

K*-Theory of Truncated Polynomial Algebras

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3.1 Introduction

In general, if A is a ring and $I \subset A$ a two-sided ideal, one defines the K -theory of A relative to I to be the mapping fiber of the map of K -theory spectra induced by the canonical projection from A to A/I . Hence, there is a natural exact triangle of spectra

$$K(A, I) \rightarrow K(A) \rightarrow K(A/I) \xrightarrow{\partial} K(A, I)[-1]$$

and an induced natural long-exact sequence of K -groups

$$\dots \rightarrow K_q(A, I) \rightarrow K_q(A) \rightarrow K_q(A/I) \xrightarrow{\partial} K_{q-1}(A, I) \rightarrow \dots .$$

If the ideal $I \subset A$ is nilpotent, the relative K -theory $K(A, I)$ can be expressed completely in terms of the cyclic homology of Connes [14] and the topological cyclic homology of Bökstedt–Hsiang–Madsen [7]. Indeed, on the one hand, Goodwillie [21] has shown that rationally

$$K_q(A, I) \otimes \mathbb{Q} \xrightarrow{\sim} \mathrm{HC}_q^-(A \otimes \mathbb{Q}, I \otimes \mathbb{Q}) \xleftarrow{\sim} \mathrm{HC}_{q-1}(A, I) \otimes \mathbb{Q} ,$$

and on the other hand, McCarthy [47] has shown that p -adically

$$K_q(A, I, \mathbb{Z}_p) \xrightarrow{\sim} \mathrm{TC}_q(A, I; p, \mathbb{Z}_p) .$$

In both cases, the argument uses the calculus of functors in the sense of Goodwillie [22, 23]. Thus, the problem of evaluating the relative K -theory is translated to the problem of evaluating the relative cyclic theories. The definitions of cyclic homology and of topological cyclic homology are given in Sect. 3.7 below.

Let A be a commutative algebra over a field k and suppose that A is a regular noetherian ring. By Popescu [50] (see also [54]), this is equivalent to A being a filtered colimit of smooth k -algebras. It is then possible by the above approach to completely evaluate the groups $K_q(A[x]/(x^e), (x))$ for truncated polynomial algebras over A relative to the ideal generated by the variable. The calculation of Connes' cyclic homology follows from Masuda and Natsume [46] and Kassel [38], but see also the Buenos Aires Cyclic Homology Group [24]. The topological cyclic homology was evaluated by Madsen and the author [28, 30].

If the field k has characteristic zero, the relative K -groups are expressed in terms of the (absolute) differential forms $\Omega_A^* = \Omega_{A/\mathbb{Z}}^*$ of the ring A . In short, Ω_A^* is the initial example of an anti-commutative differential graded ring E^* with a ring homomorphism $\lambda: A \rightarrow E^0$. The result, which we prove in Sect. 3.11 below, then is a natural (in A) isomorphism of abelian groups

$$K_{q-1}(A[x]/(x^e), (x)) \xleftarrow{\sim} \bigoplus_{m \geq 1} \left(\Omega_A^{q-2m} \right)^{e-1} .$$

Here the superscript $e - 1$ on the right indicates product. In particular, the relative K -groups are uniquely divisible groups. For $e = 2$ and $q = 3$, this was first obtained by van der Kallen [56].

If the field k has positive characteristic p , the relative K -groups are expressed in terms of the (big) de Rham–Witt differential forms $W.\Omega_A^*$ of the ring A . One defines $W.\Omega_A^*$ as the initial example of a *big Witt complex* [30]. The result, which we prove in Sect. 3.13 below, then is a natural (in A) long-exact sequence of abelian groups

$$\dots \rightarrow \bigoplus_{m \geq 1} W_m \Omega_A^{q-2m} \xrightarrow{V_e} \bigoplus_{m \geq 1} W_{me} \Omega_A^{q-2m} \rightarrow K_{q-1}(A[x]/(x^e), (x)) \rightarrow \dots$$

In particular, the relative K -groups are p -primary torsion groups. The result for A a finite field, $e = 2$, and $q \leq 4$ was obtained first by Evens and Friedlander [15] and by Aisbett, Lluís-Puebla and Snaith [2]. The big de Rham–Witt groups and the map V_e can be described in terms of the more familiar p -typical de Rham–Witt groups $W.\Omega_A^*$ of Bloch–Deligne–Illusie [5, 35]. There is a canonical decomposition

$$W_m \Omega_A^* \xrightarrow{\sim} \prod_d W_s \Omega_A^*,$$

where on the right $1 \leq d \leq m$ and prime to p , and where $s = s(m, d)$ is given by $p^{s-1}d \leq m < p^s d$. Moreover, if we write $e = p^v e'$ with e' prime to p , then the map V_e takes the factor $W_s \Omega_A^*$ indexed by $1 \leq d \leq m$ to the factor $W_{s+v} \Omega_A^*$ indexed by $1 \leq de' \leq me$ by the map

$$e' V^v: W_s \Omega_A^* \rightarrow W_{s+v} \Omega_A^*.$$

If k is *perfect* and if A is smooth of relative dimension r over k , then the groups $W_m \Omega_A^q$ are concentrated in degrees $0 \leq q \leq r$. The de Rham–Witt complex is discussed in Sect. 3.8 below.

Finally, we remark that regardless of the characteristic of the field k , the spectrum $K(A[x]/(x^e), (x))$ is a product of Eilenberg–MacLane spectra, and hence, its homotopy type is completely determined by the homotopy groups.

All rings (resp. graded rings, resp. monoids) considered in this paper are assumed to be commutative (resp. graded-commutative, resp. commutative) and unital. We denote by \mathbb{N} (resp. by \mathbb{N}_0 , resp. by I_p) the set of positive integers (resp. non-negative integers, resp. positive integers prime to p). By a pro-object of a category \mathcal{C} we mean a functor from \mathbb{N} , viewed as a category with one arrow from $n + 1$ to n , to \mathcal{C} , and by a *strict* map between pro-objects we mean a natural transformation. A general map between pro-objects X and Y of \mathcal{C} is an element of

$$\text{Hom}_{\text{pro-}\mathcal{C}}(X, Y) = \lim_n \text{colim}_m \text{Hom}_{\mathcal{C}}(X_m, Y_n).$$

We view objects of \mathcal{C} as constant pro-objects of \mathcal{C} . We denote by \mathbb{T} the multiplicative group of complex numbers of modulus one and by $C_r \subset \mathbb{T}$ the subgroup of order r . A map of \mathbb{T} -spaces (resp. \mathbb{T} -spectra) is an \mathcal{F} -equivalence if the induced map of C_r -fixed points is a weak equivalence of spaces (resp. spectra), for all $C_r \subset \mathbb{T}$.

