On the K-theory of planar cuspical curves and a new family of polytopes

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Introduction

We let k be a regular $\mathbb{F}_p$-algebra, let $a, b \geqslant 2$ be relatively prime integers, and consider the coordinate ring $A = k[x,y]/(x^b - y^a)$ of the planar cuspical curve $y^a = x^b$. The algebraic K-groups of the ring A decompose as the direct sum

$$K_q(A) = K_q(k) \oplus K_q(A, \mathfrak{a})$$

of the algebraic K-groups of the ground ring k and the relative algebraic K-groups of A with respect to the ideal $\mathfrak{a} = (x, y)$. The purpose of this paper is to evaluate the relative groups $K_q(A, \mathfrak{a})$ in terms of the big de Rham-Witt groups of k. At the moment, the calculation depends on a conjecture of a combinatorial nature that we formulate below. We prove the conjecture in some low-dimensional cases. This leads to new unconditional results for K_2 and K_3.

To state the result, we first define

$$\ell(a,b,m) = \operatorname{card}\{(i,j) \in \mathbb{N} \times \mathbb{N} \mid ai + bj = m\}$$

to be the number of expressions of the positive integer m as a linear combination $m = ai + bj$ with (i, j) a pair of positive integers, and next define

$$S(a,b,r) = \{m \in \mathbb{N} \mid \ell(a,b,m) \leqslant r\} \subset \mathbb{N},$$

where r is a non-negative integer. We recall that for every subset $S \subset \mathbb{N}$ stable under division and every positive integer e, the big de Rham-Witt groups $\mathbb{W}_S\Omega_k^q$ and the Verschiebung maps $V_e \colon \mathbb{W}_{S/e}\Omega_k^q \to \mathbb{W}_S\Omega_k^q$ are defined; see [**16**].

Theorem A. *Let k be a regular $\mathbb{F}_p$-algebra and let $A = k[x,y]/(x^b - y^a)$ be the coordinate ring of a planar cuspical curve with $a, b \geqslant 2$ relatively prime integers and with p not dividing a. Let $\mathfrak{a} = (x, y) \subset A$ be the ideal defining the cusp. Assuming that Conjecture B below holds, there is a canonical long exact sequence*

$$\begin{aligned} \cdots \longrightarrow \bigoplus (\mathbb{W}_{S/b}\Omega_k^{q-2r}/V_a\mathbb{W}_{S/ab}\Omega_k^{q-2r}) &\xrightarrow{V_b} \bigoplus (\mathbb{W}_S\Omega_k^{q-2r}/V_a\mathbb{W}_{S/a}\Omega_k^{q-2r}) \\ \xrightarrow{\varepsilon} K_q(A,\mathfrak{a}) \xrightarrow{\partial} &\bigoplus (\mathbb{W}_{S/b}\Omega_k^{q-1-2r}/V_a\mathbb{W}_{S/ab}\Omega_k^{q-1-2r}) \longrightarrow \cdots, \end{aligned}$$

where the sums range over non-negative integers r and where $S = S(a,b,r)$.

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We remark that if neither a nor b are divisible by p, then the long exact sequence in the statement of Theorem A simplifies to a canonical isomorphism

$$\bigoplus_{r \geqslant 0} (\mathbb{W}_S \Omega_k^{q-2r} / (V_a \mathbb{W}_{S/a} \Omega_k^{q-2r} + V_b \mathbb{W}_{S/b} \Omega_k^{q-2r})) \xrightarrow{\varepsilon} K_q(A, \mathfrak{a})$$

with $S = S(a, b, r)$. Indeed, in this case, the maps V_a and V_b both are injective. Similarly, if k is a perfect field, whose characteristic $p > 0$ may or may not divide a or b, then Theorem A shows that for every even non-negative integer $q = 2r$, there is a canonical isomorphism

$$\mathbb{W}_S(k)/(V_a \mathbb{W}_{S/a}(k) + V_b \mathbb{W}_{S/b}(k)) \xrightarrow{\varepsilon} K_q(A, \mathfrak{a}),$$

where again $S = S(a, b, r)$, and that for every odd or negative integer q, the relative K-group in question vanishes. We show in Section 1 below that the domain of this isomorphism is a $\mathbb{W}(k)$-module of finite length $\frac{1}{2}(q+1)(a-1)(b-1)$. The precise structure of this $\mathbb{W}(k)$-module and, more generally, of the big de Rham-Witt groups that appear in Theorem A depends on the decomposition of S into "orbits" of the multiplication-by-p map.

We briefly outline the proof of Theorem A and refer to Section 4 below for more details. Let $B = k[t]$ and let $f \colon A \to B$, $g \colon A \to k$, and $h \colon B \to k$ be the k-algebra homomorphisms that map x and y to t^a and t^b, x and y to 0, and t to 0, respectively. We consider the following diagram in which the two right-hand horizontal maps are the cyclotomic trace maps from algebraic K-theory to topological cyclic homology defined by Bökstedt-Hsiang-Madsen [**2**].

$$\begin{tikzcd} K(A) \arrow[r, equal] \arrow[d, "h"] & K(A) \arrow[r, "\mathrm{tr}"] \arrow[d, "f"] & \mathrm{TC}(A;p) \arrow[d, "f"] \\ K(k) & K(B) \arrow[l, "g"'] \arrow[r, "\mathrm{tr}"] & \mathrm{TC}(B;p), \end{tikzcd}$$

Since k is regular, the map induced by g is a weak equivalence, by the fundamental theorem in algebraic K-theory [**30**, Theorem 8, Corollary], and hence the left-hand square is homotopy cartesian. Moreover, as we explain in Theorem 4.1 below, results of McCarthy [**28**] and of Geisser and the author [**10**] implies that the right-hand square, too, is homotopy cartesian. Hence, the mapping fiber $K(A, \mathfrak{a})$ of the map of K-theory spectra induced by h is canonically weakly equivalent to the mapping fiber of the map of topological cyclic homology spectra induced by f. The homotopy groups of the latter, can be evaluated by the methods developed by Madsen and the author in [**19, 21**], provided that the solution to the combinatorial problem that we now proceed to describe is as stated in Conjecture B below.

We fix a positive integer m. Let $\mathbb{T}$ be the circle group of complex numbers of modulus 1, let $C_m \subset \mathbb{T}$ be the subgroup of order m, and let $\zeta_m \in C_m$ be the generator $\exp(2\pi i/m)$, where i is a fixed square root of -1. Let $\mathbb{R}[C_m]$ be the real regular representation, and let $\Delta^{m-1} \subset \mathbb{R}[C_m]$ be the convex hull of $C_m \subset \mathbb{R}[C_m]$. We recall that Δ^{m-1} is a simplicial complex whose set of simplices is the set of all non-empty subsets $F \subset C_m$. We consider the sub-simplicial complex

$$\Sigma(a, b, m) \subset \Delta^{m-1}$$

whose set of simplices consists of the non-empty subsets $F \subset C_m$ satisfying the following condition: If $F = \{\zeta_m^{r_1}, \dots, \zeta_m^{r_k}\}$ with $0 \leqslant r_1 < \dots < r_k < m$, then the k

gaps $r_{s+1} - r_s$, where $1 \leqslant s < k$, and $r_1 + m - r_k$ all can be expressed as $ai + bj$ with (i,j) a pair of non-negative integers. We note that $\Sigma(a,b,m)$ is non-empty if and only if m can be expressed as $m = ai + bj$ with (i,j) a pair of non-negative integers. The left action by the group C_m on $\mathbb{R}[C_m]$ restricts to left C_m-actions on Δ^{m-1} and $\Sigma(a,b,m)$ and induces a left C_m-action on the quotient space

$$X(a,b,m) = \Delta^{m-1}/\Sigma(a,b,m).$$

The problem that we wish to solve is to determine the homotopy type of the induced pointed $\mathbb{T}$-space $\mathbb{T}_+ \wedge_{C_m} X(a,b,m)$.

To state our conjectured solution to this problem, we suppose that $a < b$ and choose a pair of integers (c,d) with the property that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}),$$

this being possible since a and b are relatively prime. The equation

$$\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix}$$

defines a bijection from the set of pairs of positive integers (i,j) with $m = ai + bj$ onto the set of integers $n = ci + dj$ in the open interval $(cm/a, dm/b)$. We define $\mathbb{C}(n)$ to be $\mathbb{C}$ considered as a real C_m-representation with $z \in C_m$ acting through multiplication by z^n and define $\lambda(a,b,m)$ to be the direct sum

$$\lambda(a,b,m) = \bigoplus_{n \in (cm/a, dm/b) \cap \mathbb{Z}} \mathbb{C}(n)$$

as n ranges over the integers in the open interval $(cm/a, dm/b)$. We further consider the pointed C_m-space defined as follows.

$$Y(a,b,m) = \begin{cases} S^{\lambda(a,b,m)} & \text{if } a \text{ and } b \text{ do not divide } m \\ \widetilde{C}_a \wedge S^{\lambda(a,b,m)} & \text{if } a \text{ but not } b \text{ divides } m \\ S^{\lambda(a,b,m)} \wedge \widetilde{C}_b & \text{if } b \text{ but not } a \text{ divides } m \\ \widetilde{C}_a \wedge S^{\lambda(a,b,m)} \wedge \widetilde{C}_b & \text{if both } a \text{ and } b \text{ divide } m \end{cases}$$

Here $S^{\lambda(a,b,m)}$ is the one-point compactification of $\lambda(a,b,m)$, $\widetilde{C}_a$ is the mapping cone of the map $C_{a+} \to S^0$ that collapses C_a to the non-basepoint in S^0, and C_m acts on C_a through the (m/a)th power map. We note that the real C_m-representation $\lambda(a,b,m)$ and the pointed C_m-space $Y(a,b,m)$ do not depend on the choice of (c,d).

In general, if Z is a C_m-space and $m = st$, then the subspace Z^{C_s} fixed by the subgroup $C_s \subset C_m$ is a C_m/C_s-space and we write $\rho_s^* Z^{C_s}$ for this space viewed as a C_t-space with C_t acting through the sth root $\rho_s \colon C_t \to C_m/C_s$. Now, the family of pointed C_m-spaces $X(a,b,m)$ (resp. $Y(a,b,m)$) comes equipped with canonical isomorphisms, whenever $m = st$, of pointed C_t-spaces

$$\rho_s^* X(a,b,m)^{C_s} \xrightarrow{r_{X,s}} X(a,b,t) \qquad (\text{resp. } \rho_s^* Y(a,b,m)^{C_s} \xrightarrow{r_{Y,s}} Y(a,b,t)\ ).$$

The inverse of $r_{X,s}$ takes the class of the point $\sum a_v v$ of the face $F \subset C_t$ to the class of the point $(1/s)\sum a_{z^s} z$ of the face $F^{1/s} = \{z \in C_m \mid z^s \in F\} \subset C_m$, and the inverse of $r_{Y,s}$ is induced by the isomorphism $\lambda(a,b,t) \to \rho_s^* \lambda(a,b,m)^{C_s}$ that takes the summand indexed by $n \in (ct/a, dt/b)$ to the summand indexed by $ns \in (cm/a, dm/b)$ by the identity map $\mathrm{id} \colon \mathbb{C}(n) \to \rho_s^* \mathbb{C}(ns)^{C_s}$.

CONJECTURE B. *Given relative prime integers $1 < a < b$, there exists a family indexed by positive integers m of maps of pointed C_m-spaces*

$$X(a,b,m) \xrightarrow{u(a,b,m)} Y(a,b,m)$$

with the following properties.

(1) *If $m = st$, then the diagram of pointed C_m-spaces*

$$\begin{tikzcd} \rho_s^* X(a,b,m)^{C_s} \arrow[r, "\rho_s^* u(a,b,m)^{C_s}"] \arrow[d, "r_{X,s}"] & \rho_s^* Y(a,b,m)^{C_s} \arrow[d, "r_{Y,s}"] \\ X(a,b,t) \arrow[r, "u(a,b,t)"] & Y(a,b,t) \end{tikzcd}$$

is homotopy commutative.

(2) *The induced maps of pointed $\mathbb{T}$-spaces*

$$\mathbb{T}_+ \wedge_{C_m} X(a,b,m) \xrightarrow{\mathrm{id} \wedge u(a,b,m)} \mathbb{T}_+ \wedge_{C_m} Y(a,b,m)$$

induce isomorphisms of reduced singular homology groups.

We briefly discuss Conjecture B and refer to Section 6 for details. The singular homology groups of the domain and target of the map in (2) are finitely generated and are known to be abstractly isomorphic; see Corollary 5.2. Hence, it suffices to define a family of maps $u(a,b,m)$ that satisfy (1) and to show that the induced maps in (2) induce surjections of homology groups. By contrast, we do not know how to evaluate the homology groups of the spaces $X(a,b,m)$, although computer calculations by Welker [**35**] suggest that the maps $u(a,b,m)$ are weak equivalences. In Section 6, we formulate the combinatorial Conjecture 6.2 and, assuming this conjecture, construct maps $u(a,b,m)$ that satisfy (1). We prove in Proposition 6.3 that Conjecture 6.2 holds for all positive integers m with $\ell(a,b,m) \leqslant 1$. Using this result, we prove in Proposition 6.4 that Conjecture B holds for all positive integers m such that either $\ell(a,b,m) = 0$ or $\ell(a,b,m) = 1$ and neither a nor b divides m. This, in turn, implies the following unconditional result, which extends earlier calculations by Krusemeyer [**25**, Proposition 12.1] of K_1.

THEOREM C. *The long exact sequence in Theorem A is valid for all $q \leqslant 2$. If p divides neither a nor b, then the sequence is valid for all $q \leqslant 3$.*

Proving Conjecture B would likely require an understanding of the facet structure of the stunted regular cyclic polytopes $P(a,b,m)$ that we introduce in Section 6 below. At the moment, this important problem is completely open. However, since algebraic K-theory has a tendency to suggest deep yet solvable problems, one may well hope that a complete solution can be found.

We finally mention that for k a field of characteristic zero, the cyclic homology groups of A were calculated by Geller, Reid, and Weibel [**12**, Theorem 9.2] and that, in view of the affirmation by Cortiñas [**4**] of the KABI-conjecture made in [**12**], this gives a complete calculation of the groups $K_q(A, \mathfrak{a})$.

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1. Big de Rham-Witt forms

In this section, we recall the groups $\mathbb{W}_S\Omega_k^q$ of big de Rham-Witt forms that appear in Theorem A in the introduction. These groups were introduced in [**21**], but a better and more direct construction is given in [**16**].

The big de Rham-Witt complex of a commutative ring k is the initial example of a rather complex algebraic structure called a Witt complex. To state the definition, we say that a subset $S \subset \mathbb{N}$ of the set of positive integers is a truncation set if whenever $m = st \in S$ then both $s \in S$ and $t \in S$. We consider the truncation sets as the objects of a category with a single morphism from S to T if $S \subset T$ and recall that there is a contravariant functor that to a truncation set S associates the ring $\mathbb{W}_S(k)$ of big Witt vectors in k indexed by S. If n is a positive integer and S a truncation set, then one defines $S/n = \{s \in \mathbb{N} \mid ns \in S\}$. It is a truncation set and there are natural maps $F_n \colon \mathbb{W}_S(k) \to \mathbb{W}_{S/n}(k)$ and $V_n \colon \mathbb{W}_{S/n}(k) \to \mathbb{W}_S(k)$ called the nth Frobenius and the nth Verschiebung maps, respectively.

The following definition is [**16**, Definition 4.1].

Definition 1.1. Let k be a commutative ring. A Witt complex E over k is a contravariant functor that to a truncation set S associates an anti-symmetric graded ring E_S^* and takes colimits to limits together with a natural map of rings

$$\mathbb{W}_S(k) \xrightarrow{\eta_S} E_S^0$$

and natural maps of graded abelian groups

$$E_S^q \xrightarrow{d} E_S^{q+1} \qquad E_S^q \xrightarrow{F_n} E_{S/n}^q \qquad E_{S/n}^q \xrightarrow{V_n} E_S^q \qquad (n \in \mathbb{N})$$

such that the following (1)–(5) hold.

(1) If $\omega \in E_S^q$ and $\omega' \in E_S^{q'}$ then

$$\begin{aligned} d(\omega \cdot \omega') &= d\omega \cdot \omega' + (-1)^q \omega \cdot d\omega' \\ d(d(\omega)) &= d\log \eta_S([-1]_S) \cdot d\omega. \end{aligned}$$

(2) If m and n are positive integers, then

$$\begin{aligned} &F_1 = V_1 = \mathrm{id}; \qquad F_m F_n = F_{mn}; \qquad V_n V_m = V_{mn}; \\ &F_n V_n = n\,\mathrm{id}; \qquad F_m V_n = V_n F_m \quad \text{if } (m,n) = 1; \\ &F_n \eta_S = \eta_{S/n} F_n; \qquad V_n \eta_{S/n} = \eta_S V_n. \end{aligned}$$

(3) If n is a positive integer, $\omega \in E_S^q$, $\omega' \in E_S^{q'}$, and $\omega \in E_{S/n}^{q''}$, then

$$F_n(\omega \cdot \omega') = F_n(\omega) \cdot F_n(\omega'), \qquad \omega \cdot V_n(\omega'') = V_n(F_n(\omega) \cdot \omega'').$$

(4) If n is a positive integer and $\omega \in E_S^q$, then

$$F_n d V_n(\omega) = d\omega + (n-1) d\log \eta_{S/n}([-1]_{S/n}) \cdot \omega.$$

(5) If n is a positive integer, S a truncation set, and $a \in k$, then

$$F_n d\eta_S([a]_S) = \eta_{S/n}([a]_{S/n}^{n-1}) d\eta_{S/n}([a]_{S/n}),$$

where $[a]_S \in \mathbb{W}_S(k)$ is the Teichmüller representative.

A map of Witt complexes over k is a natural map $f_S \colon E_S^* \to E_S'^*$ of graded rings such that $f_S \eta_S = \eta_S'$, $d' f_S = f_S d$, $f_{S/n} F_n = F_n' f_S$, and $f_S V_n = V_n' f_{S/n}$.

Remark 1.2. If m is positive integer, then we define $\langle m\rangle$ to be the truncation set of divisors of m. We note that if also n is a positive integer, then $\langle m\rangle/n = \langle m/n\rangle$, if n divides m, and $\langle m\rangle/n = \emptyset$, otherwise. Hence, it makes sense to restrict a Witt complex E to the truncation sets of the form $\langle m\rangle$. This restricted Witt complex, in turn, determines E up to unique isomorphism. Indeed, every truncation set S is equal to the union of the truncation sets $\langle m\rangle$ with $m \in S$, and hence,

$$E_S^q = \lim_{m\in S} E^q_{\langle m\rangle}$$

with S ordered under division.

The big de Rham-Witt complex $\mathbb{W}_S\Omega_k^*$ is defined, up to unique isomorphism, to be an initial object in the category of Witt complexes over k. An explicit construction is given in [**16**, Section 4]. It is proved in loc. cit., Addendum 4.8, that the canonical maps $\eta_{\{1\}} \colon \Omega_k^* \to \mathbb{W}_{\{1\}}\Omega_k^*$ and $\eta_S \colon \mathbb{W}_S(k) \to \mathbb{W}_S\Omega_k^0$ from the differential graded ring of de Rham forms and the ring of big Witt vectors, respectively, are isomorphisms; whence the naming. The element $d\log \eta_S([-1]_S)$ is a manifestation of the Hopf class η. It vanishes if 2 is either invertible or nilpotent in k. Finally, we write $R^S_T \colon \mathbb{W}_S\Omega_k^* \to \mathbb{W}_T\Omega_k^*$ for the map of graded rings induced by the inclusion of truncation sets $T \subset S$ and call it the restriction map. It is a surjective map.

