $\mathrm{in}_{1*}+\mathrm{in}_{2*}$ is the identity map on the first Witt component. To see that the right-hand square commutes, we consider the following larger diagram,

$$
\begin{array}{ccccc}
\mathbb{W}(\mathbb{W}(A\ltimes M)) & & \xleftarrow{\qquad\Delta_{A\ltimes M}\qquad} & & \mathbb{W}(A\ltimes M) \\
\uparrow{\scriptstyle \mathbb{W}(\mathrm{in}_{1*}+\mathrm{in}_{2*})} & & & & \uparrow{\scriptstyle \mathrm{in}_{1*}+\mathrm{in}_{2*}} \\
\mathbb{W}(\mathbb{W}(A)\ltimes\mathbb{W}(M)) & \xleftarrow{\mathrm{in}_{1*}+\mathrm{in}_{2*}} & \mathbb{W}(\mathbb{W}(A))\ltimes\mathbb{W}(\mathbb{W}(M)) & \xleftarrow{\Delta_A\oplus\Delta_M} & \mathbb{W}(A)\ltimes\mathbb{W}(M) \\
\uparrow{\scriptstyle \mathbb{W}(\lambda_A\oplus\lambda_M)} & & \uparrow{\scriptstyle \mathbb{W}(\lambda_A)\oplus\mathbb{W}(\lambda_M)} & & \uparrow{\scriptstyle \lambda_A\oplus\lambda_M} \\
\mathbb{W}(A\ltimes M) & \xleftarrow{\mathrm{in}_{1*}+\mathrm{in}_{2*}} & \mathbb{W}(A)\ltimes\mathbb{W}(M) & \xleftarrow{\lambda_A\oplus\lambda_M} & A\ltimes M.
\end{array}
$$

Here, the lower right-hand square commutes, since (A,λ_A) is a λ-ring and since (M,λ_M) is an (A,λ_A)-module, and the lower left-hand square commutes by the naturality of $\mathrm{in}_{1*}+\mathrm{in}_{2*}$. To prove the upper rectangular diagram commutes, it suffices to show that the two compositions with the nth ghost map

$$
\mathbb{W}(\mathbb{W}(A\ltimes M)) \xrightarrow{w_n} \mathbb{W}(A\ltimes M)
$$

agree. This, in turn, follows from the calculation

$$\begin{aligned}
&w_n \circ \Delta_{A\ltimes M} \circ (\mathrm{in}_{1*}+\mathrm{in}_{2*}) = F_n \circ (\mathrm{in}_{1*}+\mathrm{in}_{2*}) = (\mathrm{in}_{1*}+\mathrm{in}_{2*}) \circ (F_n \oplus F_n) \\
&= (\mathrm{in}_{1*}+\mathrm{in}_{2*}) \circ (w_n \oplus w_n) \circ (\Delta_A \oplus \Delta_M) \\
&= (\mathrm{in}_{1*}+\mathrm{in}_{2*}) \circ w_n \circ (\mathrm{in}_{1*}+\mathrm{in}_{2*}) \circ (\Delta_A \oplus \Delta_M) \\
&= w_n \circ \mathbb{W}(\mathrm{in}_{1*}+\mathrm{in}_{2*}) \circ (\mathrm{in}_{1*}+\mathrm{in}_{2*}) \circ (\Delta_A \oplus \Delta_M),
\end{aligned}$$

where the first and third equalities hold by the definition of Δ, where the second and fourth equalities hold by the naturality of $\mathrm{in}_{1*}+\mathrm{in}_{2*}$, and where the fifth equality equality holds by the naturality of w_n. This proves (a). Next, if $h\colon (M,\lambda_M) \to (N,\lambda_N)$ is a map of (A,λ_A)-modules, then the following diagram commutes,

$$\begin{array}{ccccc}
A \ltimes M & \xrightarrow{\lambda_A \oplus \lambda_M} & \mathbb{W}(A) \ltimes \mathbb{W}(M) & \xrightarrow{\mathrm{in}_{1*}+\mathrm{in}_{2*}} & \mathbb{W}(A \ltimes M) \\
\downarrow{\scriptstyle \mathrm{id}\oplus h} & & \downarrow{\scriptstyle \mathrm{id}\oplus h_*} & & \downarrow{\scriptstyle (\mathrm{id}\oplus h)_*} \\
A \ltimes N & \xrightarrow{\lambda_A \oplus \lambda_N} & \mathbb{W}(A) \ltimes \mathbb{W}(N) & \xrightarrow{\mathrm{in}_{1*}+\mathrm{in}_{2*}} & \mathbb{W}(A \ltimes N).
\end{array}$$

Taking (N,λ_N) to be the trivial (A,λ_A)-module, (b) follows. We use other instances of this diagram to prove (c). The maps 0_B and $-_B$ are induced by the (A,λ_A)-module maps $0_M\colon (0,\mathrm{id}) \to (M,\lambda_M)$ and $-_M\colon (M,\lambda_M) \to (M,\lambda_M)$ that map 0 to 0 and x to $-x$, respectively. Hence, the diagram shows that both are λ-ring homomorphisms. Finally, if we define $\lambda_{M\oplus M}$ to be the composite map

$$M \oplus M \xrightarrow{\lambda_M \oplus \lambda_M} \mathbb{W}(M) \oplus \mathbb{W}(M) \xrightarrow{\mathrm{in}_{1*}+\mathrm{in}_{2*}} \mathbb{W}(M \oplus M).$$

then $(M \oplus M, \lambda_{M\oplus M})$ is a direct sum of the (A,λ_A)-module (M,λ_M) with itself. Now, the addition map $+_B$ is given by the map

$$(A \ltimes (M \oplus M), \lambda_{A\ltimes(M\oplus M)}) \xrightarrow{\mathrm{id}\oplus +_M} (A \ltimes M, \lambda_{A\ltimes M}),$$

where $+_M\colon (M\oplus M, \lambda_{M\oplus M}) \to (M,\lambda_M)$ takes (x,y) to $x+y$. To complete the proof of (c), we must verify that $+_M$ is a map of (A,λ_A)-modules, that is, that the diagram

$$\begin{array}{ccccc}
M \oplus M & \xrightarrow{\lambda_M \oplus \lambda_M} & \mathbb{W}(M) \oplus \mathbb{W}(M) & \xrightarrow{\mathrm{in}_{1*}+\mathrm{in}_{2*}} & \mathbb{W}(M \oplus M) \\
\downarrow{\scriptstyle +_M} & & \downarrow{\scriptstyle +_{\mathbb{W}(M)}} & & \downarrow{\scriptstyle \mathbb{W}(+_M)} \\
M & \xrightarrow{\lambda_M} & \mathbb{W}(M) & = & \mathbb{W}(M)
\end{array}$$

commutes. But the left-hand square commutes, as λ_M is an additive map, and the right-hand square commutes, since the addition in $\mathbb{W}(M)$ is given by adding the Witt components of vectors, so (c) follows. This completes the proof that the functor G^λ is well-defined. We also note that, by construction, we have $U_{(A,\lambda_A)} \circ G^\lambda = G \circ U'$.

Finally, we claim that there are unique natural isomorphisms

$$G^\lambda \circ F^\lambda \overset{\varepsilon^\lambda}{\Longrightarrow} \mathrm{id} \qquad \mathrm{id} \overset{\eta^\lambda}{\Longrightarrow} F^\lambda \circ G^\lambda$$

such that $U_{(A,\lambda_A)}(\varepsilon^{\lambda}) = \varepsilon \circ U_{(A,\lambda_A)}$ and $U'(\eta^{\lambda}) = \eta \circ U'$. Indeed, this amounts to the following diagrams being commutative, where, in the bottom diagram, $i\colon M \to B$ is the (chosen) of kernel of $f\colon B \to A$;

$$\begin{array}{ccccc}
M & \xrightarrow{\lambda_M} & \mathbb{W}(M) & = & \mathbb{W}(M) \\
\downarrow{\scriptstyle \mathrm{in}_2} & & \downarrow{\scriptstyle \mathrm{in}_2} & & \downarrow{\scriptstyle \mathrm{in}_{2*}} \\
A \ltimes M & \xrightarrow{\lambda_A \oplus \lambda_M} & \mathbb{W}(A) \ltimes \mathbb{W}(M) & \xrightarrow{\mathrm{in}_{1*} + \mathrm{in}_{2*}} & \mathbb{W}(A \ltimes M)
\end{array}$$

$$\begin{array}{ccccc}
A \ltimes M & \xrightarrow{\lambda_A \oplus \lambda_M} & \mathbb{W}(A) \ltimes \mathbb{W}(M) & \xrightarrow{\mathrm{in}_{1*} + \mathrm{in}_{2*}} & \mathbb{W}(A \ltimes M) \\
\downarrow{\scriptstyle 0_B + i} & & \downarrow{\scriptstyle 0_{B*} + i_*} & & \downarrow{\scriptstyle (0_B + i)_*} \\
B & \xrightarrow{\lambda_B} & \mathbb{W}(B) & = & \mathbb{W}(B).
\end{array}$$

The left-hand squares in the two diagrams commute by naturality and the right-hand squares commute by the universal property of the direct sum. ☐

Remark 2.11. A map of λ-rings $f\colon (B,\lambda_B) \to (A,\lambda_A)$ gives rise to a functor

$$f_*\colon \mathscr{M}(A,\lambda_A) \to \mathscr{M}(B,\lambda_B)$$

defined by viewing an (A,λ_A)-module (N,λ_N) as a (B,λ_B)-module $f_*(N,\lambda_N)$ via the map f. The functor f_* has a left adjoint functor f^* that to a (B,λ_B)-module (M,λ_M) associates the (A,λ_A)-module $f^*(M,\lambda_M) = (A,\lambda_A) \otimes_{(B,\lambda_B)} (M,\lambda_M)$ defined by

$$(A,\lambda_A) \otimes_{(B,\lambda_B)} (M,\lambda_M) = (A \otimes_B M, \lambda_{A \otimes_B M})$$

where $\lambda_{A \otimes_B M}$ is given by the composition of $\lambda_A \otimes_{\lambda_B} \lambda_M$ and the map

$$\mathbb{W}(A) \otimes_{\mathbb{W}(B)} \mathbb{W}(M) \to \mathbb{W}(A \otimes_B M),$$

that to $a \otimes x$ associates the vector whose nth Witt component is $w_n(a) \otimes x_n$.

Definition 2.12. Let (A,λ_A) be a λ-ring and let (M,λ_M) be an (A,λ_A)-module. A derivation from (A,λ_M) to (M,λ_M) is a map of sets

$$D\colon (A,\lambda_A) \to (M,\lambda_M)$$

such that the following (1)–(3) hold.

(1) For all $a,b \in A$, $D(a+b) = D(a) + D(b)$.
(2) For all $a,b \in A$, $D(ab) = bD(a) + aD(b)$.
(3) For all $a \in A$ and $n \in \mathbb{N}$, $\lambda_{M,n}(D(a)) = \sum_{e|n} \lambda_{A,e}(a)^{(n/e)-1} D(\lambda_{A,e}(a))$.

The set of derivations from (A,λ_A) to (M,λ_M) is denoted by $\mathrm{Der}((A,\lambda_A),(M,\lambda_M))$.

We next show that, under the equivalence of categories given in Proposition 2.10, Definition 2.12 agrees with Beck's general definition of a derivation.

Proposition 2.13. *Let (A,λ_A) be a λ-ring, let (M,λ_M) be an (A,λ_A)-module, and let $f\colon (A\ltimes M,\lambda_{A\ltimes M}) \to (A,\lambda_A)$ be the canonical projection. The map*

$$\mathrm{Der}((A,\lambda_A),(M,\lambda_M)) \longrightarrow \mathrm{Hom}_{\mathscr{A}_\lambda/(A,\lambda_A)}(\mathrm{id}_{(A,\lambda_A)},f)$$

that to D assigns (id_A,D) is a bijection.

Proof. The map from $\mathrm{Der}(A,M)$ to $\mathrm{Hom}_{\mathscr{A}/A}(\mathrm{id}_A,f)$ that takes D to (id_A,D) is a bijection, as is well-known and readily verified. We must show that D satisfies (3) if and only if $(\mathrm{id}_A,D)\colon A\to A\ltimes M$ is a λ-ring homomorphism, and the latter means that the following diagram commutes.

$$\begin{array}{ccccc}
A & & \xrightarrow{\quad\lambda_A\quad} & & \mathbb{W}(A) \\
\big\downarrow{\scriptstyle(\mathrm{id}_A,D)} & & & & \big\downarrow{\scriptstyle(\mathrm{id}_A,D)_*} \\
A\ltimes M & \xrightarrow{\lambda_A\oplus\lambda_M} & \mathbb{W}(A)\ltimes\mathbb{W}(M) & \xrightarrow{\mathrm{in}_{1*}+\mathrm{in}_{2*}} & \mathbb{W}(A\ltimes M)
\end{array}$$

Now, on the one hand, the map (id_A,D) takes a to (a,Da) which by $\lambda_A\oplus\lambda_M$ is mapped to $(\lambda_A,\lambda_M(Da))$ whose nth Witt component is $(\lambda_{A,n}(a),\lambda_{M,n}(Da))$ and, on the other hand, the eth Witt component of the image of a by the composite map $(\mathrm{id}_A,D)_*\circ\lambda_A$ is equal to $(\lambda_{A,e}(a),D\lambda_{A,e}(a))$. Hence, Lemma 2.2 and Addendum 2.3 show that the diagram commutes if and only if D satisfies (3). ☐

Lemma 2.14. *Let (A,λ_A) be a λ-ring. There exists a derivation*

$$(A,\lambda_A) \xrightarrow{\ d\ } (\Omega_{(A,\lambda_A)},\lambda_{\Omega_{(A,\lambda_A)}})$$

which corepresents the functor that to an (A,λ_A)-module (M,λ_M) assigns the set of derivations $\mathrm{Der}((A,\lambda_A),(M,\lambda_M))$.

Proof. We define the target of the map d to be the quotient of the free (A,λ_A)-module (F,λ_F) generated by $\{d(a)\mid a\in A\}$ by the sub-(A,λ_A)-module $(R,\lambda_R)\subset(F,\lambda_F)$ generated by $d(a+b)-d(a)-d(b)$ with $a,b\in A$; by $d(ab)-bd(a)-ad(b)$ with $a,b\in A$; and by $\lambda_{F,n}(da)-\sum_{e\mid n}\lambda_{A,e}(a)^{(n/e)-1}d(\lambda_{A,e}(a))$ with $a\in A$ and $n\in\mathbb{N}$. The map d takes $a\in A$ to the class of $d(a)$ in $\Omega_{(A,\lambda_A)}$. It is clear from the construction that given a derivation $D\colon (A,\lambda_A)\to(M,\lambda_M)$, there is a well-defined map of (A,λ_A)-modules $f\colon (\Omega_{(A,\lambda_A)},\lambda_{\Omega_{(A,\lambda_A)}})\to(M,\lambda_M)$ such that $D=f\circ d$ and that f is unique with this property. This proves the lemma. ☐

The map $d\colon A\to\Omega_{(A,\lambda_A)}$ in Lemma 2.14, in particular, is a derivation of the ring A, and hence, it defines a map of A-modules $\Omega_A\to\Omega_{(A,\lambda_A)}$. We call this map the canonical map and now prove Theorem A, which states that it is an isomorphism.

Proof of Theorem A. We consider the diagram of adjunctions

$$\begin{array}{ccccc}
\mathscr{A}/A & \underset{i}{\overset{(-)_{\mathrm{ab}}}{\rightleftarrows}} & (\mathscr{A}/A)_{\mathrm{ab}} & \underset{G}{\overset{F}{\rightleftarrows}} & \mathscr{M}(A) \\
{\scriptstyle U_{(A,\lambda_A)}}\uparrow\downarrow{\scriptstyle R_{(A,\lambda_A)}} & & {\scriptstyle U_{(A,\lambda_A)}}\uparrow\downarrow{\scriptstyle R_{(A,\lambda_A)}} & & {\scriptstyle U'}\uparrow\downarrow{\scriptstyle R'} \\
\mathscr{A}_\lambda/(A,\lambda_A) & \underset{i^\lambda}{\overset{(-)^\lambda_{\mathrm{ab}}}{\rightleftarrows}} & (\mathscr{A}_\lambda/(A,\lambda_A))_{\mathrm{ab}} & \underset{G^\lambda}{\overset{F^\lambda}{\rightleftarrows}} & \mathscr{M}(A,\lambda_A),
\end{array}$$

where the functors i and i^λ forget the abelian group object structure maps, and where $(-)_{\mathrm{ab}}$ and $(-)^\lambda_{\mathrm{ab}}$ are the respective left adjoint functors which we now define. In the right-hand square, the top and bottom adjunctions are adjoint equivalences of categories by Lemma 2.1 and Proposition 2.10, respectively. Hence, the composition of the top adjunctions in the diagram determine the top adjunction in the left-hand square, up to unique natural isomorphism, and similarly for the bottom adjunctions.

Now, we define an adjunction (H,K,ε,η) with $K = i \circ G$ as follows. The functor K takes the A-module M to the canonical projection $f\colon A \ltimes M \to A$, and we let H be the functor that to $f\colon B \to A$ assigns the A-module $A \otimes_B \Omega_B$, and let ε and η be the natural transformations given by $\varepsilon(1 \otimes d(a,x)) = x$ and $\eta(b) = (f(b), 1 \otimes db)$, respectively. We must show that the two composite natural transformations

$$H \xRightarrow{H\circ\eta} H \circ K \circ H \xRightarrow{\varepsilon\circ H} H, \qquad K \xRightarrow{\eta\circ K} K \circ H \circ K \xRightarrow{K\circ\varepsilon} K$$

are equal to the respective identity natural transformations. But $H \circ \eta$ maps $a \otimes db$ in $H(f\colon B \to A)$ to $a \otimes d(f(b), 1 \otimes db)$ in $(H \circ K \circ H)(f\colon B \to A)$ and $\varepsilon \circ H$, in turn, maps this element to $a \cdot (1 \otimes db) = a \otimes db$ in $H(f\colon B \to A)$; and $\eta \circ K$ maps (a,x) in $K(M)$ to $(a, 1 \otimes d(a,x))$ in $(K \circ H \circ K)(M)$ and $K \circ \varepsilon$, in turn, maps this element to (a,x) in $K(M)$. This shows that (H,K,ε,η) is an adjunction. Similarly, we define an adjunction $(H^\lambda, K^\lambda, \varepsilon^\lambda, \eta^\lambda)$ with $K^\lambda = i^\lambda \circ G^\lambda$ as follows. The functor K^λ takes the (A,λ_A)-module (M,λ_A) to the canonical projection $f\colon (A \ltimes M, \lambda_{A \ltimes M}) \to (A,\lambda_A)$, and we let H^λ be the functor that to $f\colon (B,\lambda_B) \to (A,\lambda_A)$ assigns the (A,λ_A)-module $(A,\lambda_A) \otimes_{(B,\lambda_B)} \Omega_{(B,\lambda_B)}$, and let ε^λ and η^λ be the natural transformations given by $\varepsilon^\lambda(1 \otimes d(a,x)) = x$ and $\eta^\lambda(b) = (f(b), 1 \otimes db)$, respectively. The change-of-rings functor that we use here was defined in Remark 2.11. The proof that the triangle identities hold follows mutatis mutandis from the calculation in the case of the adjunction (H,K,ε,η). This shows that $(H^\lambda, K^\lambda, \varepsilon^\lambda, \eta^\lambda)$ is an adjunction.

