Algebraic K-theory of planar cuspidal curves

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Introduction

The purpose of this paper is to evaluate the algebraic K-groups of a planar cuspidal curve over a perfect $\mathbb{F}_p$-algebra relative to the cusp point. A conditional calculation of these groups was given in [**4**, Theorem A], assuming a conjecture on the structure of certain polytopes. Our calculation here, however, is unconditional and illustrates the advantage of the new setup for topological cyclic homology by Nikolaus–Scholze [**10**], which we will be using throughout. The only input necessary for our calculation is the evaluation by the Buenos Aires Cyclic Homology group [**3**] and Larsen [**9**] of the structure of the Hochschild complex of the coordinate ring as a mixed complex, that is, as an object of the ∞-category of chain complexes with circle action.

We consider the planar cuspidal curve "$y^a = x^b$," where $a, b \geq 2$ are relatively prime integers. For $m \geq 0$, we define $\ell(a, b, m)$ to be the number of pairs (i, j) of positive integers such that $ai + bj = m$, and for $r \geq 0$, define $S(a, b, r)$ to be the set of positive integers m such that $\ell(a, b, m) \leq r$. The subset $S = S(a, b, r) \subset \mathbb{Z}_{>0}$ is a truncation set in the sense that if $m \in S$ and d divides m, then $d \in S$, and hence, the ring of (big) Witt vectors $\mathbb{W}_S(k)$ with underlying set k^S is defined. We refer to [**5**, Section 1] for a detailed introduction to Witt vectors.

THEOREM A. *Let k be a perfect $\mathbb{F}_p$-algebra, and let $a, b \geq 2$ be relatively prime integers. There is a canonical isomorphism*

$$K_j(k[x,y]/(y^a - x^b), (x,y)) \simeq \mathbb{W}_S(k)/(V_a\mathbb{W}_{S/a}(k) + V_b\mathbb{W}_{S/b}(k)),$$

if $j = 2r \geq 0$ with $S = S(a, b, r)$, and the remaining K-groups are zero.

We remark that recently Angeltveit [**1**] has given a different proof of this result that, unlike ours, employs equivariant homotopy theory. We also remark that the strategy employed in this paper was used by Speirs [**11**] to significantly simplify the calculation in [**6**] of the relative algebraic K-groups of a truncated polynomial algebra over a perfect $\mathbb{F}_p$-algebra.

We recall from [**4**, Section 1] that the group in the statement is a module over the ring $\mathbb{W}(k)$ of big Witt vectors in k of finite length $\frac{1}{2}(2r+1)(a-1)(b-1)$. The calculation of the length goes back to Sylvester [**12**]. Moreover, it admits a

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p-typical product decomposition indexed by positive integers m' not divisible by p. To state this, we let $s = s(a,b,r,p,m')$ be the unique integer such that

$$\ell(a,b,p^{s-1}m') \leq r < \ell(a,b,p^s m'),$$

if such an integer exists, and 0, otherwise, and assume (without loss of generality) that p does not divide b and write $a = p^u a'$ with a' not divisible by p. Then

$$\mathbb{W}_S(k)/(V_a\mathbb{W}_{S/a}(k) + V_b\mathbb{W}_{S/b}(k)) \simeq \textstyle\prod_{m' \in \mathbb{N}'} W_h(k)$$

by a canonical isomorphism, where

$$h = h(a,b,r,p,m') = \begin{cases} s, & \text{if neither } a' \text{ nor } b \text{ divides } m', \\ \min\{s,u\}, & \text{if } a' \text{ but not } b \text{ divides } m', \\ 0, & \text{if } b \text{ divides } m'. \end{cases}$$

Here we write $\mathbb{N}'$ for the set of positive integers not divisible by p.