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Topological Hochschild Homology

We first recall the Hochschild complex associated with the ring A . This is the cyclic abelian group $\mathrm{HH}(A)$, with k -simplices

$$\mathrm{HH}(A)_k = A \otimes \dots \otimes A \quad (k+1 \text{ factors})$$

and with the cyclic structure maps

$$\begin{aligned} d_r(a_0 \otimes \dots \otimes a_k) &= a_0 \otimes \dots \otimes a_r a_{r+1} \otimes \dots \otimes a_k, & 0 \leq r < k, \\ &= a_k a_0 \otimes a_1 \otimes \dots \otimes a_{k-1}, & r = k, \\ s_r(a_0 \otimes \dots \otimes a_k) &= a_0 \otimes \dots \otimes a_r \otimes 1 \otimes a_{r+1} \otimes \dots \otimes a_k, & 0 \leq r \leq k, \\ t_k(a_0 \otimes \dots \otimes a_k) &= a_k \otimes a_0 \otimes a_1 \otimes \dots \otimes a_{k-1}. \end{aligned}$$

The Hochschild homology groups $\mathrm{HH}_*(A)$ are defined as the homology groups of the associated chain complex (with differential given by the alternating sum of the face maps d_r), or equivalently [58, theorem 8.4.1] as the homotopy groups of the geometric realization of the underlying simplicial set

$$\mathrm{HH}(A) = |[k] \mapsto \mathrm{HH}(A)_k|.$$

It was discovered by Connes that the action of the cyclic group of order $k+1$ on the set of k -simplices $\mathrm{HH}(A)_k$ gives rise to a continuous \mathbb{T} -action on the space $\mathrm{HH}(A)$, see [41, 7.1.9] or [37].

We next recall the topological Hochschild space $\mathrm{THH}(A)$. The idea in the definition is to change the ground ring for the tensor product in the Hochschild complex from the ring of integers to the sphere spectrum. This was carried out by Bökstedt [6] and, as it turns out, before him by Breen [10]. To give the definition, we first associate a commutative symmetric ring spectrum \tilde{A} in the sense of Hovey–Shikey–Smith [34] to the ring A . Let $S^1 = \Delta^1 / \partial \Delta^1$ be the standard simplicial circle, and let $S^i = S^1 \wedge \dots \wedge S^1$ be the i -fold smash product. Then

$$\tilde{A}_i = |[k] \mapsto A\{S_k^i\} / A\{s_{0,k}\}|$$

is an Eilenberg–MacLane space for A concentrated in degree i . Here $A\{S_k^i\}$ is the free A -module generated by the set of k -simplices S_k^i , and $A\{s_{0,k}\}$ is the sub- A -module generated by the base-point. The action of the symmetric group Σ_i by permutation of the i smash factors of S^i induces a Σ_i -action on \tilde{A}_i . In addition, there are natural multiplication and unit maps

$$\mu_{i,i'}: \tilde{A}_i \wedge \tilde{A}_{i'} \rightarrow \tilde{A}_{i+i'}, \quad \eta_i: S^i \rightarrow \tilde{A}_i,$$

which are $\Sigma_i \times \Sigma_{i'}$ -equivariant and Σ_i -equivariant, respectively.

Let I be the category with objects the finite sets

$$\underline{i} = \{1, 2, \dots, i\}, \quad i \geq 1,$$

and the empty set $\underline{0}$, and with morphisms all injective maps. We note that every morphism in I can be written (non-uniquely) as the standard inclusion followed by an automorphism. Concatenation of sets and maps defines a strict monoidal (but not symmetric monoidal) structure on I . There is a functor $G_k(A; X)$ from I^{k+1} to pointed spaces that on objects is given by

$$G_k(A) (\underline{i}_0, \dots, \underline{i}_k) = F (S^{i_0} \wedge \dots \wedge S^{i_k}, \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_k}), \quad (3.1)$$

where the right-hand side is the space of continuous base-point preserving maps with the compact-open topology. Let $\iota: \underline{i}_r \rightarrow \underline{i}'_r$ be the standard inclusion and write $i'_r = i_r + j_r$. Then $G_k(A) (\underline{i}_0, \dots, \underline{i}, \dots, \underline{i}_k)$ takes the map

$$S^{i_0} \wedge \dots \wedge S^{i_r} \wedge \dots \wedge S^{i_k} \xrightarrow{f} \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_r} \wedge \dots \wedge \tilde{A}_{i_k}$$

to the composite

$$\begin{aligned} S^{i_0} \wedge \dots \wedge S^{i'_r} \wedge \dots \wedge S^{i_k} &\xrightarrow{\sim} S^{i_0} \wedge \dots \wedge S^{i_r} \wedge \dots \wedge S^{i_k} \wedge S^{j_r} \\ &\xrightarrow{f \wedge \eta_{j_r}} \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_r} \wedge \dots \wedge \tilde{A}_{i_k} \wedge \tilde{A}_{j_r} \\ &\xrightarrow{\sim} \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_r} \wedge \tilde{A}_{j_r} \wedge \dots \wedge \tilde{A}_{i_k} \\ &\xrightarrow{\mu_{i_r, j_r}} \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i'_r} \wedge \dots \wedge \tilde{A}_{i_k}, \end{aligned}$$

where the first and third maps are the canonical isomorphisms (in the symmetric monoidal category of pointed spaces and smash product [43]), and where we have suppressed identity maps. The symmetric group $\Sigma_{i'_r}$ acts on $S^{i'_r}$ and $\tilde{A}_{i'_r}$ and by conjugation on $G_k(\underline{i}_0, \dots, \underline{i}'_r, \dots, \underline{i}_k)$. This defines the functor G_k on morphisms.