Having established the diagram of adjunctions at the beginning of the proof, we note that the composite functors $R_{(A,\lambda_A)} \circ K$ and $K^\lambda \circ R'$ agree, up to unique natural isomorphism. Indeed, the following diagram is cartesian,

$$\begin{array}{ccccc}
A \ltimes \lambda_{A*}\mathbb{W}(M) & \xrightarrow{\lambda_A \oplus \mathrm{id}} & \mathbb{W}(A) \ltimes \mathbb{W}(M) & \xrightarrow{\mathrm{in}_{1*} + \mathrm{in}_{2*}} & \mathbb{W}(A \ltimes M) \\
\downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \mathrm{pr}_1} \\
A & \xrightarrow{\lambda_A} & \mathbb{W}(A) & = & \mathbb{W}(A).
\end{array}$$

By the uniqueness of left adjoint functors, up to unique natural isomorphism, we conclude that also the composite functors $H \circ U_{(A,\lambda_A)}$ and $U' \circ H^{\lambda}$ agree, up to unique natural isomorphism. It follows that the canonical natural transformation

$$A \otimes_B \Omega_B \longrightarrow U'((A,\lambda_A) \otimes_{(B,\lambda_B)} \Omega_{(B,\lambda_B)})$$

is an isomorphism, and taking $(B,\lambda_B) = (A,\lambda_A)$, the theorem follows. □

Theorem 2.15. *Let A be a ring. There are natural maps* $F_n \colon \Omega_{\mathbb{W}(A)} \to \Omega_{\mathbb{W}(A)}$ *that are* F_n*-linear and satisfy that for all* $a \in \mathbb{W}(A)$,

$$F_n(da) = \sum_{e|n} \Delta_{A,e}(a)^{(n/e)-1} d\Delta_{A,e}(a).$$

Moreover, the following (1)–(3) *hold.*

(1) *For all* $m,n \in \mathbb{N}$, $F_m F_n = F_{mn}$, *and* $F_1 = \mathrm{id}$.
(2) *For all* $n \in \mathbb{N}$ *and* $a \in \mathbb{W}(A)$, $dF_n(a) = nF_n(da)$.
(3) *For all* $n \in \mathbb{N}$ *and* $a \in A$, $F_n(d[a]) = [a]^{n-1}d[a]$.

Proof. Applying Theorem A to the universal λ-ring $(\mathbb{W}(A), \Delta_A)$, we conclude that the canonical map $\Omega_{\mathbb{W}(A)} \to \Omega_{(\mathbb{W}(A),\Delta_A)}$ is an isomorphism. Since the target of this map is a $(\mathbb{W}(A), \Delta_A)$-module, we have the natural map

$$F_n = \lambda_{\Omega_{(\mathbb{W}(A),\Delta_A)},n} \colon \Omega_{(\mathbb{W}(A),\Delta_A)} \to \Omega_{(\mathbb{W}(A),\Delta_A)}$$

defined to be the nth Witt component of the $(\mathbb{W}(A), \Delta_A)$-module structure map; compare Remark 2.6. It is an $F_n = w_n \circ \Delta_A$-linear map and Definition 2.12 (3) implies that it is given by the stated formula. Properties (1) and (2) follow immediately from the definition of a $(\mathbb{W}(A), \Delta_A)$-module and from the calculation

$$dF_n(a) = d(\sum_{e|n} e\Delta_{A,e}(a)^{n/e}) = \sum_{e|n} n\Delta_{A,e}(a)^{(n/e)-1} d\Delta_{A,e}(a) = nF_n(da),$$

where the first and last equality follow from the definition of Δ_A. Finally, to prove property (3), it suffices to show that $\Delta_{A,e}([a])$ is equal to $[a]$, if $e = 1$, and is equal to 0, if $e > 1$, or equivalently, that $\Delta([a]) = [[a]]$, and this was proved in Remark 1.20. □

3 The anticommutative graded algebras $\hat{\Omega}^{\cdot}_{\mathbb{W}(A)}$ and $\check{\Omega}^{\cdot}_{\mathbb{W}(A)}$

We next introduce the anticommutative graded $\mathbb{W}(A)$-algebra $\hat{\Omega}^{\cdot}_{\mathbb{W}(A)}$. It agrees with the alternating algebra $\Omega^{\cdot}_{\mathbb{W}(A)} = \bigwedge_{\mathbb{W}(A)} \Omega^1_{\mathbb{W}(A)}$, if the element

$$d\log[-1] = [-1]^{-1}d[-1] \in \Omega^1_{\mathbb{W}(A)}$$

is zero, but is different, in general. We note that $2d\log[-1] = d\log[1] = 0$ and that, by Lemma 1.5 (v) and Theorem 2.15 (3), $F_n(d\log[-1]) = d\log[-1]$ for all $n \in \mathbb{N}$.

Definition 3.1. Let A be a ring. The graded $\mathbb{W}(A)$-algebra

$$\hat{\Omega}^{\cdot}_{\mathbb{W}(A)} = T^{\cdot}_{\mathbb{W}(A)} \Omega^1_{\mathbb{W}(A)} / J^{\cdot}$$

is defined to be the quotient of the tensor algebra of the $\mathbb{W}(A)$-module $\Omega^1_{\mathbb{W}(A)}$ by the graded ideal generated by all elements of the form

$$da \otimes da - d\log[-1] \otimes F_2(da)$$

with $a \in \mathbb{W}(A)$.

We remark that the defining relation $da \cdot da = d\log[-1] \cdot F_2(da)$ is analogous to the relation $\{a,a\} = \{-1,a\}$ in Milnor K-theory.

Lemma 3.2. *The graded $\mathbb{W}(A)$-algebra $\hat{\Omega}^{\cdot}_{\mathbb{W}(A)}$ is anticommutative.*

Proof. It suffices to show that the sum $da \cdot db + db \cdot da \in \hat{\Omega}^2_{\mathbb{W}(A)}$ is equal to zero for all $a, b \in \mathbb{W}(A)$. Now, on the one hand, we have

$$d(a+b) \cdot d(a+b) = d\log[-1] \cdot F_2 d(a+b) = d\log[-1] \cdot F_2 da + d\log[-1] \cdot F_2 db,$$

since $F_2 d$ is additive, and on the other hand, we have

$$\begin{aligned} d(a+b) \cdot d(a+b) &= da \cdot da + da \cdot db + db \cdot da + db \cdot db \\ &= d\log[-1] \cdot F_2 da + da \cdot db + db \cdot da + d\log[-1] \cdot F_2 db \end{aligned}$$

This shows that $da \cdot db + db \cdot da$ is zero as desired. □

Proposition 3.3. *There exists a unique graded derivation*

$$d \colon \hat{\Omega}^{\cdot}_{\mathbb{W}(A)} \to \hat{\Omega}^{\cdot}_{\mathbb{W}(A)}$$

that extends the derivation $d \colon \mathbb{W}(A) \to \Omega^1_{\mathbb{W}(A)}$ and satisfies the formula

$$dd\omega = d\log[-1] \cdot d\omega$$

for all $\omega \in \hat{\Omega}^{\cdot}_{\mathbb{W}(A)}$. Moreover, the element $d\log[-1]$ is a cycle.

Proof. The relation $dd\omega = d\log[-1] \cdot d\omega$ implies that $d\log[-1]$ is a cycle for the desired derivation d. Indeed,

$$\begin{aligned} d(d\log[-1]) = d([-1]d[-1]) &= d[-1] \cdot d[-1] + [-1] \cdot dd[-1] \\ &= d\log[-1] \cdot F_2 d[-1] + [-1] d\log[-1] \cdot d[-1] \\ &= d\log[-1] \cdot [-1]d[-1] + d\log[-1] \cdot [-1]d[-1] \end{aligned}$$

which is zero by Lemma 3.2. This proves that the desired derivation d necessarily is unique in that for all $a_0, a_1, \dots, a_q \in \mathbb{W}(A)$, the following formula must hold,

$$d(a_0 da_1 \dots da_q) = da_0 da_1 \cdots da_q + q d\log[-1] \cdot a_0 da_1 \dots da_q.$$

Here $qd\log[-1]$ is equal to either $d\log[-1]$ or 0 as q is odd or even. To complete the proof, it remains to prove that the map d given by this formula is (a) well-defined, (b) a graded derivation, and (c) satisfies $dd\omega = d\log[-1]\cdot d\omega$. First, we have

$$\begin{aligned}
&d(a_0da_1\dots da_p\cdot b_0db_1\dots db_q) = d(a_0b_0da_1\dots da_pdb_1\dots db_q)\\
&= d(a_0b_0)da_1\dots da_pdb_1\dots db_q + (p+q)d\log[-1]\cdot a_0b_0da_1\dots da_pdb_1\dots db_q\\
&= da_0da_1\dots da_p\cdot b_0db_1\dots db_q + pd\log[-1]\cdot a_0da_1\dots da_p\cdot b_0db_1\dots db_q\\
&\quad + (-1)^p(a_0da_1\dots da_p\cdot db_0db_1\dots db_q + a_0da_1\dots da_p\cdot qd\log[-1]\cdot b_0db_1\dots db_q)\\
&= d(a_0da_1\dots da_p)\cdot b_0db_1\dots db_q + (-1)^p a_0da_1\dots da_p\cdot d(b_0db_1\dots db_q)
\end{aligned}$$

which proves (b). Next, using that q^2+q is always even, we find that

$$\begin{aligned}
&dd(a_0da_1\dots da_q) = d(da_0da_1\dots da_q + qd\log[-1]\cdot a_0da_1\dots da_q)\\
&= (q+1)d\log[-1]\cdot da_0da_1\dots da_q - qd\log[-1]\cdot da_0da_1\dots da_q\\
&\qquad - qd\log[-1]\cdot qd\log[-1]\cdot a_0da_1\dots da_q\\
&= d\log[-1]\cdot(da_0da_1\dots da_q + qd\log[-1]\cdot a_0da_1\dots da_q)\\
&= d\log[-1]\cdot d(a_0da_1\dots da_q)
\end{aligned}$$

which proves (c). Finally, to prove (a), we must show that for all $a,b\in\mathbb{W}(A)$, the elements $d(d(ab)-bda-adb)$ and $d(dada-d\log[-1]\cdot F_2da)$ of $\hat{\Omega}^{\cdot}_{\mathbb{W}(A)}$ are zero. First, using Lemma 3.2 together with (b) and (c), we find that

$$\begin{aligned}
&d(d(ab)-bda-adb) = dd(ab) - dbda - bdda - dadb - addb\\
&= d\log[-1]\cdot d(ab) - d\log[-1]\cdot bda - d\log[-1]\cdot adb\\
&= d\log[-1]\cdot(d(ab)-bda-adb)
\end{aligned}$$

which is zero, since $d\colon\mathbb{W}(A)\to\hat{\Omega}^1_{\mathbb{W}(A)}$ is a derivation. This shows that the first type of elements are zero. Next, (b) and (c) show that

$$d(dada) = 2d\log[-1]\cdot dada$$

which is zero as is

$$\begin{aligned}
d(d\log[-1]\cdot F_2da) &= d\log[-1]\cdot dF_2da = d\log[-1]\cdot d(ada + d\Delta_2(a))\\
&= d\log[-1]\cdot(dada + d\log[-1]\cdot F_2da)
\end{aligned}$$

by the definition of $\hat{\Omega}^2_{\mathbb{W}(A)}$. Hence also the second type of elements are zero. This completes the proof of (a) and hence of the proposition. □

Remark 3.4. In general, there is no $\mathbb{W}(A)$-algebra map $f\colon\hat{\Omega}^{\cdot}_{\mathbb{W}(A)}\to\Omega^{\cdot}_{\mathbb{W}(A)}$ that is compatible with the derivations.

Proposition 3.5. *Let A be a ring and let n be a positive integer. There is a unique homomorphism of graded rings*

$$F_n\colon\hat{\Omega}^{\cdot}_{\mathbb{W}(A)}\to\hat{\Omega}^{\cdot}_{\mathbb{W}(A)}$$

that is given by the maps $F_n \colon \mathbb{W}(A) \to \mathbb{W}(A)$ *and* $F_n \colon \Omega^1_{\mathbb{W}(A)} \to \Omega^1_{\mathbb{W}(A)}$ *in degrees* 0 *and* 1*, respectively. In addition, the following formula holds.*

$$dF_n = nF_n d$$

Proof. The uniqueness statement is clear: The map F_n is necessarily given by

$$F_n(a_0 da_1 \dots da_q) = F_n(a_0)F_n(da_1)\dots F_n(da_q)$$

where $a_0, \dots, a_q \in \mathbb{W}(A)$. We show that this formula gives a well-defined map. To prove this, we must show that for every $a \in \mathbb{W}(A)$,

$$F_n(da)F_n(da) = F_n(d\log[-1])F_n(F_2 da).$$

It will suffice to let $n = p$ be a prime number. In this case, we find that

$$\begin{aligned}
&F_p(da)F_p(da) = (a^{p-1}da + d\Delta_p(a)) \cdot (a^{p-1}da + d\Delta_p(a)) \\
&= (a^{p-1})^2 da \cdot da + d\Delta_p(a) \cdot d\Delta_p(a) = d\log[-1] \cdot ((a^{p-1})^2 F_2 da + F_2 d\Delta_p(a)) \\
&= d\log[-1] \cdot (F_2(a^{p-1})F_2 da + F_2 d\Delta_p(a)) = d\log[-1] \cdot F_2 F_p da \\
&= F_p(d\log[-1] \cdot F_2 da)
\end{aligned}$$

where we have used that $F_2(a)$ is congruent to a^2 modulo $2\mathbb{W}(A)$. This shows that the map F_n is well-defined. It is a graded ring homomorphism by definition.

We next prove the formula $dF_n = nF_n d$. Again, we may assume that $n = p$ is a prime number. We already know from the definition of $F_n \colon \Omega^1_{\mathbb{W}(A)} \to \Omega^1_{\mathbb{W}(A)}$ that for all $a \in \mathbb{W}(A)$, $dF_p(a) = pF_p(a)$. Now, for all $a \in \mathbb{W}(A)$,

$$dF_p(da) = d(a^{p-1}da + d\Delta_p(a)) = (p-1)a^{p-2}dada + d\log[-1] \cdot F_p da$$

which is equal to zero for $p = 2$, and equal to $d\log[-1] \cdot F_p da$ for p odd. Hence, for every prime p and every $a \in \mathbb{W}(A)$, we have

$$dF_p(da) = pd\log[-1] \cdot F_p da = pF_p(d\log[-1] \cdot da) = pF_p d(da)$$

as desired. Now, let $a_0, \dots, a_q \in \mathbb{W}(A)$. We find

$$\begin{aligned}
&dF_p(a_0 da_1 \dots da_q) = d(F_p(a_0)F_p da_1 \dots F_p da_q) \\
&= dF_p(a_0)F_p da_1 \dots F_p da_q + \sum_{1 \leqslant i \leqslant q} (-1)^{i-1} F_p(a_0)F_p da_1 \dots dF_p da_i \dots F_p da_q \\
&= pF_p d(a_0)F_p da_1 \dots F_p da_q + \sum_{1 \leqslant i \leqslant q} (-1)^{i-1} F_p(a_0)F_p da_1 \dots pF_p d(da_i) \dots F_p da_q \\
&= pF_p d(a_0 da_1 \dots da_q)
\end{aligned}$$

as stated. This completes the proof. □

We next define the quotient graded algebra $\check{\Omega}^{\cdot}_{\mathbb{W}(A)}$ of the graded algebra $\hat{\Omega}^{\cdot}_{\mathbb{W}(A)}$ and show that the Frobenius F_n and derivation d descend to this quotient.

Definition 3.6. Let A be a ring. The graded $\mathbb{W}(A)$-algebra

$$\check{\Omega}^{\cdot}_{\mathbb{W}(A)} = \hat{\Omega}^{\cdot}_{\mathbb{W}(A)}/K^{\cdot}$$

is defined to be the quotient by the graded ideal $K^{\cdot}$ generated by the elements

$$F_p dV_p(a) - da - (p-1)d\log[-1]\cdot a$$

where p ranges over all prime numbers and a over all elements of $\mathbb{W}(A)$.

We remark that the element $F_p dV_p(a) - da - (p-1)d\log[-1]\cdot a$ is annihilated by p. In particular, it is zero, if p is invertible in A, and hence, in $\mathbb{W}(A)$.