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1. Some recollections necessary for the proof

We first recall the Nikolaus–Scholze formula for topological cyclic homology from [**10**]; see also [**7**]. We write $\mathbb{T}$ for the circle group and $C_p \subset \mathbb{T}$ for the subgroup of order p. If R is a ring, then we write

$$\mathrm{TC}^-(R) \xrightarrow{\mathrm{can}} \mathrm{TP}(R)$$

for the canonical map from the homotopy fixed points to the Tate construction of the spectrum with $\mathbb{T}$-action $\mathrm{THH}(R)$. The Frobenius map

$$\mathrm{THH}(R) \xrightarrow{\varphi} \mathrm{THH}(R)^{tC_p}$$

is $\mathbb{T}$-equivariant, provided that we let $\mathbb{T}$ act on the target through the isomorphism $\rho\colon \mathbb{T} \to \mathbb{T}/C_p$ given by the pth root, and therefore, it induces a map

$$\mathrm{TC}^-(R) = \mathrm{THH}(R)^{h\mathbb{T}} \xrightarrow{\varphi^{h\mathbb{T}}} (\mathrm{THH}(R)^{tC_p})^{h(\mathbb{T}/C_p)}.$$

The Tate-orbit lemma [**10**, I.2.1, II.4.2] identifies the p-completion of the target of this map with that of $\mathrm{TP}(R)$, and Nikolaus–Scholze show that, after p-completion, the topological cyclic homology of R is the equalizer

$$\mathrm{TC}(R) \longrightarrow \mathrm{TC}^-(R) \underset{\mathrm{can}}{\overset{\varphi}{\rightrightarrows}} \mathrm{TP}(R)$$

of these two parallel maps. If p is nilpotent in R, as is the case in the situation that we consider, then the spectra in question are already p-complete.

The normalization of $A = k[x,y]/(y^a - x^b)$ is the k-algebra homomorphism to $B = k[t]$ that to x and y assigns t^a and t^b, respectively, and this homomorphism identifies A with sub-k-algebra $k[t^a, t^b] \subset k[t] = B$. In this situation, the square

$$\begin{array}{ccc} K(A) & \longrightarrow & \mathrm{TC}(A) \\ \downarrow & & \downarrow \\ K(B) & \longrightarrow & \mathrm{TC}(B) \end{array}$$

is cartesian. This follows from the birelative theorem, which has now been given a very satisfying conceptual proof by Land–Tamme [**8**]. (By contrast, the original proof in the rational case by Cortiñas [**2**] and the subsequent proof in the p-adic case by Geisser–Hesselholt both required rather elaborate calculational input.) Now, the map $K(B) \to K(k)$ induced by the k-algebra homomorphism that to t assigns 0 is an equivalence, and hence, the relative K-groups that we wish to determine are canonically identified with the homotopy groups of the common fibers of the vertical maps in the diagram above.

The k-algebras A and B are both monoid algebras. In general, if $k[\Pi]$ is the monoid algebra of an $\mathbb{E}_1$-monoid Π in spaces, then, as cyclotomic spectra,

$$\mathrm{THH}(k[\Pi]) \simeq \mathrm{THH}(k \otimes \mathbb{S}[\Pi]) \simeq \mathrm{THH}(k) \otimes B^{\mathrm{cy}}(\Pi)_+,$$

where $B^{\mathrm{cy}}(\Pi)$ denotes the unstable cyclic bar-construction of Π. In addition, on the right-hand side, the Frobenius map factors as a composition

$$\begin{aligned} \mathrm{THH}(k) \otimes B^{\mathrm{cy}}(\Pi)_+ &\xrightarrow{\varphi \otimes \tilde{\varphi}} \mathrm{THH}(k)^{tC_p} \otimes B^{\mathrm{cy}}(\Pi)_+^{hC_p} \\ &\xrightarrow{\mathrm{can}} (\mathrm{THH}(k) \otimes B^{\mathrm{cy}}(\Pi)_+)^{tC_p} \end{aligned}$$

of the map induced by the Frobenius $\varphi\colon \mathrm{THH}(k) \to \mathrm{THH}(k)^{tC_p}$ and the unstable Frobenius $\tilde{\varphi}\colon B^{\mathrm{cy}}(\Pi) \to B^{\mathrm{cy}}(\Pi)^{hC_p}$ and a canonical map. Importantly, we have a map of spectra with $\mathbb{T}$-action

$$\mathbb{Z} \longrightarrow \mathbb{Z}_p \simeq \tau_{\geq 0} \mathrm{TC}(k) \longrightarrow \mathrm{THH}(k)$$

from $\mathbb{Z}$ with (necessarily) trivial $\mathbb{T}$-action, and therefore, we can rewrite

$$\mathrm{THH}(k) \otimes B^{\mathrm{cy}}(\Pi)_+ \simeq \mathrm{THH}(k) \otimes_{\mathbb{Z}} \mathbb{Z} \otimes B^{\mathrm{cy}}(\Pi)_+.$$