One now defines a cyclic space $\text{THH}(A)$. with k -simplices the homotopy colimit (see [9, §XII] for the definition of homotopy colimits)

$$\text{THH}(A)_k = \text{hocolim}_{I^{k+1}} G_k(A). \quad (3.2)$$

Although this is not a filtered homotopy colimit, it still has the desired homotopy type in that the canonical map

$$G_k(A) (\underline{i}_0, \dots, \underline{i}_k) \rightarrow \text{THH}(A)_k \quad (3.3)$$

is $(i_0 + \dots + i_k - 1)$ -connected. The proof of this *approximation lemma* is similar to the proof of the lemma in the proof of Quillen's theorem B [51] and can be found in [44, lemma 2.3.7]. We define the face maps

$$d_r: \text{THH}(A)_k \rightarrow \text{THH}(A)_{k-1}, \quad 0 \leq r \leq k; \quad (3.4)$$

the degeneracies and the cyclic operator are defined in a similar manner. We let $\partial_r: I^{k+1} \rightarrow I^k$ be the functor given by

$$\begin{aligned} \partial_r(\underline{i}_0, \dots, \underline{i}_k) 0 &= (\underline{i}_0, \dots, \underline{i}_r \sqcup \underline{i}_{r+1}, \dots, \underline{i}_k), \quad 0 \leq r < k, \\ &= (\underline{i}_k \sqcup \underline{i}_0, \underline{i}_1, \dots, \underline{i}_{k-1}), \quad r = k, \end{aligned}$$

on objects and similarly on morphisms, and let

$$\delta_r: G_k(A) \rightarrow G_{k-1} \circ \partial_r$$

be the natural transformation that takes the map

$$S^{i_0} \wedge \dots \wedge S^{i_r} \wedge \dots \wedge S^{i_k} \xrightarrow{f} \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_r} \wedge \dots \wedge \tilde{A}_{i_k}$$

to the composite

$$\begin{aligned} S^{i_0} \wedge \dots \wedge S^{i_{r+i_{r+1}}} \wedge \dots \wedge S^{i_k} &\xrightarrow{\sim} S^{i_0} \wedge \dots \wedge S^{i_r} \wedge S^{i_{r+1}} \wedge \dots \wedge S^{i_k} \\ &\xrightarrow{f} \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_r} \wedge \tilde{A}_{i_{r+1}} \wedge \dots \wedge \tilde{A}_{i_k} \\ &\xrightarrow{\mu_{i_r, i_{r+1}}} \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_r+i_{r+1}} \wedge \dots \wedge \tilde{A}_{i_k}, \end{aligned}$$

if $0 \leq r < k$, and to the composite

$$\begin{aligned} S^{i_k+i_0} \wedge S^{i_1} \wedge \dots \wedge S^{i_{k-1}} &\xrightarrow{\sim} S^{i_0} \wedge \dots \wedge S^{i_{k-1}} \wedge S^{i_k} \\ &\xrightarrow{f} \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_{k-1}} \wedge \tilde{A}_{i_k} \\ &\xrightarrow{\sim} \tilde{A}_{i_k} \wedge \tilde{A}_{i_0} \wedge \tilde{A}_{i_1} \wedge \dots \wedge \tilde{A}_{i_{k-1}} \\ &\xrightarrow{\mu_{i_k, i_0}} \tilde{A}_{i_k+i_0} \wedge \tilde{A}_{i_1} \wedge \dots \wedge \tilde{A}_{i_{k-1}}, \end{aligned}$$

if $r = k$. Here again the unnamed isomorphisms are the canonical ones. The face map d_r in (3.4) is now defined to be the composite

$$\mathrm{hocolim}_{I^{k+1}} G_k(A) \xrightarrow{\delta_r} \mathrm{hocolim}_{I^{k+1}} G_{k-1}(A) \circ \partial_r \xrightarrow{\partial_r^*} \mathrm{hocolim}_{I^k} G_{k-1}(A).$$

Then Bökstedt's topological Hochschild space is the \mathbb{T} -space

$$\mathrm{THH}(A) = |[k] \mapsto \mathrm{THH}(A)_k|. \quad (3.5)$$

The homotopy groups $\mathrm{THH}_*(A)$ can be defined in several other ways. Notably, Pirashvili–Waldhausen [49] have shown that these groups are canonically isomorphic to the homology groups of the category $\mathcal{P}(A)$ of finitely generated projective A -modules with coefficients in the bifunctor Hom , as defined by Jibladze–Pirashvili [36].

The Topological Hochschild Spectrum

It is essential for understanding the topological cyclic homology of truncated polynomial algebras that topological Hochschild homology be defined not only as a \mathbb{T} -space $\mathrm{THH}(A)$ but as a \mathbb{T} -spectrum $T(A)$.

In general, if G is a compact Lie group, the G -stable category is a triangulated category and a closed symmetric monoidal category, and the two structures are compatible [40, II.3.13]. The objects of the G -stable category are called G -spectra. A monoid for the smash product is called a ring G -spectrum. We denote the set of maps between two G -spectra T and T' by $[T, T']_G$. Associated with a pointed G -space X one has the suspension G -spectrum which we denote again by X . If λ is a finite dimensional orthogonal G -representation, we denote by S^λ its one-point compactification. Then the G -stable category is stable in the strong sense that the suspension homomorphism

$$[T, T']_G \xrightarrow{\sim} [T \wedge S^\lambda, T' \wedge S^\lambda]_G \tag{3.6}$$

is an isomorphism [40, I.6.1]. As a model for the G -stable category we use symmetric orthogonal G -spectra; see [45] and [33, theorem 5.10].

Let X be a pointed space. Then there is a functor $G_k(A; X)$ from I^{k+1} to pointed spaces that is defined on objects by

$$G_k(A; X) (\underline{i}_0, \dots, \underline{i}_k) = F(S^{i_0} \wedge \dots \wedge S^{i_k}, \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_k} \wedge X)$$

and on morphisms in a manner similar to (3.1). If n is a non-negative integer, we let (n) be the finite ordered set $\{1, 2, \dots, n\}$ and define $I^{(n)}$ to be the product category. (The category $I^{(0)}$ is the category with one object and one morphism.) The category $I^{(n)}$ is a strict monoidal category under component-wise concatenation of sets and maps. In addition, there is a functor

$$\sqcup_n: I^{(n)} \rightarrow I$$

given by the concatenation of sets and maps according to the ordering of (n) . (The functor \sqcup_0 takes the unique object to $\underline{0}$.) Let $G_k^{(n)}(A; X)$ be the functor from $(I^{(n)})^{k+1}$ to pointed spaces defined as the composition

$$G_k^{(n)}(A, X) = G_k(A, X) \circ (\sqcup_n)^{k+1} .$$

There is a cyclic space $\mathrm{THH}_k^{(n)}(A; X)$ with k -simplices the homotopy colimit

$$\mathrm{THH}_k^{(n)}(A; X) = \mathrm{hocolim}_{(I^{(n)})^{k+1}} G_k^{(n)}(A; X)$$

and with the cyclic structure maps defined by the same formulas as in (3.4), the only difference being that the concatenation functor \sqcup in the formula for the functor ∂_r must be replaced by the component-wise concatenation functor $\sqcup^{(n)}$. Then we define the \mathbb{T} -space

$$\mathrm{THH}^{(n)}(A; X) = \left| [k] \mapsto \mathrm{THH}_k^{(n)}(A; X) \right| .$$

An ordered inclusion $\iota: (n) \rightarrow (n')$ gives rise to a \mathbb{T} -equivariant map

$$\iota_*: \mathrm{THH}^{(n)}(A; X) \rightarrow \mathrm{THH}^{(n')}(A; X),$$

which, by the approximation lemma (3.3), is an equivalence of \mathbb{T} -spaces, provided that $n \geq 1$. For $n = 0$, there is canonical \mathbb{T} -equivariant homeomorphism

$$N^{\mathrm{cy}}(A) \wedge X \xrightarrow{\sim} \mathrm{THH}^{(0)}(A; X),$$

where the first smash factor on the left is the cyclic bar-construction of the pointed monoid given by the ring A which we now recall.