Lemma 3.7. *The Frobenius $F_n\colon \hat{\Omega}^{\cdot}_{\mathbb{W}(A)} \to \hat{\Omega}^{\cdot}_{\mathbb{W}(A)}$ induces a map of graded algebras*

$$F_n\colon \check{\Omega}^{\cdot}_{\mathbb{W}(A)} \to \check{\Omega}^{\cdot}_{\mathbb{W}(A)}.$$

Proof. It will suffice to let $n = \ell$ be a prime number and show that for all prime numbers p and all $a \in \mathbb{W}(A)$, the element

$$F_\ell(F_p dV_p(a) - da - (p-1)d\log[-1]\cdot a) \in \Omega^1_{\mathbb{W}(A)}$$

maps to zero in $\check{\Omega}^1_{\mathbb{W}(A)}$. To this end, we will repeatedly use that for every prime number p and every $b \in \mathbb{W}(A)$, Theorem 2.15 and Remark 1.20 shows that

$$F_p db = b^{p-1}db + d\Delta_p(b) = b^{p-1}db + d(\frac{F_p(b) - b^p}{p})$$

as elements of $\Omega^1_{\mathbb{W}(A)}$. We also use that, by Lemma 1.5 (ii)–(iii), we have

$$V_m(a)^n = m^{n-1}V_m(a^n),$$

for all $m,n \in \mathbb{N}$ and $a \in \mathbb{W}(A)$. Now, suppose first that $\ell = p$. For p odd, we find that

$$\begin{aligned}
&F_p(F_p dV_p(a) - da) = F_p(V_p(a)^{p-1}dV_p(a) + d(\frac{F_pV_p(a) - V_p(a)^p}{p}) - da) \\
&= F_p(p^{p-2}V_p(a^{p-1})dV_p(a) - p^{p-2}dV_p(a^p)) \\
&= p^{p-1}a^{p-1}F_p dV_p(a) - p^{p-2}F_p dV_p(a^p),
\end{aligned}$$

where we also use that $F_pV_p = p\cdot \mathrm{id}$. But this element maps to zero $\check{\Omega}^1_{\mathbb{W}(A)}$, since, as maps from $\mathbb{W}(A)$ to $\check{\Omega}^1_{\mathbb{W}(A)}$, we have $F_p dV_p = d$, and since the common map is a derivation. Similarly, for $p = 2$, we find that

$$\begin{aligned}
&F_2(F_2dV_2(a) - da - d\log[-1]\cdot a) = F_2(V_2(a)dV_2(a) - dV_2(a^2) - d\log[-1]\cdot a) \\
&= 2aF_2dV_2(a) - F_2dV_2(a^2) - d\log[-1]\cdot F_2(a),
\end{aligned}$$

where we further use that $F_m(d\log[-1]) = d\log[-1]$, for every $m \in \mathbb{N}$. The image of this element in $\check{\Omega}^1_{\mathbb{W}(A)}$ is equal to

$$2ada - d(a^2) - d\log[-1]\cdot(F_2(a) - a^2),$$

which, in turn, is zero, since d is a derivation and since $F_2(a) - a^2$ is divisible by 2 by Lemma 1.18. We next suppose that $p \neq \ell$. In this case, we further use that ℓ divides $p^{\ell-1} - 1$ and that, by Lemma 1.5 (ii), $F_\ell V_p = V_p F_\ell$. If p and ℓ both are odd, then

$$\begin{aligned} &F_\ell(F_p dV_p(a) - da) = F_p(F_\ell dV_p(a)) - F_\ell da \\ &= F_p(V_p(a)^{\ell-1} dV_p(a) + d(\frac{F_\ell V_p(a) - V_p(a)^\ell}{\ell})) - a^{\ell-1} da - d(\frac{F_\ell(a) - a^\ell}{\ell}) \\ &= p^{\ell-1} a^{\ell-1} F_p dV_p(a) + F_p dV_p(\frac{F_\ell(a) - p^{\ell-1} a^\ell}{\ell}) - a^{\ell-1} da - d(\frac{F_\ell(a) - a^\ell}{\ell}), \end{aligned}$$

and the image of this element in $\check{\Omega}^1_{\mathbb{W}(A)}$ is equal to

$$(p^{\ell-1} - 1) a^{\ell-1} da - \frac{p^{\ell-1} - 1}{\ell} d(a^\ell),$$

which is zero since d is a derivation. Similarly, if $p = 2$ and $\ell \neq p$, then

$$\begin{aligned} &F_\ell(F_2 dV_2(a) - da - d\log[-1] \cdot a) = F_2(F_\ell dV_2(a)) - F_\ell da - d\log[-1] \cdot F_\ell(a) \\ &= F_2(V_2(a)^{\ell-1} dV_2(a) + d(\frac{F_\ell V_2(a) - V_2(a)^\ell}{\ell})) \\ &\qquad - a^{\ell-1} da - d(\frac{F_\ell(a) - a^\ell}{\ell}) - d\log[-1] \cdot F_\ell(a) \\ &= 2^{\ell-1} a^{\ell-1} F_2 dV_2(a) + F_2 dV_2(\frac{F_\ell(a) - 2^{\ell-1} a^\ell}{\ell}) \\ &\qquad - a^{\ell-1} da - d(\frac{F_\ell(a) - a^\ell}{\ell}) - d\log[-1] \cdot F_\ell(a), \end{aligned}$$

and the image of this element in $\check{\Omega}^1_{\mathbb{W}(A)}$ is equal to

$$(2^{\ell-1} - 1) a^{\ell-1} da - \frac{2^{\ell-1} - 1}{\ell} d(a^\ell) + d\log[-1] \cdot (\frac{F_\ell(a) - 2^{\ell-1} a^\ell}{\ell} - F_\ell(a)),$$

which is zero since d is a derivation and since ℓ is congruent to 1 modulo 2. Finally, if $\ell = 2$ and $p \neq \ell$, then we find that

$$\begin{aligned} &F_2(F_p dV_p(a) - da) = F_p(F_2 dV_p(a)) - F_2 da \\ &= F_p(V_p(a) dV_p(a) + d(\frac{F_2 V_p(a) - V_p(a)^2}{2})) - a da - d(\frac{F_2(a) - a^2}{2}) \\ &= p a F_p dV_p(a) + F_p dV_p(\frac{F_2(a) - p a^2}{2}) - a da - d(\frac{F_2(a) - a^2}{2}) \end{aligned}$$

whose image in $\check{\Omega}^1_{\mathbb{W}(A)}$ is equal to

$$(p-1) a da - \frac{p-1}{2} d(a^2),$$

which again is zero since d is a derivation. □

Lemma 3.8. *For all positive integers n and $a \in \mathbb{W}(A)$, the relation*

$$F_n dV_n(a) = da + (n-1)d\log[-1] \cdot a$$

holds in $\check{\Omega}^1_{\mathbb{W}(A)}$.

Proof. We argue by induction on the number r of prime factors in n that the stated relation holds for all $a \in \mathbb{W}(A)$. The case $r = 1$ follows from Definition 3.6. So we let n be a positive integer with $r > 1$ prime factors and assume that the relation has been proved for all positive integers with less than r prime factors. We write $n = pm$ with p a prime number and use the inductive hypothesis to conclude that

$$\begin{aligned} F_n dV_n(a) &= F_p F_m dV_m V_p(a) = F_p(dV_p(a) + (m-1)d\log[-1] \cdot V_p(a)) \\ &= F_p dV_p(a) + (m-1)d\log[-1] \cdot F_p V_p(a) \\ &= da + (p-1)d\log[-1] \cdot a + p(m-1)d\log[-1] \cdot a \\ &= da + (n-1)d\log[-1] \cdot a \end{aligned}$$

which proves the induction step. ☐

Lemma 3.9. *The graded derivation $d\colon \hat{\Omega}^{\cdot}_{\mathbb{W}(A)} \to \hat{\Omega}^{\cdot}_{\mathbb{W}(A)}$ induces a graded derivation*

$$d\colon \check{\Omega}^{\cdot}_{\mathbb{W}(A)} \to \check{\Omega}^{\cdot}_{\mathbb{W}(A)}.$$

Proof. We must show that for all prime numbers p and $a \in \mathbb{W}(A)$, the element

$$d(F_p dV_p(a) - da - (p-1)d\log[-1] \cdot a) \in \hat{\Omega}^2_{\mathbb{W}(A)}$$

maps to zero in $\check{\Omega}^2_{\mathbb{W}(A)}$. First, for $p = 2$, we have

$$\begin{aligned} &d(F_2 dV_2(a) - da - d\log[-1] \cdot a) = dF_2 dV_2(a) - dda + d\log[-1] \cdot da \\ &= 2F_2 ddV_2(a) = 2d\log[-1] \cdot F_2 dV_2(a) \end{aligned}$$

which is even zero in $\hat{\Omega}^2_{\mathbb{W}(A)}$. For p odd, we recall from Theorem 2.15 that

$$F_p dV_p(a) - da = V_p(a)^{p-1} dV_p(a) + d\Delta_p V_p(a) - da,$$

and using that d is a derivation, we find that

$$\begin{aligned} &d(F_p dV_p(a) - da) = d(V_p(a)^{p-1} dV_p(a) + d\Delta_p V_p(a)) - dda \\ &= (p-1)V_p(a)^{p-2} dV_p(a) dV_p(a) + V_p(a)^{p-1} ddV_p(a) + dd\Delta_p V_p(a) - dda. \end{aligned}$$

Now, the first summand in the bottom line vanishes, since $p-1$ is even and

$$dV_p(a) dV_p(a) = d\log[-1] \cdot F_2 dV_p(a),$$

and by Proposition 3.3, the sum of the remaining three summands is equal to

$$d\log[-1] \cdot (V_p(a)^{p-1} dV_p(a) + d\Delta_p V_p(a) - da) = d\log[-1] \cdot (F_p dV_p(a) - da),$$

which maps to zero in $\check{\Omega}^2_{\mathbb{W}(A)}$, since $p-1$ is even. ☐

Definition 3.10. Let A be a ring, let $S \subset \mathbb{N}$ be a truncation set, and let $I_S(A) \subset \mathbb{W}(A)$ be the kernel of the restriction map $R_S^{\mathbb{N}}\colon \mathbb{W}(A) \to \mathbb{W}_S(A)$. The maps

$$\hat{\Omega}^{\cdot}_{\mathbb{W}(A)} \xrightarrow{R_S^{\mathbb{N}}} \hat{\Omega}^{\cdot}_{\mathbb{W}_S(A)}, \qquad \check{\Omega}^{\cdot}_{\mathbb{W}(A)} \xrightarrow{R_S^{\mathbb{N}}} \check{\Omega}^{\cdot}_{\mathbb{W}_S(A)}$$

are defined to be the quotient maps that annihilate the respective graded ideals generated by $I_S(A)$ and $dI_S(A)$.

Remark 3.11. The kernel $K_S^{\cdot}$ of the canonical projection $\hat{\Omega}^{\cdot}_{\mathbb{W}_S(A)} \to \check{\Omega}^{\cdot}_{\mathbb{W}_S(A)}$ is equal to the graded ideal generated by the elements

$$p^{p-2}(V_p(R^S_{S/p}(a^{p-1}))dV_pR^S_{S/p}(a) - dV_pR^S_{S/p}(a^p)) - (p-1)d\log[-1]_S \cdot a$$

with p a prime number and $a \in \mathbb{W}_S(A)$. Indeed, letting $b = V_p(a)$, the formula for F_pdb in the beginning of the proof of Lemma 3.7 shows that for all prime numbers p and $a \in \mathbb{W}(A)$, the following identity holds in $\Omega^1_{\mathbb{W}(A)}$,

$$F_pdV_p(a) - da = p^{p-2}(V_p(a^{p-1})dV_p(a) - dV_p(a^p)).$$

Remark 3.12. If p is a prime number and A a $\mathbb{Z}_{(p)}$-algebra, then for every truncation set S, the ideal $V_p\mathbb{W}_{S/p}(A) \subset \mathbb{W}_S(A)$ has a divided power structure defined by

$$V_p(a)^{[n]} = \frac{p^{n-1}}{n!}V_p(a^n).$$

If p is *odd*, then $d\colon \mathbb{W}_S(A) \to \check{\Omega}^1_{\mathbb{W}_S(A)}$ is a divided power derivation in the sense that

$$d(V_p(a)^{[n]}) = V_p(a)^{[n-1]}dV_p(a)$$

and it is universal with this property; see [22, Lemma 1.2].

Lemma 3.13. *The derivation, restriction, and Frobenius induce maps*

$$\begin{aligned}
d\colon \hat{\Omega}^{\cdot}_{\mathbb{W}_S(A)} &\to \hat{\Omega}^{\cdot}_{\mathbb{W}_S(A)} && (\textit{resp. } d\colon \check{\Omega}^{\cdot}_{\mathbb{W}_S(A)} \to \check{\Omega}^{\cdot}_{\mathbb{W}_S(A)})\\
R^S_T\colon \hat{\Omega}^{\cdot}_{\mathbb{W}_S(A)} &\to \hat{\Omega}^{\cdot}_{\mathbb{W}_T(A)} && (\textit{resp. } R^S_T\colon \check{\Omega}^{\cdot}_{\mathbb{W}_S(A)} \to \check{\Omega}^{\cdot}_{\mathbb{W}_T(A)})\\
F_n\colon \hat{\Omega}^{\cdot}_{\mathbb{W}_S(A)} &\to \hat{\Omega}^{\cdot}_{\mathbb{W}_{S/n}(A)} && (\textit{resp. } F_n\colon \check{\Omega}^{\cdot}_{\mathbb{W}_S(A)} \to \check{\Omega}^{\cdot}_{\mathbb{W}_{S/n}(A)})
\end{aligned}$$

Moreover, the maps d are graded derivations; the maps R^S_T and F_n are graded ring homomorphisms; the maps R^S_T and d commute; and $dF_n = nF_nd$.

Proof. To prove the statement for d, we note that as d is a derivation, it suffices to show that $R_S^{\mathbb{N}}(ddI_S(A)) \subset \hat{\Omega}^2_{\mathbb{W}_S(A)}$ is zero. But if $x \in I_S(A)$, then

$$R_S^{\mathbb{N}}(ddx) = R_S^{\mathbb{N}}(d\log[-1] \cdot dx) = R_S^{\mathbb{N}}(d\log[-1]) \cdot R_S^{\mathbb{N}}(dx)$$

which is zero as desired. It follows that $R_T^{\mathbb{N}}(dI_S(A)) = dR_T^{\mathbb{N}}(I_S(A)$. Hence, also the statement for R^S_T follows as $R_T^{\mathbb{N}}(I_S(A)$ is trivially zero. Finally, to prove the statement for F_n, we show that both $R^{\mathbb{N}}_{S/n}(F_n(I_S(A)))$ and $R^{\mathbb{N}}_{S/n}(F_n(dI_S(A)))$ are zero. For the former, this follows immediately from Lemma 1.4, and for the latter, it will suffice to show that for divisors e of n, $\Delta_e(I_S(A)) \subset I_{S/e}(A)$. Moreover, to prove this, we may assume that A is torsion free. So let e be a divisor of n and assume that for all proper divisors d of e, $\Delta_d(I_S(A)) \subset I_{S/d}(A)$. Since $F_e(I_S(A)) \subset I_{S/e}(A)$, the formula

$$F_e(a) = \sum_{d|e} d\Delta_d(a)^{e/d}$$

shows that $e\Delta_e(I_S(A)) \subset I_{S/e}(A)$. This completes the proof of the first part of the statement and the second part is clear. □

Finally, we record the following result concerning the case $S = \{1\}$.

Lemma 3.14. *For every ring A, the differential graded algebras* $\hat{\Omega}^{\cdot}_A$ *and* $\Omega^{\cdot}_A$ *are equal and the canonical projection* $\hat{\Omega}^{\cdot}_A \to \check{\Omega}^{\cdot}_A$ *is an isomorphism.*

Proof. Since $d\log[-1]_{\{1\}}$ is zero, $\hat{\Omega}^{\cdot}_A = \Omega^{\cdot}_A$ as stated. Moreover, Remark 3.11 shows that the kernel $K_{\{1\}}$ of the canonical projection $\hat{\Omega}^{\cdot}_A \to \check{\Omega}^{\cdot}_A$ is zero. □

4 The big de Rham-Witt complex

In this section, we construct the big de Rham-Witt complex. We let J be the category with objects the truncation sets $S \subset \mathbb{N}$ and with a single morphism from T to S if $T \subset S$. If A is a ring, then there is a contravariant functor from J to the category of rings that to S assigns $\mathbb{W}_S(A)$ and that to $T \subset S$ assigns $R^S_T \colon \mathbb{W}_S(A) \to \mathbb{W}_T(A)$; it takes colimits in J to limits in the category of rings. For every $n \in \mathbb{N}$, there is an endofunctor on J that takes S to S/n, and the ring homomorphism $F_n \colon \mathbb{W}_S(A) \to \mathbb{W}_{S/n}(A)$ and the abelian group homomorphism $V_n \colon \mathbb{W}_{S/n}(A) \to \mathbb{W}_S(A)$ are natural transformations with respect to S.

We proceed to define the notion of a Witt complex over A. The original definition given in [15, Definition 1.1.1] is not quite correct unless the prime 2 is either invertible or zero in A. The correct definition of a 2-typical Witt complex was given first by Costeanu [9, Definition 1.1]. The definition given below was also inspired by [25].

Definition 4.1. A *Witt complex* over A is a contravariant functor that to every truncation set $S \subset \mathbb{N}$ assigns an anticommutative graded ring $E^{\cdot}_S$ and that takes colimits to limits together with a natural ring homomorphism

$$\mathbb{W}_S(A) \xrightarrow{\eta_S} E^0_S$$

and natural maps of graded abelian groups

$$E^q_S \xrightarrow{d} E^{q+1}_S \qquad E^q_S \xrightarrow{F_n} E^q_{S/n} \qquad E^q_{S/n} \xrightarrow{V_n} E^q_S \qquad (n \in \mathbb{N})$$

such that the following (i)–(v) hold.

(i) For all $x \in E_S^q$ and $x' \in E_S^{q'}$,

$$d(x \cdot x') = d(x) \cdot x' + (-1)^q x \cdot d(x'),$$
$$d(d(x)) = d\log \eta_S([-1]_S) \cdot d(x),$$

where $d\log \eta_S([-1]_S) = \eta_S([-1])^{-1} d\eta_S([-1]_S)$.

(ii) For all positive integers m and n,

$$F_1 = V_1 = \mathrm{id}, \quad F_m F_n = F_{mn}, \quad V_n V_m = V_{mn},$$
$$F_n V_n = n \cdot \mathrm{id}, \quad F_m V_n = V_n F_m \quad \text{if } (m,n) = 1,$$
$$F_n \eta_S = \eta_{S/n} F_n, \qquad \eta_S V_n = V_n \eta_{S/n}.$$

(iii) For all positive integers n, the map F_n is a ring homomorphism and the maps F_n and V_n satisfy the projection formula that for all $x \in E_S^q$ and $y \in E_{S/n}^{q''}$,

$$x \cdot V_n(y) = V_n(F_n(x)y).$$

(iv) For all positive integers n and all $y \in E_{S/n}^q$,

$$F_n d V_n(y) = d(y) + (n-1) d\log \eta_{S/n}([-1]_{S/n}) \cdot y.$$

(v) For all positive integers n and $a \in A$,

$$F_n d\eta_S([a]_S) = \eta_{S/n}([a]_{S/n}^{n-1}) d\eta_{S/n}([a]_{S/n}).$$

A map of Witt complexes is a natural map of graded rings $f \colon E_S^{\cdot} \to E_S'^{\cdot}$ such that $f\eta = \eta'$, $fd = d'f$, $fF_n = F_n'f$, and $fV_n = V_n'f$.