Accordingly, we do not need to understand the homotopy type of the space with $\mathbb{T}$-action $B^{\mathrm{cy}}(\Pi)$. It suffices to understand the homotopy type of the chain complex with $\mathbb{T}$-action $\mathbb{Z} \otimes B^{\mathrm{cy}}(\Pi)_+$, which, in the case at hand, is exactly what the Buenos Aires Cyclic Homology group [**3**] and Larsen [**9**] have done for us.

To state their result, we let $\langle t^a, t^b \rangle \subset \langle t \rangle$ be the free monoid on a generator t and the submonoid generated by t^a and t^b, respectively, and set

$$B^{\mathrm{cy}}(\langle t \rangle, \langle t^a, t^b \rangle) = B^{\mathrm{cy}}(\langle t \rangle) / B^{\mathrm{cy}}(\langle t^a, t^b \rangle).$$

Counting powers of t gives a $\mathbb{T}$-equivariant decomposition of pointed spaces

$$B^{\mathrm{cy}}(\langle t \rangle, \langle t^a, t^b \rangle) \simeq \textstyle\bigvee_{m \in \mathbb{Z}_{>0}} B^{\mathrm{cy}}(\langle t \rangle, \langle t^a, t^b \rangle; m).$$

We can now state the result of the calculation by the Buenos Aires Cyclic Homology group [**3**] and by Larsen [**9**] as follows.

THEOREM 1. *Let $a, b \geq 2$ be relatively prime integers, and let $m \geq 1$ be an integer. In the ∞-category $D(\mathbb{Z})^{B\mathbb{T}}$ of chain complexes with $\mathbb{T}$-action, there is a canonical equivalence between*

$$\mathbb{Z} \otimes B^{\mathrm{cy}}(\langle t \rangle, \langle t^a, t^b \rangle; m)$$

and the total cofiber of the square

$$\begin{array}{ccc} \mathbb{Z} \otimes (\mathbb{T}/C_{m/ab})_+[2\ell(a,b,m)] & \longrightarrow & \mathbb{Z} \otimes (\mathbb{T}/C_{m/a})_+[2\ell(a,b,m)] \\ \downarrow & & \downarrow \\ \mathbb{Z} \otimes (\mathbb{T}/C_{m/b})_+[2\ell(a,b,m)] & \longrightarrow & \mathbb{Z} \otimes (\mathbb{T}/C_{m})_+[2\ell(a,b,m)]. \end{array}$$

Here, all maps in the square are induced by the respective canonical projections, and if c does not divide m, then $\mathbb{T}/C_{m/c}$ is understood to be the empty space.

The result is not stated in this from in op. cit., and therefore, some explanation is in order. The ∞-category $D(\mathbb{Z})^{B\mathbb{T}}$ is equivalent to the ∞-category of modules over the $\mathbb{E}_1$-algebra $C_*(\mathbb{T}, \mathbb{Z})$ given by the singular chains on the circle. This $\mathbb{E}_1$-algebra, in turn, is a Postnikov section of a free $\mathbb{E}_1$-algebra over $\mathbb{Z}$, and therefore, it is formal in the sense that, as an $\mathbb{E}_1$-algebra over $\mathbb{Z}$, it is equivalent to the Pontryagin ring $H_*(\mathbb{T}, \mathbb{Z})$ given by its homology. As a model for the ∞-category of modules over the latter, we may use the dg-category of dg-modules over $H_*(\mathbb{T}, \mathbb{Z})$. But a dg-module over $H_*(\mathbb{T}, \mathbb{Z}) = \mathbb{Z}[d]/(d^2)$ is precisely what is called a mixed complex in op. cit. This shows that $D(\mathbb{Z})^{B\mathbb{T}}$ is equivalent to the ∞-category of mixed complexes. Some additional translation is necessary to bring the results in the above form for which we refer to [**4**, Section 5].