We now let p be a prime number and assume that k is a $\mathbb{Z}_{(p)}$-algebra. In this case, the big de Rham-Witt groups admit the following p-typical decomposition. Let $P \subset \mathbb{N}$ be the truncation set consisting of the powers of p. We say that a truncation set S is p-typical if $S \subset P$. The finite p-typical truncation sets are the empty set $\emptyset$ and the sets $\langle p^{v-1}\rangle = \{1, p, \dots, p^{v-1}\}$ with v a positive integer. If S is any truncation set, then we define $S' \subset S$ to be the sub-truncation set consisting of the elements $e \in S$ that are not divisible by p. In this situation, the map

$$\mathbb{W}_S\Omega_k^q \xrightarrow{\gamma} \prod\nolimits_{e\in S'} \mathbb{W}_{(S/e)\cap P}\Omega_k^q \tag{1.3}$$

whose eth component is the composite map

$$\mathbb{W}_S\Omega_k^q \xrightarrow{F_e} \mathbb{W}_{S/e}\Omega_k^q \xrightarrow{R} \mathbb{W}_{(S/e)\cap P}\Omega_k^q$$

is an isomorphism; see [**21**, Corollary 1.2.6]. The decomposition may be memorized by noting that S is the disjoint union of the orbits $S \cap eP$ of multiplication by p and that multiplication by e defines a bijection of the p-typical truncation set $(S/e) \cap P$ onto the orbit $S \cap eP$. We also spell out the p-typical decomposition of the maps F_s, V_s, and R^S_T. We write $s = p^v s'$ with s' not divisible by p. The following three square diagrams with F_s^γ, V_s^γ, and $(R^S_T)^\gamma$ defined below commute.

$$\begin{tikzcd}
\mathbb{W}_S\Omega_k^q \arrow[r, "\gamma"] \arrow[d, shift right, "F_s"'] & \prod \mathbb{W}_{(S/e)\cap P}\Omega_k^q \arrow[d, shift right, "F_s^\gamma"'] \\
\mathbb{W}_{S/s}\Omega_k^q \arrow[r, "\gamma"] \arrow[u, shift right, "V_s"'] & \prod \mathbb{W}_{(S/se)\cap P}\Omega_k^q \arrow[u, shift right, "V_s^\gamma"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{W}_S\Omega_k^q \arrow[r, "\gamma"] \arrow[d, "R^S_T"] & \prod \mathbb{W}_{(S/e)\cap P}\Omega_k^q \arrow[d, "(R^S_T)^\gamma"] \\
\mathbb{W}_T\Omega_k^q \arrow[r, "\gamma"] & \prod \mathbb{W}_{(T/e)\cap P}\Omega_k^q
\end{tikzcd}$$

Here, the map F_s^γ takes the factor indexed by $e \in S' \cap s'\mathbb{N}$ to the factor indexed by $e/s' \in (S/s)'$ by the map F_{p^v} and annihilates the factors indexed by $e \in S' \smallsetminus s'\mathbb{N}$. The map V_s^γ takes the factor indexed by $e \in (S/s)'$ to the factor indexed by $s'e \in S'$ by the map $s'V_{p^v}$. Finally, the map $(R^S_T)^\gamma$ takes the factor indexed by $e \in T' \subset S'$ to the factor indexed by $e \in T'$ by the map $R^{(S/e)\cap P}_{(T/e)\cap P}$ and annihilates the factors indexed by $e \in S' \smallsetminus T'$.

We also remark that for any commutative ring k, the group

$$W_v\Omega_k^q = \mathbb{W}_{\{1,p,\dots,p^{v-1}\}}\Omega_k^q$$

agrees, up to unique isomorphism, with the p-typical de Rham-Witt group defined in [**23**] for p odd and in [**5**] for $p = 2$. If k is an $\mathbb{F}_p$-algebra, then the common group further agrees, up to unique isomorphism, with the classical p-typical de Rham-Witt group defined in [**24**]. If k is a regular $\mathbb{F}_p$-algebra, then the structure of the wth graded piece $\operatorname{gr}^w W_v\Omega_k^q$ for the descending filtration of $W_v\Omega_k^q$ given by the kernels $\operatorname{Fil}^w W_v\Omega_k^q = V^w W_{v-w}\Omega_k^q + dV^w W_{v-w}\Omega_k^q \subset W_v\Omega_k^q$ of the restriction maps $W_v\Omega_k^q \to W_w\Omega_k^q$ is determined in [**24**, Corollary I.3.9] in terms of the groups of de Rham forms Ω_k^q and the Cartier operator. We also remark that if k is a perfect $\mathbb{F}_p$-algebra, then $W_v\Omega_k^q$ vanishes for all $q > 0$, by op. cit., Proposition I.1.6.

We will now examine the truncation sets $S(a,b,r)$ defined in the introduction and the corresponding groups of big de Rham-Witt forms $\mathbb{W}_{S(a,b,r)}\Omega_k^q$ in more detail. So let again $1 < a < b$ be a fixed pair of relatively prime integers. We choose a pair of integers (c,d) with $ad - bc = 1$ and recall that the number $\ell(a,b,m)$ of ways in which m can be written as $m = ai + bj$ with (i,j) a pair of positive integers is equal to the number of integers in the open interval $(cm/a, dm/b)$. We first establish some basic properties of the function $\ell(a,b,m)$ and the truncation set $S(a,b,r)$.

Lemma 1.4. *The function that to a positive integer m associates the non-negative integer $\ell(a,b,m)$ has the following properties.*

(1) $\ell(a,b,m+ab) = \ell(a,b,m) + 1$.
(2) *If $rab+1 \leqslant m \leqslant (r+1)ab$ with r a non-negative integer, then $\ell(a,b,m) = r$ or $\ell(a,b,m) = r+1$. If, in addition, m is divisible by a or b, then $\ell(a,b,m) = r$.*
(3) *There are exactly ab positive integers m with $\ell(a,b,m) = r \geqslant 1$.*
(4) *There are exactly $\frac{1}{2}(a+1)(b+1) - 1$ positive integers m with $\ell(a,b,m) = 0$.*
(5) *If s is a divisor in m, then $\ell(a,b,s) \leqslant \ell(a,b,m)$.*

Proof. (1) The values $\ell(a,b,m)$ and $\ell(a,b,m+ab)$ are the numbers of integers in the open intervals $(cm/a, dm/b)$ and $(cm/a + bc, dm/b + ad)$, respectively. Since bc and ad are consecutive integers, $\ell(a,b,m+ab) = \ell(a,b,m) + 1$ as stated.

(2) The length of the open interval $(cm/a, dm/b)$ is equal to m/ab which is strictly larger than r and less than or equal to $r+1$. Therefore, the number $\ell(a,b,m)$ of integers in this interval is either r or $r+1$. If m is divisible by either a or b, then one or both end points of the interval are integers. Therefore, in this case, it contains precisely r integers.

(3) By (2), the positive integers m that satisfies $\ell(a,b,m) = r$ are among the $2ab$ integers $(r-1)ab + 1 \leqslant m \leqslant (r+1)ab$. Moreover, on the first half (resp. the second half), the value of $\ell(a,b,m)$ is either $r-1$ or r (resp. either r or $r+1$). Finally, by (1), the number of integers $(r-1)ab+1 \leqslant m \leqslant rab$ with $\ell(a,b,m) = r-1$ is equal to the number of integers $rab + 1 \leqslant m \leqslant (r+1)ab$ with $\ell(a,b,m) = r$. Therefore, the total number of integers m with $\ell(a,b,m) = r$ is equal to ab as stated.

(4) It follows from (2) that all positive integers m with $\ell(a,b,m) = 0$ are among the ab integers $1 \leqslant m \leqslant ab$. Moreover, on said ab integers, the value $\ell(a,b,m)$ is either 0 or 1 and is equal to the coefficient c_m in the polynomial

$$f(t) = \sum_{0<m<2ab} c_m t^m = (t^a + t^{2a} + \dots + t^{a(b-1)})(t^b + t^{2b} + \dots + t^{(a-1)b}).$$

The coefficients in this polynomial satisfy $c_m = c_{2ab-m}$ because $f(t)t^{-2ab} = f(t^{-1})$. Since $c_{ab} = \ell(a,b,ab) = 0$, we find by setting $t = 1$ that

$$\sum_{0<m<ab} c_m = \frac{1}{2} \sum_{0<m<2ab} c_m = \frac{1}{2}(a-1)(b-1).$$

But this is the number of integers $1 \leqslant m \leqslant ab$ with $\ell(a,b,m) = 1$, and therefore, the number of integers $1 \leqslant m \leqslant ab$ with $\ell(a,b,m) = 0$ is equal to

$$ab - \frac{1}{2}(a-1)(b-1) = \frac{1}{2}(a+1)(b+1) - 1$$

as stated.

(5) Let s be a divisor in m and let $t = m/s$. If (i,j) is a pair of positive integers with $s = ai + bj$, then (it, jt) is a pair of positive integers with $m = ait + bit$, and hence, $\ell(a,b,s) \leqslant \ell(a,b,m)$ as stated. $\square$

Remark 1.5. We let $f\colon A \to B$ be the k-algebra homomorphism considered in the introduction. The conductor ideal from A to B is defined to be the largest ideal $I \subset A$ such that also $f(I) \subset B$ is an ideal. It is generated by t^v, where v is smallest with the property that $t^m \in f(A)$ for all $m \geqslant v$. This, in turn, means that v is smallest with the property that every $m \geqslant v$ can be expressed as a linear combination $m = ai + bj$ with (i,j) a pair of non-negative integers, or equivalent, that $v + a + b$ is smallest with the property that every $m \geqslant v + a + b$ can be expressed as a linear combination $m = ai + bj$ with (i,j) a pair of positive integers. Lemma 1.4 (2) shows, in particular, that $v + a + b = ab + 1$, and therefore, we conclude that $v = (a-1)(b-1)$ as was first noted by Sylvester [**33**].

It follows from Lemma 1.4 (5) that the subsets

$$S(a,b,r) = \{m \in \mathbb{N} \mid l(a,b,m) \leqslant r\} \subset \mathbb{N}$$

considered in the introduction are truncation sets. We record their cardinalities.

Corollary 1.6. *Let r be a non-negative integer.*

(1) $\operatorname{card}(S(a,b,r)) = \frac{1}{2}(a+1)(b+1) - 1 + rab.$
(2) $\operatorname{card}(S(a,b,r)/a) = (r+1)b.$
(3) $\operatorname{card}(S(a,b,r)/b) = (r+1)a.$
(4) $\operatorname{card}(S(a,b,r)/ab) = r+1.$

Proof. The statement (1) follows from Lemma 1.4, (3)–(4). To prove (2), we note that multiplication by a defines a bijection of $S(a,b,r)/a$ onto $S(a,b,r) \cap a\mathbb{N}$. Lemma 1.4 (2) shows immediately that the latter set has cardinality $(r+1)b$, which proves (2). The proofs of (3) and (4) are analogous. $\square$

Corollary 1.7. *Let r be a non-negative integer, let $S = S(a,b,r)$, and let k be a commutative ring. The $\mathbb{W}(k)$-module $\mathbb{W}_S(k)/(V_a\mathbb{W}_{S/a}(k) + V_b\mathbb{W}_{S/b}(k))$ has finite length $\frac{1}{2}(2r+1)(a-1)(b-1)$.*

Proof. In general, if T is a finite truncation set, then

$$\operatorname{length}_{\mathbb{W}(k)}(\mathbb{W}_T(k)) = \operatorname{card}(T).$$

Here $\mathbb{W}_T(k)$ is viewed as a $\mathbb{W}(k)$-module via $R^{\mathbb{N}}_T\colon \mathbb{W}(k) \to \mathbb{W}_T(k)$. In the case at hand, if $T \subset S$ is the sub-truncation set of elements $m \in S$ that are not divisible by either a or b, then the restriction map R^S_T induces an isomorphism

$$\mathbb{W}_S(k)/(V_a\mathbb{W}_{S/a}(k) + V_b\mathbb{W}_{S/b}(k)) \xrightarrow{\sim} \mathbb{W}_T(k),$$

and by Corollary 1.6,

$$\operatorname{card}(T) = \operatorname{card}(S) - \operatorname{card}(S/a) - \operatorname{card}(S/b) + \operatorname{card}(S/ab)$$

is equal to $\frac{1}{2}(2r+1)(a-1)(b-1)$ as stated. $\square$

Example 1.8. To illustrate the above, we let k be a regular $\mathbb{F}_p$-algebra and use Theorem A to evaluate the groups $K_q(A,\mathfrak{a})$ in low degrees for $(a,b) = (2,3)$. Our results agree with and extend earlier results of Krusemeyer [**25**, Proposition 12.1]. We proceed to evaluate the long exact sequence in Theorem A in low degrees, beginning with the case $p \neq 2$. Since $S(2,3,0) = \{1,2,3,4,6\}$, the sequence begins

$$\begin{aligned} K_2(A,\mathfrak{a}) &\xrightarrow{\partial} \mathbb{W}_{\{1\}}\Omega^1_k \xrightarrow{V_3} \mathbb{W}_{\{1,3\}}\Omega^1_k \xrightarrow{\varepsilon} K_1(A,\mathfrak{a}) \\ &\xrightarrow{\partial} \mathbb{W}_{\{1\}}\Omega^0_k \xrightarrow{V_3} \mathbb{W}_{\{1,3\}}\Omega^0_k \xrightarrow{\varepsilon} K_0(A,\mathfrak{a}) \longrightarrow 0. \end{aligned}$$

In the bottom line, the map V_3 is injective with cokernel

$$\mathbb{W}_{\{1,3\}}\Omega^0_k \xrightarrow{R^{\{1,3\}}_{\{1\}}} \mathbb{W}_{\{1\}}\Omega^0_k = k.$$

If also $p \neq 3$, then the map V_3 in the top line is injective with the canonical retraction $\frac{1}{3}F_3$ and with cokernel the restriction map

$$\mathbb{W}_{\{1,3\}}\Omega^1_k \xrightarrow{R^{\{1,3\}}_{\{1\}}} \mathbb{W}_{\{1\}}\Omega^1_k = \Omega^1_k.$$

Indeed, in general, the kernel of the restriction map is

$$V_3\mathbb{W}_{\{1\}}\Omega^1_k + dV_3\mathbb{W}_{\{1\}}\Omega^0_k \subset \mathbb{W}_{\{1,3\}}\Omega^1_k,$$

and since $p \neq 3$, we have $dV_3 = \frac{1}{3}V_3 d$. However, if $p = 3$, then the map V_3 in the top line of the long exact sequence above need not be injective and its cokernel may be larger than Ω^1_k.

We next assume that $p \neq 3$. Interchanging the rôle of a and b, the long exact sequence in Theorem A now begins

$$\begin{aligned} K_2(A,\mathfrak{a}) &\xrightarrow{\partial} \mathbb{W}_{\{1,2\}}\Omega^1_k \xrightarrow{V_2} \mathbb{W}_{\{1,2,4\}}\Omega^1_k \xrightarrow{\varepsilon} K_1(A,\mathfrak{a}) \\ &\xrightarrow{\partial} \mathbb{W}_{\{1,2\}}\Omega^0_k \xrightarrow{V_2} \mathbb{W}_{\{1,2,4\}}\Omega^0_k \xrightarrow{\varepsilon} K_0(A,\mathfrak{a}) \longrightarrow 0. \end{aligned}$$

The map V_2 is the bottom line is injective with cokernel

$$\mathbb{W}_{\{1,2,4\}}\Omega^0_k \xrightarrow{R^{\{1,2,4\}}_{\{1\}}} \mathbb{W}_{\{1\}}\Omega^0_k = k.$$

Moreover, if also $p \neq 2$, then the map V_2 in the top line is injective with the canonical retraction $\frac{1}{2}F_2$ and with cokernel

$$\mathbb{W}_{\{1,2,4\}}\Omega^1_k \xrightarrow{R^{\{1,2,4\}}_{\{1\}}} \mathbb{W}_{\{1\}}\Omega^1_k = \Omega^1_k.$$

However, if $p = 2$, then the map V_2 in the top line is not necessarily injective and its cokernel may be larger than Ω^1_k.

2. The groups $\mathrm{TR}^m_{q-\lambda}(k)$

The purpose of this section is to determine the structure of the equivariant homotopy groups $\mathrm{TR}^m_{q-\lambda}(k)$ of the topological Hochschild $\mathbb{T}$-spectrum $T(k)$. This is mostly a recollection of results from [**20, 21, 18**], but as a new contribution, we exhibit an explicit generator of the cyclic $\mathbb{W}(\mathbb{F}_p)$-module $\mathrm{TR}^m_{q-\lambda}(\mathbb{F}_p)$.

Let k be a unital associative ring and let $T(k)$ be the associated topological Hochschild $\mathbb{T}$-spectrum. For every positive integer m, every integer q, and every finite dimensional orthogonal $\mathbb{T}$-representation λ, we define

$$\mathrm{TR}^m_{q-\lambda}(k) = [S^q \wedge (\mathbb{T}/C_m)_+, S^\lambda \wedge T(k)]_{\mathbb{T}}$$

to be the abelian group of maps in the $\mathbb{T}$-stable homotopy category. If $m = st$, then the canonical projection of $\mathbb{T}/C_t$ onto $\mathbb{T}/C_m$ and the associated equivariant transfer $S^q \wedge (\mathbb{T}/C_m)_+ \to S^q \wedge (\mathbb{T}/C_t)_+$ induce group homomorphisms

$$\mathrm{TR}^m_{q-\lambda}(k) \xrightarrow{F_s} \mathrm{TR}^t_{q-\lambda}(k) \qquad \mathrm{TR}^t_{q-\lambda}(k) \xrightarrow{V_s} \mathrm{TR}^m_{q-\lambda}(k)$$

called the sth Frobenius and the sth Verschiebung. Similarly, Connes' operator

$$\mathrm{TR}^m_{q-\lambda}(k) \xrightarrow{d} \mathrm{TR}^m_{q+1-\lambda}(k)$$

is induced by a map $\delta\colon S^{q+1} \wedge (\mathbb{T}/C_m)_+ \to S^q \wedge (\mathbb{T}/C_m)_+$ in the $\mathbb{T}$-stable category; see [**23**, Section 2.2]. The restriction map

$$\mathrm{TR}^m_{q-\lambda}(k) \xrightarrow{R_s} \mathrm{TR}^t_{q-\lambda'}(k)$$

is more complicated and uses the cyclotomic structure of $T(k)$; see [**23**, Section 2.3]. Here, and below, we use the abbreviation

$$\lambda' = \rho_s^*(\lambda^{C_s}).$$

If the ring k is commutative, then $T(k)$ inherits the structure of a commutative ring $\mathbb{T}$-spectrum. Hence, for fixed m, the groups $\mathrm{TR}^m_q(k)$ form an anti-symmetric graded ring, and the groups $\mathrm{TR}^m_{q-\lambda}(k)$ form a graded (right) module over this graded ring. It is proved in [**17**, Section 1] that if $m = st$, then $F_s, R_s\colon \mathrm{TR}^m_*(k) \to \mathrm{TR}^t_*(k)$ are graded ring homomorphisms, and $V_s\colon \mathrm{TR}^t_{*-\lambda}(k) \to \mathrm{TR}^m_{*-\lambda}(k)$ becomes a map of graded $\mathrm{TR}^m_*(k)$-modules when the domain is viewed as a graded $\mathrm{TR}^m_*(k)$-module via $F_s\colon \mathrm{TR}^m_*(k) \to \mathrm{TR}^t_*(k)$. Moreover, by [**20**, Addendum 3.3], there are canonical natural ring isomorphisms $\eta_{\langle m\rangle}\colon \mathbb{W}_{\langle m\rangle}(k) \to \mathrm{TR}^m_0(k)$ which are compatible with the respective restriction, Frobenius, and Verschiebung maps. In fact, the groups

$$E^q_{\langle m\rangle} = \mathrm{TR}^m_q(k)$$

and the various structure maps defined above define a Witt complex E over k, unique up to unique isomophism; see [**17**, Section 1] and Remark 1.2. Hence, by the universal property of the big de Rham-Witt complex, there is a unique map

$$\mathbb{W}_{\langle m\rangle}\Omega^q_k \xrightarrow{\eta_{\langle m\rangle}} \mathrm{TR}^m_q(k) \tag{2.1}$$

of Witt complexes over k. This map is of a nature similar to the canonical map from the Milnor K-theory of a field to the Quillen K-theory of the field.