Remark 4.2. (a) For $T \subset S$ a pair of truncation sets, we write $R_T^S \colon E_S^{\cdot} \to E_T^{\cdot}$ for the map of graded rings that is part of the structure of a Witt complex and call it the restriction from S to T.

(b) Every Witt complex is determined, up to canonical isomorphism, by its value on finite truncation sets. Indeed, for every truncation set S and non-negative integer q, the maps in (a) defined a bijection from E_S^q to the limit with respect to the restriction maps of the E_T^q, where $T \subset S$ ranges over the finite sub-truncation sets. In particular, if we write $a \in \mathbb{W}(A)$ as a convergent sum $a = \sum_{n \in S} V_n([a_n]_{S/n})$ as in Lemma 1.5 (i), then the element $d\eta_S(a) \in E_S^1$, too, admits the convergent sum expression

$$d\eta_S(a) = \sum_{n \in S} dV_n([a_n]_{S/n}).$$

(c) The element $d\log \eta_S([-1]_S)$ is annihilated by 2. Indeed, since d is a derivation,

$$2d\log \eta_S([-1]_S) = d\log \eta_S([1]_S) = 0.$$

Therefore, $d\log \eta_S([-1]_S)$ is zero if 2 is invertible in A and hence in $\mathbb{W}_S(A)$. It is also zero if $2 = 0$ in A since, in this case, $[-1]_S = [1]_S$. Finally, it follows from the general formula $[-1]_S = -[1]_S + V_2([1]_{S/2})$ proved in Addendum 1.7 that $d\log \eta_S([-1]_S)$ is zero if every $n \in S$ is odd.

(d) Let A be a ring and let $E_S^\cdot$ be a Witt complex over A. For every non-negative integer q, the pair $(E_{\mathbb{N}}^q, \lambda_{E^q})$ consisting of $E_{\mathbb{N}}^q$ considered as a $\mathbb{W}(A)$-module via the ring homomorphism $\eta_{\mathbb{N}} \colon \mathbb{W}(A) \to E_{\mathbb{N}}^0$ and of the maps $\lambda_{E^q,n} = F_n \colon E_{\mathbb{N}}^q \to E_{\mathbb{N}}^q$ is a module over the λ-ring $(\mathbb{W}(A), \Delta_A)$ in the sense of Definition 2.5. Moreover, we may substitute the axiom (v) in Definition 4.1 by the statement (v') that the map

$$(\mathbb{W}(A), \Delta_A) \xrightarrow{\ d\ } (E_{\mathbb{N}}^1, \lambda_{E^1})$$

is a derivation in the sense of Definition 2.12. Indeed, it follows from Theorem 2.15 that (i)–(iv) and (v') imply (v), and we will show in Proposition 4.4 below that (i)–(v) imply (v').

Lemma 4.3. *Let m and n be positive integers, let $c = (m,n)$ be the greatest common divisor, and let i and j be any pair of integers such that $mi + nj = c$. The following relations hold in every Witt complex.*

$$dF_n = nF_n d, \quad V_n d = ndV_n,$$
$$F_m dV_n = idF_{m/c}V_{n/c} + jF_{m/c}V_{n/c}d + (c-1)d\log\eta_{S/m}([-1]_{S/m}) \cdot F_{m/c}V_{n/c},$$
$$d\log\eta_S([-1]_S) = \textstyle\sum_{r\geqslant 1} 2^{r-1} dV_{2^r}\eta_{S/2^r}([1]_{S/2^r}),$$
$$d\log\eta_S([-1]_S) \cdot d\log\eta_S([-1]_S) = 0, \quad dd\log\eta_S([-1]_S) = 0,$$
$$F_n(d\log\eta_S([-1]_S)) = d\log\eta_{S/n}([-1]_{S/n}),$$

Proof. The following calculation verifies the first two relations.

$$\begin{aligned} dF_n(x) &= F_n dV_n F_n(x) - (n-1)d\log\eta[-1] \cdot F_n(x) \\ &= F_n d(V_n\eta([1]) \cdot x) - (n-1)d\log\eta([-1]) \cdot F_n(x) \\ &= F_n(dV_n\eta([1]) \cdot x + V_n\eta([1]) \cdot dx) - (n-1)d\log\eta([-1]) \cdot F_n(x) \\ &= F_n dV_n\eta([1]) \cdot F_n(x) + F_n V_n\eta([1]) \cdot F_n d(x) - (n-1)d\log\eta([-1]) \cdot F_n(x) \\ &= (n-1)d\log\eta([-1]) \cdot F_n(x) + nF_n d(x) - (n-1)d\log\eta([-1]) \cdot F_n(x) \\ &= nF_n d(x) \end{aligned}$$

$$\begin{aligned} V_n d(x) &= V_n(F_n dV_n(x) - (n-1)d\log\eta([-1]) \cdot x) \\ &= V_n\eta([1]) \cdot dV_n(x) - (n-1)V_n(d\log\eta([-1]) \cdot x) \\ &= d(V_n\eta([1]) \cdot V_n(x)) - dV_n\eta([1]) \cdot V_n(x) - (n-1)V_n(d\log\eta([-1]) \cdot x) \\ &= dV_n(F_n V_n\eta([1]) \cdot x) - V_n(F_n dV_n\eta([1]) \cdot x) - (n-1)V_n(d\log\eta([-1]) \cdot x) \\ &= ndV_n(x) - 2(n-1)V_n(d\log\eta([-1]) \cdot x) \\ &= ndV_n(x) \end{aligned}$$

Next, the last formula follows from $F_m([-1]) = [-1]^m$ and the calculation

$$\begin{aligned} F_m(d\log\eta([-1]_S)) &= F_m(\eta([-1]^{-1})d\eta([-1])) = F_m\eta([-1]^{-1})F_m d\eta([-1]) \\ &= \eta([-1]^{-m})\eta([-1]^{m-1})d\eta([-1]) \\ &= \eta([-1]^{-1})d\eta([-1]) = d\log\eta([-1]). \end{aligned}$$

Using the three relations proved thus far together with the projection formula, we find

$$\begin{aligned}
F_m dV_n(x) &= F_{m/c}F_c dV_c V_{n/c}(x) \\
&= F_{m/c}dV_{n/c}(x) + (c-1)d\log\eta([-1]) \cdot F_{m/c}V_{n/c}(x) \\
&= ((m/c)i + (n/c)j)F_{m/c}dV_{n/c}(x) + (c-1)d\log\eta([-1]) \cdot F_{m/c}V_{n/c}(x) \\
&= idF_{m/c}V_{n/c}(x) + jF_{m/c}V_{n/c}d(x) + (c-1)d\log\eta([-1]) \cdot F_{m/c}V_{n/c}(x).
\end{aligned}$$

Next, to prove the stated formula for $d\log\eta([-1]_S)$, we use that, by Addendum 1.7, we have $[-1]_S = -[1]_S + V_2([1]_{S/2})$ to see that

$$\begin{aligned}
d\log\eta([-1]_S) &= \eta([-1]_S)d\eta([-1]_S) \\
&= \eta(-[1]_S + V_2([1]_{S/2})d\eta(-[1]_S + V_2([1]_{S/2}) \\
&= -dV_2\eta([1]_{S/2}) + V_2(F_2 dV_2\eta([1]_{S/2})) \\
&= -dV_2\eta([1]_{S/2}) + V_2(d\log\eta([-1]_{S/2})) \\
&= dV_2\eta([1]_{S/2}) + V_2(d\log\eta([-1]_{S/2})),
\end{aligned}$$

from which the stated formula follows by easy induction. Here, in the last equality, we have used that $2dV_2\eta([1]_{S/2}) = V_2 d\eta([1]_{S/2}) = 0$. Using this, we find that

$$\begin{aligned}
dV_2(d\log\eta([-1]_{S/2})) &= \sum_{r\geqslant 1} 2^r ddV_{2^{r+1}}\eta([1]_{S/2^{r+1}}) \\
&= \sum_{r\geqslant 1} 2^r d\log\eta([-1]_S) \cdot dV_{2^{r+1}}\eta([1]_{S/2^{r+1}})
\end{aligned}$$

which is zero, since $2d\log\eta([-1]_S) = 0$. Now, using Addendum 1.7, we find that

$$\begin{aligned}
(d\log\eta([-1]_S))^2 &= (d\eta([-1]_S))^2 = (dV_2\eta([1]_{S/2}))^2 \\
&= d(V_2\eta([1]_{S/2}) \cdot dV_2\eta([1]_{S/2})) - V_2\eta([1]_{S/2}) \cdot ddV_2\eta([1]_{S/2}) \\
&= dV_2(d\log\eta([-1]_{S/2})) - V_2\eta([1]_{S/2}) \cdot dV_2 d\log\eta([-1]_{S/2}) \\
&= dV_2(d\log\eta([-1]_{S/2})) \cdot \eta([1]_S - V_2([1]_{S/2})),
\end{aligned}$$

which is zero, since the first factor in the bottom line is zero, by what was just proved. This, in turn, shows that $(d\eta([-1]_S))^2 = 0$, from which we find that

$$\begin{aligned}
dd\log\eta([-1]_S) &= d(\eta([-1]_S) \cdot d\eta([-1]_S)) \\
&= d\eta([-1]_S) \cdot d\eta([-1]_S) + \eta([-1]_S) \cdot dd\eta([-1]_S) \\
&= \eta([-1]_S)dd\eta([-1]_S) \\
&= \eta([-1]_S)d\log\eta([-1]_S)d\eta([-1]_S) \\
&= d\eta([-1]_S)d\eta([-1]_S) = 0.
\end{aligned}$$

This completes the proof. □

Proposition 4.4. *For every Witt complex $E_S^{\cdot}$ over A and every $m \in \mathbb{N}$, the diagram*

$$\begin{array}{ccc} \Omega^1_{\mathbb{W}_S(A)} & \xrightarrow{\eta_S} & E^1_S \\ \downarrow F_m & & \downarrow F_m \\ \Omega^1_{\mathbb{W}_{S/m}(A)} & \xrightarrow{\eta_{S/m}} & E^1_{S/m} \end{array}$$

commutes. Here, the horizontal maps take $a_0 da_1$ to $\eta(a_0)d\eta(a_1)$.

Proof. Since the restriction map $R^{\mathbb{N}}_S \colon \Omega^1_{\mathbb{W}(A)} \to \Omega^1_{\mathbb{W}_S(A)}$ is surjective and satisfies that both $F_m R^{\mathbb{N}}_S = R^{\mathbb{N}}_{S/m} F_m$ and $\eta_S R^{\mathbb{N}}_S = R^{\mathbb{N}}_S \eta_{\mathbb{N}}$, we may assume that $S = \mathbb{N}$. In addition, by Remark 4.2 (b), it will suffice to show that for every $n \in \mathbb{N}$ and $a \in A$,

$$F_p dV_n \eta_{\mathbb{N}}([a]_{\mathbb{N}}) = \eta_{\mathbb{N}} F_p dV_n([a]_{\mathbb{N}})$$

as elements of $E^1_{\mathbb{N}}$. To ease notation, we suppress the subscript $\mathbb{N}$. We first suppose that p does not divide n and set $k = (1 - n^{p-1})/p$ and $l = n^{p-2}$ such that, in particular, $kp + ln = 1$. By Lemma 4.3, we have

$$\begin{aligned} F_p dV_n \eta([a]) &= k \cdot dV_n F_p \eta([a]) + l \cdot V_n F_p d\eta([a]) \\ &= k \cdot dV_n \eta([a]^p) + l \cdot V_n(\eta([a])^{p-1} d\eta([a])) \\ &= k \cdot dV_n \eta([a]^p) + l \cdot V_n \eta([a]^{p-1} d[a]) \end{aligned}$$

Moreover, arguing as in the proof of Lemma 3.7 above, we find that

$$\begin{aligned} \eta F_p dV_n([a]) &= \eta(V_n([a])^{p-1} \cdot dV_n([a]) + d\Delta_p V_n([a])) \\ &= \eta(l \cdot V_n([a]^{p-1}) \cdot dV_n([a]) + k \cdot dV_n([a]^p)) \\ &= l \cdot V_n \eta([a]^{p-1}) \cdot dV_n \eta([a]) + k \cdot dV_n \eta([a]^p) \\ &= l \cdot V_n(\eta([a]^{p-1}) \cdot F_n dV_n \eta([a])) + k \cdot dV_n \eta([a]^p) \\ &= l \cdot V_n \eta([a]^{p-1} d[a]) + k \cdot dV_n \eta([a]^p) \end{aligned}$$

where the last equality uses that $n^{p-2}(n-1) d\log \eta([-1])$ is zero. This proves that the desired equality holds if p does not divide n. Suppose next that p divides n and write $n = pr$. We consider the cases $p = 2$ and p odd separately. First, for p odd,

$$\begin{aligned} F_p dV_n \eta([a]) &= dV_r \eta([a]) \\ \eta F_p dV_n([a]) &= \eta(V_n([a])^{p-1} \cdot dV_n([a]) + d(\frac{F_p V_n([a]) - V_n([a])^p}{p})) \\ &= \eta(V_n([a])^{p-1} \cdot dV_n([a]) + dV_r([a]) - p^{p-2} r^{p-1} dV_n([a]^p)) \\ &= V_n(\eta([a]))^{p-1} \cdot dV_n \eta([a]) + dV_r \eta([a]) - p^{p-2} r^{p-1} dV_n(\eta([a])^p), \end{aligned}$$

and the following calculation shows that the first and third terms cancel,

$$\begin{aligned} p^{p-2}r^{p-1}dV_n(\eta([a])^p) &= p^{p-3}r^{p-2}V_nd(\eta([a])^p) \\ &= p^{p-2}r^{p-2}V_n(\eta([a])^{p-1}d\eta([a])) \\ &= n^{p-2}V_n(\eta([a])^{p-1})\cdot dV_n\eta([a]) \\ &= V_n(\eta([a]))^{p-1}\cdot dV_n\eta([a]). \end{aligned}$$

Here, the third equality follows from Definition 4.1 (iii)–(iv) and from the fact that $n^{p-2}(n-1)d\log[-1]$ vanishes. Finally, if $p=2$, then

$$\begin{aligned} F_2dV_n\eta([a]) &= dV_r\eta([a]) + d\log\eta([-1])\cdot V_r\eta([a]) \\ \eta F_2dV_n([a]) &= \eta(V_n([a])\cdot dV_n([a]) + d(\frac{F_2V_n([a])-V_n([a])^2}{2})) \\ &= \eta(V_n([a])\cdot dV_n([a]) + dV_r([a]) - rdV_n([a]^2)) \\ &= V_n\eta([a])\cdot dV_n\eta([a]) + dV_r\eta([a]) - rdV_n(\eta([a])^2), \end{aligned}$$

and hence, we must show that

$$d\log\eta([-1])\cdot V_r\eta([a]) = V_n\eta([a])\cdot dV_n\eta([a]) - rdV_n(\eta([a])^2).$$

Suppose first that $r=1$. By Addendum 1.7, $[-1] = -[1] + V_2([1])$, so that

$$d\log\eta([-1]) = V_2(\eta([1]))\cdot dV_2\eta([1]) - dV_2\eta([1]),$$

and hence, using the Witt complex axioms, we find

$$\begin{aligned} &d\log\eta([-1])\cdot\eta([a]) = V_2(\eta([a])^2)\cdot dV_2\eta([1]) - \eta([a])\cdot dV_2\eta([1]) \\ &= V_2(\eta([a])^2\cdot F_2dV_2\eta([1])) - d(\eta([a])\cdot V_2\eta([1])) + d(\eta([a]))\cdot V_2\eta([1]) \\ &= V_2(\eta([a])^2\cdot d\log\eta([-1])) - dV_2(\eta([a])^2) + V_2(\eta([a])d\eta([a])) \\ &= V_2(\eta([a]))\cdot dV_2\eta([a]) - dV_2(\eta([a])^2) \end{aligned}$$

as desired. In general, we apply V_r to the formula that we just proved. This gives

$$\begin{aligned} &d\log\eta([-1])\cdot V_r(\eta([a])) = V_r(V_2(\eta([a]))\cdot dV_2\eta([a])) - rdV_n(\eta([a])^2) \\ &= V_n(\eta([a])\cdot F_2dV_2\eta([a])) - rdV_n(\eta([a])^2) \\ &= V_n(\eta([a])\cdot F_ndV_n\eta([a])) - rdV_n(\eta([a])^2) \\ &= V_n\eta([a])\cdot dV_n\eta([a]) - rdV_n(\eta([a])^2), \end{aligned}$$

where we have $F_2dV_2 = F_ndV_n$, since n is even. This completes the proof. □

Corollary 4.5. *Let $E_S^\cdot$ be a Witt complex over the ring A. There is a unique natural homomorphism of graded rings*

$$\check{\Omega}^\cdot_{\mathbb{W}_S(A)} \xrightarrow{\eta_S} E_S^\cdot$$

that extends the natural ring homomorphism $\eta_S\colon \mathbb{W}_S(A) \to E^0_S$ and commutes with the derivations. In addition, for every positive integer m, the diagram

$$\begin{array}{ccc} \check{\Omega}^{\cdot}_{\mathbb{W}_S(A)} & \xrightarrow{\eta_S} & E^{\cdot}_S \\ \downarrow{\scriptstyle F_m} & & \downarrow{\scriptstyle F_m} \\ \check{\Omega}^{\cdot}_{\mathbb{W}_{S/m}(A)} & \xrightarrow{\eta_{S/m}} & E^{\cdot}_{S/m} \end{array}$$

commutes.