In the following, we will use the abbreviation

$$B_m = B^{\mathrm{cy}}(\langle t \rangle, \langle t^a, t^b \rangle; m).$$

Theorem 1 shows, in particular, that the connectivity of B_m tends to infinity with m. Hence, tensoring with $\mathrm{THH}(k)$, we obtain an equivalence

$$\mathrm{THH}(k) \otimes B^{\mathrm{cy}}(\langle t \rangle, \langle t^a, t^b \rangle) \simeq \bigoplus_{m \in \mathbb{Z}_{>0}} \mathrm{THH}(k) \otimes B_m \simeq \prod_{m \in \mathbb{Z}_{>0}} \mathrm{THH}(k) \otimes B_m,$$

which, in turn, implies a product decomposition

$$(\mathrm{THH}(k) \otimes B^{\mathrm{cy}}(\langle t \rangle, \langle t^a, t^b \rangle))^{tC_p} \simeq \prod_{m \in \mathbb{Z}_{>0}} (\mathrm{THH}(k) \otimes B_m)^{tC_p}.$$

We will use the following result repeatedly below.

LEMMA 2. *The unstable Frobenius induces an equivalence*

$$\mathrm{THH}(k)^{tC_p} \otimes B_{m/p} \xrightarrow{\mathrm{id} \otimes \tilde{\varphi}} (\mathrm{THH}(k) \otimes B_m)^{tC_p},$$

where the left-hand side is understood to be zero if p does not divide m.

PROOF. We recall two facts from [**4**, Section 3]. The first is that the pointed space $B_{m/p}$ is finite, and the second is that the cofiber of the composition

$$B_{m/p} \xrightarrow{\tilde{\varphi}} (B_m)^{hC_p} \longrightarrow B_m$$

of the unstable Frobenius map and the canonical "inclusion" is a finite colimit of free pointed C_p-cells. Here C_p acts trivially on the left-hand and middle terms.

Now, the map in the statement factors as the composition

$$\mathrm{THH}(k)^{tC_p} \otimes B_{m/p} \longrightarrow (\mathrm{THH}(k) \otimes B_{m/p})^{tC_p} \longrightarrow (\mathrm{THH}(k) \otimes B_m)^{tC_p}$$

of the canonical colimit interchange map, where $B_{m/p}$ is equipped with the trivial C_p-action, and the map of Tate spectra induced from the composite map above. The first fact implies that the left-hand map is an equivalence, and the second fact implies that the right-hand map is an equivalence. □

Proposition 3. *Let G be a compact Lie group, let $H \subset G$ be a closed subgroup, let $\lambda = T_H(G/H)$ be the tangent space at $H = eH$ with the adjoint left H-action, and let S^λ the one-point compactification of λ. For every spectrum with G-action X, there are canonical natural equivalences*

$$(X \otimes (G/H)_+)^{hG} \simeq (X \otimes S^\lambda)^{hH},$$
$$(X \otimes (G/H)_+)^{tG} \simeq (X \otimes S^\lambda)^{tH}.$$

Proof. We recall that for every map of spaces $f\colon S \to T$, the restriction functor $f^*\colon \mathrm{Sp}^T \to \mathrm{Sp}^S$ has both a left adjoint $f_!$ and a right adjoint f_*. In the case of the unique map $p\colon BG \to \mathrm{pt}$, we have $p_!(X) \simeq X_{hG}$ and $p_*(X) \simeq X^{hG}$. We now consider the following diagram of spaces.

$$\begin{array}{ccc} BH & \xrightarrow{\quad f \quad} & BG \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & \mathrm{pt} & \end{array}$$

The top horizontal map is the map induced by the inclusion of H in G. It is a fiber bundle, whose fibers are compact manifolds.[1] Therefore, by parametrized Atiyah duality, its relative dualizing spectrum $Df \in \mathrm{Sp}^{BH}$ is given by the sphere bundle associated with the fiberwise normal bundle, which is $Df \simeq S^{-\lambda}$. By definition of the dualizing spectrum, we have for all $Y \in \mathrm{Sp}^{BH}$, a natural equivalence

$$f_!(Y \otimes S^{-\lambda}) \simeq f_*(Y)$$

in Sp^{BG}. It follows that for all $Y \in \mathrm{Sp}^{BH}$, we have a natural equivalence