We define a *pointed monoid* to be a monoid in the symmetric monoidal category of pointed spaces and smash product. The ring A determines a pointed monoid, which we also denote A , with A considered as a pointed set with basepoint 0 and with the multiplication and unit maps given by multiplication and unit maps from the ring structure. Let $N^{\mathrm{cy}}(\Pi)$ be the cyclic space with k -simplices

$$N_k^{\mathrm{cy}}(\Pi) = \Pi \wedge \dots \wedge \Pi \quad (k+1 \text{ times}) \quad (3.7)$$

and with the Hochschild-type cyclic structure maps

$$\begin{aligned} d_i(\pi_0 \wedge \dots \wedge \pi_k) &= \pi_0 \wedge \dots \wedge \pi_i \pi_{i+1} \wedge \dots \wedge \pi_k, & 0 \leq i < k, \\ &= \pi_k \pi_0 \wedge \pi_1 \wedge \dots \wedge \pi_{k-1}, & i = k, \\ s_i(\pi_0 \wedge \dots \wedge \pi_k) &= \pi_0 \wedge \dots \wedge \pi_i \wedge 1 \wedge \pi_{i+1} \wedge \dots \wedge \pi_k, & 0 \leq i \leq k, \\ t_k(\pi_0 \wedge \dots \wedge \pi_k) &= \pi_k \wedge \pi_0 \wedge \pi_1 \wedge \dots \wedge \pi_{k-1}. \end{aligned}$$

Then $N^{\mathrm{cy}}(\Pi)$ is the geometric realization.

We define of the symmetric orthogonal \mathbb{T} -spectrum $T(A)$. Let n be a non-negative integer, and let λ be a finite-dimensional orthogonal \mathbb{T} -representation. Then the (n, λ) th space of $T(A)$ is defined to be the space

$$T(A)_{n, \lambda} = \mathrm{THH}^{(n)}(A; S^n \wedge S^\lambda) \quad (3.8)$$

with the diagonal \mathbb{T} -action induced by the $\mathbb{T} \times \mathbb{T}$ -action, where the action by one factor is induced by the \mathbb{T} -action on λ , and where the other is the canonical action by \mathbb{T} on the realization of a cyclic space. Similarly, we give $T(A)_{n, \lambda}$ the diagonal Σ_n -action induced by the $\Sigma_n \times \Sigma_n$ -action, where the two factors act by permutation respectively on (n) and on the smash factors of $S^n = S^1 \wedge \dots \wedge S^1$. In particular, $T(A)_{0,0} = N^{\mathrm{cy}}(A)$. The definition of the $\Sigma_n \times \Sigma_{n'} \times \mathbb{T}$ -equivariant spectrum structure maps

$$\sigma_{n, n', \lambda, \lambda'}: T(A)_{n, \lambda} \wedge S^{n'} \wedge S^{\lambda'} \rightarrow T(A)_{n+n', \lambda \oplus \lambda'}$$

is straightforward, but can be found in [20, section 2.2]. We also refer to *op. cit.*, appendix, for the definition of the ring spectrum product maps

$$\mu_{n, \lambda, n', \lambda'}: T(A)_{n, \lambda} \wedge T(A)_{n', \lambda'} \rightarrow T(A)_{n+n', \lambda \oplus \lambda'},$$

which are $\Sigma_n \times \Sigma_{n'} \times \mathbb{T}$ -equivariant with \mathbb{T} acting diagonally on the left. With this product $T(A)$, becomes a commutative \mathbb{T} -ring spectrum. The following result is proved in [29, proposition 2.4].

Proposition 1 Suppose that $n \geq 1$. Then the adjoint of the structure map

$$\tilde{\sigma}_{n,n',\lambda,\lambda'} : T(A)_{n,\lambda} \rightarrow F\left(S^{n'} \wedge S^{\lambda'}, T(A)_{n+n',\lambda \oplus \lambda'}\right)$$

is an \mathcal{F} -equivalence of pointed \mathbb{T} -spaces.

As a corollary of the proposition, we have a canonical isomorphism

$$\mathrm{THH}_q(A) \xrightarrow{\sim} [S^q \wedge \mathbb{T}_+, T(A)]_{\mathbb{T}} .$$

We also define a Hochschild \mathbb{T} -spectrum $H(A)$ and a map of \mathbb{T} -spectra

$$\ell : T(A) \rightarrow H(A) ,$$

which is called the linearization map. The construction of $H(A)$ is completely analogous to that of $T(A)$ but with the functor $G_k(A)$ replaced by the functor $G'_k(A)$ that is defined on objects by

$$G'_k(A) (\dot{i}_0, \dots, \dot{i}_k) = F\left(S^{i_0} \wedge \dots \wedge S^{i_k}, (A \otimes \dots \otimes A)_{i_0+\dots+i_k}\right)$$

and on morphisms in a manner similar to the formulas following (3.1). There are $k + 1$ tensor factors on the right and the ground ring for the tensor products is the ring of rational integers.

Equivariant Homotopy Theory

Before we proceed, we discuss a few concepts and elementary results from the homotopy theory of G -spaces. We refer the reader to Adams [1] for an introduction to this material.