Proof. The map η_S necessarily is given by

$$\eta_S(a_0 da_1 \dots da_q) = \eta_S(a_0) d\eta_S(a_1) \dots d\eta_S(a_q).$$

We show that this formula gives a well-defined map. First, from Proposition 4.4, we find that for all $a \in \mathbb{W}(A)$,

$$F_2 d\eta_{\mathbb{N}}(a) = \eta_{\mathbb{N}} F_2 d(a) = \eta_{\mathbb{N}}(ada + d\Delta_2(a)) = \eta_{\mathbb{N}}(a) d\eta_{\mathbb{N}}(a) + d\eta_{\mathbb{N}}\Delta_2(a).$$

Applying d to this equation, the left-hand side becomes

$$dF_2 d\eta_{\mathbb{N}}(a) = 2F_2 dd\eta_{\mathbb{N}}(a) = 0$$

while the right-hand side becomes

$$\begin{aligned} & d\eta_{\mathbb{N}}(a) d\eta_{\mathbb{N}}(a) + d\log \eta_{\mathbb{N}}([-1])_{\mathbb{N}} \cdot (\eta_{\mathbb{N}}(a) d\eta_{\mathbb{N}}(a) + d\eta_{\mathbb{N}}\Delta_2(a)) \\ & = d\eta_{\mathbb{N}}(a) d\eta_{\mathbb{N}}(a) + d\log \eta_{\mathbb{N}}([-1])_{\mathbb{N}} \cdot F_2 d\eta_{\mathbb{N}}(a). \end{aligned}$$

Hence, there is a well-defined map of graded rings $\eta_S\colon \hat{\Omega}^{\cdot}_{\mathbb{W}_S(A)} \to E^{\cdot}_S$ given by the formula stated at the beginning of the proof, and by axiom (iv) in Definition 4.1, this map factors through the canonical projection from $\hat{\Omega}^{\cdot}_{\mathbb{W}_S(A)}$ onto $\check{\Omega}^{\cdot}_{\mathbb{W}_S(A)}$. Finally, Proposition 4.4 shows that the diagram in the statement commutes. □

Proof of Theorem B. We recall that, in the diagrams in the statement, the left-hand vertical maps were defined in Lemma 3.13. We define maps of graded rings

$$\check{\Omega}^{\cdot}_{\mathbb{W}_S(A)} \xrightarrow{\eta_S} \mathbb{W}_S\Omega^{\cdot}_A$$

as quotients by graded ideals $N^{\cdot}_S$ and verify that, in the diagrams in the statement, the right-hand vertical maps R^S_T, F_n, and d making the respective diagrams commute exist. We further define maps of graded abelian groups

$$\mathbb{W}_{S/n}\Omega^{\cdot}_A \xrightarrow{V_n} \mathbb{W}_S\Omega^{\cdot}_A$$

that make the following diagrams commute,

$$\begin{array}{ccc} \mathbb{W}_{S/n}(A) & \xrightarrow{\eta_{S/n}} & \mathbb{W}_{S/n}\Omega_A^0 \\ \downarrow{\scriptstyle V_n} & & \downarrow{\scriptstyle V_n} \\ \mathbb{W}_S(A) & \xrightarrow{\eta_S} & \mathbb{W}_S\Omega_A^0 \end{array} \qquad \begin{array}{ccc} \mathbb{W}_{S/n}\Omega_A^{\cdot} & \xrightarrow{V_n} & \mathbb{W}_S\Omega_A^{\cdot} \\ \downarrow{\scriptstyle R_{T/n}^{S/n}} & & \downarrow{\scriptstyle R_T^S} \\ \mathbb{W}_{T/n}\Omega_A^{\cdot} & \xrightarrow{V_n} & \mathbb{W}_T\Omega_A^{\cdot} \end{array}$$

$$\begin{array}{ccccc} \mathbb{W}_{S/n}\Omega_A^{\cdot} \otimes \mathbb{W}_{S/n}\Omega_A^{\cdot} & \xleftarrow{\mathrm{id}\otimes F_n} & \mathbb{W}_{S/n}\Omega_A^{\cdot} \otimes \mathbb{W}_S\Omega_A^{\cdot} & \xrightarrow{V_n\otimes \mathrm{id}} & \mathbb{W}_S\Omega_A^{\cdot} \otimes \mathbb{W}_S\Omega_A^{\cdot} \\ \downarrow{\scriptstyle \mu} & & & & \downarrow{\scriptstyle \mu} \\ \mathbb{W}_{S/n}\Omega_A^{\cdot} & & \xrightarrow{\qquad\qquad V_n \qquad\qquad} & & \mathbb{W}_S\Omega_A^{\cdot}. \end{array}$$

The definition of these maps, as S ranges over all finite truncation sets, $T \subset S$ over all sub-truncation sets, and n over all positive integers, will be by induction on the cardinality of S and will take up most of the proof. Once this is completed, we will show that the combined structure is a Witt complex over A and that it is initial among Witt complexes over A.

We define $\mathbb{W}_\emptyset\Omega_A^{\cdot}$ to be the terminal graded ring, which is zero in all degrees, and define $\eta_\emptyset$ to be the unique map of graded rings. So let S be a finite non-empty truncation set and assume, inductively, that the maps η_T, R_U^T, F_n, d, and V_n have been defined, for all proper truncation sets $T \subset S$, all truncation sets $U \subset T$, and all positive integers n, with the properties listed at the beginning of the proof. In this situation, we define $\eta_S \colon \check{\Omega}_{\mathbb{W}_S(A)}^{\cdot} \to \mathbb{W}_S\Omega_A^{\cdot}$ to be the quotient map that annihilates the graded ideal $N_S^{\cdot}$ generated by all sums

$$\sum_\alpha V_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha}, \qquad d(\sum_\alpha V_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha}),$$

where the Witt vectors $x_\alpha \in \mathbb{W}_{S/n}(A)$ and $y_{1,\alpha},\dots,y_{q,\alpha} \in \mathbb{W}_S(A)$ and the integers $n \geqslant 2$ and $q \geqslant 1$ are such that the sum

$$\eta_{S/n}(\sum_\alpha x_\alpha F_n dy_{1,\alpha}\dots F_n dy_{q,\alpha})$$

in $\mathbb{W}_{S/n}\Omega_A^q$ is zero; and for every positive integer n, we define the map of graded abelian groups $V_n \colon \mathbb{W}_{S/n}\Omega_A^{\cdot} \to \mathbb{W}_S\Omega_A^{\cdot}$ by

$$V_n\eta_{S/n}(xF_ndy_1\dots F_ndy_q) = \eta_S(V_n(x)dy_1\dots dy_q).$$

Here we use that every element of $\mathbb{W}_{S/n}\Omega_A^q$ can be written as a sum of elements of the form $\eta_{S/n}(xF_ndy_1\dots F_ndy_q)$ with $x \in \mathbb{W}_{S/n}(A)$ and $y_1,\dots,y_q \in \mathbb{W}_S(A)$. Indeed,

$$dx = F_ndV_n(x) - (n-1)d\log[-1]_{S/n}\cdot x = F_ndV_n(x) - (n-1)xF_nd([-1]_S).$$

To prove the existence of the necessarily unique right-hand vertical maps R_T^S, d, and F_n making the diagrams in the statement of the theorem commute, we must show that the left-hand vertical maps in these diagrams satisfy $\eta_T(R_T^S(N_S^q)) = 0$, $\eta_S(d(N_S^q)) = 0$, and $\eta_{S/m}(F_m(N_S^q)) = 0$, respectively, and to this end, we use the properties of the latter maps established in Lemma 3.13. So we fix a positive integer n and an element

$$\omega = \sum_\alpha V_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha} \in \check{\Omega}^q_{\mathbb{W}_S(A)}$$

with

$$\eta_{S/n}(\sum_\alpha x_\alpha F_n dy_{1,\alpha}\dots F_n dy_{q,\alpha}) \in \mathbb{W}_{S/n}\Omega^q_A$$

equal to zero and show that $\eta_T R^S_T(\omega)$, $\eta_S(dd\omega)$, $\eta_{S/m}F_m(\omega)$, and $\eta_{S/m}F_m(d\omega)$ all are zero. First, in order to show that

$$\eta_T R^S_T(\omega) = \eta_T(\sum_\alpha V_n R^{S/n}_{T/n}(x_\alpha)dR^S_T(y_{1,\alpha})\dots dR^S_T(y_{q,\alpha}))$$

is zero, it suffices by the definition of the ideal $N^{\cdot}_T$ to show that

$$\eta_{T/n}(\sum_\alpha R^{S/n}_{T/n}(x_\alpha)F_n dR^S_T(y_{1,\alpha})\dots F_n dR^S_T(y_{q,\alpha}))$$

is zero. But this element is equal to

$$\eta_{T/n}R^{S/n}_{T/n}(\sum_\alpha x_\alpha F_n dy_{1,\alpha}\dots F_n dy_{q,\alpha})$$

which, by the inductive hypothesis, is equal to

$$R^{S/n}_{T/n}\eta_{S/n}(\sum_\alpha x_\alpha F_n dy_{1,\alpha}\dots F_n dy_{q,\alpha})$$

which we assumed to be zero. Similarly, we have

$$\begin{aligned}\eta_S(dd\omega) &= \eta_S(d\log([-1]_S)\cdot d\omega) = \eta_S(d(d\log([-1]_S)\cdot\omega))\\ &= \eta_S(d(\sum_\alpha V_n(x_\alpha[-1]^{-n}_{S/n})d([-1]_S)dy_{1,\alpha}\dots dy_{q,\alpha})),\end{aligned}$$

and by the definition of $N^{\cdot}_S$, this element is zero, since

$$\begin{aligned}&\eta_{S/n}(\sum_\alpha x_\alpha[-1]^{-n}_{S/n}F_n d([-1]_S)F_n dy_{1,\alpha}\dots F_n dy_{q,\alpha})\\ &= d\log\eta_{S/n}([-1]_{S/n})\cdot\eta_{S/n}(\sum_\alpha x_\alpha F_n dy_{1,\alpha}\dots F_n dy_{q,\alpha})\end{aligned}$$

is zero. Next, to prove that $\eta_{S/m}F_m(\omega)$ and $\eta_{S/m}F_m(d\omega)$ are zero, we may assume that $m = p$ is a prime number. Indeed, if $m = kp$, then, by the inductive hypothesis, we have $\eta_{S/m}F_m = \eta_{S/m}F_kF_p = F_k\eta_{S/p}F_p$. Suppose first that $n = lp$ is divisible by p. We remark that we have

$$\begin{aligned}&\eta_{S/p}(\sum_\alpha V_l(x_\alpha)F_p dy_{1,\alpha}\dots F_p dy_{q,\alpha})\\ &= \sum_\alpha V_l(\eta_{S/n}(x_\alpha))\eta_{S/p}(F_p dy_{1,\alpha}\dots F_p dy_{q,\alpha})\\ &= \sum_\alpha V_l(\eta_{S/n}(x_\alpha)F_l(\eta_{S/p}(F_p dy_{1,\alpha}\dots F_p dy_{q,\alpha})))\\ &= V_l\eta_{S/n}(\sum_\alpha x_\alpha F_n dy_{1,\alpha}\dots F_n dy_{q,\alpha})\end{aligned} \tag{4.6}$$

which is zero. Indeed, the three equalities hold since, by the inductive hypothesis, the maps $V_l\colon \mathbb{W}_{S/n}\Omega^q_A \to \mathbb{W}_{S/p}\Omega^q_A$ and $F_l\colon \mathbb{W}_{S/p}\Omega^q_A \to \mathbb{W}_{S/n}\Omega^q_A$ exist and have the properties listed at the beginning of the proof. Now,

$$\begin{aligned}\eta_{S/p}F_p(\omega) &= \eta_{S/p}(\sum_\alpha F_pV_n(x_\alpha)F_pdy_{1,\alpha}\dots F_pdy_{q,\alpha})\\ &= p\eta_{S/p}(\sum_\alpha V_l(x_\alpha)F_pdy_{1,\alpha}\dots F_pdy_{q,\alpha})\end{aligned}$$

which is zero by (4.6). Similarly, using Proposition 3.3, we have

$$\eta_{S/p}F_p(d\omega) = \eta_{S/p}F_p(\sum_\alpha dV_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha}) + \varepsilon\cdot\eta_{S/p}F_p(\omega)$$

with $\varepsilon = qd\log\eta_{S/p}([-1]_{S/p})$, and we have just proved that $\eta_{S/p}F_p(\omega)$ is zero. By Lemma 3.8, we may rewrite the first summand as the sum

$$\eta_{S/p}(\sum_\alpha dV_l(x_\alpha)F_pdy_{1,\alpha}\dots F_pdy_{q,\alpha}) + \varepsilon\cdot\eta_{S/p}(\sum_\alpha V_l(x_\alpha)F_pdy_{1,\alpha}\dots F_pdy_{q,\alpha})$$

with $\varepsilon = (p-1)d\log\eta_{S/p}([-1]_{S/p})$. Here, the second term is zero by (4.6), and we rewrite the first summand as

$$\begin{aligned}&\eta_{S/p}(d(\sum_\alpha V_l(x_\alpha)F_pdy_{1,\alpha}\dots F_pdy_{q,\alpha}) - \sum_\alpha V_l(x_\alpha)d(F_pdy_{1,\alpha}\dots F_pdy_{q,\alpha}))\\ &= \eta_{S/p}d(\sum_\alpha V_l(x_\alpha)F_pdy_{1,\alpha}\dots F_pdy_{q,\alpha}) - \varepsilon\cdot\eta_{S/p}(\sum_\alpha V_l(x_\alpha)F_pdy_{1,\alpha}\dots F_pdy_{q,\alpha})\\ &= d\eta_{S/p}(\sum_\alpha V_l(x_\alpha)F_pdy_{1,\alpha}\dots F_pdy_{q,\alpha}) - \varepsilon\cdot\eta_{S/p}(\sum_\alpha V_l(x_\alpha)F_pdy_{1,\alpha}\dots F_pdy_{q,\alpha})\end{aligned}$$

with $\varepsilon = pqd\log\eta_{S/p}([-1]_{S/p})$. Here, the last equality uses that $\eta_{S/p}$, by definition, commutes with d. It follows from (4.6) that both summands in the last line vanish, so $\eta_{S/p}F_p(d\omega) = 0$ as desired. Next, suppose that p does not divide n. We have

$$\begin{aligned}\eta_{S/p}F_p(\omega) &= \sum_\alpha \eta_{S/p}V_nF_p(x_\alpha)\cdot\eta_{S/p}F_p(dy_{1,\alpha}\dots dy_{q,\alpha})\\ &= \sum_\alpha V_nF_p\eta_{S/n}(x_\alpha)\cdot\eta_{S/p}F_p(dy_{1,\alpha}\dots dy_{q,\alpha})\\ &= \sum_\alpha V_n(F_p\eta_{S/n}(x_\alpha)\cdot F_n\eta_{S/p}F_p(dy_{1,\alpha}\dots dy_{q,\alpha}))\\ &= \sum_\alpha V_n(F_p\eta_{S/n}(x_\alpha)\cdot\eta_{S/np}F_{np}(dy_{1,\alpha}\dots dy_{q,\alpha}))\\ &= \sum_\alpha V_n(F_p\eta_{S/n}(x_\alpha)\cdot F_p\eta_{S/n}F_n(dy_{1,\alpha}\dots dy_{q,\alpha}))\\ &= V_nF_p\eta_{S/n}(\sum_\alpha x_\alpha F_ndy_{1,\alpha}\dots F_ndy_{q,\alpha})\end{aligned}$$

which is zero. Here the second, third, and forth equalities use that, by the inductive hypothesis, the maps $F_n\colon \mathbb{W}_{S/p}\Omega^q_A \to \mathbb{W}_{S/np}\Omega^q_A$ and $V_n\colon \mathbb{W}_{S/np}\Omega^q_A \to \mathbb{W}_{S/p}\Omega^q_A$ exist and have the properties listed at the beginning of the proof, and similarly, the fifth equality uses that the map $F_p\colon \mathbb{W}_{S/n}\Omega^q_A \to \mathbb{W}_{S/np}\Omega^q_A$ with $F_p\eta_{S/n} = \eta_{S/np}F_p$ exists.

We proceed to show that also $\eta_{S/p}F_p(d\omega)$ vanishes, and to this end, it suffices to show that both $p\eta_{S/p}F_P(d\omega)$ and $n\eta_{S/p}F_p(d\omega)$ vanish. First,

$$p\eta_{S/p}F_pd(\omega) = \eta_{S/p}dF_p(\omega) = d\eta_{S/p}F_p(\omega),$$

which is zero by what was just proved. Here the two equalities hold by Proposition 3.5 and by the definition of $\eta_{S/p}$, respectively. Next,

$$\begin{aligned} n\eta_{S/p}F_pd(\omega) &= \sum_\alpha n\eta_{S/p}F_pd(V_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha}) \\ &= \sum_\alpha n\eta_{S/p}F_p(dV_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha}) + \varepsilon\cdot\eta_{S/p}F_p(\omega) \end{aligned}$$

with $\varepsilon = nqd\log\eta_{S/p}([-1])_{S/p}$, and we have already proved that $\eta_{S/p}F_p(\omega)$ is zero. Moreover, we may rewrite the first term in the lower line as

$$\begin{aligned} &\sum_\alpha \eta_{S/p}F_p(dV_n([1]_{S/n})V_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha}) \\ &+\sum_\alpha \eta_{S/p}F_p(V_n([1]_{S/n})dV_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha}), \end{aligned}$$

since, by Lemma 1.5 and by d being a derivation,

$$\begin{aligned} ndV_n(x) &= dV_nF_nV_n(x) = d(V_n([1]_{S/n})\cdot V_n(x)) \\ &= dV_n([1]_{S/n})\cdot V_n(x) + V_n([1]_{S/n})\cdot dV_n(x). \end{aligned}$$

Now, since both $\eta_{S/p}$ and F_p are graded ring homomorphisms, we have

$$\sum_\alpha \eta_{S/p}F_p(dV_n([1]_{S/n})V_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha}) = \eta_{S/p}F_pdV_n([1]_{S/n})\cdot\eta_{S/p}F_p(\omega)$$

which is zero, since $\eta_{S/p}F_p(\omega)$ is zero, and

$$\begin{aligned} &\sum_\alpha \eta_{S/p}F_p(V_n([1]_{S/n})dV_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha}) \\ &= \sum_\alpha \eta_{S/p}F_pV_n([1]_{S/n})\cdot\eta_{S/p}F_p(dV_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha}) \\ &= \sum_\alpha \eta_{S/p}V_nF_p([1]_{S/n})\cdot\eta_{S/p}F_p(dV_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha}) \\ &= \sum_\alpha V_n\eta_{S/np}F_p([1]_{S/n})\cdot\eta_{S/p}F_p(dV_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha}) \\ &= \sum_\alpha V_n(\eta_{S/np}F_p([1]_{S/n})\cdot F_n\eta_{S/p}F_p(dV_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha})) \\ &= \sum_\alpha V_n(\eta_{S/np}F_p([1]_{S/n})\cdot\eta_{S/np}F_{np}(dV_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha})) \\ &= \sum_\alpha V_n(\eta_{S/np}F_p([1]_{S/n})\cdot F_p\eta_{S/n}F_n(dV_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha})), \end{aligned}$$

where the third and forth equalities hold, since, by the inductive hypothesis, both the maps $F_n\colon \mathbb{W}_{S/p}\Omega^q_A \to \mathbb{W}_{S/np}\Omega^q_A$ and $V_n\colon \mathbb{W}_{S/np}\Omega^q_A \to \mathbb{W}_{S/p}\Omega^q_A$ exist and have the properties listed at the beginning of the proof, and where the fifth and sixth equalities hold, since the maps $F_p\colon \mathbb{W}_{S/n}\Omega_A^q \to \mathbb{W}_{S/np}\Omega_A^q$ and $V_p\colon \mathbb{W}_{S/np}\Omega_A^q \to \mathbb{W}_{S/n}\Omega_A^q$ exist and have the properties listed at the beginning of the proof. Since $\eta_{S/np}F_p([1]_{S/n})$ is the identity and $F_n dV_n(x_\alpha) = dx_\alpha + (n-1)d\log([-1]_S)\cdot x_\alpha$, this becomes

$$V_nF_p\eta_{S/n}(\sum_\alpha d(x_\alpha)F_ndy_{1,\alpha}\dots F_ndy_{q,\alpha}) + \varepsilon\cdot V_nF_p\eta_{S/n}(\sum_\alpha x_\alpha F_ndy_{1,\alpha}\dots F_ndy_{q,\alpha})$$

with $\varepsilon = (n-1)d\log\eta_{S/p}([-1]_{S/p})$, where the second term is zero. Finally, since d is a derivation, we may rewrite the first term as

$$V_nF_p\eta_{S/n}d(\sum_\alpha x_\alpha F_ndy_{1,\alpha}\dots F_ndy_{q,\alpha}) - \varepsilon\cdot V_nF_p\eta_{S/n}(\sum_\alpha x_\alpha F_ndy_{1,\alpha}\dots F_ndy_{q,\alpha})$$

with $\varepsilon = nqd\log\eta_{S/p}([-1]_{S/p})$, and these terms both are zero. Hence, $nF_pd(\omega)$ is zero, and therefore, we conclude that $F_pd(\omega)$ is zero as desired.