$$p_* f_!(Y \otimes S^{-\lambda}) \simeq p_* f_*(Y) \simeq q_*(Y)$$

in Sp, which we also write as

$$((Y \otimes S^{-\lambda}) \otimes_H G_+)^{hG} \simeq Y^{hH}.$$

By [**10**, Theorem I.4.1 (3)], we further deduce a natural equivalence

$$((Y \otimes S^{-\lambda}) \otimes_H G_+)^{tG} \simeq Y^{tH}.$$

Indeed, the left-hand side vanishes for $Y = \Sigma^\infty H_+$, and the fiber of the map

$$((Y \otimes S^{-\lambda}) \otimes_H G_+)^{hG} \xrightarrow{\mathrm{can}} ((Y \otimes S^{-\lambda}) \otimes_H G_+)^{tG}$$

preserves colimits in Y.

Finally, given $X \in \mathrm{Sp}^{BG}$, we set $Y = f^*(X) \otimes S^\lambda$ to obtain the equivalences in the statement. □

[1] Strictly speaking this statement does not make sense, since BH and BG are only defined as homotopy types. What we mean is that the map is classified by a map $BG \to B\,\mathrm{Diff}(G/H)$, where the latter is the diffeomorphism group of the compact manifold G/H.

2. Proof of Theorem A

We consider the diagram with horizontal equalizers

$$\begin{array}{ccccc} \mathrm{TC}(A) & \longrightarrow & \mathrm{TC}^-(A) & \underset{\mathrm{can}}{\overset{\varphi}{\rightrightarrows}} & \mathrm{TP}(A) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{TC}(B) & \longrightarrow & \mathrm{TC}^-(B) & \underset{\mathrm{can}}{\overset{\varphi}{\rightrightarrows}} & \mathrm{TP}(B) \end{array}$$

and wish to evaluate the cofiber of the left-hand vertical map. We have

$$\mathrm{cofiber}(\mathrm{THH}(A) \to \mathrm{THH}(B)) \simeq \textstyle\bigoplus_{m \geq 1} \mathrm{THH}(k) \otimes B_m \simeq \prod_{m \geq 1} \mathrm{THH}(k) \otimes B_m.$$

The right-hand equivalence follows from the fact that the connectivity of B_m tends to infinity with m. Therefore, the cofiber of $\mathrm{TC}(A) \to \mathrm{TC}(B)$ is identified with the equalizer of the induced maps

$$\textstyle\prod_{m \geq 1} (\mathrm{THH}(k) \otimes B_m)^{h\mathbb{T}} \underset{\mathrm{can}}{\overset{\varphi}{\rightrightarrows}} \prod_{m \geq 1} (\mathrm{THH}(k) \otimes B_m)^{t\mathbb{T}}.$$

While the canonical map "can" preserves this product decomposition, the Frobenius map "φ" takes the factor indexed by m to the factor factor indexed pm. Therefore, we write $m = p^v m'$ with m' not divisible by p and rewrite the diagram as

$$\textstyle\prod_{m'} \prod_{v \geq 0} (\mathrm{THH}(k) \otimes B_{p^v m'})^{h\mathbb{T}} \underset{\mathrm{can}}{\overset{\varphi}{\rightrightarrows}} \prod_{m'} \prod_{v \geq 0} (\mathrm{THH}(k) \otimes B_m)^{t\mathbb{T}},$$

where both maps now preserve the outer product decomposition indexed by positive integers m' not divisible by p. We will abbreviate and write

$$\mathrm{TC}(m') \longrightarrow \mathrm{TC}^-(m') \underset{\mathrm{can}}{\overset{\varphi}{\rightrightarrows}} \mathrm{TP}(m')$$

for the equalizer diagram given by the factors indexed by m'. To complete the proof, we evaluate the induced diagram on homotopy groups.