We now let p be a prime number and assume that the ring k is a $\mathbb{Z}_{(p)}$-algebra, not necessarily commutative. In this case, the groups and maps considered above admit the following p-typical decomposition. We define

$$\mathrm{TR}^v_{q-\lambda}(k;p) = \mathrm{TR}^{p^{v-1}}_{q-\lambda}(k)$$

and abbreviate $F = F_p$, $V = V_p$, and $R = R_p$. We write $m = p^{v-1}m'$ with m' not divisible by p. In this situation, we recall from [**1**, Proposition 2.1] that the map

$$\mathrm{TR}^m_{q-\lambda}(k) \xrightarrow{\gamma} \prod_{e\in\langle m'\rangle} \mathrm{TR}^v_{q-\lambda'}(k;p) \tag{2.2}$$

whose eth component is the composite map

$$\mathrm{TR}^m_{q-\lambda}(k) \xrightarrow{F_e} \mathrm{TR}^{m/e}_{q-\lambda}(k) \xrightarrow{R_h} \mathrm{TR}^{p^{v-1}}_{q-\lambda'}(k)$$

is an isomorphism. Here $h = m'/e$. We caution that λ' depends on e. Suppose that $m = st$ and write $s = p^w s'$ and $t = p^{v-w-1}t'$ with s' and t' not divisible by p. We also recall from loc. cit. that there are three commutative square diagrams

$$\begin{tikzcd}
\mathrm{TR}^m_{q-\lambda}(k) \arrow[r, "\gamma"] \arrow[d, shift right, "F_s"'] & \prod \mathrm{TR}^v_{q-\lambda'}(k;p) \arrow[d, shift right, "F^\gamma_s"'] \\
\mathrm{TR}^t_{q-\lambda}(k) \arrow[r, "\gamma"] \arrow[u, shift right, "V_s"'] & \prod \mathrm{TR}^{v-w}_{q-\lambda'}(k;p) \arrow[u, shift right, "V^\gamma_s"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{TR}^m_{q-\lambda}(k) \arrow[r, "\gamma"] \arrow[d, "R_s"] & \prod \mathrm{TR}^v_{q-\lambda'}(k;p) \arrow[d, "R^\gamma_s"] \\
\mathrm{TR}^t_{q-\lambda'}(k) \arrow[r, "\gamma"] & \prod \mathrm{TR}^{v-w}_{q-\lambda''}(k;p)
\end{tikzcd}$$

with F^γ_s, V^γ_s, and R^γ_s defined as follows. The map F^γ_s takes the factor indexed by $e \in \langle m'\rangle \cap s'\,\mathbb{N}$ to the factor indexed by $e/s' \in \langle t'\rangle$ by the map F^v and annihilates the factors indexed by $e \in \langle m'\rangle \smallsetminus s'\,\mathbb{N}$. The map V^γ_s takes the factor indexed by $e \in \langle t'\rangle$ to the factor indexed by $s'e \in \langle m'\rangle$ by the map $s'\,V^v$. Finally, the map R^γ_s takes the factor indexed $e \in \langle t'\rangle \subset \langle m'\rangle$ to the factor indexed by $e \in \langle t'\rangle$ by the map R^v and annihilates the factors indexed by $e \in \langle m'\rangle \smallsetminus \langle t'\rangle$.

Suppose that the $\mathbb{Z}_{(p)}$-algebra k is commutative. We recall from (1.3) that in this case the big de Rham-Witt groups admit a similar p-typical decomposition. In fact, since the map (2.1) is a map of Witt complexes and since $\langle m\rangle' = \langle m'\rangle$, the following diagram, where we abbreviate $\eta_v = \eta_{\langle p^{v-1}\rangle}$, commutes.

$$\begin{tikzcd}
\mathbb{W}_{\langle m\rangle}\Omega^q_k \arrow[r, "\gamma"] \arrow[d, "\eta_{\langle m\rangle}"] & \prod_{e\in\langle m'\rangle} W_v\Omega^q_k \arrow[d, "\prod \eta_v"] \\
\mathrm{TR}^m_q(k) \arrow[r, "\gamma"] & \prod_{e\in\langle m'\rangle} \mathrm{TR}^v_q(k;p)
\end{tikzcd}$$

It follows that the $\mathbb{W}_{\langle m\rangle}(k)$-module $\mathrm{TR}^m_{q-\lambda}(k)$ is completely determined by the collection of $W_v(k)$-modules $\mathrm{TR}^v_{q-\lambda'}(k;p)$ with $e \in \langle m'\rangle$.

Given a $\mathbb{T}$-representation λ and a positive integer v, we write

$$\lambda^{(v)} = \rho^*_{p^v}(\lambda^{C_{p^v}}).$$

The following result is proved in [**18**, Theorem 11]. The canonical map in the statement is induced by the map (2.1) and the map of TR-groups induced by the unit map $\iota\colon \mathbb{F}_p \to k$.

THEOREM 2.3. *Let k be a regular $\mathbb{F}_p$-algebra and let λ be a finite dimensional complex $\mathbb{T}$-representation. The canonical map*

$$W_v\Omega_k^* \otimes_{W_v(\mathbb{F}_p)} \mathrm{TR}^v_{*-\lambda}(\mathbb{F}_p;p) \longrightarrow \mathrm{TR}^v_{*-\lambda}(k;p)$$

factors through an isomorphism

$$\bigoplus_{r\geqslant 0} W_{v-w}\Omega_k^{q-2r} \otimes_{W_v(\mathbb{F}_p)} \mathrm{TR}^v_{2r-\lambda}(\mathbb{F}_p;p) \longrightarrow \mathrm{TR}^v_{q-\lambda}(k;p)$$

where $w = w(r,\lambda,v)$ is equal to 0, if $\dim_{\mathbb{R}}(\lambda) \leqslant 2r$; is equal to w, if there exists an integer $1 \leqslant w < v$ such that $\dim_{\mathbb{R}}(\lambda^{(w)}) \leqslant 2r < \dim_{\mathbb{R}}(\lambda^{(w-1)})$; and is equal to v, if $2r < \dim_{\mathbb{R}}(\lambda^{(v-1)})$.

It was proved in [**20**, Proposition 9.1] that the group $\mathrm{TR}^v_{2r-\lambda}(\mathbb{F}_p;p)$ is cyclic of order p^{v-w} with $w = w(r,\lambda,v)$ as in the statement. Below, we will give a preferred generator of this group, and for this purpose, we first recall the Tate spectrum and the Tate spectral sequence following [**22**, Section 4].

Let E be a free $\mathbb{T}$-CW-complex which, non-equivariantly, is contractible; if E' is another $\mathbb{T}$-CW-complex with this property, then there exists a unique $\mathbb{T}$-homotopy class of $\mathbb{T}$-homotopy equivalences from E to E'. We define $\tilde{E}$ to be the pointed $\mathbb{T}$-CW-complex given by the mapping cone of the pointed map $\pi\colon E_+ \to S^0$ that collapses E onto the non-basepoint in S^0. We note that the definition of the mapping cone and of the cofibration sequence of pointed $\mathbb{T}$-CW-complexes

$$E_+ \xrightarrow{\pi} S^0 \xrightarrow{\iota} \tilde{E} \xrightarrow{\delta} \Sigma E_+$$

depend on choices and that we always use the choices made in [**22**, Section 2]. Now, the $(-q)$th Tate cohomology group of C_{p^v} with coefficients in the $\mathbb{T}$-spectrum T is defined to be the equivariant homotopy group

$$\hat{\mathbb{H}}^{-q}(C_{p^v},T) = [S^q \wedge (\mathbb{T}/C_{p^v})_+, \tilde{E} \wedge F(E_+,T)^c]_{\mathbb{T}},$$

where $X^c \to X$ is a $\mathbb{T}$-CW-replacement. The Tate cohomology groups of C_{p^v} with coefficients in T are the abutment of the Tate spectral sequence

$$E^2_{s,t} = \hat{H}^{-s}(C_{p^v},\pi_t(T)) \Rightarrow \hat{\mathbb{H}}^{-s-t}(C_{p^v},T)$$

from the corresponding Tate cohomology groups of C_{p^v} with coefficients in the trivial C_{p^u}-modules $\pi_t(T) = [S^t \wedge \mathbb{T}_+, T]_{\mathbb{T}}$. If T is a ring $\mathbb{T}$-spectrum, then the spectral sequence is multiplicative.

The definition of the cyclotomic trace map given by Dundas and McCarthy in [**7**] gives a map in the stable homotopy category $\mathrm{tr}\colon K(\mathbb{F}_p) \to T(\mathbb{F}_p)^{\mathbb{T}}$. This map, in turn, determines a map in the $\mathbb{T}$-stable homotopy category that we also write

$$K(\mathbb{F}_p) \xrightarrow{\mathrm{tr}} T(\mathbb{F}_p).$$

It follows from [**9**, Appendix] that this map is a map of ring $\mathbb{T}$-spectra.

LEMMA 2.4. *Let p be a prime number, let v be a positive integer, and let μ be a complex $\mathbb{T}$-representation of real dimension $2d$.*

(1) *The graded ring $\hat{H}^*(C_{p^v},\mathbb{Z})$ is a Laurent polynomial algebra over $\mathbb{Z}/p^v\mathbb{Z}$ on a preferred generator t of degree 2.*
(2) *The graded $\hat{H}^*(C_{p^v},\mathbb{Z})$-module $\hat{H}^*(C_{p^v},\pi_{2d}(S^\mu \wedge K(\mathbb{F}_p)))$ is free of rank one generated by the product $[S^\mu]\iota$ of the fundamental class $[S^\mu] \in \pi_{2d}(S^\mu)$ and the multiplicative unit element $\iota \in \pi_0(K(\mathbb{F}_p))$.*

(3) *The edge homomorphism of the Tate spectral sequence*

$$\hat{\mathbb{H}}^{-q}(C_{p^v}, S^\mu \wedge K(\mathbb{F}_p)) \xrightarrow{\varepsilon} \hat{H}^{2d-q}(C_{p^v}, \pi_{2d}(S^\mu \wedge K(\mathbb{F}_p)))$$

is an isomorphism for all integers q.

(4) *The map induced by the cyclotomic trace map*

$$\hat{\mathbb{H}}^{-q}(C_{p^v}, S^\mu \wedge K(\mathbb{F}_p)) \xrightarrow{\mathrm{tr}} \hat{\mathbb{H}}^{-q}(C_{p^v}, S^\mu \wedge T(\mathbb{F}_p))$$

is an isomorphism for all integers q.

Proof. The statement (1) is proved in [**22**, Section 4] with the preferred generator t defined in loc. cit. to be the class of the cycle $y_0 \otimes Nx_1^*$, and (2) follows immediately from $\pi_{2d}(S^\mu \wedge K(\mathbb{F}_p)) = \mathbb{Z} \cdot [S^\mu]\iota$ being a trivial C_{p^v}-module. For the proof of (3), we recall that for $t > 0$, the group $K_t(\mathbb{F}_p)$ is finite of order prime to p. It follows that the (multiplicative) Tate spectral sequence

$$\hat{E}^2_{s,t} = \hat{H}^{-s}(C_{p^v}, \pi_t(S^\mu \wedge K(\mathbb{F}_p))) \Rightarrow \hat{\mathbb{H}}^{-s-t}(C_{p^v}, S^\mu \wedge K(\mathbb{F}_p))$$

collapses and that the edge homomorphism is an isomorphism as stated. Finally, to prove (4), we first consider the (multiplicative) Tate spectral sequence for $T(\mathbb{F}_p)$, which takes the form

$$\hat{E}^2 = \Lambda_{\mathbb{Z}/p\mathbb{Z}}\{u\} \otimes S_{\mathbb{Z}/p\mathbb{Z}}\{t, t^{-1}\} \otimes S_{\mathbb{Z}/p\mathbb{Z}}\{\sigma\} \Rightarrow \hat{\mathbb{H}}^{-*}(C_{p^v}, T(\mathbb{F}_p))$$

with $\deg(u) = (-1, 0)$, $\deg(t) = (-2, 0)$, and $\deg(\sigma) = (0, 2)$; the precise definition of the generators u, t, and σ is given in [**22**, Section 4], and the spectral sequence was evaluated in [**20**, Section 5]. The result is that the elements t and σ both are infinite cycles and that the non-zero differentials are multiplicatively generated from $d^{2v+1}(u)$ being equal to $t^{v+1}\sigma^v$ times a unit in $\mathbb{F}_p$. It follows that

$$\hat{E}^\infty = S_{\mathbb{Z}/p\mathbb{Z}}\{t, t^{-1}\} \otimes S_{\mathbb{Z}/p\mathbb{Z}}\{\sigma\}/(\sigma^v).$$

Next, the Tate spectral sequence for $S^\mu \wedge T(\mathbb{F}_p)$ is a module spectral sequence over the spectral sequence for $T(\mathbb{F}_p)$ generated by the infinite cycle $\mathrm{tr}([S^\mu]\iota) \in \hat{E}^r_{0,2d}$. Moreover, it was proved in loc. cit. that the extensions in passing from E^∞ to the abutment are maximally non-trivial. We conclude that the domain and target of the map (4) are abstractly isomorphic abelian groups, and since the map takes a generator of the domain to a generator of the target, it is an isomorphism. □

The following result is a refinement of [**20**, Proposition 9.1]. We recall the function $w = w(r, \lambda, v)$ defined in the statement of Theorem 2.3 and recall the abbreviation $\lambda' = \rho_p^*(\lambda^{C_p})$.

Proposition 2.5. *Let p be a prime number, let v be a positive integer, and let λ be a finite dimensional complex $\mathbb{T}$-representation. Let μ be a choice of a finite dimensional complex $\mathbb{T}$-representation with $\mu' = \lambda$.*

(1) *The group $\mathrm{TR}^v_{q-\lambda}(\mathbb{F}_p; p)$ is trivial, if q is odd, and is cyclic of order p^{v-w} with a preferred generator $\sigma(r, \mu, v)$, if $q = 2r$ is even. Here $w = w(r, \lambda, v)$.*

(2) *If $q = 2r$ is even and $q < \dim_{\mathbb{R}}(\lambda)$, then the restriction map*

$$\mathrm{TR}^v_{q-\lambda}(\mathbb{F}_p; p) \xrightarrow{R} \mathrm{TR}^{v-1}_{q-\lambda'}(\mathbb{F}_p; p)$$

is an isomorphism and takes $\sigma(r, \mu, v)$ to $\sigma(r, \mu', v-1)$.

(3) *If $q = 2r$ is even, then the Frobenius map*

$$\mathrm{TR}^v_{q-\lambda}(\mathbb{F}_p;p) \xrightarrow{F} \mathrm{TR}^{v-1}_{q-\lambda}(\mathbb{F}_p;p)$$

is surjective and takes $\sigma(r,\mu,v)$ to $\sigma(r,\mu,v-1)$.

Proof. We first consider the case $\dim_{\mathbb{R}}(\lambda) \leqslant q$. We recall that, in this case, it is proved in [**20**, Addendum 9.1] that the map

$$\mathrm{TR}^v_{q-\lambda}(\mathbb{F}_p;p) \xrightarrow{\hat{\Gamma}_v} \hat{\mathbb{H}}^{-q}(C_{p^v}, S^\mu \wedge T(\mathbb{F}_p))$$

is an isomorphism. Moreover, Lemma 2.4 shows that the common group is zero, if q is odd, and a cyclic group of order p^v, if $q = 2r$ is even. To define the preferred generator $\sigma(r,\mu,v)$, we consider (the left-hand column of) the commutative diagram

$$\begin{tikzcd}
\mathrm{TR}^v_{q-\lambda}(\mathbb{F}_p;p) \arrow[r, "F"] \arrow[d, "\hat{\Gamma}_v"] & \mathrm{TR}^{v-1}_{q-\lambda}(\mathbb{F}_p;p) \arrow[d, "\hat{\Gamma}_{v-1}"] \\
\hat{\mathbb{H}}^{-q}(C_{p^v}, S^\mu \wedge T(\mathbb{F}_p)) \arrow[r, "F"] & \hat{\mathbb{H}}^{-q}(C_{p^{v-1}}, S^\mu \wedge T(\mathbb{F}_p)) \\
\hat{\mathbb{H}}^{-q}(C_{p^v}, S^\mu \wedge K(\mathbb{F}_p)) \arrow[r, "F"] \arrow[u, "\mathrm{tr}"] \arrow[d, "\varepsilon"] & \hat{\mathbb{H}}^{-q}(C_{p^{v-1}}, S^\mu \wedge K(\mathbb{F}_p)) \arrow[u, "\mathrm{tr}"] \arrow[d, "\varepsilon"] \\
\hat{H}^{2d-q}(C_{p^v}, \pi_{2d}(S^\mu \wedge K(\mathbb{F}_p))) \arrow[r, "F"] & \hat{H}^{2d-q}(C_{p^{v-1}}, \pi_{2d}(S^\mu \wedge K(\mathbb{F}_p))),
\end{tikzcd}$$

in which the vertical maps are isomorphisms and $2d = \dim_{\mathbb{R}}(\mu)$. We define the preferred generator $\sigma(r,\mu,v)$ of the top left-hand group to be the unique class that corresponds under the isomorphisms in the left-hand column to the preferred generator $t^{d-r}[S^\mu]\iota$ of the lower left-hand group. The lower horizontal map F is the map of Tate cohomology groups induced by the canonical inclusion of $C_{p^{v-1}}$ in C_{p^v}. One readily verifies that it is surjective and that it maps the generator $t^{d-r}[S^\mu]\iota$ in the domain to the generator $t^{d-r}[S^\mu]\iota$ in the target. It follows that the top horizontal map F is surjective and takes the generator $\sigma(r,\mu,v)$ to the generator $\sigma(r,\mu,v-1)$ as stated. This proves the proposition in the case $\dim_{\mathbb{R}}(\lambda) \leqslant q$.

Suppose next that $\dim_{\mathbb{R}}(\lambda^{(w)}) \leqslant q < \dim_{\mathbb{R}}(\lambda^{(w-1)})$ with $1 \leqslant w < v$. In this case, it follows from [**20**, Theorem 2.2] that the restriction map

$$\mathrm{TR}^v_{q-\lambda}(\mathbb{F}_p;p) \xrightarrow{R^w} \mathrm{TR}^{v-w}_{q-\lambda^{(w)}}(\mathbb{F}_p;p)$$

is an isomorphism, and the target was determined above. We conclude that the domain is zero, if q is odd, and a cyclic group of order p^{v-w}, if $q = 2r$ is even. In the latter case, we define the generator $\sigma(r,\mu,v)$ of the domain to be the unique class that is mapped to generator $\sigma(r,\mu^{(w)},v-w)$ of the target. This proves (1) and (2), and (3) follows from the commutativity of the diagram

$$\begin{tikzcd} \mathrm{TR}^{v}_{q-\lambda}(\mathbb{F}_p;p) \arrow[r, "F"] \arrow[d, "R^w"] & \mathrm{TR}^{v-1}_{q-\lambda}(\mathbb{F}_p;p) \arrow[d, "R^w"] \\ \mathrm{TR}^{v-w}_{q-\lambda^{(w)}}(\mathbb{F}_p;p) \arrow[r, "F"] & \mathrm{TR}^{v-w-1}_{q-\lambda^{(w)}}(\mathbb{F}_p;p) \end{tikzcd}$$

and from what was proved above.