In order to complete the recursive definition of the maps η_S, R^S_T, F_n, d, and V_n, we must show that the three diagrams at the beginning of the proof commute. The top left-hand diagram commutes by the definition of V_n, and the calculation

$$\begin{aligned}
&R^S_TV_n\eta_{S/n}(xF_ndy_1\dots F_ndy_q) = R^S_T\eta_S(V_n(x)dy_1\dots dy_q)\\
&= \eta_TR^S_T(V_n(x)dy_1\dots dy_q) = \eta_T(V_nR^{S/n}_{T/n}(x)dR^S_T(y_1)\dots dR^S_T(y_q))\\
&= V_n\eta_{T/n}(R^{S/n}_{T/n}(x)F_ndR^S_T(y_1)\dots F_ndR^S_T(y_q)) = V_nR^{S/n}_{T/n}\eta_{S/n}(xF_ndy_1\dots F_ndy_q)
\end{aligned}$$

shows that the top right-hand diagram commutes. Finally, the following calculation shows that the bottom diagram commutes,

$$\begin{aligned}
&V_n\eta_{S/n}(xF_ndy_1\dots F_ndy_q)\cdot\eta_S(zdw_1\dots dw_r)\\
&= \eta_S(V_n(x)dy_1\dots dy_q)\cdot\eta_S(zdw_1\dots dw_r)\\
&= \eta_S(V_n(x)dy_1\dots dy_q\cdot zdw_1\dots dw_r)\\
&= \eta_S(V_n(xF_n(z))dy_1\dots dy_qdw_1\dots dw_r)\\
&= V_n\eta_{S/n}(xF_n(z)F_ndy_1\dots F_ndy_qF_ndw_1\dots F_ndw_r)\\
&= V_n(\eta_{S/n}(xF_ndy_1\dots F_ndy_q)\cdot\eta_{S/n}F_n(zdw_1\dots dw_r))\\
&= V_n(\eta_{S/n}(xF_ndy_1\dots F_ndy_q)\cdot F_n\eta_S(zdw_1\dots dw_r)).
\end{aligned}$$

Here the first and fourth equalities hold by the definition of the map V_n; the second and fifth equalties hold by the multiplicativity of the maps η_S, $\eta_{S/n}$, and F_n; the third equality holds by Lemma 1.5; and the sixth equality holds by the existence of the map F_n with $F_n\eta_S = \eta_{S/n}F_n$. This completes the recursive definition of the graded rings $\mathbb{W}_S\Omega_A^{\cdot}$ and the maps η_S, R^S_T, F_n, d, and V_n for finite truncation sets S. We extend to infinite truncation sets as discussed in Remark 4.2 (b).

To show that the structure defined above forms a Witt complex over A, it remains to prove that $V_1 = \mathrm{id}$; $V_nV_m = V_{mn}$; $F_nV_n = n\cdot\mathrm{id}$; and $F_mV_n = V_nF_m$, if $(m,n) = 1$. The first identity holds by definition, and the second identity holds, since

$$\begin{aligned}
&V_{mn}\eta_{S/mn}(xF_{mn}dy_1\dots F_{mn}dy_q) = \eta_S(V_{mn}(x)dy_1\dots dy_q)\\
&= \eta_S(V_m(V_n(x))dy_1\dots dy_q) = V_m\eta_{S/m}(V_n(x)F_m dy_1\dots F_m dy_q)\\
&= V_m(V_n(\eta_{S/mn}(x))F_m d\eta_S(y_1)\dots F_m d\eta_S(y_q))\\
&= V_m(V_n(\eta_{S/mn}(x)F_{mn}d\eta_S(y_1)\dots F_{mn}d\eta_S(y_q)))\\
&= V_m(V_n\eta_{S/mn}(xF_{mn}dy_1\dots F_{mn}dy_q)).
\end{aligned}$$

Here the first and third equalities hold by the definition of V_{mn} and V_m, respectively; the fourth equality holds by the existence of the map F_m with $\eta_{S/m}F_m = F_m\eta_S$; the fifth equality holds by the inductive hypothesis; and the last equality holds by the existence of the map F_{mn} with $\eta_{S/mn}F_{mn} = F_{mn}\eta_S$. Similarly, we have

$$\begin{aligned}
&F_nV_n\eta_{S/n}(xF_ndy_1\dots F_ndy_q) = F_n\eta_S(V_n(x)dy_1\dots dy_q)\\
&= \eta_{S/n}F_n(V_n(x)dy_1\dots dy_q) = n\eta_{S/n}(xF_ndy_1\dots F_ndy_q),
\end{aligned}$$

which shows that $F_nV_n = n\cdot\mathrm{id}$; and finally, if $(m,n) = 1$, we have

$$\begin{aligned}
&F_mV_n\eta_{S/n}(x\cdot F_ndy_1\dots F_ndy_q) = F_m\eta_S(V_n(x)\cdot dy_1\dots dy_q)\\
&= \eta_{S/m}F_m(V_n(x)\cdot dy_1\dots dy_q) = \eta_{S/m}F_m(V_n(x))\cdot\eta_{S/m}F_m(dy_1\dots dy_q)\\
&= \eta_{S/m}V_n(F_m(x))\cdot\eta_{S/m}F_m(dy_1\dots dy_q) = V_n(\eta_{S/mn}F_m(x))\cdot\eta_{S/m}F_m(dy_1\dots dy_q)\\
&= V_n(\eta_{S/mn}F_m(x)\cdot F_n\eta_{S/m}F_m(dy_1\dots dy_q)) = V_n\eta_{S/mn}F_m(x\cdot F_n(dy_1\dots dy_q))\\
&= V_nF_m\eta_{S/n}(x\cdot F_n(dy_1\dots dy_q)) = V_nF_m\eta_{S/n}(x\cdot F_ndy_1\dots F_ndy_q),
\end{aligned}$$

which shows that $F_mV_n = V_nF_m$ as desired.

Finally, let $E_S^{\cdot}$ be a Witt complex over A and let $\eta_S^E\colon \check{\Omega}^{\cdot}_{\mathbb{W}_S(A)} \to E_S^{\cdot}$ be the map in Corollary 4.5. We claim that this map factors as

$$\check{\Omega}^{\cdot}_{\mathbb{W}_S(A)} \xrightarrow{\eta_S} \mathbb{W}_S\Omega^{\cdot}_A \xrightarrow{f_S} E_S^{\cdot}.$$

Since the left-hand map η_S is surjective, the right-hand map f_S necessarily is unique. We may further assume that the truncation set S is finite. To prove the claim, we proceed by induction on the cardinality of S, the case $S=\emptyset$ being trivial as $\eta_\emptyset$ is a bijection. So we let S be a finite non-empty truncation set and assume that for every proper sub-truncation set $T\subset S$, the factorization $\eta_T^E = f_T\eta_T$ exists. To prove that also the factorization $\eta_S^E = f_S\eta_S$ exists, we must show that whenever n is a positive integer and $x_\alpha\in\mathbb{W}_{S/n}(A)$ and $y_{1,\alpha},\dots,y_{q,\alpha}\in\mathbb{W}_S(A)$ are Witt vectors such that

$$\eta_{S/n}(\sum_\alpha x_\alpha F_ndy_{1,\alpha}\dots F_ndy_{q,\alpha}) \in \mathbb{W}_{S/n}\Omega_A^q$$

vanishes, then so does

$$\eta_S^E(\sum_\alpha V_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha}) \in E_S^q.$$

Now, using that $E_S^{\cdot}$ is a Witt complex over A, we find

$$\begin{aligned}
&\eta_S^E(\sum_\alpha V_n(x_\alpha)\cdot dy_{1,\alpha}\dots dy_{q,\alpha}) = \sum_\alpha \eta_S^E(V_n(x_\alpha))\cdot\eta_S^E(dy_{1,\alpha}\dots dy_{q,\alpha})\\
&= \sum_\alpha V_n(\eta_{S/n}^E(x_\alpha))\cdot\eta_S^E(dy_{1,\alpha}\dots dy_{q,\alpha}) = \sum_\alpha V_n(\eta_{S/n}^E(x_\alpha)\cdot F_n\eta_S^E(dy_{1,\alpha}\dots dy_{q,\alpha}))\\
&= \sum_\alpha V_n(\eta_{S/n}^E(x_\alpha)\cdot\eta_{S/n}^E F_n(dy_{1,\alpha}\dots dy_{q,\alpha})) = V_n\eta_{S/n}^E(\sum_\alpha x_\alpha F_n dy_{1,\alpha}\dots F_n dy_{q,\alpha})\\
&= V_n f_{S/n}\eta_{S/n}(\sum_\alpha x_\alpha F_n dy_{1,\alpha}\dots F_n dy_{q,\alpha}),
\end{aligned}$$

which vanishes as required. Here the last equality holds by the inductive hypothesis. This shows that, for every truncation set S, the map η_S^E factors as $f_S\eta_S$. Finally, to show that $\mathbb{W}_S\Omega_A^{\cdot}$ is initial among Witt complexes over A, it remains to verify that the maps f_S constitute a map of Witt complexes. In view of Corollary 4.5, the only statement that needs proof is that for every truncation set S and for every positive integer n, we have $f_S V_n = V_n f_{S/n}$, and this follows from the calculation

$$\begin{aligned}
&f_S V_n\eta_{S/n}(x\cdot F_n dy_1\dots F_n dy_q) = f_S\eta_S(V_n(x)\cdot dy_1\dots dy_q)\\
&= \eta_S^E(V_n(x)\cdot dy_1\dots dy_q) = \eta_S^E(V_n(x))\cdot\eta_S^E(dy_1\dots dy_q)\\
&= V_n(\eta_{S/n}^E(x))\cdot\eta_S^E(dy_1\dots dy_q) = V_n(\eta_{S/n}^E(x)\cdot F_n\eta_S^E(dy_1\dots dy_q))\\
&= V_n(\eta_{S/n}^E(x)\cdot\eta_{S/n}^E F_n(dy_1\dots dy_q)) = V_n\eta_{S/n}^E(x\cdot F_n dy_1\dots F_n dy_q)\\
&= V_n f_{S/n}\eta_{S/n}(x\cdot F_n dy_1\dots F_n dy_q),
\end{aligned}$$

since every element in $\mathbb{W}_S\Omega_A^q$ can be written as a convergent sum of elements of the form $\eta_{S/n}(x\cdot F_n dy_1\dots F_n dy_q)$ with $n\in\mathbb{N}$, and $x\in\mathbb{W}_S(A)$, and $y_1,\dots,y_q\in\mathbb{W}_{S/n}(A)$. This completes the proof of Theorem B. □

Definition 4.7. The initial Witt complex $\mathbb{W}_S\Omega_A^{\cdot}$ over the ring A is called the big de Rham-Witt complex of A.

Addendum 4.8. (i) *For all q, the map* $\eta_{\{1\}}\colon \Omega_A^q\to\mathbb{W}_{\{1\}}\Omega_A^q$ *is an isomorphism.*
(ii) *For all S, the map* $\eta_S\colon \mathbb{W}_S(A)\to\mathbb{W}_S\Omega_A^0$ *is an isomorphism.*

Proof. This follows immediately from the proof of Theorem B and Lemma 3.14. □

The statement (i) in Addendum 4.8 is a special case of the question raised at the top of page 133 in [15]. The explicit construction of the big de Rham-Witt complex given in the proof of Theorem B answers this question in the affirmative.

Remark 4.9. Suppose that (k,λ) is a λ-ring and that $f\colon k\to A$ is a k-algebra. We let $f_S\colon k\to\mathbb{W}_S(A)$ be the composite ring homomorphism $R_S^{\mathbb{N}}\circ\mathbb{W}(f)\circ\lambda$ and define the big de Rham-Witt complex of A relative to (k,λ) to be the quotient

$$\mathbb{W}_S\Omega_{A/(k,\lambda)}^{\cdot} = \mathbb{W}_S\Omega_A^{\cdot}/R_S^{\cdot}$$

of the big de Rham-Witt complex $\mathbb{W}_S\Omega_A^{\cdot}$ by the graded ideal $R_S^{\cdot}$ generated by the images of $\eta_S\circ f_{S*}\colon \Omega_k^1\to\mathbb{W}_S\Omega_A^1$ and $d\circ\eta_S\circ f_{S*}\colon \Omega_k^1\to\mathbb{W}_S\Omega_A^2$. It is initial among Witt complexes over A in which the map d is k-linear when its domain and target are viewed as k-modules via the map $\eta_S f_S \colon k \to E_S^0$. In the particular case, where (k,λ) is $(\mathbb{W}(R), \Delta_R)$ and where A is an R-algebra viewed as a k-algebra via $\varepsilon_R \colon k \to R$, we obtain (a big version of) the Langer-Zink relative de Rham-Witt complex [22].

5 Étale morphisms

The functor that to the ring A associates the $\mathbb{W}_S(A)$-module $\mathbb{W}_S\Omega_A^q$ defines a presheaf of $\mathbb{W}_S(\mathscr{O})$-modules on the category of affine schemes. In this section, we use the theorem of Borger [6] and van der Kallen [27] which we recalled as Theorem 1.25 to show that for S finite, this presheaf is a quasi-coherent sheaf of $\mathbb{W}_S(\mathscr{O})$-modules for the étale topology. This is the statement of Theorem C which we now prove.

Proof of Theorem C. We fix an étale morphism $f \colon A \to B$ and consider the map

$$\mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega_A^{\cdot} \xrightarrow{\alpha} \mathbb{W}_S\Omega_B^{\cdot}$$

that to $b \otimes \omega$ assigns $b \cdot f_*(\omega)$. To show that this map is an isomorphism, we define a structure of Witt complex over B on the domain $E_S^{\cdot}$ of α. By Theorem 1.25, the map

$$\mathbb{W}_S(f) \colon \mathbb{W}_S(A) \to \mathbb{W}_S(B)$$

is étale. Hence, the graded derivation $d \colon \mathbb{W}_s\Omega_A^q \to \mathbb{W}_S\Omega_A^{q+1}$ extends uniquely to a graded derivation $d^E \colon E_S^q \to E_S^{q+1}$ defined by

$$d^E(b \otimes x) = (d'b)x + b \otimes dx$$

with $d'b$ the image of b by the composition

$$\mathbb{W}_S(B) \xrightarrow{d} \Omega^1_{\mathbb{W}_S(B)} \xleftarrow{\sim} \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \Omega^1_{\mathbb{W}_S(A)} \xrightarrow{\mathrm{id} \otimes \eta_S} \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega_A^1,$$

where the middle map is the canonical isomorphism. We further define the maps $R_T^{E,S} \colon E_S^q \to E_T^q$ and $F_n^E \colon E_S^q \to E_{S/n}^q$ to be $R_T^S \otimes R_T^S$ and $F_n^E = F_n \otimes F_n$, respectively. Next, to define the map $V_n^E \colon E_{S/n}^q \to E_S^q$, we use that, since the square in the statement of Theorem 1.25 is cocartesian, the map

$$\mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_{S/n}\Omega_A^q \xrightarrow{F_n \otimes \mathrm{id}} \mathbb{W}_{S/n}(B) \otimes_{\mathbb{W}_{S/n}(A)} \mathbb{W}_{S/n}\Omega_A^q$$

is an isomorphism, and we then define V_n^E to be the composition of the inverse of this isomorphism and the map

$$\mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_{S/n}\Omega_A^q \xrightarrow{\mathrm{id} \otimes V_n} \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega_A^q.$$

Finally, we define the map $\eta_S^E \colon \mathbb{W}_S(B) \to E_B^0$ to be the composition

$$\mathbb{W}_S(B) \longrightarrow \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S(A) \longrightarrow \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega_A^0$$

of the canonical isomorphism and the map $\mathrm{id}\otimes\eta_S$. We proceed to show that the maps defined above make $E_S^{\cdot}$ a Witt complex over B. The axioms (i)–(iii) of Definition 4.1 are readily verified. For example, we have $d^E d^E(-) = d\log\eta_S^E([-1]_S)d^E(-)$, since both sides are derivations which agree on $\mathbb{W}_S\Omega_A^q$; and the calculation

$$\begin{aligned}&V_n^E(F_n(b)\otimes\omega)\cdot b'\otimes\omega' = b\otimes V_n(\omega)\cdot b'\otimes\omega' = bb'\otimes V_n(\omega)\omega'\\&= bb'\otimes V_n(\omega F_n(\omega')) = V_n^E(F_n(bb')\otimes\omega F_n(\omega')) = V_n^E(F_n(b)\otimes\omega\cdot F_n^E(b'\otimes\omega'))\end{aligned}$$

verifies axiom (iii), since every element of $E_{S/n}^q$ can be written as a sum of elements of the form $F_n(b)\otimes\omega$ with $b\in\mathbb{W}_S(B)$ and $\omega\in\mathbb{W}_{S/n}\Omega_A^q$.