We fix $m = p^v m'$. It follows from Theorem 1 that

$$\mathrm{THH}(k) \otimes B_m \simeq \mathrm{THH}(k) \otimes_{\mathbb{Z}} (\mathbb{Z} \otimes B_m)$$

agrees, up to canonical equivalence, with the total cofiber of the square[2]

$$\begin{array}{ccc} \mathrm{THH}(k) \otimes (\mathbb{T}/C_{m/ab})_+[2\ell(a,b,m)] & \longrightarrow & \mathrm{THH}(k) \otimes (\mathbb{T}/C_{m/a})_+[2\ell(a,b,m)] \\ \downarrow & & \downarrow \\ \mathrm{THH}(k) \otimes (\mathbb{T}/C_{m/b})_+[2\ell(a,b,m)] & \longrightarrow & \mathrm{THH}(k) \otimes (\mathbb{T}/C_{m})_+[2\ell(a,b,m)]. \end{array}$$

By Proposition 3, the induced square of $\mathbb{T}$-homotopy fixed points takes the form

$$\begin{array}{ccc} \mathrm{THH}(k)^{hC_{m/ab}}[2\ell(a,b,m)+1] & \longrightarrow & \mathrm{THH}(k)^{hC_{m/a}}[2\ell(a,b,m)+1] \\ \downarrow & & \downarrow \\ \mathrm{THH}(k)^{hC_{m/b}}[2\ell(a,b,m)+1] & \longrightarrow & \mathrm{THH}(k)^{hC_{m}}[2\ell(a,b,m)+1] \end{array}$$

with the maps in the diagram given by the corestriction maps on homotopy fixed points. Indeed, the adjoint representation $\lambda = T_{C_r}(\mathbb{T}/C_r)$ is a trivial one-dimensional

[2] Here we use in an essential way that, as a spectrum with $\mathbb{T}$-action, $\mathrm{THH}(k)$ is a $\mathbb{Z}$-module.

real C_r-representation. We now write $a = p^u a'$ with a' not divisible by p and assume that p does not divide b. If a and b both do not divide m, then

$$\begin{aligned}(\mathrm{THH}(k) \otimes B_m)^{h\mathbb{T}} &\simeq \mathrm{THH}(k)^{hC_m}[2\ell(a,b,m)+1] \\ &\simeq \mathrm{THH}(k)^{hC_{p^v}}[2\ell(a,b,m)+1],\end{aligned}$$

and if $a = p^u a'$ but not b divides m, then

$$\begin{aligned}(\mathrm{THH}(k) \otimes B_m)^{h\mathbb{T}} &\simeq \mathrm{cofiber}(\mathrm{THH}(k)^{hC_{m/a}} \to \mathrm{THH}(k)^{hC_m})[2\ell(a,b,m)+1] \\ &\simeq \mathrm{cofiber}(\mathrm{THH}(k)^{hC_{p^{v-u}}} \to \mathrm{THH}(k)^{hC_{p^v}})[2\ell(a,b,m)+1].\end{aligned}$$

Similarly, if b but not a divides m, then

$$\begin{aligned}(\mathrm{THH}(k) \otimes B_m)^{h\mathbb{T}} &\simeq \mathrm{cofiber}(\mathrm{THH}(k)^{hC_{m/b}} \to \mathrm{THH}(k)^{hC_m})[2\ell(a,b,m)+1] \\ &\simeq \mathrm{cofiber}(\mathrm{THH}(k)^{hC_{p^v}} \to \mathrm{THH}(k)^{hC_{p^v}})[2\ell(a,b,m)+1] \\ &\simeq 0,\end{aligned}$$

and if a and b both divide m, then

$$(\mathrm{THH}(k) \otimes B_m)^{h\mathbb{T}} \simeq 0,$$

since $B_m \simeq 0$. By the same reasoning, we find that

$$(\mathrm{THH}(k) \otimes B_m)^{t\mathbb{T}} \simeq \mathrm{THH}(k)^{tC_{p^v}}[2\ell(a,b,m)+1],$$

if a and b both do not divide m, that

$$(\mathrm{THH}(k) \otimes B_m)^{t\mathbb{T}} \simeq \mathrm{cofiber}(\mathrm{THH}(k)^{tC_{p^{v-u}}} \to \mathrm{THH}(k)^{tC_{p^v}})[2\ell(a,b,m)+1],$$

if $a = p^u a'$ divides m but b does not divide m, and that

$$(\mathrm{THH}(k) \otimes B_m)^{t\mathbb{T}} \simeq 0,$$

otherwise.