Finally, if $q < \dim_{\mathbb{R}}(\lambda^{(v-1)})$, then [**20**, Theorem 2.2] shows similarly that the group $\mathrm{TR}^{v}_{q-\lambda}(\mathbb{F}_p;p)$ is isomorphic to the group $\mathrm{TR}^{1}_{q-\lambda^{(v-1)}}(\mathbb{F}_p;p)$. But this group is zero, since $T(\mathbb{F}_p)$ is connective. □

Remark 2.6. In the proof of Proposition 2.5, we used that the map

$$\mathrm{TR}^{v}_{q-\lambda}(\mathbb{F}_p;p) \xrightarrow{\hat{\Gamma}_v} \hat{\mathbb{H}}^{-q}(C_{p^v}, S^{\mu} \wedge T(\mathbb{F}_p))$$

is an isomorphism if $\dim_{\mathbb{R}}(\lambda) \leqslant q$. For general q, the map was determined, up to a unit, in [**13**, Proposition 5.1]. We note that, contrary to what one might first expect, the map is generally not injective.

3. The cyclic bar-construction

The k-algebras $A = k[x,y]/(x^b - y^a)$ and $B = k[t]$ both are monoid algebras, and moreover, the k-algebra map $f\colon A \to B$ that takes x to t^a and y to t^b is induced by a monoid map. In general, if M is a monoid and $k[M]$ the associated monoid k-algebra, then there is a canonical natural map of cyclotomic spectra

$$N^{\mathrm{cy}}(M)_+ \wedge T(k) \xrightarrow{\alpha} T(k[M])$$

where $N^{\mathrm{cy}}(M)$ is the cyclic bar-construction of M whose definition we recall below. It follows from [**20**, Theorem 7.1] that this map induces isomorphisms of equivariant homotopy groups, and hence, one may hope to evaluate the groups $\mathrm{TR}^m_q(k[M])$ by understanding the structure of the $\mathbb{T}$-space $N^{\mathrm{cy}}(M)$. In this section, we will examine the structure of the cyclic bar-construction of the monoids corresponding to A and B. We refer the reader to Loday's book [**27**] for an introduction to cyclic sets and their geometric realization, but see also [**6, 31**].

The cyclic bar-construction of a monoid M is the cyclic set $N^{\mathrm{cy}}(M)[-]$ whose set of q-simplices is the $(q+1)$-fold product

$$N^{\mathrm{cy}}(M)[q] = M \times \cdots \times M$$

and whose cyclic structure maps are defined as follows.

$$\begin{aligned} d_i(x_0,\dots,x_q) &= \begin{cases} (x_0,\dots,x_ix_{i+1},\dots,x_q) & \text{for } 0 \leqslant i < q \\ (x_qx_0,x_1,\dots,x_{q-1}) & \text{for } i = q \end{cases} \\ s_i(x_0,\dots,x_q) &= (x_0,\dots,x_i,1,x_{i+1},\dots,x_q) \qquad \text{for } 0 \leqslant i \leqslant q \\ t_q(x_0,\dots,x_q) &= (x_q,x_0,\dots,x_{q-1}) \end{aligned}$$

We note that the q-simplices of the form $(1,x_1,\dots,x_q)$ are not degenerate, unless one or more of $x_1,\dots,x_q$ are equal to the identity element $1 \in M$. We write

$$N^{\mathrm{cy}}(M) = |N^{\mathrm{cy}}(M)[-]|$$

for its geometric realization. It has a canonical left $\mathbb{T}$-action, as does the geometric realization of any cyclic set, but it has an additional structure that we now explain. The subspace $N^{\mathrm{cy}}(M)^{C_s}$ fixed by $C_s \subset \mathbb{T}$ is canonically a $\mathbb{T}/C_s$-space and we write $\rho_s^* N^{\mathrm{cy}}(M)^{C_s}$ for this space considered as a $\mathbb{T}$-space via the sth root $\rho_s \colon \mathbb{T} \to \mathbb{T}/C_s$. The additional structure is a canonical isomorphism of $\mathbb{T}$-spaces

$$\rho_s^* N^{\mathrm{cy}}(M)^{C_s} \xrightarrow{r_s} N^{\mathrm{cy}}(M).$$

It is defined in [**2**, Section 2] to be the composition the canonical (non-simplicial) isomorphism of $\mathbb{T}$-spaces $D_s \colon \rho_s^* N^{\mathrm{cy}}(M)^{C_s} \to |(\mathrm{sd}_s\, N^{\mathrm{cy}}(M)[-])^{C_s}|$, whose target is the geometric realization of the cyclic set defined by the C_s-fixed set of the s-fold edgewise subdivision of $N^{\mathrm{cy}}(M)[-]$, and the inverse of the map induced by the isomorphism of cyclic sets $N^{\mathrm{cy}}(M)[-] \to (\mathrm{sd}_s\, N^{\mathrm{cy}}(M)[-])^{C_s}$ that to the q-simplex $(x_0, \dots, x_q)$ associates its s-fold repetition $(x_0, \dots, x_q, \dots, x_0, \dots, x_q)$.

We now define $\langle x, y\rangle$ to be the submonoid of the underlying multiplicative monoid of A given by the elements of the form $x^i y^j$, where (i, j) is a pair of non-negative integers. Similarly, we define $\langle t\rangle$ to be the submonoid of the underlying multiplicative monoid of B given by the elements of the form t^m, where m is a non-negative integer. The k-algebras A and B are the monoid k-algebras of $\langle x, y\rangle$ and $\langle t\rangle$, respectively, and the k-algebra homomorphism $f \colon A \to B$ is induced by a monoid map $f \colon \langle x, y\rangle \to \langle t\rangle$. We define $N^{\mathrm{cy}}(f)$ to be the mapping cone of the induced map of pointed $\mathbb{T}$-spaces $f \colon N^{\mathrm{cy}}(\langle x, y\rangle)_+ \to N^{\mathrm{cy}}(\langle t\rangle)_+$. We also define

$$N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle) = |N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle)[-]|$$

to be the geometric realization of the quotient of the cyclic set $N^{\mathrm{cy}}(\langle t\rangle)[-]$ by the image $N^{\mathrm{cy}}(\langle t^a, t^b\rangle)[-]$ of the map $f \colon N^{\mathrm{cy}}(\langle x, y\rangle)[-] \to N^{\mathrm{cy}}(\langle t\rangle)[-]$. Since the latter map is injective, it follows that the canonical projection

$$N^{\mathrm{cy}}(f) \xrightarrow{q} N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle)$$

is a weak equivalence of pointed $\mathbb{T}$-spaces.

The cyclic sets considered above admit the following decompositions into wedge sums indexed by non-negative integers m,

$$\begin{tikzcd}
\bigvee N^{\mathrm{cy}}(\langle x, y\rangle; m)[-]_+ \arrow[r, "\bigvee f"] \arrow[d, "\sim"] & \bigvee N^{\mathrm{cy}}(\langle t\rangle; m)[-]_+ \arrow[r, "\bigvee q"] \arrow[d, "\sim"] & \bigvee N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle; m)[-] \arrow[d, "\sim"] \\
N^{\mathrm{cy}}(\langle x, y\rangle)[-]_+ \arrow[r, "f"] & N^{\mathrm{cy}}(\langle t\rangle)[-]_+ \arrow[r, "q"] & N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle)[-].
\end{tikzcd}$$

Here $N^{\mathrm{cy}}(\langle t\rangle; m)[-]$ is the cyclic subset of $N^{\mathrm{cy}}(\langle t\rangle)[-]$ whose q-simplices are the $(q+1)$-tuples $(t^{m_0}, \dots, t^{m_q})$ such that $m_0 + \dots + m_q = m$, and the cyclic subsets $N^{\mathrm{cy}}(\langle x, y\rangle; m)[-]$ and $N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle; m)[-]$ are respectively the pre-image by f and the image by q of this cyclic subset. These wedge decompositions of cyclic sets induce the following decompositions of the pointed $\mathbb{T}$-spaces considered above into wedge sums indexed by non-negative integers m,

$$\begin{tikzcd}
\bigvee N^{\mathrm{cy}}(f; m) \arrow[r, "\bigvee q"] \arrow[d, "\sim"] & \bigvee N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle; m) \arrow[d, "\sim"] \\
N^{\mathrm{cy}}(f) \arrow[r, "q"] & N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle).
\end{tikzcd}$$

Moreover, if $m = st$, then the canonical isomorphism r_s defined above induces the vertical isomorphisms in the following commutative diagram of pointed $\mathbb{T}$-spaces,

$$\begin{array}{ccc}
\rho_s^* N^{\mathrm{cy}}(f;m)^{C_s} & \xrightarrow{\rho_s^* q^{C_s}} & \rho_s^* N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle; m)^{C_s} \\
\downarrow{\scriptstyle r_s} & & \downarrow{\scriptstyle r_s} \\
N^{\mathrm{cy}}(f;t) & \xrightarrow{\quad q \quad} & N^{\mathrm{cy}}(\langle t\rangle; \langle t^a, t^b\rangle; t).
\end{array}$$

We also note that if the positive integer s does not divide m, then the pointed $\mathbb{T}$-spaces in the top line of this diagram both are singletons.

Let $m = st$ be a positive integer. We recall the pointed C_m-space $X(a,b,m)$ and the isomorphism of pointed C_t-spaces $r_{X,s}\colon \rho_s^* X(a,b,m)^{C_s} \to X(a,b,t)$ defined in the introduction. We precompose the induced isomorphism of pointed $\mathbb{T}$-spaces

$$\mathbb{T}_+ \wedge_{C_t} \rho_s^* X(a,b,m)^{C_s} \xrightarrow{\mathrm{id} \wedge r_{X,s}} \mathbb{T}_+ \wedge_{C_t} X(a,b,t)$$

with the inverse of the isomorphism of pointed $\mathbb{T}$-spaces

$$\rho_s^*(\mathbb{T}_+ \wedge_{C_m} X(a,b,m))^{C_s} \longleftarrow \mathbb{T}_+ \wedge_{C_t} \rho_s^* X(a,b,m)^{C_s}$$

that to the class of (z,x) associates the class of $(\rho_s(z), x)$ to obtain the isomorphism

$$\rho_s^*(\mathbb{T}_+ \wedge_{C_m} X(a,b,m))^{C_s} \xrightarrow{r_s} \mathbb{T}_+ \wedge_{C_t} \rho_s^* X(a,b,t).$$

Proposition 3.1. *The pointed $\mathbb{T}$-space $N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle; 0)$ is a singleton, and for every positive integer m, there is a canonical isomorphism of pointed $\mathbb{T}$-spaces*

$$\mathbb{T}_+ \wedge_{C_m} X(a,b,m) \xrightarrow{e_m} N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle; m).$$

Moreover, if $m = st$, then the diagram

$$\begin{array}{ccc}
\rho_s^*(\mathbb{T}_+ \wedge_{C_m} X(a,b,m))^{C_s} & \xrightarrow{\rho_s^* e_m^{C_s}} & \rho_s^* N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle; m)^{C_s} \\
\downarrow{\scriptstyle r_s} & & \downarrow{\scriptstyle r_s} \\
\mathbb{T}_+ \wedge_{C_t} X(a,b,t) & \xrightarrow{\quad e_t \quad} & N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle; t)
\end{array}$$

commutes.

Proof. The case $m = 0$ is clear, so we suppose that m is a positive integer. The cyclic set $N^{\mathrm{cy}}(\langle t\rangle; m)[-]$ is generated by the $(m-1)$-simplex $(t, \dots, t)$ subject to the relations generated by the identity $t_{m-1}(t, \dots, t) = (t, \dots, t)$. Hence, it follows from [**20**, Section 7.2] that there is a canonical $\mathbb{T}$-equivariant homeomorphism

$$\mathbb{T} \times_{C_m} \Delta^{m-1} \xrightarrow{e_m} N^{\mathrm{cy}}(\langle t\rangle; m)$$

from the left $\mathbb{T}$-space induced from the left C_m-space Δ^{m-1}. We claim that it restricts to a $\mathbb{T}$-equivariant homeomorphism of the sub-$\mathbb{T}$-space $\mathbb{T} \times_{C_m} \Sigma(a,b,m)$ of the domain onto the sub-$\mathbb{T}$-space $N^{\mathrm{cy}}(\langle t^a, t^b\rangle; m)$ of the target. To see this, let us say that a q-simplex $(t^{m_0}, \dots, t^{m_q})$ in the cyclic set $N^{\mathrm{cy}}(\langle t^a, t^b\rangle; m)[-]$ is positive if the exponents $m_0, \dots, m_q$ all are positive. Now, if $F = \theta^*(\Delta^{m-1})$ is a q-simplex in $\Sigma(a,b,m)$ indexed by $\theta\colon [q] \to [m-1]$, then

$$\theta^*(t, \dots, t) = (t^{m_0}, \dots, t^{m_q})$$

is a positive q-simplex in $N^{\mathrm{cy}}(\langle t^a, t^b\rangle; m)[-]$, and conversely, every positive simplex in $N^{\mathrm{cy}}(\langle t^a, t^b\rangle; m)[-]$ arises (non-uniquely) in this way. It follows, that e_m restricts to a necessarily injective $\mathbb{T}$-equivariant map

$$\mathbb{T} \times_{C_m} \Sigma(a,b,m) \xrightarrow{e_m} N^{\mathrm{cy}}(\langle t^a, t^b\rangle; m).$$

But this map also is surjective, since $N^{\mathrm{cy}}(\langle t^a, t^b\rangle; m)[-]$ is generated as a cyclic set (but not as a simplicial set) by the positive simplices. It follows that e_m induces the stated isomorphism of the quotient $\mathbb{T}$-spaces. $\square$

REMARK 3.2. The C_m-action on $X(a,b,m)$ does not extend to a $\mathbb{T}$-action, and therefore, $\mathbb{T}_+ \wedge_{C_m} X(a,b,m)$ does not admit untwisting into a smash product of two pointed $\mathbb{T}$-spaces. By contrast, the C_m-action on $S^{\lambda(a,b,m)}$ extends to a $\mathbb{T}$-action, and hence, we obtain the untwisting isomorphism

$$\mathbb{T}_+ \wedge_{C_m} S^{\lambda(a,b,m)} \xrightarrow{\xi} (\mathbb{T}/C_m)_+ \wedge S^{\lambda(a,b,m)}$$

that maps the class of (z,v) to (zC_m, zv). Here, the target is given the diagonal $\mathbb{T}$-action.

4. Proof of Theorem A

In this section we complete the proof of Theorem A in the introduction. We first use the comparison theorems between K-theory and topological cyclic homology proved by McCarthy [**28**] and by Geisser and the author [**10, 11**] to prove the following general result.

THEOREM 4.1. *Let k be a unital associative ring, let $1 < a < b$ be relatively prime integers, let $A = k[x,y]/(x^b - y^a)$, let $B = k[t]$, and let $f\colon A \to B$ be the k-algebra homomorphism that maps x and y to t^a and t^b, respectively. If the prime number p is nilpotent in k, then the diagram of symmetric spectra*

$$\begin{tikzcd}
K(A) \arrow[r, "\mathrm{tr}"] \arrow[d, "f"] & \mathrm{TC}(A) \arrow[d, "f"] \\
K(B) \arrow[r, "\mathrm{tr}"] & \mathrm{TC}(B)
\end{tikzcd}$$

is homotopy cartesian. The diagram becomes homotopy cartesian upon pro-finite completion for every unital associative ring k.

PROOF. Let $I \subset A$ be the conductor ideal from A to B, which we calculated in Remark 1.5, and let $J = f(I) \subset B$. Let $\mathfrak{a} \subset A$ be the ideal generated by x and y, and let $\mathfrak{b} \subset B$ be the ideal generated by t. We consider the following diagram of spectra, where all vertical maps are induced by $f\colon A \to B$, where all maps from the back rectangular diagram to the front rectangular diagram are the cyclotomic trace maps, and where all horizontal maps are induced by the respective canonical projection maps of k-algebras.

$$
\begin{tikzcd}
K(A) \arrow[rr] \arrow[dr] \arrow[dd] & & K(A/I) \arrow[rr] \arrow[dr] \arrow[dd] & & K(A/\mathfrak{a}) \arrow[dr] \arrow[dd] & \\
& \mathrm{TC}(A) \arrow[rr] \arrow[dd] & & \mathrm{TC}(A/I) \arrow[rr] \arrow[dd] & & \mathrm{TC}(A/\mathfrak{a}) \arrow[dd] \\
K(B) \arrow[rr] \arrow[dr] & & K(B/J) \arrow[rr] \arrow[dr] & & K(B/\mathfrak{b}) \arrow[dr] & \\
& \mathrm{TC}(B) \arrow[rr] & & \mathrm{TC}(B/J) \arrow[rr] & & \mathrm{TC}(B/\mathfrak{b})
\end{tikzcd}
$$

The right-hand vertical square is homotopy cartesian for the trivial reason that the map $\bar{f}\colon A/\mathfrak{a} \to B/\mathfrak{b}$ induced by $f\colon A \to B$ is an isomorphism. Moreover, it follows from [**11**, Theorem B] that the top and bottom right-hand horizontal squares both are homotopy cartesian, since the ideals $\mathfrak{a}/I \subset A/I$ and $\mathfrak{b}/J \subset B/J$ are nilpotent. Therefore, also the middle vertical square is homotopy cartesian. Finally, it follows from [**11**, Theorem D] that the left-hand cubical diagram is homotopy cartesian. This shows that the left-hand vertical square is homotopy cartesian as stated. The statement for a general ring k is proved similarly, substituting [**28**, Main Theorem] for [**11**, Theorem B] and [**10**, Theorem 1] for [**11**, Theorem D]. $\square$

We consider the diagram in the statement of Theorem 4.1. The theorem shows that the cyclotomic trace map induces an weak equivalence of the homotopy fiber of the left-hand vertical map to the homotopy fiber of the right-hand vertical map. In view of Proposition 3.1, if we assume Conjecture B, then we may repeat the argument of [**20**, §8] to obtain the following result.

Proposition 4.2. *Let $1 < a < b$ be relatively prime integers, let k be an arbitrary ring, let $A = k[x,y]/(x^b - y^a)$, let $B = k[t]$, and let $f\colon A \to B$ be the k-algebra map that takes x to t^a and y to t^b. Assuming Conjecture B, the profinite completions of the mapping fiber of the map of topological cyclic homology spectra*

$$\mathrm{TC}(A) \xrightarrow{f} \mathrm{TC}(B)$$

induced by f and the iterated mapping cone of the diagram of spectra

$$
\begin{tikzcd}
\operatorname{holim}_R (S^{\lambda(a,b,m)} \wedge T(k))^{C_{m/ab}} \arrow[r, "V_b"] \arrow[d, "V_a"] & \operatorname{holim}_R (S^{\lambda(a,b,m)} \wedge T(k))^{C_{m/a}} \arrow[d, "V_a"] \\
\operatorname{holim}_R (S^{\lambda(a,b,m)} \wedge T(k))^{C_{m/b}} \arrow[r, "V_b"] & \operatorname{holim}_R (S^{\lambda(a,b,m)} \wedge T(k))^{C_{m}}
\end{tikzcd}
$$

are canonically weakly equivalent. Here we index all four homotopy limits by the set $\mathbb{N}$ of positive integers ordered under division with the understanding that the mth term in the upper right-hand limit (resp. lower left-hand limit, resp. upper left-hand limit) is trivial if m is not divisible by a (resp. by b, resp. by ab).