The homotopy category of pointed spaces is equivalent to the category of pointed CW-complexes and pointed homotopy classes of pointed cellular maps. Similarly, the homotopy category of pointed G -spaces is equivalent to the category of pointed G -CW-complexes and pointed G -homotopy classes of pointed cellular G -maps. Let G be a compact Lie group. Then a pointed G -CW-complex is a pointed (left) G -space X together with a sequence of pointed sub- G -spaces

$$* = \mathrm{sk}_{-1} X \subset \mathrm{sk}_0 X \subset \mathrm{sk}_1 X \subset \dots \subset \mathrm{sk}_n X \subset \dots \subset X ,$$

and for all $n \geq 0$, a push-out square of (un-pointed) G -spaces

$$\begin{array}{ccc}
 \coprod_{\alpha} \partial D^n \times G/H_{\alpha} & \xrightarrow{\varphi_n|_{\partial}} & \text{sk}_{n-1} X \\
 \downarrow & & \downarrow \\
 \coprod_{\alpha} D^n \times G/H_{\alpha} & \xrightarrow{\varphi_n} & \text{sk}_n X,
 \end{array}$$

with each $H_{\alpha} \subset G$ a closed subgroup, such that the canonical pointed G -map

$$\text{colim}_n \text{sk}_n X \rightarrow X$$

is a homeomorphism. The map φ_n restricts to an embedding on the interior of $D^n \times G/H_{\alpha}$. We say that the image of this embedding is a cell of dimension n and orbit-type G/H_{α} .

A pointed G -map $f: X \rightarrow X'$ between pointed G -CW-complexes is cellular if $f(\text{sk}_n X) \subset \text{sk}_n X'$, for all $n \geq -1$.

2

Lemma 2 Let G be a finite group, let X be a pointed G -CW-complex of finite dimension, and let $d(H)$ be the supremum of the dimension of the cells of X of orbit-type G/H . Let Y be a pointed G -space such that Y^H is $n(H)$ -connected. Then the equivariant mapping space $F(X, Y)^G$ is m -connected with

$$m = \inf\{n(H) - d(H) \mid H \in \mathcal{O}_G(X)\}.$$

Here $\mathcal{O}_G(X)$ denotes the set of subgroups $H \subset G$ for which X has a cell of orbit-type G/H .

Proof We show by induction on $n \geq -1$ that $F(\text{sk}_n X, Y)^G$ is m -connected. The case $n = -1$ is trivial, so we assume the statement for $n - 1$. One shows as usual that the map $\text{sk}_{n-1} X \hookrightarrow \text{sk}_n X$ has the G -homotopy extension property, and that the induced map

$$F(\text{sk}_n X, Y)^G \twoheadrightarrow F(\text{sk}_{n-1} X, Y)^G$$

has the homotopy lifting property [52]. The fiber of the latter map over the base-point is canonically homeomorphic to

$$\begin{aligned}
 F(\text{sk}_n X / \text{sk}_{n-1} X, Y)^G &\simeq F\left(\bigvee_{\alpha} S^n \wedge G/H_{\alpha+}, Y\right)^G \\
 &\simeq \prod_{\alpha} F(S^n \wedge G/H_{\alpha+}, Y)^G \simeq \prod_{\alpha} F(S^n, Y^{H_{\alpha}}),
 \end{aligned}$$

where the first map is induced by the map φ_n . But the space $F(S^n, Y^{H_{\alpha}})$ is $(n(H_{\alpha}) - n)$ -connected and $n(H_{\alpha}) - n \geq m$, so the induction step follows. Since $X = \text{sk}_n X$, for some n , we are done.

Pointed Monoid Algebras

Let A be a ring, and let Π be a discrete pointed monoid, that is, a monoid in the symmetric monoidal category of pointed sets and smash product. Then the pointed monoid algebra $A(\Pi)$ is defined to be the quotient of the monoid algebra $A[\Pi]$ by the ideal generated by the base-point of Π . For example,

$$A[x]/(x^e) = A(\Pi_e) ,$$

where $\Pi_e = \{0, 1, x, \dots, x^{e-1}\}$ considered as a pointed monoid with base-point 0 and with the multiplication given by $x^e = 0$. There is a canonical maps $\phi: A \rightarrow A(\Pi)$ and $\iota: \Pi \rightarrow A(\Pi)$ of rings and pointed monoids, respectively, given by $\phi(a) = a \cdot 1$ and $\iota(\pi) = 1 \cdot \pi$. The following is [29, theorem 7.1].

Proposition 3 Let A be a ring and Π a pointed monoid. Then the composite

$$T(A) \wedge N^{cy}(\Pi) \xrightarrow{\phi^{\wedge \iota}} T(A(\Pi)) \wedge N^{cy}(A(\Pi)) \xrightarrow{\mu} T(A(\Pi))$$

is an \mathcal{F} -equivalence of \mathbb{T} -spectra.

Before we give the proof, we mention the analogous result in the linear situation. The derived category of abelian groups is a triangulated category and a symmetric monoidal category. A monoid for the tensor product is called a differential graded ring. If C is a simplicial abelian group, we write C_* for the associated chain complex. If R is a simplicial ring, then R_* is a differential graded ring with product given by the composite

$$R_* \otimes R_* \xrightarrow{\theta} (R \otimes R)_* \xrightarrow{\mu_*} R_* ,$$

where the left-hand map is the Eilenberg–Zilber shuffle map [58, 8.5.4]. If A is a (commutative) ring, then $\text{HH}(A)$ is a simplicial ring with the product

$$(a_0 \otimes \dots \otimes a_k) \cdot (a'_0 \otimes \dots \otimes a'_k) = a_0 a'_0 \otimes \dots \otimes a_k a'_k ,$$

and hence $\text{HH}(A)_*$ is a differential graded ring. We claim that the composite

$$\text{HH}(A)_* \otimes \mathbb{Z}(N^{cy}(\Pi))_* \rightarrow \text{HH}(A(\Pi))_* \otimes \text{HH}(A(\Pi))_* \xrightarrow{\mu_*} \text{HH}(A(\Pi))_*$$

is a quasi-isomorphism. Indeed, this map is equal to the composite of the Eilenberg–Zilber map

$$\text{HH}(A)_* \otimes \mathbb{Z}(N^{cy}(\Pi))_* \xrightarrow{\theta} (\text{HH}(A) \otimes \mathbb{Z}(N^{cy}(\Pi)))_* ,$$

which is a quasi-isomorphism [58, theorem 8.5.1], and the map of simplicial abelian groups

$$\text{HH}(A) \otimes \mathbb{Z}(N^{cy}(\Pi)) \rightarrow \text{HH}(A(\Pi)) \otimes \text{HH}(A(\Pi)) \xrightarrow{\mu} \text{HH}(A(\Pi)) ,$$

which is an isomorphism, since the tensor product of simplicial abelian groups is formed degree-wise.