It follows from [**10**, Section IV.4] that, on homotopy groups, the diagram

$$\begin{array}{ccc}\mathrm{THH}(k)^{hC_{p^u}} & \xrightarrow{\mathrm{can}} & \mathrm{THH}(k)^{tC_{p^u}} \\ \downarrow{\scriptstyle\mathrm{cor}} & & \downarrow{\scriptstyle\mathrm{cor}} \\ \mathrm{THH}(k)^{hC_{p^v}} & \xrightarrow{\mathrm{can}} & \mathrm{THH}(k)^{tC_{p^v}}\end{array}$$

becomes

$$\begin{array}{ccc}W(k)[t,x]/(tx-p,p^u t) & \longrightarrow & W(k)[t^{\pm 1},x]/(tx-p,p^u t) \\ \downarrow & & \downarrow \\ W(k)[t,x]/(tx-p,p^v t) & \longrightarrow & W(k)[t^{\pm 1},x]/(tx-p,p^v t),\end{array}$$

where $\deg(t) = -2$ and $\deg(x) = 2$, where the horizontal maps are the unique graded $W(k)$-algebra homomorphisms that take t to t and x to $x = pt^{-1}$, and where the vertical maps are the unique maps of graded $W(k)[t,x]$-modules that take 1 to p^{v-u}.

We have now determined the diagram

$$\mathrm{TC}(m') \longrightarrow \mathrm{TC}^-(m') \underset{\mathrm{can}}{\overset{\varphi}{\rightrightarrows}} \mathrm{TP}(m')$$

at the level of homotopy groups, the Frobenius given by Lemma 2. Hence, it is merely a matter of bookkeeping to see that the statement of the theorem ensues. We recall the functions $s = s(a,b,r,p,m')$ and $h = h(a,b,r,p,m')$ from the p-typical decomposition recalled in the introduction,

$$\mathbb{W}_S(k)/(V_a\mathbb{W}_{S/a}(k) + V_b\mathbb{W}_{S/b}(k)) \simeq \textstyle\prod_{m'\in\mathbb{N}'} W_h(k).$$

Suppose first that neither a' nor b divides m'. Then

$$\pi_{2r+1}((\mathrm{THH}(k)\otimes B_{p^v m'})^{h\mathbb{T}}) \simeq \begin{cases} W_{v+1}(k), & \text{if } 0 \le v < s, \\ W_v(k), & \text{if } s \le v, \end{cases}$$
$$\pi_{2r+1}((\mathrm{THH}(k)\otimes B_{p^v m'})^{t\mathbb{T}}) \simeq W_v(k),$$

with $s = s(a,b,r,p,m')$. Also, we remark that the corresponding homotopy groups in even degree $2r$ are zero. The Frobenius map

$$\pi_{2r+1}((\mathrm{THH}(k)\otimes B_{p^v m'})^{h\mathbb{T}}) \xrightarrow{\varphi} \pi_{2r+1}((\mathrm{THH}(k)\otimes B_{p^{v+1} m'})^{t\mathbb{T}})$$

is an isomorphism for $0 \le v < s$, and the canonical map

$$\pi_{2r+1}((\mathrm{THH}(k)\otimes B_{p^v m'})^{h\mathbb{T}}) \xrightarrow{\mathrm{can}} \pi_{2r+1}((\mathrm{THH}(k)\otimes B_{p^v m'})^{t\mathbb{T}})$$

is an isomorphism for $s \le v$. Hence, we have a map of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \prod_{s\le v} W_v(k) & \longrightarrow & \mathrm{TC}^-_{2r+1}(m') & \longrightarrow & \prod_{0\le v<s} W_{v+1}(k) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi-\mathrm{can}} & & \downarrow{\scriptstyle \varphi-\mathrm{can}} & & \downarrow{\scriptstyle \overline{\varphi-\mathrm{can}}} & & \\ 0 & \longrightarrow & \prod_{s\le v} W_v(k) & \longrightarrow & \mathrm{TP}_{2r+1}(m') & \longrightarrow & \prod_{0\le v<s} W_v(k) & \longrightarrow & 0, \end{array}$$

where the left-hand vertical map is an isomorphism, and where the right-hand vertical map is an epimorphism with kernel $W_s(k)$. Since $h = s$, we conclude that $\mathrm{TC}_{2r+1}(m') \simeq W_h(k)$ and $\mathrm{TC}_{2r}(m') \simeq 0$ as desired.