We remark that by [**1**, Proposition 1.1], the canonical projections from the four terms in the diagram in Proposition 4.2 to the mth terms of the corresponding limit systems are $\dim_{\mathbb{R}}(\lambda(a,b,2m))$-connected. In particular, the induced diagram of qth homotopy groups maps isomorphically onto the diagram

$$
\begin{tikzcd}
\lim_R \mathrm{TR}^{m/ab}_{q-\lambda(a,b,m)}(k) \arrow[r, "V_b"] \arrow[d, "V_a"] & \lim_R \mathrm{TR}^{m/a}_{q-\lambda(a,b,m)}(k) \arrow[d, "V_a"] \\
\lim_R \mathrm{TR}^{m/b}_{q-\lambda(a,b,m)}(k) \arrow[r, "V_b"] & \lim_R \mathrm{TR}^{m}_{q-\lambda(a,b,m)}(k)
\end{tikzcd}
\tag{4.3}
$$

and the limits in the diagram stabilize. Here we again index all four limits by the set $\mathbb{N}$ ordered under division. As explained in Section 2, the group $\mathrm{TR}^m_{q-\lambda}(k)$ has a canonical module structure over the ring $\mathrm{TR}^m_0(k)$, and hence, we can view it as a module over the ring $\mathbb{W}(k)$ via the composite ring homomorphism

$$\mathbb{W}(k) \xrightarrow{R^{\mathbb{N}}_{\langle m\rangle}} \mathbb{W}_{\langle m\rangle}(k) \xrightarrow{\eta_{\langle m\rangle}} \mathrm{TR}^m_0(k).$$

The structure maps in the four limit systems are $\mathbb{W}(k)$-linear with respect to this $\mathbb{W}(k)$-module structure, since the map η is compatible with restriction maps. This defines $\mathbb{W}(k)$-module structures on the four limits in the diagram above. However, the maps V_a are not linear with respect to this module structure but instead are F_a-linear in the sense that the projection formula $x \cdot V_a(y) = V_a(F_a(x)\cdot y)$ holds, and similarly for the maps V_b.

Proof of Theorem A. Let k be a regular $\mathbb{F}_p$-algebra. We construct, for every positive integer n, a canonical isomorphism of $\mathbb{W}(k)$-modules

$$\bigoplus_{r\geqslant 0} \mathbb{W}_{S(a,b,r)/n}\Omega^{q-2r}_k \xrightarrow{f_n} \lim_R \mathrm{TR}^{m/n}_{q-\lambda(a,b,m)}(k)$$

compatible with the respectible Verschiebung operators. These isomorphisms define a canonical isomorphism from the diagram

$$
\begin{tikzcd}
\bigoplus_{r\geqslant 0} \mathbb{W}_{S(a,b,r)/ab}\Omega^{q-2r}_k \arrow[r, "V_b"] \arrow[d, "V_a"] & \bigoplus_{r\geqslant 0} \mathbb{W}_{S(a,b,r)/a}\Omega^{q-2r}_k \arrow[d, "V_a"] \\
\bigoplus_{r\geqslant 0} \mathbb{W}_{S(a,b,r)/b}\Omega^{q-2r}_k \arrow[r, "V_b"] & \bigoplus_{r\geqslant 0} \mathbb{W}_{S(a,b,r)}\Omega^{q-2r}_k
\end{tikzcd}
$$

onto the diagram (4.3), and this, in turn, proves the theorem. Indeed, since the prime number p does not divide a, the vertical maps V_a in the diagram are injective with canonical retraction $\frac{1}{a}F_a$, and therefore, the stated long exact sequence follows from Theorem 4.1 and Proposition 4.2.

First, in the case $k = \mathbb{F}_p$, we must construct, for every positive integer n and even non-negative integer $q = 2r$, a canonical isomorphism of $\mathbb{W}(\mathbb{F}_p)$-modules

$$\mathbb{W}_{S(a,b,r)/n}(\mathbb{F}_p) \xrightarrow{f_n} \lim_R \mathrm{TR}^{m/n}_{q-\lambda(a,b,r)}(\mathbb{F}_p)$$

compatible with Verschiebung maps. We must also show that, for q odd or negative, the right-hand side vanishes. To this end, we consider the p-typical decompositions of the two $\mathbb{W}(\mathbb{F}_p)$-modules in question. We write $n = p^w n'$ with n' prime to p. On the one hand, the canonical ring isomorphism (1.3) takes the form

$$\mathbb{W}_{S(a,b,r)/n}(\mathbb{F}_p) \xrightarrow{\gamma} \prod_{e\in\mathbb{N}'} \mathbb{W}_{(S(a,b,r)/ne)\cap P}(\mathbb{F}_p).$$

Moreover, it follows readily from the definition of $S(a,b,r)$ that

$$\mathrm{card}((S(a,b,r)/ne) \cap P) = s - w$$

with $s = s(a,b,r,n,e)$ defined to be the unique integer $s > w$ such that

$$\ell(a,b,p^{s-1}n'e) \leqslant r < \ell(a,b,p^s n'e),$$

if such an integer exists, and w, otherwise. On the other hand, taking limits over $m \in \mathbb{N}$ of the isomorphisms (2.2), we obtain a canonical isomorphism

$$\lim_R \mathrm{TR}^{m/n}_{q-\lambda(a,b,m)}(\mathbb{F}_p) \xrightarrow{\gamma} \prod_{e \in \mathbb{N}'} \lim_R \mathrm{TR}^{v-w}_{q-\lambda(a,b,p^{v-1}n'e)}(\mathbb{F}_p;p)$$

with the limit systems on the right-hand side indexed by the set $\mathbb{N}$ ordered under addition. Moreover, Proposition 2.5 shows that if $q = 2r$ is non-negative, then

$$\lim_R \mathrm{TR}^{v-w}_{q-\lambda(a,b,p^{v-1}n'e)}(\mathbb{F}_p) = \mathbb{W}_{(S(a,b,r)/ne)\cap P}(\mathbb{F}_p) \cdot \sigma(a,b,r,n,e)$$

with the preferred generator $\sigma(a,b,r,n,e)$ defined to be the unique class such that

$$\mathrm{pr}_v(\sigma(a,b,r,n,e)) = \sigma(r, \lambda(a,b,p^v n'e), v-w)$$

for $v \geqslant s(a,b,r,n,e)$. It also follows from Proposition 2.5 that the limit in question is zero, if q is odd or negative. We conclude that, if $q = 2r$ is non-negative and even, then there is an isomorphism of $\mathbb{W}(\mathbb{F}_p)$-modules

$$\mathbb{W}_{S(a,b,r)/n}(\mathbb{F}_p) \xrightarrow{f_n} \lim_R \mathrm{TR}^{m/n}_{q-\lambda(a,b,r)}(\mathbb{F}_p)$$

that to 1 associates the class $\sigma(a,b,r,n)$ defined by

$$\gamma(\sigma(a,b,r,n)) = (\sigma(a,b,r,n,e))_{e \in \mathbb{N}'},$$

and that if q is odd or negative, then the limit is zero. It remains to show that the isomorphisms f_n are comtable with Verschiebung operators. By the projection formula, this is equivalent to showing that if $n = st$ then the map

$$\lim_R \mathrm{TR}^{m/t}_{2r-\lambda(a,b,m)}(\mathbb{F}_p) \xrightarrow{F_s} \lim_R \mathrm{TR}^{m/n}_{2r-\lambda(a,b,m)}(\mathbb{F}_p)$$

takes $\sigma(a,b,r,t)$ to $\sigma(a,b,r,n)$. We write $s = p^i s'$ and $t = p^j t'$ with s' and t' prime to p such that $n' = s't'$ and $w = i + j$. It suffices to show that for all $e \in s'\mathbb{N}'$,

$$\lim_R \mathrm{TR}^{v-j}_{2r-\lambda(a,b,p^{v-1}n'e)}(\mathbb{F}_p;p) \xrightarrow{F^i} \lim_R \mathrm{TR}^{v-w}_{2r-\lambda(a,b,p^{v-1}n'e)}(\mathbb{F}_p;p)$$

takes $\sigma(a,b,r,t,e)$ to $\sigma(a,b,r,n,e/s')$ which, in turn, translates to showing that

$$\mathrm{TR}^{v-j}_{2r-\lambda(a,b,p^{v-1}n'e)}(\mathbb{F}_p;p) \xrightarrow{F^i} \mathrm{TR}^{v-w}_{2r-\lambda(a,b,p^{v-1}n'e)}(\mathbb{F}_p;p)$$

takes $\sigma(r, \lambda(a,b,p^v t'e), v-j)$ to $\sigma(r, \lambda(a,b,p^v t'e), v-w)$. But this was proved in Proposition 2.5, so the proof of the theorem for $k = \mathbb{F}_p$ is complete.

Next, if k is a regular $\mathbb{F}_p$-algebra, we consider $\mathrm{TR}^{m/n}_{*-\lambda(a,b,m)}(k)$ as a graded module over the graded ring $\mathbb{W}\,\Omega^*_k = \lim_R \mathbb{W}_S \Omega^*_k$ via the composition

$$\mathbb{W}\,\Omega^*_k \longrightarrow \mathbb{W}_{\langle m/n \rangle}\Omega^*_k \longrightarrow \mathrm{TR}^{m/n}_*(k)$$

of the canonical projection and the unique map of Witt complexes. Since the latter is compatible with restriction maps, the limit $\lim_R \mathrm{TR}^{m/n}_{*-\lambda(a,b,m)}(k)$ inherits a graded $\mathbb{W}\,\Omega^*_k$-module structure. Hence, the map of graded $\mathbb{W}(\mathbb{F}_p)$-modules

$$\lim_R \mathrm{TR}^{m/n}_{*-\lambda(a,b,m)}(\mathbb{F}_p) \xrightarrow{\iota} \lim_R \mathrm{TR}^{m/n}_{*-\lambda(a,b,m)}(k)$$

induced by the unit map $\iota\colon \mathbb{F}_p \to k$ extends to a map of graded $\mathbb{W}\,\Omega^*_k$-modules

$$\mathbb{W}\,\Omega^*_k \otimes_{\mathbb{W}(\mathbb{F}_p)} \lim_R \mathrm{TR}^{m/n}_{*-\lambda(a,b,m)}(\mathbb{F}_p) \longrightarrow \lim_R \mathrm{TR}^{m/n}_{*-\lambda(a,b,m)}(k).$$

Considering the p-typical decomposition of this map, we conclude from Theorem 2.3 and from the same cardinality counting argument as above that it factors through an isomorphism of graded $\mathbb{W}\,\Omega^*_k$-modules

$$\bigoplus_{r\geqslant 0} \mathbb{W}_{S(a,b,r)/n}\Omega^{*-2r}_k \xrightarrow{f_n} \lim_R \mathrm{TR}^{m/n}_{*-\lambda(a,b,m)}(k),$$

the rth component of which is given by

$$f_{n,r}(\omega) = \tilde{\omega}\cdot\iota(\sigma(a,b,r,n))$$

with $\tilde{\omega} \in \mathbb{W}\,\Omega^{q-2r}_k$ any choice of lift of $\omega \in \mathbb{W}_{S(a,b,m)/n}\Omega^{q-2r}_k$. Finally, the projection formula again shows that the isomorphisms f_n are compatible with the respective Verschiebung maps. This completes the proof. □

5. Homology calculations

In this section, we calculate the reduced (cellular) homology groups of the pointed $\mathbb{T}$-spaces $N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle; m)$ and determine Connes' operator

$$\tilde{H}_q(N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle; m)) \xrightarrow{d} \tilde{H}_{q+1}(N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle; m))$$

that is defined to be the composition of the cross product with the fundamental class $[\mathbb{T}]$ and the map of reduced homology groups induced by the action map

$$\mathbb{T}_+ \wedge N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle; m) \xrightarrow{\mu} N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle; m).$$

This amounts to a reinterpretation of the calculation in [**14, 26**] of the Hochschild homology groups of the rings $A = \mathbb{Z}[x,y]/(x^b - y^a)$ and $B = \mathbb{Z}[t]$ and of Connes' operator on these groups. The final outcome of the calculation is Corollary 5.2 which shows that Conjecture B holds as far as homology is concerned. We begin by recalling the situation in more generality, following the choices concerning signs etc. made in [**22**, Section 2, Appendix].

Let k be a commutative ring and let R be a k-algebra. The associated Hochschild complex is defined to be the cyclic k-module $HH(R/k)[-]$ whose k-module of q-simplices is the $(q+1)$-fold tensor product

$$\mathrm{HH}(R/k)[q] = R\otimes_k \cdots \otimes_k R$$

and whose cyclic structure maps are given as follows.

$$d_i(a_0\otimes\cdots\otimes a_q) = \begin{cases} a_0\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_q & \text{for } 0\leqslant i<q\\ a_qa_0\otimes a_1\otimes\cdots\otimes a_{q-1} & \text{for } i=q\end{cases}$$
$$s_i(a_0\otimes\cdots\otimes a_q) = a_0\otimes\cdots\otimes a_i\otimes 1\otimes a_{i+1}\otimes\cdots\otimes a_q \qquad \text{for } 0\leqslant i\leqslant q$$
$$t_q(a_0\otimes\cdots\otimes a_q) = a_q\otimes a_0\otimes\cdots\otimes a_{q-1}$$

The homology groups of the associated chain complex $\mathrm{HH}(R/k)$ are called the Hochschild homology groups of R/k and denoted $\mathrm{HH}_q(R/k)$. Let

$$R^e = R \otimes_k R^{\mathrm{op}}$$

be the enveloping algebra of R over k. The two-sided bar-construction of R over k is defined to be the simplicial left R^e-module $B(R,R,R)[-]$ whose R^e-module of q-simplices is the $(q+2)$-fold tensor product

$$B(R,R,R)[q] = R \otimes_k \cdots \otimes_k R$$

with the left multiplication by $a \otimes a' \in R^e$ defined by

$$(a \otimes a') \cdot a_0 \otimes a_1 \otimes \cdots \otimes a_q \otimes a_{q+1} = aa_0 \otimes a_1 \otimes \cdots \otimes a_q \otimes a_{q+1}a'$$

and whose simplicial structure maps are given by

$$\begin{aligned} d_i(a_0 \otimes \cdots \otimes a_{q+1}) &= a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{q+1} \\ s_i(a_0 \otimes \cdots \otimes a_{q+1}) &= a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_{q+1} \end{aligned}$$

with $0 \leqslant i \leqslant q$. We view R as a right R^e-module (resp. left R^e-module) with the multiplication by $a \otimes a' \in R^e$ defined by $x \cdot (a \otimes a') = a'xa$ (resp. by $(a \otimes a') \cdot x = axa'$) and recall the canonical isomorphism of simplicial k-modules

$$R \otimes_{R^e} B(R,R,R)[-] \xrightarrow{v} \mathrm{HH}(R/k)[-]$$

that to $x \otimes_{R_e} (a_0 \otimes \cdots \otimes a_{q+1})$ associates $a_{q+1}xa_0 \otimes a_1 \otimes \cdots \otimes a_q$. The homology of the associated chain complex $R \otimes_{R^e} B(R,R,R)$ of the left-hand simplicial k-module is interpreted homologically as follows. The augmentation

$$B(R,R,R)[-] \xrightarrow{\epsilon} R$$

that takes $a_0 \otimes a_1 \in B(R,R,R)[0]$ to $a_0a_1 \in R$ is a weak equivalence of simplicial left R^e-modules, where R is considered as a constant simplicial left R^e-module. Indeed, the map of the underlying simplicial k-modules

$$R \xrightarrow{\eta} B(R,R,R)[-]$$

defined by $\eta(a) = 1 \otimes a$ satisfies $\varepsilon \circ \eta = \mathrm{id}_R$ and the k-linear map

$$B(R,R,R)[q] \xrightarrow{s_{-1}} B(R,R,R)[q+1]$$

that takes $a_0 \otimes \cdots \otimes a_{q+1}$ to $1 \otimes a_0 \otimes \cdots \otimes a_{q+1}$ is a k-linear chain homotopy from the identity map of $B(R,R,R)[-]$ to the composite $\eta \circ \epsilon$. Moreover, if R is flat over k, then the left R^e-modules $B(R,R,R)[q]$ are flat. Therefore, in this case, the isomorphism v above gives rise to a canonical isomorphism

$$\mathrm{Tor}_q^{R^e}(R,R) \xrightarrow{v} \mathrm{HH}_q(R/k).$$

We also recall from [**3**, Exposé 7] that if R is commutative, then the chain complex $B(R,R,R)$ equipped with the shuffle product is a strictly anti-symmetric differential graded R^e-algebra. Moreover, the subalgebra of elements of even degrees carries a canonical divided power structure on the ideal of elements of positive degree. This makes $\mathrm{HH}_*(R/k)$ a strictly anti-symmetric graded R-algebra such that the subalgebra of even degree elements carries a canonical divided power structure on the ideal of elements of positive degree. Finally, if R is any k-algebra, we define

$$\mathrm{HH}(R/k)[q] \xrightarrow{d} \mathrm{HH}(R/k)[q+1]$$

to be the k-linear map given by

$$d(a_0 \otimes \cdots \otimes a_q) = \sum_{0 \leqslant i \leqslant n} (-1)^{ni} 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}.$$

The induced map of homology groups is Connes' operator

$$\mathrm{HH}_q(R/k) \xrightarrow{d} \mathrm{HH}_{q+1}(R/k).$$

It is a differential and, for R commutative, a graded derivation. Hence, if R is a commutative k-algebra, then we have the unique map

$$\Omega^q_{R/k} \xrightarrow{\eta} \mathrm{HH}_q(R/k)$$

from the initial strictly anti-symmetric differential graded k-algebra with underlying k-algebra R. It is an isomorphism for $q \leqslant 1$, and if the unit map $k \to R$ is a smooth (or, more generally, regular) morphism, then it is an isomorphism for all integers q by the Hochschild-Kostant-Rosenberg theorem [**34**, Theorem 9.4.7] (and by the approximation theorem of Popescu [**29, 32**]).

If M is a monoid, then there is a canonical isomorphism of cyclic k-modules

$$k[N^{\mathrm{cy}}(M)[-]] \xrightarrow{w} \mathrm{HH}(k[M]/k)[-]$$

that takes the generator $(x_0, \dots, x_q)$ to the element $x_0 \otimes \cdots \otimes x_q$. It induces a canonical isomorphism from the cellular homology groups of with k-coefficients of $N^{\mathrm{cy}}(M)$ to the Hochschild homology groups of $k[M]/k$, and the diagram

$$\begin{tikzcd} \tilde{H}_q(N^{\mathrm{cy}}(M); k) \arrow[r, "w"] \arrow[d, "d"] & \mathrm{HH}_q(k[M]/k) \arrow[d, "d"] \\ \tilde{H}_{q+1}(N^{\mathrm{cy}}(M); k) \arrow[r, "w"] & \mathrm{HH}_{q+1}(R/k) \end{tikzcd}$$

commutes. In fact, the corresponding diagram of associated normalized chain groups and maps commutes; see the proof of [**17**, Proposition 1.4.5].

We now return to the task at hand where we consider the following cofibration sequence of pointed $\mathbb{T}$-spaces.

$$N^{\mathrm{cy}}(\langle x, y \rangle)_+ \xrightarrow{f} N^{\mathrm{cy}}(\langle t \rangle)_+ \xrightarrow{p} N^{\mathrm{cy}}(\langle t \rangle, \langle t^a, t^b \rangle) \xrightarrow{\partial} \Sigma N^{\mathrm{cy}}(\langle x, y \rangle)_+$$

The canonical isomorphism w defined above identifies the cellular homology groups with k-coefficients of the two left-hand terms with the Hochschild homology groups of the k-algebras $A = k[x, y]/(y^a - x^b)$ and $B = k[t]$, respectively, and identifies the map of cellular homology groups induced by f with the map of Hochschild homology groups induced the k-algebra homomorphism $f \colon A \to B$. Moreover, the cofibration sequence above decomposes as a wedge sum indexed by non-negative integers m of the cofibration sequences of the corresponding weight m pieces.