Proof (*Proof of proposition 3*) We shall use the following criterion for a map $f: X \rightarrow Y$ of symmetric orthogonal \mathbb{T} -spectra to be an \mathcal{F} -equivalence. Suppose that for all integers $n \geq 0$, all finite dimensional orthogonal \mathbb{T} -representations λ , and all finite subgroups $C \subset \mathbb{T}$, the induced map

$$(f_{n,\lambda})^C: (X_{n,\lambda})^C \rightarrow (Y_{n,\lambda})^C$$

is $(n + \dim_{\mathbb{R}}(\lambda^C) + \varepsilon_C(n, \lambda))$ -connected, where $\varepsilon_C(n, \lambda)$ tends to infinity with n and λ . Then $f: X \rightarrow Y$ is an \mathcal{F} -equivalence of \mathbb{T} -spectra. This follows directly from the definition of \mathcal{F} -equivalence [33, 45].

We note that a C -equivariant isometric isomorphism $\lambda \xrightarrow{\sim} \lambda'$ between two finite dimensional orthogonal \mathbb{T} -representations induces a natural C -equivariant homeomorphism $T(A)_{n,\lambda} \xrightarrow{\sim} T(A)_{n,\lambda'}$. Hence, for a given finite subgroup $C \subset \mathbb{T}$, it suffices to consider the map

$$(f_{n,\lambda})^C: (T(A)_{n,\lambda})^C \wedge N^{\text{cy}}(\Pi)^C \rightarrow (T(A(\Pi))_{n,\lambda})^C$$

induced by the map of the statement in the case where λ is a direct sum of copies of the regular representation ϱ_C . To prove the proposition, we show that the map $(f_{n,m\varrho_C})^C$ is $(n + m + m - 1)$ -connected. We first unravel the definition of this map.

The composite of canonical maps

$$\begin{aligned} & F(S^{i_0} \wedge \dots \wedge S^{i_k}, \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_k} \wedge S^{n+m\varrho_C}) \wedge \Pi^{\wedge(k+1)} \\ & \rightarrow F(S^{i_0} \wedge \dots \wedge S^{i_k}, \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_k} \wedge \Pi^{\wedge(k+1)} \wedge S^{n+m\varrho_C}) \\ & \rightarrow F(S^{i_0} \wedge \dots \wedge S^{i_k}, \tilde{A}(\Pi)_{i_0} \wedge \dots \wedge \tilde{A}(\Pi)_{i_k} \wedge S^{n+m\varrho_C}) \end{aligned} \tag{3.9}$$

defines a natural transformation of functors from I^{k+1} to pointed spaces

$$G_k(A; S^{n+m\varrho_C}) \wedge N_k^{\text{cy}}(\Pi) \rightarrow G_k(A(\Pi); S^{n+\varrho_C}) .$$

If we pre-compose on both sides by the functor $(\sqcup_n)^{k+1}$, we get a similar natural transformation with $G_k^{(n)}$ in place of G_k . Taking homotopy colimits over $(I^{(n)})^{k+1}$, we obtain a map

$$\text{THH}_k^{(n)}(A; S^{n+m\varrho_C}) \wedge N_k^{\text{cy}}(\Pi) \rightarrow \text{THH}_k^{(n)}(A(\Pi); S^{n+m\varrho_C}) .$$

Here we have used that taking homotopy colimits and smashing by a (fixed) pointed space commute up to canonical homeomorphism. Finally, as k varies, these maps constitute a map of cyclic spaces, and hence we get an induced map after geometric realization

$$\text{THH}^{(n)}(A; S^{n+m\varrho_C}) \wedge N^{\text{cy}}(\Pi) \rightarrow \text{THH}^{(n)}(A(\Pi); S^{n+m\varrho_C}) .$$

This is the map $f_{n,m\varrho_C}$ of the statement. Here we have used that the geometric realization of a smash product of pointed simplicial spaces is canonically homeomorphic to the smash product of their geometric realizations.

We next give a similar description of the induced map of C -fixed points $(f_{n,m\varrho_C})^C$. The C -action on the domain and target of the map $f_{n,m\varrho_C}$, we recall, is the diagonal action induced from a natural $C \times C$ -action, where the action by one factor is induced by the C -action on ϱ_C , and where the other is the canonical action by the cyclic group $C \subset \mathbb{T}$ on the realization of a cyclic space. The latter C -action is not induced from simplicial C -action. However, this can be achieved by edge-wise subdivision, which now we recall.

Let X . be a simplicial space, and let r be the order of C . Then by [7, lemma 1.1], there is a canonical (non-simplicial) homeomorphism

$$D_r: |[k] \mapsto (\text{sd}_r X)_k| \xrightarrow{\sim} |[k] \mapsto X_k| ,$$

where $\text{sd}_r X$. is the simplicial space with k -simplices given by

$$\text{sd}_r(X)_k = X_{r(k+1)-1}$$

and simplicial structure maps, for $0 \leq i \leq k$, given by

$$d'_i: \text{sd}_r(X_s)_k \rightarrow \text{sd}_r(X_s)_{k-1} , \quad d'_i = d_i \circ d_{i+(k+1)} \circ \dots \circ d_{i+(r-1)(k+1)} ,$$

$$s'_i: \text{sd}_r(X_s)_k \rightarrow \text{sd}_r(X_s)_{k+1} , \quad s'_i = s_{i+(r-1)(k+2)} \circ \dots \circ s_{i+(k+2)} \circ s_i .$$

If X . is a cyclic space, then the action by C on $\text{sd}_r(X)_k$, where the generator $e^{2\pi i/r}$ acts as the operator $(t_{r(k+1)-1})^{k+1}$, is compatible with the simplicial structure maps, and hence induces a C -action on the geometric realization. Moreover, the homeomorphism D_r is C -equivariant, if we give the domain and target the C -action induced from the simplicial C -action and from the canonical \mathbb{T} -action, respectively.

In the case at hand, we now consider

$$\text{sd}_r \text{THH} . (A; S^{n+m\varrho_C})_k = \text{THH}_{r(k+1)-1} (A; S^{n+m\varrho_C})$$

with the diagonal C -action induced by the $C \times C$ -action, where the generator $e^{2\pi i/r}$ of one C -factor acts as the operator $(t_{r(k+1)-1})^{k+1}$ on the right, and where the action by the other C -factor is induced from the C -action on ϱ_C . This action is not induced from an action on the individual terms of the homotopy colimit that defines the right-hand side. However, if we let $\Delta_{r,k}: I^{k+1} \rightarrow I^{(k+1)r}$ be the diagonal functor given by

$$\Delta_{r,k}(i_0, \dots, i_k) = (i_0, \dots, i_k, \dots, \dots, i_0, \dots, i_k) ,$$

then the canonical map of homotopy colimits

$$\text{hocolim}_{I^{k+1}} G_{r(k+1)-1} (A; S^{n+m\varrho_C}) \circ \Delta_{r,k} \rightarrow \text{hocolim}_{I^{(k+1)r}} G_{r(k+1)-1} (A; S^{n+m\varrho_C})$$

induces a homeomorphism of C -fixed sets. On the left, the group C acts trivially on the index category, and hence the action is induced from an action on the individual terms of the homotopy colimit. A typical term is canonically homeomorphic to

$$F((S^{i_0} \wedge \dots \wedge S^{i_k})^{\wedge r}, (\tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_k})^{\wedge r} \wedge S^{n+m\varrho_C}) .$$