It remains to verify axioms (iv)–(v) of Definition 4.1. To prove axiom (iv), we must show that for all $\omega\in E_{S/n}^q$, the equality

$$F_n^E d^E V_n^E(\omega) + (n-1)d\log\eta_{S/n}^E([-1]_{S/n})\cdot\omega = d^E(\omega)$$

holds in $E_{S/n}^{q+1}$. On the right-hand side, d^E, by definition, is the unique graded derivation on $E_{S/n}^{\cdot}$ that extends the graded derivation d on $\mathbb{W}_{S/n}\Omega_A^{\cdot}$; and, on the left-hand side, D also extends d. Hence, it will suffice to show that D, too, is a graded derivation. Moreover, since the square diagram of rings in Theorem 1.25 is cocartesian, and since D is an additive function, it suffices to show that D is a graded derivation on elements of the form $\omega = F_n(b)\otimes\tau$ with $b\in\mathbb{W}_S(B)$ and $\tau\in\mathbb{W}_{S/n}\Omega_A^{\cdot}$. We claim that

$$D(F_n(b)\otimes\tau) = F_n^E(d^E b)\cdot n\tau + F_n(b)\otimes d\tau$$

as elements of $E_{S/n}^{\cdot}$. Granting this, it follows that axiom (iv) holds, as the right-hand side clearly is a graded derivation of $F_n(b)\otimes\tau$. Now,

$$\begin{aligned}D(F_n(b)\otimes\tau) &= F_n^E d^E V_n^E(F_n(b)\otimes\tau) + (n-1)d\log\eta_{S/n}^E([-1]_{S/n})\cdot F_n(b)\otimes\tau\\&= F_n^E d^E(b\otimes V_n(\tau)) + F_n(b)\otimes(n-1)d\log\eta_{S/n}([-1])\tau\\&= F_n^E(d^E(b)\cdot V_n(\tau) + b\otimes dV_n(\tau)) + F_n(b)\otimes(n-1)d\log\eta_{S/n}([-1])\tau\\&= F_n^E(d^E b)\cdot n\tau + F_n(b)\otimes(F_n dV_n(\tau) + (n-1)d\log\eta_{S/n}([-1])\tau)\\&= F_n^E(d^E b)\cdot n\tau + F_n(b)\otimes d\tau,\end{aligned}$$

which proves the claim. Here the first two equalities follow from the definitions; the third equality holds, since d^E is a derivation; the fourth equality holds, since F_n^E is a ring homomorphism and satisfies $F_n^E V_n^E = n\,\mathrm{id}$; and the last equality holds, since axiom (iv) holds in the de Rham-Witt complex over A.

In order to prove axiom (v), we consider the following diagram, where the left-hand horizontal maps are the canonical isomorphisms,

$$\begin{array}{ccccc}\Omega^1_{\mathbb{W}_S(B)} & \longleftarrow & \mathbb{W}_S(B)\otimes_{\mathbb{W}_S(A)}\Omega^1_{\mathbb{W}_S(A)} & \xrightarrow{\mathrm{id}\otimes\eta_S} & E_S^1\\ \downarrow{\scriptstyle F_n} & & \downarrow{\scriptstyle F_n\otimes F_n} & & \downarrow{\scriptstyle F_n^E}\\ \Omega^1_{\mathbb{W}_{S/n}(B)} & \longleftarrow & \mathbb{W}_{S/n}(B)\otimes_{\mathbb{W}_{S/n}(A)}\Omega^1_{\mathbb{W}_{S/n}(A)} & \xrightarrow{\mathrm{id}\otimes\eta_{S/n}} & E_{S/n}^1.\end{array}$$

Here, the left-hand square commutes, since $F_n\colon \Omega^1_{\mathbb{W}(B)} \to \Omega^1_{\mathbb{W}(B)}$ is F_n-linear and a natural transformation; and right-hand square commutes by Proposition 4.4. Hence, also the outer square commutes and this immediately implies axiom (v); compare Remark 4.2 (d).

We have proved that the domains of the canonical map α at the beginning of the proof form a Witt complex over B. Therefore, there exists a unique map

$$\beta\colon \mathbb{W}_S\Omega^q_B \to \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega^q_A$$

of Witt complexes over B. The composition $\alpha \circ \beta$ is a selfmap of the initial object $\mathbb{W}_S\Omega^q_B$, and therefore, is the identity map. The composition $\beta \circ \alpha$ is a map of Witt complexes over B. In particular, it is a map of $\mathbb{W}_S(B)$-modules, and therefore, is determined by the composition with the map of Witt complexes

$$\iota\colon \mathbb{W}_S\Omega^q_A \to \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega^q_A$$

that takes x to $[1]_S \otimes x$. But ι and $\beta \circ \alpha \circ \iota$ both are maps of Witt complexes over A with domain the initial Witt complex over A. Therefore, the two maps are equal, and hence, also $\beta \circ \alpha$ is the identity map. This completes the proof. □

6 The big de Rham-Witt complex of the ring of integers

We finally evaluate the absolute de Rham-Witt complex of the ring of integers. If m and n are positive integers, we write (m,n) and $[m,n]$ for the greatest common divisor and least common multiple of m and n, respectively. We define $(m,n]$ to be the unique integer modulo $[m,n]$ such that $(m,n] \equiv 0$ modulo m and $(m,n] \equiv (m,n)$ modulo n, and define $\{m,n\}$ to the unique integer modulo 2 that is non-zero if and only if both m and n are even. We note that $(m,n] + (n,m] \equiv (m,n)$ modulo $[m,n]$. We also remark that, by Lemma 4.3 and by d being a derivation, in any Witt complex, the element $dV_n\eta_{S/n}([1]_{S/n})$ is annihilated by n.

Theorem 6.1. *The big de Rham-Witt complex of $\mathbb{Z}$ is given as follows:*

$$\mathbb{W}_S\Omega^0_{\mathbb{Z}} = \prod_{n\in S} \mathbb{Z} \cdot V_n\eta_{S/n}([1]_{S/n})$$
$$\mathbb{W}_S\Omega^1_{\mathbb{Z}} = \prod_{n\in S} \mathbb{Z}/n\mathbb{Z} \cdot dV_n\eta_{S/n}([1]_{S/n})$$

and the groups in degrees $q \geqslant 2$ are zero. The multiplication is given by

$$V_m\eta_{S/m}([1]_{S/m}) \cdot V_n\eta_{S/n}([1]_{S/n}) = (m,n) \cdot V_{[m,n]}\eta_{S/[m,n]}([1]_{S/[m,n]})$$
$$V_m\eta_{S/m}([1]_{S/m}) \cdot dV_n\eta_{S/n}([1]_{S/n}) = (m,n] \cdot dV_{[m,n]}\eta_{S/[m,n]}([1]_{S/[m,n]}) + \{m,n\} \sum_{r\geqslant 1} 2^{r-1}[m,n] \cdot dV_{2^r[m,n]}\eta([1]_{S/2^r[m,n]}),$$

and the mth Frobenius and Verschiebung maps are given by

$$\begin{aligned}
F_m V_n \eta_{S/n}([1]_{S/n}) &= (m,n) \cdot V_{[m,n]/m} \eta_{S/[m,n]}([1]_{S/[m,n]}) \\
F_m dV_n \eta_{S/n}([1]_{S/n}) &= (m,n]/m \cdot dV_{[m,n]/m} \eta_{S/[m,n]}([1]_{S/[m,n]}) \\
&\quad + \{m,n\} \sum_{r \geqslant 1} (2^{r-1}[m,n]/m) \cdot dV_{2^r[m,n]/m} \eta_{S/2^r[m,n]}([1]_{S/2^r[m,n]}) \\
V_m(V_n \eta_{S/mn}([1]_{S/mn})) &= V_{mn} \eta_{S/mn}([1]_{S/mn}) \\
V_m(dV_n \eta_{S/mn}([1]_{S/mn})) &= m \cdot dV_{mn} \eta_{S/mn}([1]_{S/mn}).
\end{aligned}$$

Proof. We claim that there is a Witt complex $E_S^{\cdot}$ over $\mathbb{Z}$ with

$$\begin{aligned}
E_S^0 &= \prod_{n \in S} \mathbb{Z} \cdot V_n \eta_{S/n}([1]_{S/n}), \\
E_S^1 &= \prod_{n \in S} \mathbb{Z}/n\mathbb{Z} \cdot dV_n \eta_{S/n}([1]_{S/n}),
\end{aligned}$$

with $E_S^q = 0$ for $q \geqslant 2$, and with the Witt complex structure maps defined to be the unique additive maps satisfying the formulas listed in the statement. For instance, the map $\eta_S \colon \mathbb{W}_S(\mathbb{Z}) \to E_S^0$ is defined to be the unique additive map that to $V_n([1]_{S/n})$ assigns $V_n \eta_{S/n}([1]_{S/n})$; it is a ring isomorphism by Proposition 1.6. Granting this claim, the map η_S extends uniquely to a map

$$\mathbb{W}_S \Omega_{\mathbb{Z}}^{\cdot} \xrightarrow{\eta_S} E_S^{\cdot}$$

of Witt complex over $\mathbb{Z}$. It is an isomorphism in degree $q = 0$, as noted above, and it is also an isomorphism in degree $q = 1$. For it is clearly surjective in degree $q = 1$, and since Lemma 4.3 shows that, in every Witt complex over $\mathbb{Z}$,

$$ndV_n \eta_{S/n}([1]_{S/n}) = V_n d\eta_{S/n}([1]_{S/n}) = 0,$$

it is also injective in degree $q = 1$. Finally, to prove that η_S is an isomorphism in degrees $q \geqslant 2$, we must show that $\mathbb{W}_S \Omega_{\mathbb{Z}}^2$ is zero, and to this end, it suffices to show that, for every finite truncation set S and every $n \in S$, the element $ddV_n \eta_{S/n}([1]_{S/n})$ of $\mathbb{W}_S \Omega_{\mathbb{Z}}^2$ vanishes. Now, using Lemma 4.3 and the projection formula, we find

$$\begin{aligned}
ddV_n \eta_{S/n}([1]_{S/n}) &= d\log \eta_S([-1]_S) \cdot dV_n \eta_{S/n}([1]_{S/n}) \\
&= d(d\log \eta_S([-1]_S) \cdot V_n \eta_{S/n}([1]_{S/n})) = dV_n(d\log \eta_{S/n}([-1]_{S/n})) \\
&= \sum_{r \geqslant 1} 2^{r-1} dV_n dV_{2^r} \eta_{S/2^r n}([1]_{S/2^r n}) = nddV_{2n} \eta_{S/2n}([1]_{S/2n}),
\end{aligned}$$

and since S was assumed to be finite, this furnishes an induction argument showing that $ddV_n \eta([1]_{S/n})$ is zero.

It remains to prove the claim. For notational convenience, we will suppress the subscript S. We first show that the product on $E_S^{\cdot}$ is associative. Since $[-1]$ is a square root of one in $\mathbb{W}_S(\mathbb{Z})$ which, by Addendum 1.7, is equal to $-[1] + V_2([1])$, the formula defining the product in $E_S^{\cdot}$ shows that, as elements of E_S^1,

$$\begin{aligned}
d\log \eta([-1]) &= (-\eta([1]) + V_2\eta([1])) d(-\eta([1]) + V_2\eta([1])) \\
&= (-\eta([1]) + V_2\eta([1])) dV_2\eta([1]) = \sum_{r \geqslant 1} 2^{r-1} dV_{2^r} \eta([1]).
\end{aligned} \tag{6.2}$$

Using this formula, we find

$$\begin{aligned}
&V_e\eta([1])\cdot d\log\eta([-1]) = \sum_{r\geqslant 1} 2^{r-1}V_e\eta([1])dV_{2^r}\eta([1])\\
&= \sum_{r\geqslant 1} 2^{r-1}(e,2^r]dV_{[e,2^r]}\eta([1]) + \{e,2\}\sum_{s\geqslant 1} 2^{s-1}[e,2]dV_{2^s[e,2]}\eta([1]).
\end{aligned}$$

Moreover, $2^{r-1}(m,2^r]$ is congruent to $2^{r-1}m$ modulo $[m,2^r]$. For $2^{r-1}m$ is congruent to 0 modulo m and to $2^{r-1}(m,2^r)$ modulo 2^r. Hence, if e is odd, the lower left-hand summand is equal to $\sum_{r\geqslant 1} 2^{r-1}edV_{2^r e}\eta([1])$ and the lower right-hand summand is zero, and if e is even, the lower left-hand summand is zero and the lower right-hand summand is equal to $\sum_{s\geqslant 1} 2^{s-1}edV_{2^s e}\eta([1])$. So for any positive integer e, we have

$$V_e\eta([1])\cdot d\log\eta([-1]) = \sum_{r\geqslant 1} 2^{r-1}edV_{2^r e}\eta([1]). \tag{6.3}$$

We conclude that the product in $E_S^{\cdot}$ satisfies

$$\begin{aligned}
V_m\eta([1])\cdot dV_n\eta([1]) &= (m,n]\cdot dV_{[m,n]}\eta([1])\\
&\quad + \{m,n\}\cdot V_{[m,n]}\eta([1])\cdot d\log\eta([-1]),
\end{aligned} \tag{6.4}$$

where we use (6.3) to identify the second term on the right-hand side. A similar calculation shows that for all positive integers a and b,

$$V_a\eta([1])\cdot(V_b\eta([1])\cdot d\log\eta([-1])) = (V_a\eta([1])\cdot V_b\eta([1]))\cdot d\log\eta([-1]).$$

Using this identity, we find, on the one hand, that

$$\begin{aligned}
&V_l\eta([1])\cdot(V_m([1])\cdot dV_n\eta([1])) = (l,[m,n]](m,n]\cdot dV_{[l,[m,n]]}\eta([1])\\
&\quad + (\{l,[m,n]\}(m,n] + (l,[m,n])\{m,n\})\cdot V_{[l,[m,n]]}\eta([1])d\log\eta([-1]),
\end{aligned}$$

and, on the other hand, that

$$\begin{aligned}
&(V_l\eta([1])\cdot V_m([1]))\cdot dV_n\eta([1]) = (l,m)([l,m],n]\cdot dV_{[[l,m],n]}\eta([1])\\
&\quad + (l,m)\{[l,m],n\}\cdot V_{[[l,m],n]}\eta([1])d\log\eta([1]).
\end{aligned}$$

Here $[l,[m,n]] = [[l,m],n]$ and to prove that $(l,[m,n]](m,n]$ and $(l,m)([l,m],n]$ are congruent modulo $[l,[m,n]]$, we use that $[l,[m,n]]\mathbb{Z}$ is the kernel of the map

$$\mathbb{Z}\to\mathbb{Z}/l\mathbb{Z}\times\mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}$$

that takes a to $(a+l\mathbb{Z}, a+m\mathbb{Z}, a+n\mathbb{Z})$. So it will suffice to verify that the desired congruence holds modulo l, m, and n, respectively. By definition, both numbers are zero modulo l and m, and the congruence modulo n follows from the identity

$$(l,[m,n])\cdot(m,n) = (l,m)\cdot([l,m],n)$$

which is readily verified by multiplying by $[l,[m,n]] = [[l,m],n]$ on both sides. We also note that $\{l,[m,n]\}(m,n] + (l,[m,n])\{m,n\}$ and $(l,m)\{[l,m],n\}$ are well-defined integers modulo 2 which are non-zero if and only if n and exactly one of l and m are even. This shows that the product in $E_S^{\cdot}$ is associative.

We proceed to verify the axioms (i)–(v) of Definition 4.1. First, we note that since the sum $(m,n]+(n,m]$ is congruent to (m,n) modulo $[m,n]=[n,m]$, we have

$$\begin{aligned}
&dV_m\eta([1])\cdot V_n\eta([1])+V_m\eta([1])\cdot dV_n\eta([1])\\
&=(n,m]dV_{[n,m]}\eta([1])+\{n,m\}V_{[n,m]}\eta([1])d\log\eta([-1])\\
&\quad+(m,n]dV_{[m,n]}\eta([1])+\{m,n\}V_{[m,n]}\eta([1])d\log\eta([-1])\\
&=(m,n)dV_{[m,n]}\eta([1])=d(V_m\eta([1])\cdot V_n\eta([1]))
\end{aligned}$$

which verifies axiom (i).

To verify axiom (iv), we first show that for all positive integers m,

$$F_m(d\log\eta([-1]))=d\log([-1]). \tag{6.5}$$

It follows from formula (6.2) that

$$\begin{aligned}
&F_m(d\log\eta([-1]))=\sum_{r\geqslant 1}2^{r-1}F_m dV_{2^r}\eta([1])\\
&=\sum_{r\geqslant 1}2^{r-1}(m,2^r]/m\cdot dV_{[m,2^r]/m}\eta([1])+\{m,2\}\sum_{s\geqslant 1}2^{s-1}[m,2]/m\cdot dV_{2^s[m,2]/m}\eta([1])\\
&=\sum_{r\geqslant 1}2^{r-1}\cdot dV_{[m,2^r]/m}\eta([1])+\{m,2\}\sum_{s\geqslant 1}2^{s-1}[m,2]/m\cdot dV_{2^s[m,2]/m}\eta([1])
\end{aligned}$$

where the second equality follows from the definition of F_m and the last equality uses that $2^{r-1}(m,2^r]$ is congruent to $2^{r-1}m$ modulo $[m,2^r]$. Now, if the integer m is even, then the lower the left-hand term is zero, since $[m,2^r]/m=2^t$ with $t<r$, and the lower right-hand term is equal to $\sum_{s\geqslant 1}2^{s-1}dV_{2^s}\eta([1])$; and if m is odd, then the lower left-hand term is equal to $\sum_{r\geqslant 1}2^{r-1}dV_{2^r}\eta([1])$ and the lower right-hand term is zero. Hence, using (6.2) again, we conclude that (6.5) holds. By using this equality, we may restate the definition of the Frobenius on E_S^1 in the form

$$\begin{aligned}
F_m dV_n\eta([1])&=(m,n]/m\cdot dV_{[m,n]/m}\eta([1])\\
&\quad+\{m,n\}\cdot V_{[m,n]/m}\eta([1])\cdot d\log\eta([-1]),
\end{aligned} \tag{6.6}$$

and taking $n=mk$ and $y=V_k\eta([1])$, this verifies axiom (iv).