The Hochschild homology groups $\mathrm{HH}_q(A/k)$ were evaluated in [**14**] and Connes' operator on these groups was evaluated in [**26**]. We recall the result, but follow the choices of signs and definitions made in [**22**, Section 2, Appendix] which differ from those of [**14**] and [**26**] which, in turn, differ from one another. We consider the differential graded algebra A^e-algebra

$$R(A) = A^e \otimes_k \Lambda_k\{dx, dy\} \otimes_k \Gamma_k\{z\},$$

where dx and dy are exterior generators of degree 1 and z a divided power generator of degree 2, and where the differential δ maps

$$\delta(dx) = x \otimes 1 - 1 \otimes x, \qquad \delta(dy) = y \otimes 1 - 1 \otimes y,$$
$$\delta(z) = \frac{x^b \otimes 1 - 1 \otimes x^b}{x \otimes 1 - 1 \otimes x} \cdot dx - \frac{y^a \otimes 1 - 1 \otimes y^a}{y \otimes 1 - 1 \otimes y} \cdot dy$$

and satisfies $\delta(z^{[r]}) = z^{[r-1]}\delta(z)$ for all $r \geqslant 1$. Here $z^{[r]}$ is the rth divided power of z. The augmentation $\varepsilon_R \colon R(A) \to A$ that takes $a \otimes a'$ to aa' is a resolution of the left A^e-module A by free left A^e-modules. Therefore, the groups $\mathrm{HH}_*(A/k)$ are canonically isomorphic to the homology groups of the differential graded A-algebra $\bar{R}(A) = A \otimes_{A^e} R(A)$, where the differential δ annihilates dx and dy, maps z to

$$\delta(z) = bx^{b-1}dx - ay^{a-1}dy,$$

and satisfies $\delta(z^{[r]}) = z^{[r-1]}\delta(z)$ for all $r \geqslant 1$. The isomorphism is given as follows. We choose a map of chain complexes of left A^e-modules

$$R(A) \xrightarrow{h} B(A, A, A)$$

such that $h \circ \varepsilon = \varepsilon_R$. The map h is uniquely determined, up to chain homotopy, and is a chain homotopy equivalence. Hence, the composite chain map

$$\bar{R}(A) = A \otimes_{A^e} R(A) \xrightarrow{\mathrm{id} \otimes h} A \otimes_{A^e} B(A, A, A) \xrightarrow{v} \mathrm{HH}(A/k)$$

induces an isomorphism of homology groups which is independent of the choice of the map h. The standard choice of h is the unique map of differential graded A^e-algebras that preserves divided powers and satisfies

$$h(dx) = s_{-1}(x \otimes 1 - 1 \otimes x), \qquad h(dy) = s_{-1}(y \otimes 1 - 1 \otimes y),$$
$$h(z) = s_{-1}\big(\frac{x^b \otimes 1 - 1 \otimes x^b}{x \otimes 1 - 1 \otimes x} \cdot h(dx) - \frac{y^a \otimes 1 - 1 \otimes y^a}{y \otimes 1 - 1 \otimes y} \cdot h(dy)\big)$$

with s_{-1} the k-linear chain homotopy defined earlier. Following [**26**, (2.8.5)], we also define a k-linear map

$$\bar{R}(A)[q] \xrightarrow{d_R} \bar{R}(A)[q+1]$$

as follows. We first choose the basis of the free k-module $\bar{R}(A)$ that consists of the elements $x^iy^jz^{[r]}$, $x^iy^jdxz^{[r]}$, $x^iy^jdyz^{[r]}$, and $x^iy^jdxdyz^{[r]}$, where (i,j,r) is a triple of non-negative integers and $i < b$, and next define d_R on these elements by

$$d_R(x^iy^jz^{[r]}) = \begin{cases} 0 & \text{if } i = 0 \text{ and } j = 0 \\ (i+br)x^{i-1}dxz^{[r]} & \text{if } i \geqslant 1 \text{ and } j = 0 \\ jy^{j-1}dyz^{[r]} & \text{if } i = 0 \text{ and } j \geqslant 1 \\ (i+br)x^{i-1}y^jdxz^{[r]} + jx^iy^{j-1}dyz^{[r]} & \text{if } i \geqslant 1 \text{ and } j \geqslant 1 \end{cases}$$
$$d_R(x^iy^j(dx)^{\varepsilon_x}(dy)^{\varepsilon_y}z^{[r]}) = d_R(x^iy^jz^{[r]})(dx)^{\varepsilon_x}(dy)^{\varepsilon_y}.$$

The reader will notice the lack in symmetry in the definition of the map d_R and, not surprisingly, the map is not a derivation and the diagram

$$\begin{tikzcd} \bar{R}(A)[q] \arrow[r, "v\circ(\mathrm{id}\otimes h)"] \arrow[d, "d_R"] & \mathrm{HH}(A/k)[q] \arrow[d, "d"] \\ \bar{R}(A)[q+1] \arrow[r, "v\circ(\mathrm{id}\otimes h)"] & \mathrm{HH}(A/k)[q+1] \end{tikzcd}$$

does not commute. Nevertheless, by using the notion of strongly homotopy k-linear maps introduced in [**15**], it is proved in [**26**, §1] that the map of homology groups induced by d_R is a derivation and that the diagram of homology groups

$$\begin{tikzcd} H_q(\bar{R}(A)) \arrow[r, "v\circ(\mathrm{id}\otimes h)"] \arrow[d, "d_R"] & \mathrm{HH}_q(A/k) \arrow[d, "d"] \\ H_{q+1}(\bar{R}(A)) \arrow[r, "v\circ(\mathrm{id}\otimes h)"] & \mathrm{HH}_{q+1}(A/k) \end{tikzcd}$$

does commute. Hence, we can use the map d_R to evaluate Connes' operator.

The Hochschild homology groups of $B = k[t]$ are calculated in a similar way; this is the starting point of the proof of the Hochschild-Kostant-Rosenberg theorem. To this end, we consider the differential graded B^e-algebra

$$R(B) = B^e \otimes_k \Lambda_k\{dt\},$$

where dt is of degree 1, and where the differential δ maps $\delta(dt) = t\otimes 1 - 1\otimes t$. The augmentation $\varepsilon_R\colon R(B) \to B$ defined by $\varepsilon_R(b\otimes b') = bb'$ is a resolution of the left B^e-module B by free left B^e-modules. We define

$$R(B) \xrightarrow{h} B(B,B,B)$$

to be the unique map of differential graded B^e-algebras that maps the generator dt to $s_{-1}(t\otimes 1 - 1\otimes t)$. It satisfies $\varepsilon\circ h = \varepsilon_R$, and hence, the map

$$\bar{R}(B) = B\otimes_{B^e} R(B) \xrightarrow{\mathrm{id}\otimes h} B\otimes_{B^e} B(B,B,B) \xrightarrow{v} \mathrm{HH}(B/k)$$

is a chain homotopy equivalence, the chain homotopy class of which is independent on our choice of the augmentation preserving chain map h. We also define

$$\bar{R}(B)[q] \xrightarrow{d_R} \bar{R}(B)[q+1]$$

to be the k-linear derivation given by $d_R(t^m) = mt^{m-1}dt$ and note that the following diagram of homology groups commutes, despite the failure of the corresponding diagram of chain groups and maps to do so.

$$\begin{tikzcd} H_q(\bar{R}(B)) \arrow[r, "v\circ(\mathrm{id}\otimes h)"] \arrow[d, "d_R"] & \mathrm{HH}_q(B/k) \arrow[d, "d"] \\ H_{q+1}(\bar{R}(B)) \arrow[r, "v\circ(\mathrm{id}\otimes h)"] & \mathrm{HH}_{q+1}(B/k) \end{tikzcd}$$

Since the differential δ in $\bar{R}(B)$ is trivial, we find that $\eta\colon \Omega^q_{B/k} \to H_q(\bar{R}(B))$ is an isomorphism. Here the target is considered as a strictly anti-symmetric differential graded k-algebra with respect to d_R.

Finally, we define $f_R\colon R(A) \to R(B)$ to be the unique map of differential graded k-algebras that, in degree 0, is given by $f^e\colon A^e \to B^e$ and that maps

$$f_R(dx) = \frac{t^a \otimes 1 - 1 \otimes t^a}{t \otimes 1 - 1 \otimes t} dt, \qquad f_R(dy) = \frac{t^b \otimes 1 - 1 \otimes t^b}{t \otimes 1 - t \otimes t} dt.$$

In this situation, although the diagram of chain groups and maps, which is depicted on the left-hand side below, fails to commute, the induced diagram of homology groups, depicted on the right-hand side, does commute.

$$\begin{tikzcd}
\bar{R}(A) \arrow[r, "f_R"] \arrow[d, "v\circ(\mathrm{id}\otimes h)"] & \bar{R}(B) \arrow[d, "v\circ(\mathrm{id}\otimes h)"] \\
\mathrm{HH}(A/k) \arrow[r, "f"] & \mathrm{HH}(B/k)
\end{tikzcd}
\qquad
\begin{tikzcd}
H_q(\bar{R}(A)) \arrow[r, "f_R"] \arrow[d, "v\circ(\mathrm{id}\otimes h)"] & H_q(\bar{R}(B)) \arrow[d, "v\circ(\mathrm{id}\otimes h)"] \\
\mathrm{HH}_q(A/k) \arrow[r, "f_R"] & \mathrm{HH}_q(B/k)
\end{tikzcd}$$

We use the right-hand diagram to evaluate the reduced cellular homology groups of the pointed $\mathbb{T}$-spaces $N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle; m)$ and the action of Connes' operator.

Proposition 5.1. *Let $1 < a < b$ be relatively prime integers and let m be a positive integer.*

(1) *If neither a nor b divides m, then $\tilde{H}_q(N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle; m); \mathbb{Z})$ is a free abelian group of rank 1 if $q = 2\ell(a,b,m)$ or $q = 2\ell(a,b,m)+1$, and is zero otherwise. In addition, if $q = 2\ell(a,b,m)$, then Connes' operator*

$$\tilde{H}_q(N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle; m); \mathbb{Z}) \xrightarrow{d} \tilde{H}_{q+1}(N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle; m); \mathbb{Z})$$

takes a generator of the domain to m times a generator of the target.

(2) *If a but not b divides m, then $\tilde{H}_q(N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle; m); \mathbb{Z})$ is cyclic of order a, if $q = 2\ell(a,b,m)+1$, and is zero, otherwise.*

(3) *If b but not a divides m, then $\tilde{H}_q(N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle; m); \mathbb{Z})$ is cyclic of order b, if $q = 2\ell(a,b,m)+1$, and is zero, otherwise.*

(4) *If both a and b divide m, then $\tilde{H}_q(N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle; m); \mathbb{Z})$ is zero, for all q.*

Proof. We consider the long exact sequence of reduced cellular homology groups associated with the weight m summand of the cofibration sequence

$$N^{\mathrm{cy}}(\langle x, y\rangle)_+ \xrightarrow{f} N^{\mathrm{cy}}(\langle t\rangle)_+ \xrightarrow{p} N^{\mathrm{cy}}(\langle t\rangle, \langle t^a, t^b\rangle) \xrightarrow{\partial} \Sigma N^{\mathrm{cy}}(\langle x, y\rangle)_+.$$

In this sequence, the maps induced by f and p commute with Connes' operator d and the boundary map anti-commutes with d. We evaluate the groups and maps in the long exact sequence by means of the commutative diagram

$$\begin{tikzcd}
H_q(N^{\mathrm{cy}}(\langle x, y\rangle; m); \mathbb{Z}) \arrow[r, "f"] \arrow[d, "w"] & H_q(N^{\mathrm{cy}}(\langle t\rangle; m); \mathbb{Z}) \arrow[d, "w"] \\
\mathrm{HH}_q(A; m) \arrow[r, "f"] & \mathrm{HH}_q(B; m) \\
H_q(\bar{R}(A; m)) \arrow[r, "f_R"] \arrow[u, "v\circ(\mathrm{id}\otimes h)"] & H_q(\bar{R}(B; m)) \arrow[u, "v\circ(\mathrm{id}\otimes h)"]
\end{tikzcd}$$

in which the vertical maps are isomorphisms.

We begin with the case (1) where neither a nor b divides m. If $\ell(a,b,m)=0$, then $N^{\mathrm{cy}}(\langle x,y\rangle;m)$ is the empty space, and hence, the map

$$H_q(N^{\mathrm{cy}}(\langle t\rangle;m);\mathbb{Z}) \xrightarrow{\ p\ } H_q(N^{\mathrm{cy}}(\langle t\rangle,\langle t^a,t^b\rangle;m);\mathbb{Z})$$

is an isomorphism. The left-hand group is free abelian of rank 1, if $q=0$ or $q=1$, and is zero, otherwise. Moreover, Connes' operator maps the generator t^m in degree $q=0$ to m times the generator $t^{m-1}dt$ in degree $q=1$. This proves the statement for $\ell(a,b,m)=0$. If $\ell(a,b,m)=r+1\geqslant 1$, then we write $m=ai+bj+abr$ with $0<i<b$ and $0<j<a$. In this case, the complex $\bar{R}(A;m)$ is freely generated, as a graded abelian group, by the homogeneous elements

$$x^iy^{j+as}z^{[r-s]},\ \ x^{i-1}y^{j+as}dxz^{[r-s]},\ \ x^iy^{j+as-1}dyz^{[r-s]},\ \ x^{i-1}y^{j+as-1}dxdyz^{[r-s]}$$

with $0\leqslant s\leqslant r$. If $r=0$, then the complex has zero differential, and hence, the homology groups are free abelian on the homology classes of the cycles

$$x^iy^j,\ \ x^{i-1}y^jdx,\ \ x^iy^{j-1}dy,\ \ x^{i-1}y^{j-1}dxdy$$

of degrees 0, 1, 1, and 2, respectively. Similarly, the complex $\bar{R}(B;m)$ has free abelian homology groups generated by the classes of the cycles t^m and $t^{m-1}dt$ of degrees 0 and 1, respectively. Moreover, the map

$$H_q(\bar{R}(A;m)) \xrightarrow{\ f_R\ } H_q(\bar{R}(B;m))$$

takes x^iy^j to t^m and $x^{i-1}y^jdx$ and $x^iy^{j-1}dy$ to $at^{m-1}dt$ and $bt^{m-1}dt$, respectively. Hence, for $q=0$, this map is an isomorphism, and for $q=1$, it is a surjection whose kernel is generated by the class of $b\cdot x^{i-1}y^jdx-a\cdot x^iy^{j-1}dy$. In addition,

$$d_R(b\cdot x^{i-1}y^jdx-a\cdot x^iy^{j-1}dy)=-m\cdot x^{i-1}y^{j-1}dxdy.$$

The statement (1) for $\ell(a,b,m)=1$ follows. Finally, if $r\geqslant 1$, then we find that the groups $H_q(\bar{R}(A;m))$ are free abelian generated by the classes of the cycles

$$\begin{aligned}&x^iy^{j+ar},\ \ d\cdot x^{i-1}y^{j+ar}dx-c\cdot x^iy^{j+ar-1}dy,\\ &b\cdot x^{i-1}y^jdxz^{[r]}-a\cdot x^iy^{j-1}dyz^{[r]},\ \ x^{i-1}y^{j-1}dxdyz^{[r]}\end{aligned}$$

whose degrees are 0, 1, $2r+1$, and $2r+2$, respectively. Here, we have chosen a pair of integers (c,d) such that $ad-bc=1$, but we note that the homology class of the cycle $d\cdot x^{i-1}y^{j+ar}dx-c\cdot x^iy^{j+ar-1}dy$ is independent of this choice. The map

$$H_q(\bar{R}(A;m)) \xrightarrow{\ f_R\ } H_q(\bar{R}(B;m))$$

takes x^iy^{j+ar} to t^m and $d\cdot x^{i-1}y^{j+ar}dx-c\cdot x^iy^{j+ar-1}dy$ to $t^{m-1}dt$, and hence, is an isomorphism for $q=0$ and $q=1$. Moreover,

$$d_R(b\cdot x^{i-1}y^jdxz^{[r]}-a\cdot x^iy^{j-1}dyz^{[r]})=-m\cdot x^{i-1}y^{j-1}dxdyz^{[r]}.$$

The statement (1) for $\ell(a,b,m)\geqslant 2$ follows.

We next consider the case (2), where a but not b divides m, the case (3) being similar. If we write $m=ai+abr$ with $0<i<b$, then $\ell(a,b,m)=r$. As a graded abelian group, $\bar{R}(A)$ is freely generated by the homogeneous elements

$$\begin{aligned}&x^iy^{as}z^{[r-s]},\ \ x^{i-1}y^{as}dxz^{[r-s]} &&\text{with } 0\leqslant s\leqslant r;\\ &x^iy^{as-1}dyz^{[r-s]},\ \ x^{i-1}y^{as-1}dxdyz^{[r-s]} &&\text{with } 1\leqslant s\leqslant r.\end{aligned}$$

If $r = 0$, then $\bar{R}(A)$ has zero differential, and therefore, its homology groups are free abelian generated by the classes of the cycles x^i and $x^{i-1}dx$. Moreover,

$$H_q(\bar{R}(A;m)) \xrightarrow{f_R} H_q(\bar{R}(B;m))$$

maps x^i to t^m and $x^{i-1}dx$ to $a \cdot t^{m-1}dt$, respectively, so the statement (2) in the case $\ell(a,b,m) = 0$ follows. If $r \geqslant 1$, then the homology groups of $\bar{R}(A;m)$ are concentrated in degrees $q = 0$, $q = 1$, and $q = 2r$. The first two groups are free abelian of rank 1 generated by the classes of the cycles

$$x^i y^{ar}, \quad d \cdot x^{i-1} y^{ar} dx - c \cdot x^i y^{ar-1} dy,$$

respectively, and we conclude as before that $f_R \colon H_q(\bar{R}(A;m)) \to H_q(\bar{R}(B;m))$ is an isomorphism in degrees $q = 0$ and $q = 1$. The third group is cyclic of order a generated by the class of the cycle $x^{i-1}y^{a-1}dxdyz^{[r-1]}$. Hence, we conclude that the statement (2) holds also for $\ell(a,b,m) \geqslant 1$.

Finally, we consider the case (4), where a and b both divide m. We write $m = abr$ with $r \geqslant 1$ and note that $\ell(a,b,m) = r - 1$. As a graded free abelian group, the complex $\bar{R}(A;m)$ is generated by the homogeneous elements

$$\begin{array}{ll} y^{as}z^{[r-s]} & \text{with } 0 \leqslant s \leqslant r; \\ x^{bs-1}dxz^{[r-s]}, y^{as-1}dyz^{[r-s]} & \text{with } 1 \leqslant s \leqslant r; \\ x^{b-1}y^{a(s-1)-1}dxdyz^{[r-s]} & \text{with } 2 \leqslant s \leqslant r. \end{array}$$

The homology groups are free abelian generated by the classes of the cycles

$$y^{ar}, \quad d \cdot x^{br-1}dx - c \cdot y^{a-1}dy,$$

and we see as before that $f_R \colon H_q(\bar{R}(A;m)) \to H_q(\bar{R}(B;m))$ is an isomorphism. It follows that the groups $\tilde{H}_q(N^{\mathrm{cy}}(\langle t \rangle, \langle t^a, t^b \rangle; m))$ are all trivial, as stated. □

We may view the reduced homology of a pointed left $\mathbb{T}$-space as a graded left module over the Pontryagin ring $H_*(\mathbb{T};\mathbb{Z})$ with left multication by the fundamental class $[\mathbb{T}]$ given by Connes' operator.