Moreover, the group C acts on the mapping space by the conjugation action induced from the action on the two r -fold smash products by cyclic permutation of the smash factors and from the action on $S^{n+m\varrho_C}$ induced from the one on ϱ_C . The canonical maps

$$\begin{aligned} & F((S^{i_0} \wedge \dots \wedge S^{i_k})^{\wedge r}, (\tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_k})^{\wedge r} \wedge S^{n+m\varrho_C}) \wedge (\Pi^{\wedge(k+1)})^{\wedge r} \\ & \rightarrow F((S^{i_0} \wedge \dots \wedge S^{i_k})^{\wedge r}, (\tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_k} \wedge \Pi^{\wedge(k+1)})^{\wedge r} \wedge S^{n+m\varrho_C}) \quad (3.10) \\ & \rightarrow F((S^{i_0} \wedge \dots \wedge S^{i_k})^{\wedge r}, (\tilde{A}(\Pi)_{i_0} \wedge \dots \wedge \tilde{A}(\Pi)_{i_k})^{\wedge r} \wedge S^{n+m\varrho_C}) \end{aligned}$$

are C -equivariant and their composite defines a natural transformation of functors from I^{k+1} to pointed C -spaces

$$\begin{aligned} & (G_{(k+1)r-1}(A; S^{n+m\varrho_C}) \wedge N_{(k+1)r-1}^{\text{cy}}(\Pi)) \circ \Delta_{r,k} \\ & \rightarrow G_{(k+1)r-1}(A(\Pi); S^{n+m\varrho_C}) \circ \Delta_{r,k} . \end{aligned}$$

If we pre-compose both sides by the functor $(\sqcup_n)^{k+1}$, we get a similar natural transformation with $G_k^{(n)}$ in place of G_k . Taking C -fixed points and homotopy colimits over $(I^{(n)})^{k+1}$, we obtain the map

$$\begin{aligned} & \left(\text{sd}_r(\text{THH}^{(n)}(A; S^{n+m\varrho_C}) \wedge N^{\text{cy}}(\Pi))_k \right)^C \\ & \rightarrow \left(\text{sd}_r \text{THH}^{(n)}(A(\Pi); S^{n+m\varrho_C})_k \right)^C . \end{aligned}$$

As k varies, these maps constitute a map of simplicial spaces, and hence we get an induced map of the associated geometric realizations. Finally, this map and the canonical homeomorphism D_r determine a map

$$(\text{THH}^{(n)}(A; S^{n+m\varrho_C}) \wedge N^{\text{cy}}(\Pi))^C \rightarrow \text{THH}^{(n)}(A(\Pi); S^{n+m\varrho_C})^C .$$

This is the map of C -fixed points induced by the map $f_{n,m\varrho_C}$ of the statement. Here we have used that geometric realization and finite limits (such as C -fixed points) commute [16, chap. 3, §3].

It remains to show that $(f_{n,m\varrho_C})^C$ is $(n+2m-1)$ -connected as stated. As we have just seen, this is the geometric realization of a map of simplicial spaces. It suffices to show that for the latter map, the induced map of k -simplices is $(n+2m-1)$ -connected, for all $k \geq 0$. Finally, by the approximation lemma (3.3), it suffices to show that for i_0, \dots, i_k large, the maps of C -fixed points induced by the maps (3.10) are $(n+2m-1)$ -connected. Let $i = i_0 + \dots + i_k$.

We first consider the second map in (3.10). The canonical map

$$\tilde{A}_j \wedge \Pi \rightarrow \tilde{A}(\Pi)_j$$

is $(2j - 1)$ -connected. Indeed, this map is the inclusion of a wedge in the corresponding (weak) product. It follows that the canonical map

$$\begin{aligned} & (\tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_k} \wedge \Pi^{\wedge(k+1)})^{\wedge r} \wedge S^{n+m\varrho_C} \\ & \rightarrow (\tilde{A}(\Pi)_{i_0} \wedge \dots \wedge \tilde{A}(\Pi)_{i_k})^{\wedge r} \wedge S^{n+m\varrho_C} \end{aligned}$$

is $(2ir + n + mr - 1)$ -connected. Let $C_s \subset C_r$ and $r = st$. Then the induced map of C_s -fixed points is $n(C_s)$ -connected with $n(C_s) = 2it + n + mt - 1$. The supremum of the dimension of the cells of $(S^{i_0} \wedge \dots \wedge S^{i_k})^{\wedge r}$ of orbit-type C_s is $d(C_s) = it$. Hence, by lemma 2, the map of C -fixed points induced from the second map in (3.10) is n -connected with

$$n = \inf \{n(C_s) - d(C_s) \mid C_s \subset C_r\} = i + n + m - 1 .$$

Hence, this map is $(n + 2m - 1)$ -connected, if $i \geq m$.

Finally, we consider the first map in (3.10). We abbreviate

$$\begin{aligned} X &= S^{i_0} \wedge \dots \wedge S^{i_k}, & Y &= \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_k}, \\ Z &= S^{n+m\varrho_C}, & P &= \Pi^{\wedge(k+1)}, \end{aligned}$$

and consider the following diagram.

$$\begin{array}{ccc} F(X^{\wedge r}, Y^{\wedge r} \wedge Z) \wedge P^{\wedge r} & \longrightarrow & F(X^{\wedge r}, Y^{\wedge r} \wedge Z \wedge P^{\wedge r}) & (3.11) \\ \downarrow \tilde{\delta} & & \downarrow \tilde{\delta}_* & \\ F(P^{\wedge r}, F(X^{\wedge r}, Y^{\wedge r} \wedge Z)) & \xrightarrow{\sim} & F(X^{\wedge r}, F(P^{\wedge r}, Y^{\wedge r} \wedge Z)) & \end{array}$$

where the map $\tilde{\delta}$ is the adjoint of the (pointed) Kronecker delta function

$$\delta: P^{\wedge r} \wedge P^{\wedge r} \rightarrow S^0 .$$

The top horizontal map in (3.11) is equal to the first map in (3.10). We wish to show that it induces an $(n + 2m - 1)$ -connected map of C -fixed points. We prove that the map of C -fixed points induced by the left-hand vertical map in (3.11) is $(n + 2m - 1)$ -connected and leave the analogous case of the right-hand vertical map to the reader. So consider the following diagram, where the map in question is the top horizontal map.