We next consider axiom (ii) which is easily verified on E^0. We first show that the identity $F_lF_m=F_{lm}$ holds on E^1. Using (6.6), we have, on the one hand, that

$$\begin{aligned}
&F_l(F_m dV_n\eta([1]))=(l,[m,n]/m]/l\cdot(m,n]/m\cdot dV_{[l,[m,n]/m]/l}\eta([1])\\
&+(\{l,[m,n]/m\}(m,n]/m+(l,[m,n]/m)\{m,n\})\cdot V_{[l,[m,n]/m]/l}\eta([1])\cdot d\log\eta([-1]),
\end{aligned}$$

and, on the other hand, that

$$\begin{aligned}
F_{lm}dV_n\eta([1])&=(lm,n]/lm\cdot dV_{[lm,n]/lm}\eta([1])\\
&\quad+\{lm,n\}\cdot V_{[lm,n]/lm}\eta([1])\cdot d\log\eta([-1]).
\end{aligned}$$

Here, we have $[l,[m,n]/m]/l = [lm,n]/lm$, since both are equal to $n/(lm,n)$, and moreover, $(lm,n]$ and $(l,[m,n]/m](m,n]$ are congruent modulo $[lm,n]$, since both are congruent to 0 modulo lm and congruent to $(lm,n) = (l,[m,n]/m)(m,n)$ modulo n. Finally, the two factors $\{lm,n\}$ and $\{l,[m,n]/m\}(m,n]/m + (l,[m,n]/m)\{m,n\}$ are well-defined integers modulo 2 which are non-zero if and only if lm and n are even. Indeed, for the first factor, this is the definition, and for the second factor, it is seen as follows. If n is odd, then both summands in this factor are zero, so suppose that n is even. If lm is odd, then again both summand are zero; if l is odd and m is even, then the first summand is zero and the second summand is non-zero; if l is even and m is odd, then the first summand is non-zero and the second summand is zero; if l and m are both even and if the 2-adic valuation of m is strictly less than that of n, then $[m,n]/m$ is even and $(m,n]/m$ is not divisible by 2 modulo $[m,n]/m$, so the first summand is non-zero and the second summand is zero; and, finally, if l and m are both even and the 2-adic valuation of m is greater than or equal to that of n, then $[m,n]/m$ is odd, so the first summand is zero and the second summand is non-zero. This completes the proof that $F_lF_m = F_{lm}$. The formulas $V_lV_m = V_{lm}$ and $F_mV_m = m\cdot\mathrm{id}$ are readily verified, so we next show that $F_lV_m = V_mF_l$ if l and m are relatively prime. To this end, we first note that by (6.3) and by the definition of V_m on E_S^1, we have

$$V_m(V_e\eta([1])\cdot d\log\eta([-1])) = V_{me}\eta([1])\cdot d\log\eta([-1]), \tag{6.7}$$

for all positive integers m and e. Using this identity, (6.5), and (6.6), we find that

$$\begin{aligned} F_lV_mdV_n\eta([1]) &= m(l,mn]/l\cdot dV_{[l,mn]/l}\eta([1]) \\ &\quad + m\{l,mn\}\cdot V_{[l,mn]/l}\eta([1])\cdot d\log\eta([-1]) \\ V_mF_ldV_n\eta([1]) &= m(l,n]/l\cdot dV_{m[l,n]/l}\eta([1]) \\ &\quad + \{l,n\}\cdot V_{m[l,n]/l}\eta([1])\cdot d\log\eta([-1]). \end{aligned}$$

But if l and m are relatively prime, then $[l,mn]$ and $m[l,n]$ are equal; $m(l,mn]$ and $m(l,n]$ are congruent modulo $[l,mn] = m[l,n]$, as both are congruent to 0 modulo lm and to $m(l,mn) = m(l,n)$ modulo mn; and $m\{l,mn\} = \{l,n\}$, as is easily checked. This shows that $F_lV_m = V_mF_l$, concluding the proof of axiom (ii).

To verify axiom (iii), we first note that for all positive integers l, m, and n,

$$V_l\eta([1])\cdot F_mdV_n\eta([1]) = V_l(\eta([1])\cdot F_{lm}dV_n\eta([1])).$$

Indeed, by (6.4) and (6.6), the identity becomes

$$\begin{aligned} &(l,[m,n]/m](m,n]/m\cdot dV_{[l,[m,n]/m]}\eta([1]) \\ &+ (\{l,[m,n]/m\}(m,n]/m + (l,[m,n]/m)\{m,n\})\cdot V_{[l,[m,n]/m]}\eta([1])\cdot d\log\eta([-1]) \\ &= (lm,n]/lm\cdot dV_{[lm,n]/m}\eta([1]) + \{lm,n\}\cdot V_{[lm,n]/m}\eta([1])\cdot d\log\eta([-1]), \end{aligned}$$

and hence, the proof of the identity $F_lF_m = F_{lm}$ above shows that the two sides are equal. Now, from this equation and from the definition of V_m on $E_S^{\cdot}$, we find that

$$V_l(V_m\eta([1]))\cdot dV_n\eta([1]) = V_l(V_m\eta([1])\cdot F_ldV_n\eta([1])).$$

It remains to prove that also

$$V_l(dV_m\eta([1]))\cdot V_n\eta([1]) = V_l(dV_m\eta([1])\cdot F_lV_n\eta([1])).$$

Using (6.4), the left-hand side becomes

$$l(n,lm]\cdot dV_{[n,lm]}\eta([1]) + l\{n,lm\}\cdot V_{[n,lm]}\eta([1])\cdot d\log\eta([-1])$$

and the right-hand sides becomes

$$\begin{aligned}&l(l,n)([l,n]/l,m]\cdot dV_{l[[l,n]/l,m]}\eta([1])\\ &\quad+(l,n)\{[l,n]/l,m\}\cdot V_{l[[l,n]/l,m]}\eta([1])\cdot d\log\eta([-1]).\end{aligned}$$

We have seen above that $[n,lm] = l[[l,n]/l,m]$, and $l(n,lm]$ and $l(l,n)([l,n]/l,m]$ are congruent modulo $[n,lm]$, since both are congruent to 0 modulo n and congruent to $l(n,lm)$ modulo lm. Here we use that $([l,n]/l,m]$ is congruent to $([l,n]/l,m)$ modulo m and that $(l,n)([l,n]/l,m) = (n,lm)$. Moreover, $l\{n,lm\}$ and $(l,n)\{[l,n]/l,m\}$ are well-defined integers modulo 2 which are non-zero if and only if l is odd and m and n are both even. This completes the proof that axiom (iii) holds. Indeed, it is clear that axiom (iii) holds on E^0.

Finally, to verify axiom (v), it suffices to consider the case $S = \mathbb{N}$. Since the formula for $[a]$ in Addendum 1.7 is quite complicated, it would be a rather onerous task to verify this axiom directly. By axiom (i), the map $d\eta\colon \mathbb{W}(A)\to E^1_{\mathbb{N}}$ is a derivation, provided that we view $E^1_{\mathbb{N}}$ as a $\mathbb{W}(A)$-module via $\eta^0 = \eta\colon \mathbb{W}(A)\to E^0_{\mathbb{N}}$, and hence, there is a unique $\mathbb{W}(A)$-linear map $\eta^1\colon \Omega^1_{\mathbb{W}(A)}\to E^1_{\mathbb{N}}$ such that $d\eta^0 = \eta^1 d$. We will show that, for every positive integer m, the diagram

$$\begin{array}{ccc}\Omega^1_{\mathbb{W}(A)} & \xrightarrow{\eta^1} & E^1_{\mathbb{N}}\\ \downarrow{\scriptstyle F_m} & & \downarrow{\scriptstyle F_m}\\ \Omega^1_{\mathbb{W}(A)} & \xrightarrow{\eta^1} & E^1_{\mathbb{N}}\end{array}$$

commutes. Granting this, we find that

$$\begin{aligned}&F_md\eta([a]) = F_md\eta^0([a]) = F_m\eta^1d([a]) = \eta^1F_md([a]) = \eta^1([a]^{m-1}d[a])\\ &= \eta^0([a]^{m-1})\eta^1d([a]) = \eta^0([a])^{m-1}d\eta^0([a]) = \eta([a])^{m-1}d\eta([a]),\end{aligned}$$

which verifies axiom (v). Here, the first and last equalities are identities; the third equality holds by the commutativity of the diagram above; the fourth equality holds by Theorem 2.15; and the remaining equalities hold by the properties of the maps η^0 and η^1. It remains to prove that the diagram above commutes, and by Theorem 2.15 and axiom (i), we may assume that $m = p$ is a prime number. It further suffices to show that for every positive integer n, the image of the element $dV_n([1])$ by the two composites in the diagram are equal. We consider three cases separately. First, if p is odd and $n = ps$ is divisible by p, then

$$F_p\eta^1dV_n([1]) = F_pdV_n\eta^0([1]) = dV_s\eta^0([1]),$$

while

$$\begin{aligned}\eta^1 F_p dV_n([1]) &= \eta^1(V_n([1])^{p-1}dV_n([1]) + d(\frac{F_pV_n([1]) - V_n([1])^p}{p}))\\ &= \eta^1(n^{p-2}(V_n([1])dV_n([1]) + dV_s([1]) - n^{p-2}sdV_n([1])))\\ &= \eta^1 dV_s([1]) = dV_s\eta^0([1])\end{aligned}$$

as desired. Here we used that $ndV_n([1]) = 0$. Second, if $p = 2$ and $n = 2s$ is even, then

$$F_2\eta^1 dV_n([1]) = F_2 dV_n\eta^0([1]) = dV_s\eta^0([1]) + \sum_{r\geqslant 1} 2^{r-1} s dV_{2^r s}\eta^0([1])$$

while

$$\begin{aligned}\eta^1 F_2 dV_n([1]) &= \eta^1(dV_s([1]) + V_n([1])dV_n([1]) - sdV_n([1]))\\ &= dV_s\eta^0([1]) + V_n\eta^0([1])dV_n\eta^0([1]) - sdV_n\eta^0([1])\\ &= dV_s\eta^0([1]) + \sum_{t\geqslant 1} 2^{t-1} n dV_{2^t n}\eta^0([1]) - sdV_n\eta^0([1]),\end{aligned}$$

so the desired equality holds in this case, too, since $2sdV_n\eta^0([1]) = 0$. Third, if n is not divisible by p, then $(p,n]$ is congruent to $1 - n^{p-1}$ modulo $[p,n] = pn$, since both are congruent to 0 modulo p and to 1 modulo n, and $\{p,n\}$ is zero. Hence,

$$F_p\eta^1 dV_n([1]) = F_p dV_n\eta^0([1]) = \frac{1 - n^{p-1}}{p} dV_n\eta^0([1]).$$

We wish to prove that this is equal to

$$\begin{aligned}\eta^1 F_p dV_n([1]) &= \eta^1(V_n([1])^{p-1}dV_n([1]) + d(\frac{F_pV_n([1]) - V_n([1])^p}{p}))\\ &= \eta^1(n^{p-2}V_n([1])dV_n([1]) + \frac{1 - n^{p-1}}{p} dV_n([1]))\\ &= n^{p-2}V_n\eta^0([1])dV_n\eta^0([1]) + \frac{1 - n^{p-1}}{p} dV_n\eta^0([1]),\end{aligned}$$

or equivalently, that $n^{p-2}V_n\eta^0([1])dV_n\eta^0([1])$ is zero. If p is odd, then this holds, since $ndV_n\eta^0([1])$ is zero; and if $p = 2$, then it holds, since $V_n\eta^0([1])dV_n\eta^0([1])$ is zero, for n odd. This completes the proof of the claim made at the beginning of the proof, and hence, of the theorem. □

Addendum 6.8. *Let S be a finite truncation set. The kernel of the canonical map*

$$\eta_S\colon \hat{\Omega}^{\cdot}_{\mathbb{W}_S(Z)} \to \mathbb{W}_S\Omega^{\cdot}_Z$$

is equal to the graded ideal generated by the following elements (i)–(ii) *together with their images by the derivation d.*

(i) *For all $m,n \in S$, the element*

$$\begin{aligned}&V_m([1]_{S/m})dV_n([1]_{S/n}) - (m,n]dV_{[m,n]}([1]_{S/[m,n]})\\ &\quad - \{m,n\}\sum_{r\geqslant 1} 2^{r-1}[m,n]dV_{2^r[m,n]}([1]_{S/2^r[m,n]})\end{aligned}$$

(ii) *For all $n \in S$, the element $ndV_n([1]_{S/n})$.*

Proof. This follows from the proof of Theorem 6.1. □

We remark that in Addendum 6.8, the graded ring $\hat{\Omega}^{\cdot}_{\mathbb{W}_S(\mathbb{Z})}$ may be replaced by the graded ring $\Omega^{\cdot}_{\mathbb{W}_S(\mathbb{Z})}$.

Acknowledgements It is a pleasure to gratefully acknowledge the hospitality and financial support of the following institutions that I have visited while working on this paper: University of Rennes, Australian National University, the Lorentz Institute at Leiden, and the Mathematical Sciences Research Institute at Berkeley. I would also like to acknowledge the mathematical debt of the work presented here to my collaboration with Ib Madsen and to conversations with Jim Borger. Finally, I would like to thank an anonymous referee for many valuable suggestions on improving the exposition.

References

1. M. F. Atiyah and D. O. Tall, *Group representations, λ-rings and the J-homomorphism*, Topology **8** (1969), 253–297.
2. J. M. Beck, *Triples, algebras and cohomology*, Reprints in Theory and Applications of Categories **2** (2003), 1–59.
3. P. Berthelot, A. Grothendieck, and L. Illusie, *Théorie des intersections et théorème de Riemann-Roch*, Séminaire de Géometrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Lecture Notes in Math., vol. 225, Springer-Verlag, New York, 1971.
4. M. Bökstedt, W.-C. Hsiang, and I. Madsen, *The cyclotomic trace and algebraic K-theory of spaces*, Invent. Math. **111** (1993), 465–540.
5. J. Borger, *Lambda-rings and the field with one element*, arXiv:0906.3146.
6. ______, *The basic geometry of Witt vectors, I. The affine case*, Algebra Number Theory **5** (2011), 231–285.
7. J. Borger and B. Wieland, *Plethystic algebra*, Adv. Math. **194** (2005), 246–283.
8. P. Cartier, *Groupes formels associés aux anneaux de Witt*, C. R. Acad. Sci. Paris Sér. A–B **265** (1967), A49–A52.
9. V. Costeanu, *On the 2-typical de Rham-Witt complex*, Doc. Math. **13** (2008), 413–452.
10. T. Geisser and L. Hesselholt, *The de Rham-Witt complex and p-adic vanishing cycles*, J. Amer. Math. Soc. **19** (2006), 1–36.
11. A. Grothendieck, *La théorie des classes de Chern*, Bull. Soc. Math. France **86** (1958), 137–154.
12. M. Hazewinkel, *Formal groups and applications*, Pure and Applied Mathematics, vol. 78, Academic Press, 1978.
13. L. Hesselholt, *On the p-typical curves in Quillen's K-theory*, Acta Math. **177** (1997), 1–53.
14. ______, *Algebraic K-theory and trace invariants*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), Higher Ed. Press, Beijing, 2002, pp. 415–425.
15. L. Hesselholt and I. Madsen, *On the K-theory of nilpotent endomorphisms*, Homotopy methods in algebraic topology (Boulder, CO, 1999), Contemp. Math., vol. 271, Amer. Math. Soc., Providence, RI, 2001, pp. 127–140.
16. ______, *On the K-theory of local fields*, Ann. of Math. **158** (2003), 1–113.
17. ______, *On the de Rham-Witt complex in mixed characteristic*, Ann. Sci. École Norm. Sup. **37** (2004), 1–43.
18. H. L. Hiller, *λ-rings and algebraic K-theory*, J. Pure Appl. Algebra **20** (1981), 241–266.
19. L. Illusie, *Complexe de de Rham-Witt et cohomologie cristalline*, Ann. Scient. Éc. Norm. Sup. (4) **12** (1979), 501–661.
20. D. Knutson, *λ-rings and the representation theory of symmetric groups*, Lecture Notes in Math., vol. 308, Springer-Verlag, New York, 1973.
21. C. Kratzer, *λ-structure en K-théorie algébrique*, Comment. Math. Helv. **55** (1980), 233–254.
22. A. Langer and T. Zink, *De Rham-Witt cohomology for a proper and smooth morphism*, J. Inst. Math. Jussieu **3** (2004), 231–314.

23. S. MacLane, *Categories for the working mathematician*, Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1971.
24. D. Mumford, *Lectures on curves on an algebraic surface*, Annals of Mathematics Studies, vol. 59, Princeton University Press, Princeton, N.J., 1966.
25. J. Stienstra, *Cartier-Dieudonné theory for Chow groups*, J. Reine Angew. Math. **355** (1985), 1–66.
26. O. Teichmüller, *Über die Struktur diskret bewerteter perfekte Körper*, Nachr. Ges. Wiss. Göttingen, N.F. **1** (1936), 151–161.
27. W. van der Kallen, *Descent for the K-theory of polynomial rings*, Math. Z. **191** (1986), 405–415.
28. C. Wilkerson, *Lambda rings, binomial domains, and vector bundles over $CP(\infty)$*, Comm. Alg. **10** (1982), 311–328.
29. E. Witt, *Zyklische Körper und Algebren der Charakteristik p vom Grad p^n. Struktur diskret bewerteter perfekter Körper mit vollkommenem Restklassenkörper der Charakteristik p*, J. Reine Angew. Math. **176** (1936), 126–140.