Corollary 5.2. *Let $1 < a < b$ be relative prime integers. For every positive integer m, the left $H_*(\mathbb{T};\mathbb{Z})$-modules given by the reduced homology groups of the pointed $\mathbb{T}$-spaces $\mathbb{T}_+ \wedge_{C_m} X(a,b,m)$ and $\mathbb{T}_+ \wedge_{C_m} Y(a,b,m)$ are abstractly isomorphic.*

Proof. The structure of the left $H_*(\mathbb{T};\mathbb{Z})$-module given by the reduced homology groups of $\mathbb{T}_+ \wedge_{C_m} X(a,b,m)$ was determined in Propositions 3.1 and 5.1. Using the untwisting isomorphism in Remark 3.2, we may identify the pointed left $\mathbb{T}$-space $\mathbb{T}_+ \wedge_{C_m} Y(a,b,m)$ with the iterated mapping cone of the diagram

$$\begin{tikzcd} (\mathbb{T}/C_{m/ab})_+ \wedge S^{\lambda(a,b,m)} \arrow[r] \arrow[d] & (\mathbb{T}/C_{m/a})_+ \wedge S^{\lambda(a,b,m)} \arrow[d] \\ (\mathbb{T}/C_{m/b})_+ \wedge S^{\lambda(a,b,m)} \arrow[r] & (\mathbb{T}/C_m)_+ \wedge S^{\lambda(a,b,m)}, \end{tikzcd}$$

where the maps are the canonical projections, and the upper right-hand term (resp. lower left-hand term, resp. upper left-hand term) is understood to be a one-point space if m is not divisible by a (resp. by b, resp. by ab). Now one readily verifies that the left $H_*(\mathbb{T};\mathbb{Z})$-module given by the reduced homology groups of

$\mathbb{T}_+ \wedge_{C_m} Y(a,b,m)$ has the same abstract structure as that given by the reduced homology groups of $\mathbb{T}_+ \wedge_{C_m} X(a,b,m)$. $\square$

REMARK 5.3. Let v be a non-negative integer. It follows from Corollary 5.2 and the proof of Theorem A that if Conjecture B holds for all positive integers m with $\ell(a,b,m) \leqslant v$, then long exact sequence in the statement of Theorem A is valid for all $q \leqslant 2v+1$. A similar argument shows that Theorem C follows from Proposition 6.4 which we prove below.

6. Stunted regular cyclic polytopes

In this section, we formulate Conjecture 6.2 concerning a new family of polytopes that we call stunted regular cyclic polytopes. Assuming the conjecture, we construct maps $u(a,b,m)$ that satisfy part (1) of Conjecture B. Finally, we prove Conjecture 6.2 and Conjecture B for small values of m.

We recall that the regular cyclic polytope of dimension $2d$ with m vertices as defined by Gale [**8**] is the convex hull $P(d,m)$ of the subset

$$V(d,m) = \{(z, z^2, \dots, z^d) \mid z \in C_m\} \subset \mathbb{C}^{\{1,2,\dots,d\}}$$

where $C_m \subset \mathbb{C}^*$ is the group of mth roots of unity. The regular cyclic polytopes are high-dimensional generalizations of the regular polygons, and the combinational structure of their faces is completely understood. We are interested in a family of polytopes defined in a similar manner but with the interval $\{1,2,\dots,d\}$ replaced by different intervals of integers. We call these stunted regular cyclic polytopes.

DEFINITION 6.1. Let $1 < a < b$ be relatively prime integers, let (c,d) be a pair of integers with $ad - bc = 1$, and let m be a positive integer. The stunted regular cyclic polytope $P(a,b,m)$ is the convex hull of the finite subset

$$V(a,b,m) = \{(z^n \mid n \in J(a,b,m)) \mid z \in C_m\} \subset \mathbb{C}^{J(a,b,m)},$$

where $J(a,b,m)$ is the set of integers in the closed interval $[cm/a, dm/b]$.

We note that the dimension of the polytope $P(a,b,m)$ is equal to twice the number $\ell(a,b,m+a+b)$ of ways in which m can be expressed as $m = ai + bj$ with (i,j) a pair of non-negative integers.

We let $\mathbb{C}(n)$ be the C_m-representation that is given by $\mathbb{C}$ with $z \in C_m$ acting as multiplication by z^n and define $\bar{\lambda}(a,b,m)$ to be the direct sum over all integers n in the closed interval $[cm/a, dm/b]$ of the representations $\mathbb{C}(n)$. It contains the representation $\lambda(a,b,m)$ defined as the direct sum of the representations $\mathbb{C}(n)$ as n ranges over all integers in the open interval $(cm/a, dm/b)$ as a subrepresentation, and we have $\lambda(a,b,m) = \bar{\lambda}(a,b,m)$ if and only if a and b do not divide m. Indeed, if $ad - bc = 1$, then $(a,b) = (a,c) = (b,d) = (c,d) = 1$. If a divides m (resp. if b divides m), then we define $\lambda'(a,b,m)$ (resp. $\lambda''(a,b,m)$) to be the subrepresentation of $\bar{\lambda}(a,b,m)$ given by the summand $\mathbb{C}(cm/a)$ (resp. by the summand $\mathbb{C}(dm/b)$). The map $x\colon C_m \to \bar{\lambda}(a,b,m)$ with nth component $x_n(z) = z^n$ extends to a map of representations $x\colon \mathbb{R}[C_m] \to \bar{\lambda}(a,b,m)$ and the stunted regular cyclic polytope

$$P(a,b,m) \subset \bar{\lambda}(a,b,m)$$

is equal to the image by the latter map of $\Delta^{m-1} \subset \mathbb{R}[C_m]$. We remark that $P(a,b,m)$ contains the origin of $\bar{\lambda}(a,b,m)$ as an interior point. We also define

$$Q(a,b,m) \subset P(a,b,m)$$

to be the image of the sub-simplicial complex $\Sigma(a,b,m) \subset \Delta^{m-1}$ defined in the introduction; it is a sub-C_m-space of $P(a,b,m)$. To understand this subspace better, we let p be an integer in the closed interval $[cm/a, dm/b]$ and write $p = cu + dv$ for a unique pair of non-negative integers (u,v). We define an index function of weight p to be a map $g\colon \mathbb{Z} \to \mathbb{Z}$ with the property that for all $t \in \mathbb{Z}$, $g(t+u+v) - g(t) = m$ and $g(t) - g(t-1) \in \{a,b\}$, and define $Q(a,b,m;g)$ to be the polytope given by the convex hull in $\bar{\lambda}(a,b,m)$ of the finite subset

$$V(a,b,m;g) = \{(\zeta_m^{g(t)n} \mid n \in J(a,b,m)) \mid t \in \mathbb{Z}\} \subset \mathbb{C}^{J(a,b,m)}.$$

Then $Q(a,b,m)$ is equal to the union of the polytopes $Q(a,b,m;g)$ as p ranges over all integers in $[cm/a, dm/b]$ and g ranges over all index functions of weight p.

If a (resp. b) divides m, then the composition of the canonical inclusion of the subspace $\lambda'(a,b,m)$ (resp. $\lambda''(a,b,m)$) into $\bar{\lambda}(a,b,m)$ and the canonical projection of $\bar{\lambda}(a,b,m)$ onto $\mathbb{C}(cm/a)$ (resp. $\mathbb{C}(dm/b)$) is an isomorphism, and we define the subset $C_a' \subset \lambda'(a,b,m)$ (resp. $C_b'' \subset \lambda''(a,b,m)$) to be the inverse image of the subset $C_a \subset \mathbb{C}(cm/a)$ (resp. $C_b \subset \mathbb{C}(dm/b)$).

Conjecture 6.2. *If $1 < a < b$ are relatively prime integers and if m is a positive integer, then the following hold.*

(1) *The subspace $Q(a,b,m) \subset P(a,b,m)$ does not contain the origin of $\bar{\lambda}(a,b,m)$.*
(2) *If a divides m, then $Q(a,b,m) \cap \lambda'(a,b,m) = C_a' \subset \lambda'(a,b,m)$.*
(3) *If b divides m, then $Q(a,b,m) \cap \lambda''(a,b,m)) = C_b'' \subset \lambda''(a,b,m)$.*
(4) *If neither a nor b divides m, then $\partial P(a,b,m) \subset Q(a,b,m)$.*

Granting this conjecture, we define maps of pointed C_m-spaces

$$X(a,b,m) \xrightarrow{u(a,b,m)} Y(a,b,m)$$

satisfying part (1) of Conjecture B as follows. The maps are the composition

$$\Delta^{m-1}/\Sigma(a,b,m) \xrightarrow{x} P(a,b,m)/Q(a,b,m) \xrightarrow{h} Y(a,b,m)$$

of the map x and a map h which we now define. Suppose first that neither a nor b divides m. In this case, we define h to be the map of pointed C_m-spaces

$$P(a,b,m)/Q(a,b,m) \xrightarrow{h} D(\lambda(a,b,m))/S(\lambda(a,b,m)) = S^{\lambda(a,b,m)}$$

induced by a suitable radial dilation away from $0 \in P(a,b,m)$. It follows from Conjecture 6.2 (1) that the map is well-defined, provided that the dilation factor is sufficiently large. Moreover, if the map is defined, then its homotopy class is independent of the choice of dilation factor. Suppose next that a but not b divides m. We first use Conjecture 6.2 (2) to choose a small ball $B \subset \bar{\lambda}(a,b,m) \smallsetminus C_a'$ around the unique point ζ_{2a}' of $S(\lambda'(a,b,m))$ with image $\zeta_{2a} \in \mathbb{C}(cm/a)$ by the canonical projection and consider the open subset

$$U = (C_m \cdot B) \cap S(\bar{\lambda}(a,b,m)) \subset S(\bar{\lambda}(a,b,m)).$$

It follows from Conjecture 6.2 (1) that radial dilation with sufficiently large dilation factor away from $0 \in P(a,b,m)$ induces a map of pointed C_m-spaces

$$P(a,b,m)/Q(a,b,m) \xrightarrow{h'} D(\bar{\lambda}(a,b,m))/(S(\bar{\lambda}(a,b,m)) \smallsetminus U).$$

Second, let $C_a^\dagger = C_m \cdot \zeta_{2a}' \subset S(\lambda'(a,b,m))$ be the subset of translates by C_m of ζ_{2a}', and let $\widetilde{C}_a^\dagger \subset D(\lambda'(a,b,m))$ be the cone on $C_a^\dagger$ with apex 0. Then the canonical homeomorphism from $D(\lambda'(a,b,m)) \times D(\lambda(a,b,m))$ onto $D(\bar{\lambda}(a,b,m))$ induces an inclusion ι of the pointed C_m-space

$$(\widetilde{C}_a^\dagger \times D(\lambda(a,b,m)))/(\widetilde{C}_a^\dagger \times S(\lambda(a,b,m)) \cup C_a^\dagger \times D(\lambda(a,b,m))),$$

which we identify with $Y(a,b,m)$ by identifying $\zeta_{2a}' \in C_a^\dagger$ with $1 \in C_a$, into the target pointed C_m-space of the map h'. More, the inclusion ι is readily verified to be a strong deformation retract. Finally, we define the map h to be the composition $h'' \circ h'$ of the map h' and a homotopy inverse h'' of ι. Again, the homotopy class of the map of pointed C_m-spaces c is independent of the choices made. The definition of the map h in the remaining cases is analogous.

We fix a pair of relatively prime integers $1 < a < b$ and choose integers c and d with $ad - bc = 1$. Let m be a positive integer, let (i,j) be a pair of non-negative integers such that $m = ai + bj$, and let $n = ci + dj$ be the corresponding integer in the closed interval $[cm/a, bm/d]$. Let also p be an integer in $[cm/a, dm/b]$ and let $g \colon \mathbb{Z} \to \mathbb{Z}$ be an index function of weight p. We proceed to describe the images

$$Q(a,b,m,n;g) \subset Q(a,b,m,n) \subset P(a,b,m,n) \subset \mathbb{C}(n)$$

of $Q(a,b,m;g) \subset Q(a,b,m) \subset P(a,b,m) \subset \bar{\lambda}(a,b,m)$ by the canonical projection onto the summand $\mathbb{C}(n)$. Since for every pair of integers (k,l), we have

$$\begin{pmatrix} k & l \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} k & l \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix},$$

and since the square matrix on the right-hand side is invertible, we find that the greatest common divisors (m,n) and (i,j) are equal. We let q be the common value and write $m = m'q$, $n = n'q$, $i = i'q$, and $j = j'q$. Then $P(a,b,m,n)$ is equal to the regular polygon in $\mathbb{C}(n)$ with vertex set

$$V(a,b,m,n) = \{z^n \mid z \in C_m\} = C_{m'}.$$

Similarly, $Q(a,b,m,n;g)$ is the (irregular) polygon in $\mathbb{C}(n)$ with vertex set

$$V(a,b,m,n,g) = \{\zeta_m^{g(t)n} \mid t \in \mathbb{Z}\} = \{\zeta_{m'}^{g(t)n'} \mid t \in \mathbb{Z}\} \subset C_{m'},$$

and $Q(a,b,m,n)$ is the union of the polygons $Q(a,b,m,n,g)$ as p ranges over all integers in $[cm/a, dm/b]$ and g ranges over all index functions of weight p. In this connection, we note that since $i = dm - bn$ and $j = -cm + an$, we have

$$\zeta_m^{an} = \zeta_m^{j} = \zeta_{m'}^{j'}, \qquad \zeta_m^{bn} = \zeta_m^{-i} = \zeta_{m'}^{-i'}.$$

Finally, let (r,s) be a pair of integers with $ri' + sj' = 1$. We define (k,l) by

$$\begin{pmatrix} k & l \end{pmatrix} = \begin{pmatrix} r & s \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and note that since $km' + ln' = 1$, we have

$$\zeta_m^{ln} = \zeta_{m'}^{ln'} = \zeta_{m'}.$$

We proceed to describe the integer l more precisely. First, if i and j are both positive, then we may choose (r,s) with $0 \leqslant -r \leqslant j'-1$ and $1 \leqslant s \leqslant i'$. Indeed, if $1 \leqslant s \leqslant i'$, then $j' \leqslant sj' \leqslant i'j'$, and hence, $j'-1 \leqslant -ri' \leqslant i'j'-1$, which implies that $0 \leqslant -r \leqslant j'-1$ as desired. Therefore, if i and j are both positive, then we can choose $l = as + b(-r)$ to be a linear combination of a and b with non-negative coefficients that satisfies the inequalities

$$a \leqslant l \leqslant m' - b.$$

Next, if $i > 0$ and $j = 0$, then $i' = 1$, $j' = 0$, $m' = a$, and $n' = c$. We necessarily have $r = 1$ and $s = 0$, which gives $k = d$ and $l = -b$. Finally, if $i = 0$ and $j > 0$, then $i' = 0$, $j' = 1$, $m' = b$, and $n' = d$. Therefore, we have $r = 0$ and $s = 1$, which gives $k = -c$ and $l = a$. We use this to prove the following result.

PROPOSITION 6.3. *Let $1 < a < b$ be relatively prime integers. Conjecture 6.2 is true for all positive integers m with $\ell(a,b,m) \leqslant 1$.*

PROOF. We use the descriptions of $P(a,b,m,n)$ and $Q(a,b,m,n)$ established above and first assume that neither a nor b divides m. If $\ell(a,b,m) = 0$, then $P(a,b,m)$ consists of a single point and $Q(a,b,m)$ is empty, so Conjecture 6.2 holds trivially for m. Therefore, we may further assume that $\ell(a,b,m) = 1$, in which case $m = ai + bj$ with $0 < i < b$ and $0 < j < a$, and hence, $P(a,b,m) = P(a,b,m,n)$ and $Q(a,b,m) = Q(a,b,m,n)$ with $n = ci + dj$. Moreover, every index function $g\colon \mathbb{Z} \to \mathbb{Z}$ has weight n. To prove statement (4) of Conjecture 6.2 for m, it will suffice to show that there exists an index function $g\colon \mathbb{Z} \to \mathbb{Z}$ that takes both of the values 0 and l. For then $\{1, \zeta_{m'}\} \subset V(a,b,m,n,g)$, and therefore, the edge that connects 1 and $\zeta_{m'}$ is contained in $Q(a,b,m)$, which, in turn, shows that $\partial P(a,b,m) \subset Q(a,b,m)$, since $Q(a,b,m) = C_m \cdot Q(a,b,m) \subset \mathbb{C}(n)$. But we saw above that $l = as + b(-r)$ with $1 \leqslant s \leqslant i'$ and $0 \leqslant -r \leqslant j'-1$, so the required index function indeed exists. It remains to prove that statement (1) of Conjecture 6.2 holds for m. To this end we note that, since $0 < i < b$ and $0 < j < a$,

$$m' - 2ij' = ai' + bj' - ji' - ij' = (a-j)i' + (b-i)j' \geqslant i' + j' \geqslant 2,$$

which, in turn, gives the inequality

$$ij' < \lfloor m'/2 \rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Now, if $g\colon \mathbb{Z} \to \mathbb{Z}$ be an index function, then among the $i + j$ integers $\{0, 1, \dots, i+j-1\}$, there are i integers u with $g(u) - g(u-1) = a$ and j integers u with $g(u) - g(u-1) = b$, and since $\zeta_m^{an} = \zeta_{m'}^{j'}$ and $\zeta_m^{bn} = \zeta_{m'}^{-i'}$, the inequality above shows that

$$V(a,b,m,n,g) \subset \{\zeta_{m'}^{g(0)n'+u} \mid u \in \{0, 1, \dots, \lfloor m'/2 \rfloor - 1\}\}.$$

Therefore, the polygon $Q(a,b,m,n,g)$ does not contain $0 \in \mathbb{C}(n)$, and since this is true for every index function $g\colon \mathbb{Z} \to \mathbb{Z}$, neither does $Q(a,b,m)$. This proves that also statement (1) of Conjecture 6.2 holds for m.

We next assume that a but not b divides m. If $\ell(a,b,m) = 0$, then $m = ai$ with $0 < i < b$, and we have $P(a,b,m) = P(a,b,m,n)$ and $Q(a,b,m) = Q(a,b,m,n)$ with $n = ci$. The former is the regular polygon with vertex set $C_a \subset \mathbb{C}(n)$ and the latter is the subset C_a of vertices, and hence, Conjecture 6.2 holds for m. So we further assume that $\ell(a,b,m) = 1$, in which case $m = a(i+b) = ai + ba$ with $0 < i < b$, and hence, $\bar{\lambda}(a,b,m) = \mathbb{C}(n_1) \oplus \mathbb{C}(n_2)$ with $n_1 = ci + bc$ and $n_2 = ci + ad$. We find that $P(a,b,m,n_1)$ and $P(a,b,m,n_2)$ are the regular polygons with vertex sets $C_a \subset \mathbb{C}(n_1)$ and $C_{m/(a,i)} \subset \mathbb{C}(n_2)$, respectively. The index functions $g\colon \mathbb{Z} \to \mathbb{Z}$ have weight either n_1 or n_2. If g has weight n_1, then $Q(a,b,m,n_1;g)$ is a single element of $C_a \subset \mathbb{C}(n_1)$ and $Q(a,b,m,n_2;g)$ is the regular polygon with vertex set $C_{i+b} \subset \mathbb{C}(n_2)$. It follows that $Q(a,b,m;g)$ does not contain the origin of $\bar{\lambda}(a,b,m)$ and that $Q(a,b,m;g) \cap \lambda'(a,b,m)$ is a single element of $C_a' \subset \lambda'(a,b,m)$. If g has weight n_2, then $Q(a,b,m,n_1;g) = P(a,b,m,n_1)$ while $Q(a,b,m,n_2;g)$ is contained in the (irregular) polygon with vertex set

$$\{\zeta_{m/(a,i)}^{w+ta/(a,i)} \mid t \in \{0,1,\dots,i-1\}\} \subset C_{m/(a,i)} \subset \mathbb{C}(n_2),$$

for some integer w. Since

$$(ai/(a,i))/(m/(a,i)) = ai/m = ai/(ai+ab) < ai/(ai+ai) = 1/2,$$

we conclude that $Q(a,b,m,n_2;g) \subset \mathbb{C}(n_2)$ does not contain the origin. It follows that $Q(a,b,m;g) \subset \bar{\lambda}(a,b,m)$ does not contain the origin and does not intersect the subspace $\lambda'(a,b,m)$. This shows that Conjecture 6.2 holds for m. The case where b but not a divides m is completely analogous.