Suppose next that a' but not b divides m'. If $u \le s$, then

$$\pi_{2r+1}((\mathrm{THH}(k)\otimes B_{p^v m'})^{h\mathbb{T}}) \simeq \begin{cases} W_{v+1}(k), & \text{if } 0 \le v < u, \\ W_u(k), & \text{if } u \le v, \end{cases}$$
$$\pi_{2r+1}((\mathrm{THH}(k)\otimes B_{p^v m'})^{t\mathbb{T}}) \simeq \begin{cases} W_v(k), & \text{if } 0 \le v < u, \\ W_u(k), & \text{if } u \le v. \end{cases}$$

Moreover, we see as before that the Frobenius and canonical maps are isomorphisms for $0 \le v < s$ and $s \le v$, respectively, so we have a map of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \prod_{s\le v} W_u(k) & \longrightarrow & \mathrm{TC}^-_{2r+1}(m') & \longrightarrow & \prod_{0\le v<s} W_c(k) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi-\mathrm{can}} & & \downarrow{\scriptstyle \varphi-\mathrm{can}} & & \downarrow{\scriptstyle \overline{\varphi-\mathrm{can}}} & & \\ 0 & \longrightarrow & \prod_{s\le v} W_u(k) & \longrightarrow & \mathrm{TP}_{2r+1}(m') & \longrightarrow & \prod_{0\le v<s} W_d(k) & \longrightarrow & 0, \end{array}$$

where $c = \min\{u, v+1\}$ and $d = \min\{u, v\}$. The left-hand vertical map is an isomorphism, and the right-hand vertical map is an epimorphism with kernel $W_u(k)$, so again $\mathrm{TC}_{2r+1}(m') \simeq W_h(k)$, since $h = u$, and $\mathrm{TC}_{2r}(m') \simeq 0$.

If $s < u$, then

$$\pi_{2r+1}((\mathrm{THH}(k) \otimes B_{p^v m'})^{h\mathbb{T}}) \simeq \begin{cases} W_{v+1}(k), & \text{if } 0 \leq v < s, \\ W_v(k), & \text{if } s \leq v < u, \\ W_u(k), & \text{if } u \leq v, \end{cases}$$

$$\pi_{2r+1}((\mathrm{THH}(k) \otimes B_{p^v m'})^{t\mathbb{T}}) \simeq \begin{cases} W_v(k), & \text{if } 0 \leq v < u, \\ W_u(k), & \text{if } u \leq v, \end{cases}$$

and a similar argument shows that $\mathrm{TC}_{2r+1}(m') \simeq W_h(k)$, since $h = s$, and that $\mathrm{TC}_{2r}(m') \simeq 0$.

Finally, if b divides m', then $\mathrm{TC}^-_{2r+1}(m')$ and $\mathrm{TP}_{2r+1}(m')$ are both zero, and therefore, so is $\mathrm{TC}_{2r+1}(m')$ and $\mathrm{TC}_{2r}(m')$. This completes the proof.

REMARK 4. As a pointed space with $\mathbb{T}$-action, the homotopy type of B_m was described conjecturally in [**4**, Conjecture B] and this conjecture was affirmed by Angeltveit in [**1**, Theorem 2.1]. The result is that B_m is equivalent to the total cofiber of a square of pointed spaces with $\mathbb{T}$-action

$$\begin{array}{ccc} (\mathbb{T}/C_{m/ab})_+ \wedge S^{\lambda(a,b,m)} & \longrightarrow & (\mathbb{T}/C_{m/a})_+ \wedge S^{\lambda(a,b,m)} \\ \downarrow & & \downarrow \\ (\mathbb{T}/C_{m/b})_+ \wedge S^{\lambda(a,b,m)} & \longrightarrow & (\mathbb{T}/C_{m})_+ \wedge S^{\lambda(a,b,m)}. \end{array}$$

So the trivial shift $[2\ell(a,b,m)]$ that appears in Theorem 1 is replaced by a shift by a non-trivial representation $\lambda(a,b,m)$ of real dimension $2\ell(a,b,m)$.

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