It remains only to consider the cases $m = ab$ and $m = 2ab$. In the former case, we have $\ell(a,b,m) = 0$ and $\bar{\lambda}(a,b,m) = \mathbb{C}(n_1) \oplus \mathbb{C}(n_2)$ with $n_1 = bc$ and $n_2 = ad$. Hence, $P(a,b,m,n_1)$ and $P(a,b,m,n_2)$ are the regular polygons with vertex sets $C_a \subset \mathbb{C}(n_1)$ and $C_b \subset \mathbb{C}(n_2)$, respectively. An index function $g\colon \mathbb{Z} \to \mathbb{Z}$ has weight either n_1 or n_2. If g has weight n_1, then $Q(a,b,m,n_1;g)$ is a single point in $C_a \subset \mathbb{C}(n_1)$ while $Q(a,b,m,n_2;g) = P(a,b,m,n_2)$; and if g has weight n_2, then $Q(a,b,m,n_1;g) = P(a,b,m,n_1)$ while $Q(a,b,m,n_2;g)$ is a single point in $C_b \subset \mathbb{C}(n_2)$. This proves that Conjecture 6.2 holds for $m = ab$. In the latter case $m = 2ab$, we have $\ell(a,b,m) = 1$ and

$$\bar{\lambda}(a,b,m) = \mathbb{C}(n_1) \oplus \mathbb{C}(n_2) \oplus \mathbb{C}(n_3)$$

with $n_1 = 2bc$, $n_2 = ad+bc$, and $n_3 = 2ad$. Hence, $P(a,b,m,n_1)$, $P(a,b,m,n_2)$, and $P(a,b,m,n_3)$ are the regular polygons with vertex sets $C_a \subset \mathbb{C}(n_1)$, $C_m \subset \mathbb{C}(n_2)$, and $C_b \subset \mathbb{C}(n_3)$, respectively. The index functions $g\colon \mathbb{Z} \to \mathbb{Z}$ have weights n_1, n_2, or n_3. If g has weight n_1, then $Q(a,b,m,n_1;g)$ is a single point in $C_a \subset \mathbb{C}(n_1)$ while $Q(a,b,m,n_2;g)$ and $Q(a,b,m,n_3;g)$ are the regular polygons with vertex sets $C_{2b} \subset \mathbb{C}(n_2)$ and $C_b \subset \mathbb{C}(n_3)$, respectively. Moreover, the images of

$$z = x((1/2b)(\zeta_m^{g(0)} + \zeta_m^{g(1)} + \dots + \zeta_m^{g(2b-1)})) \in Q(a,b,m;g)$$

in $Q(a,b,m,n_2;g)$ and $Q(a,b,m,n_3;g)$ are the respective origins. It follows that $Q(a,b,m;g) \subset \bar{\lambda}(a,b,m)$ does not contain the origin, that $Q(a,b,m;g) \cap \lambda'(a,b,m)$ is a subset of $C'_a \subset \lambda'(a,b,m)$, and that this subset is non-empty. Likewise, if g has weight n_3, then a similar argument shows that $Q(a,b,m;g) \subset \bar{\lambda}(a,b,m)$ does not contain the origin, that the intersection $Q(a,b,m;g) \cap \lambda''(a,b,m)$ is a subset of $C''_b \subset \lambda''(a,b,m)$, and that this subset is non-empty. Finally, if g has weight n_2, then $Q(a,b,m,n_2;g)$ is contained in the convex hull of

$$\{\zeta_m^{t+u} \mid t \in \{0,1,\dots,ab\}\} \subset \mathbb{C}(n_2),$$

some some integer u, and may contain $0 \in \mathbb{C}(n_2)$. If it does, then there exists an integer u with the property that

$$g(t) = \begin{cases} g(u) + a(t-u) & \text{if } 0 \leqslant t-u < b \\ g(u) + ab + b(t-u-b) & \text{if } b \leqslant t-u < a+b \end{cases}$$

and $0 \in \mathbb{C}(n_2)$ is the image by the canonical projection of the unique point

$$z = x((1/2)(\zeta_m^{g(u)} + \zeta_m^{g(u)+ab})) \in Q(a,b,m;g).$$

But the images of z in $Q(a,b,m,n_1;g)$ and $Q(a,b,m,n_3;g)$ are equal to $1 \in \mathbb{C}(n_1)$ and $1 \in \mathbb{C}(n_3)$, respectively. It follows that $Q(a,b,m;g) \subset \bar{\lambda}(a,b,m)$ does not contain the origin and does not intersect neither $\lambda'(a,b,m)$ nor $\lambda''(a,b,m)$. This completes the proof that Conjecture 6.2 holds for $m = 2ab$. $\square$

Proposition 6.4. *If $1 < a < b$ are relatively prime integers, then Conjecture B is true for all positive integers m with $\ell(a,b,m) = 0$ and for all positive integers divisible by neither a nor b with $\ell(a,b,m) = 1$.*

Proof. We recall that for all positive integers m for which Conjecture 6.2 is true, we have already constructed maps of pointed C_m-spaces

$$X(a,b,m) \xrightarrow{u(a,b,m)} Y(a,b,m)$$

that satisfy part (1) of Conjecture B, and by Proposition 6.3, Conjecture 6.2 is true for all positive integers m with $\ell(a,b,m) \leqslant 1$. Therefore, to prove the theorem, we must show that for these m, the induced maps of pointed $\mathbb{T}$-spaces

$$\mathbb{T}_+ \wedge_{C_m} X(a,b,m) \xrightarrow{u'(a,b,m)} \mathbb{T}_+ \wedge_{C_m} Y(a,b,m)$$

induce isomorphisms of reduced singular homology groups. We already know from Corollary 5.2 that the reduced homology groups of the domain and target of $u'(a,b,m)$ are abstractly isomorphic cyclic groups. Therefore, it will suffice to show that the map $u'(a,b,m)$ induces a surjection on reduced homology groups.

We first consider the case in which neither a nor b divides m. If $\ell(a,b,m) = 0$, then $\Sigma(a,b,m)$ is empty and $u'(a,b,m)$ is the homotopy equivalence

$$\mathbb{T}_+ \wedge_{C_m} \Delta^{m-1}_+ \xrightarrow{u'(a,b,m)} \mathbb{T}_+ \wedge_{C_m} S^0$$

that collapses Δ^{m-1} onto $0 \in S^0$, and hence, induces an isomorphism of reduced homology groups. So we assume that $\ell(a,b,m) = 1$ and consider the diagram

$$\begin{tikzcd}
\tilde{H}_2(\mathbb{T}_+ \wedge_{C_m} X(a,b,m);\mathbb{Z}) \arrow[r, "u'(a,b,m)"] \arrow[d, "d"] & \tilde{H}_2(\mathbb{T}_+ \wedge_{C_m} Y(a,b,m);\mathbb{Z}) \arrow[d, "d"] \\
\tilde{H}_3(\mathbb{T}_+ \wedge_{C_m} X(a,b,m);\mathbb{Z}) \arrow[r, "u'(a,b,m)"] & \tilde{H}_3(\mathbb{T}_+ \wedge_{C_m} Y(a,b,m);\mathbb{Z})
\end{tikzcd}$$

in which the vertical maps are given by Connes' operator. The diagram commutes since $u'(a,b,m)$ is $\mathbb{T}$-equivariant. The four groups in the diagram all are infinite cyclic and are the only non-zero reduced homology groups of the pointed spaces in question. Moreover, the vertical maps are injections onto the respective subgroups of index m. Therefore, to prove that the horizontal maps are isomorphisms, it will suffice to show that the top map is surjective. We further consider the diagram

$$\begin{tikzcd}
\tilde{H}_2(X(a,b,m);\mathbb{Z}) \arrow[r, "u(a,b,m)"] \arrow[d] & \tilde{H}_2(Y(a,b,m);\mathbb{Z}) \arrow[d] \\
\tilde{H}_2(\mathbb{T}_+ \wedge_{C_m} X(a,b,m);\mathbb{Z}) \arrow[r, "u'(a,b,m)"] & \tilde{H}_2(\mathbb{T}_+ \wedge_{C_m} Y(a,b,m);\mathbb{Z})
\end{tikzcd}$$

in which the vertical maps are induced by the maps that take z to the class of $(1,z)$. Since the right-hand vertical map is an isomorphism, we are further reduced to showing that the top horizontal map in this diagram is surjective. We use the known facet structure of the polytope $P(a,b,m)$ to produce a generator of the target of the map induced by $u(a,b,m)$ and an element of the domain of this map that maps to this generator. We have $m = ai + bj$ for a unique pair (i,j) of positive integers and define the integers n, q, m', n', i', j', r, s, k, and l as in the discussion preceeding Proposition 6.3. With this notation in hand, the polytope $P(a,b,m)$ is the regular polygon in $\mathbb{C}(n)$ with vertex set

$$V(a,b,m) = \{z^n \mid z \in C_m\} = C_{m'} \subset \mathbb{C}(n).$$

We triangulate $P(a,b,m)$ by the cones with apex $1 \in C_{m'}$ and with bases given by the edges in $\partial P(a,b,m)$ that do not have 1 as a vertex. To this end, we define $F(a,b,m)$ to be the set of strictly increasing maps $\theta\colon [2] \to [m-1]$ with the property that $\theta(0) = 0$, $\theta(2) < m'$, and $\theta(2) - \theta(1) \equiv \pm l$ modulo m' and let

$$\operatorname{sgn}(\theta) = \begin{cases} +1 & \text{if } \theta(2) - \theta(1) \equiv +l \text{ modulo } m' \\ -1 & \text{if } \theta(2) - \theta(1) \equiv -l \text{ modulo } m'. \end{cases}$$

Now the cones with apex $1 \in \mathbb{C}(n)$ and bases given by the edges in $\partial P(a,b,m)$ that do not have 1 as a vertex are exactly the images of the composite maps

$$\Delta^2 \xrightarrow{\Delta^\theta} \Delta^{m-1} \xrightarrow{x} P(a,b,m)$$

with $\theta \in F(a,b,m)$. Moreover, we may orient Δ^2 and $P(a,b,m)$ in such a way that the map $x \circ \Delta^\theta$ is orientation-preserving if $\operatorname{sgn}(\theta) = +1$ and orientation-reversing if $\operatorname{sgn}(\theta) = -1$. It follows that the singular chain in $Y(a,b,m)$ defined by

$$\mathfrak{z} = \sum_{\theta \in F(a,b,m)} \operatorname{sgn}(\theta)(h \circ x \circ \Delta^\theta)$$

is a cycle whose homology class generates $\tilde{H}_2(Y(a,b,m);\mathbb{Z})$. The chain $\mathfrak{z}$ is the image by $u(a,b,m)$ of the chain in $X(a,b,m)$ defined by

$$\mathfrak{z}' = \sum_{\theta \in F(a,b,m)} \operatorname{sgn}(\theta)(\operatorname{pr} \circ \Delta^\theta),$$

where $\operatorname{pr}\colon \Delta^{m-1} \to X(a,b,m)$ is the canonical projection, and therefore, it will suffice to show that $\mathfrak{z}'$ is a cycle. Its boundary is given as follows. Let $E(a,b,m)$ be the set of strictly increasing maps $\sigma\colon [1] \to [m-1]$ with the property that $\sigma(1) < m'$ and $\sigma(1) - \sigma(0) = \pm l$ modulo m' and define $\operatorname{sgn}(\sigma)$ to be $+1$ if $\sigma(1) - \sigma(0) \equiv +l$ modulo m' and to be -1 if $\sigma(1) - \sigma(0) \equiv -l$ modulo m'. Then

$$\partial(\mathfrak{z}') = \sum_{\sigma \in E(a,b,m)} \operatorname{sgn}(\sigma)(\operatorname{pr} \circ \Delta^\sigma),$$

and each summand is zero, since l and $m' - l$ both are of the form $au + bv$ with (u,v) a pair of non-negative integers.

We next consider the case where a but not b divides m and $\ell(a,b,m) = 0$. We recall that $m' = a$ and $n' = c$. In the diagram

$$\begin{tikzcd} \tilde{H}_1(X(a,b,m);\mathbb{Z}) \arrow[r, "u(a,b,m)"] \arrow[d] & \tilde{H}_1(Y(a,b,m);\mathbb{Z}) \arrow[d] \\ \tilde{H}_1(\mathbb{T}_+ \wedge_{C_m} X(a,b,m);\mathbb{Z}) \arrow[r, "u'(a,b,m)"] & \tilde{H}_1(\mathbb{T}_+ \wedge_{C_m} Y(a,b,m);\mathbb{Z}), \end{tikzcd}$$

where the vertical maps are induced by the maps that take z to the class of $(1,z)$, the groups in the bottom row both are cyclic of order a, the upper right-hand group is a free abelian group of rank $a-1$, and the right-hand vertical map is a surjection which was evaluated in [**19**, Lemma 3.3.4]. Let $\theta\colon [1] \to [m-1]$ be the map defined by $\theta(0) = 0$ and $\theta(1) = l$, where $0 \leqslant l < a$ is a multiplicative inverse of c modulo a. Since $\Sigma(a,b,m) \subset \Delta^{m-1}$ contains the vertices, the composite map

$$\Delta^1 \xrightarrow{\Delta^\theta} \Delta^{m-1} \xrightarrow{\mathrm{pr}} X(a,b,m)$$

is a cycle in $X(a,b,m)$. Moreover, it follows from loc. cit. that the homology class of the image of this cycle by the composite map

$$X(a,b,m) \xrightarrow{u(a,b,m)} Y(a,b,m) \xrightarrow{p} \mathbb{T}_+ \wedge_{C_m} Y(a,b,m)$$

is a generator of the lower right-hand group in the diagram above as desired. The case where b but not a divides m and $\ell(a,b,m) = 0$ is proved analogously.

Finally, if a and b both divide m, then the homology groups of the domain and target of the map $u'(a,b,m)$ are both trivial, so the proposition trivially holds. □

Remark 6.5. We would have liked to prove that Conjecture B holds for all positive integers m with $\ell(a,b,m) \leqslant 1$, since this would imply that the long exact sequence in Theorem A is valid for $q \leqslant 3$. However, in the remaining cases, where m is divisible by a but not b and $\ell(a,b,m) = 1$, it appears necessary to understand the facet structure of the 4-dimensional polytope $P(a,b,m)$ in order to find a cycle whose homology class generates $\tilde{H}_3(\mathbb{T}_+ \wedge X(a,b,m); \mathbb{Z})$.

References

[1] V. Angeltveit, T. Gerhardt, and L. Hesselholt, *On the K-theory of truncated polynomial algebras over the integers*, J. Topol. **2** (2009), 277–294.

[2] M. Bökstedt, W.-C. Hsiang, and I. Madsen, *The cyclotomic trace and algebraic K-theory of spaces*, Invent. Math. **111** (1993), 465–540.

[3] H. Cartan, *Séminaire 1954–1955. Algébre d'Eilenberg-Mac lane et homologie*, Benjamin, New York, 1967.

[4] G. Cortiñas, *The obstruction to excision in K-theory and in cyclic homology*, Invent. Math. **164** (2006), 143–173.

[5] V. Costeanu, *On the 2-typical de Rham-Witt complex*, Doc. Math. **13** (2008), 413–452.

[6] V. Drinfeld, *On the notion of geometric realization*, Mosc. Math. J. **4** (2004), 619–626.

[7] B. I. Dundas and R. McCarthy, *Topological Hochschild homology of ring functors and exact categories*, J. Pure Appl. Alg. **109** (1996), 231–294.

[8] D. Gale, *Neighborly and cyclic polytopes*, Convexity (Seattle, Washington, 1961), Proc. Symp. Pure Math., vol. 7, Amer. Math. Soc., Providence, RI, 1963, pp. 225–232.

[9] T. Geisser and L. Hesselholt, *Topological cyclic homology of schemes*, K-theory (Seattle, 1997), Proc. Symp. Pure Math., vol. 67, 1999, pp. 41–87.

[10] ———, *Bi-relative algebraic K-theory and topological cyclic homology*, Invent. Math. **166** (2006), 359–395.

[11] ———, *On the relative and bi-relative K-theory of rings of finite characteristic*, J. Amer. Math. Soc. **24** (2011), 29–49.

[12] S. Geller, L. Reid, and C. A. Weibel, *The cyclic homology and K-theory of curves*, J. reine angew. Math. **393** (1989), 39–90.

[13] T. Gerhardt, *The $R(S^1)$-graded equivariant homotopy groups of* $\mathrm{THH}(\mathbb{F}_p)$, Alg. Geom. Top. **8** (2008), 1961–1987.

[14] J. A. Guccione, J. J. Guccione, M. J. Redondo, and O. E. Villamayor, *Hochschild and cyclic homology of hypersurfaces*, Adv. Math. **95** (1992), 18–60.

[15] V. K. A. M. Gugenheim and H. J. Munkholm, *On the extended functoriality of* Tor *and* Cotor, J. Pure Appl. Algebra **4** (1974), 9–29.

[16] L. Hesselholt, *The big de Rham-Witt complex*, arxiv:1006.3125.

[17] ———, *On the p-typical curves in Quillen's K-theory*, Acta Math. **177** (1997), 1–53.

[18] ———, *K-theory of truncated polynomial algebras*, Handbook of K-theory, Springer-Verlag, New York, 2005.

[19] L. Hesselholt and I. Madsen, *Cyclic polytopes and the K-theory of truncated polynomial algebras*, Invent. Math. **130** (1997), 73–97.
[20] ———, *On the K-theory of finite algebras over Witt vectors of perfect fields*, Topology **36** (1997), 29–102.
[21] ———, *On the K-theory of nilpotent endomorphisms*, Homotopy methods in algebraic topology (Boulder, CO, 1999), Contemp. Math., vol. 271, Amer. Math. Soc., Providence, RI, 2001, pp. 127–140.
[22] ———, *On the K-theory of local fields*, Ann. of Math. **158** (2003), 1–113.
[23] ———, *On the de Rham-Witt complex in mixed characteristic*, Ann. Sci. École Norm. Sup. **37** (2004), 1–43.
[24] L. Illusie, *Complexe de de Rham-Witt et cohomologie cristalline*, Ann. Scient. Éc. Norm. Sup. (4) **12** (1979), 501–661.
[25] M. I. Krusemeyer, *Fundamental groups, algebraic K-theory, and a problem of Abhyankar*, Invent. Math. **19** (1973), 15–47.
[26] M. Larsen, *Filtrations, mixed complexes, and cyclic homology in mixed characteristic*, K-Theory **9** (1995), 173–198.
[27] J.-L. Loday, *Cyclic homology*, Grundlehren der mathematischen Wissenschaften, vol. 301, Springer-Verlag, New York, 1992.
[28] R. McCarthy, *Relative algebraic K-theory and topological cyclic homology*, Acta Math. **179** (1997), 197–222.
[29] D. Popescu, *General Néron desingularization*, Nagoya Math. J. **100** (1985), 97–126.
[30] D. Quillen, *Higher algebraic K-theory I*, Algebraic K-theory I: Higher K-theories (Battelle Memorial Inst., Seattle, Washington, 1972), Lecture Notes in Math., vol. 341, Springer-Verlag, New York, 1973.
[31] S. Saito, *On the geometric realization and subdivisions of dihedral sets*, Algebr. Geom. Topol. **13** (2013), 1071–1087.
[32] R. G. Swan, *Néron-Popescu desingularization*, Algebra and geometry (Taipei, 1995), Lect. Algebra Geom., 2, Internat. Press, Cambridge, MA, 1998, pp. 135–192.
[33] J. J. Sylvester, *Question 7382*, Mathematical Questions from the Educational Times **41** (1884), 21.
[34] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.
[35] V. Welker, Private communications.

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