$$\begin{array}{ccc} F(X^{\wedge r}, Y^{\wedge r} \wedge Z)^C \wedge P & \xrightarrow{\tilde{\delta}^C} & F(P^{\wedge r}, F(X^{\wedge r}, Y^{\wedge r} \wedge Z))^C \\ \downarrow \tilde{\delta} & & \downarrow \Delta^* \\ F(P, F(X^{\wedge r}, Y^{\wedge r} \wedge Z)^C) & \xrightarrow{\sim} & F(P, F(X^{\wedge r}, Y^{\wedge r} \wedge Z))^C \end{array}$$

The left-hand vertical map is the inclusion of a wedge in the corresponding product. The summands $F(X^{\wedge r}, Y^{\wedge r} \wedge Z)^C$ are $(n + m - 1)$ -connected by lemma 2, and hence this map is $(2(n + m) - 1)$ -connected. The right-hand vertical map is induced from the inclusion $\Delta: P \rightarrow P^{\wedge r}$ of the diagonal. The map Δ^* has the homotopy lifting property, and the fiber over the base-point is canonically homeomorphic to the mapping space

$$F(P^{\wedge r}/\Delta(P), F(X^{\wedge r}, Y^{\wedge r} \wedge Z))^C .$$

The connectivity can be evaluated by using lemma 2. Since $P^{\wedge r}/\Delta(P)$ has no cells of orbit-type C/C , we find that this space is $(n + tm - 1)$ -connected, where t is the smallest non-trivial divisor in r . Since $t \geq 2$, we are done.

A similar argument shows the following result.

4 Proposition 4 Let A be a ring and Π a pointed monoid. Then the composite

$$H(A) \wedge N^{\text{cy}}(\Pi) \rightarrow H(A(\Pi)) \wedge N^{\text{cy}}(A(\Pi)) \xrightarrow{\mu} H(A(\Pi))$$

is an \mathcal{F} -equivalence of \mathbb{T} -spectra.

3.6 The Cyclic Bar-construction of Π_e

The \mathbb{T} -equivariant homotopy type of the \mathbb{T} -spaces $N^{\text{cy}}(\Pi_e)$ that occur in proposition 3 for truncated polynomial algebras was evaluated in [28]. The result, which we now recall, is quite simple, and it is this simplicity which, in turn, facilitates the understanding of the topological cyclic homology of truncated polynomial algebras.

There is a natural wedge decomposition

$$\bigvee_{i \in \mathbb{N}_0} N^{\text{cy}}(\Pi_e, i) \xrightarrow{\sim} N^{\text{cy}}(\Pi_e) ,$$

where $N^{\text{cy}}(\Pi_e, i)$ is the realization of the pointed cyclic subset $N_i^{\text{cy}}(\Pi_e, i)$ generated by the 0-simplex 1, if $i = 0$, and by the $(i - 1)$ -simplex $x \wedge \dots \wedge x$, if $i > 0$. The \mathbb{T} -space $N^{\text{cy}}(\Pi, 0)$ is homeomorphic to the discrete space $\{0, 1\}$. For $i > 0$, we let $d = [(i - 1)/e]$ be the largest integer less than or equal to $(i - 1)/e$ and consider the complex \mathbb{T} -representation

$$\lambda_d = \mathbb{C}(1) \oplus \mathbb{C}(2) \oplus \dots \oplus \mathbb{C}(d) ,$$

where $\mathbb{C}(t)$ denotes the representation of \mathbb{T} on \mathbb{C} through the t -fold power map. The following result is [28, theorem B].

Theorem 5 There is a canonical exact triangle of pointed \mathbb{T} -spaces

$$S^{\Lambda_d} \wedge \mathbb{T}/C_{i|e+} \xrightarrow{\text{id} \wedge \text{pr}} S^{\Lambda_d} \wedge \mathbb{T}/C_{i+} \rightarrow N^{\text{cy}}(\Pi_e, i) \xrightarrow{\partial} S^{\Lambda_d} \wedge \mathbb{T}/C_{i|e+}[-1],$$

where $d = [(i - 1)/e]$ and where the right and left-hand terms are understood to be a point, if e does not divide i .

We sketch the proof. By elementary cyclic theory, $N^{\text{cy}}(\Pi_e, i)$ is a quotient of the cyclic standard $(i - 1)$ -simplex $\Delta^{i-1} = \Delta^{i-1} \times \mathbb{T}$. In fact, it is not difficult to see that there is a \mathbb{T} -equivariant homeomorphism

$$(\Delta^{i-1}/C_i \cdot \Delta^{i-e}) \wedge_{C_i} \mathbb{T}_+ \xrightarrow{\sim} N^{\text{cy}}(\Pi_e, i),$$

where Δ^{i-1} is the standard $(i - 1)$ -simplex with C_i acting by cyclically permuting the vertices and $\Delta^{i-e} \subset \Delta^{i-1}$ is the face spanned by the first $i - e + 1$ vertices. It is also easy to understand the homology. Indeed, the reduced cellular complex of $N^{\text{cy}}(\Pi_e)$ is canonically isomorphic to the Hochschild complex of $\mathbb{Z}[x]/(x^e)$, and the homology of the latter was evaluated in [24]. If $e = 2$, $C_i \cdot \Delta^{i-2} = \partial \Delta^{i-1}$, and the result readily follows. If $e > 2$, however, the homological dimension $2d$ is smaller than the topological dimension $i - 1$, and the main difficulty is to produce an equivariant map of degree one from the sphere S^{Λ_d} to $N^{\text{cy}}(\Pi_e, i)$. It is the complete understanding of the combinatorial structure of the so-called cyclic polytopes [13, 17] that makes this possible.

Topological Cyclic Homology

We recall the definition of the cyclic homology of Connes [14] and the topological cyclic homology of Bökstedt–Hsiang–Madsen [7]. We first give the definition of the version of cyclic homology that was defined independently by Loday–Quillen [42] and Feigin–Tsygan [55] and that agrees with Connes’ original definition rationally.

In general, if G is a compact Lie group and $H \subset G$ a finite subgroup, then there is a canonical duality isomorphism in the G -stable category

$$[X \wedge G/H_+, Y \wedge S^{\mathfrak{g}}]_G \xrightarrow{\sim} [X, Y \wedge G/H_+]_G, \tag{3.12}$$

where \mathfrak{g} denotes the Lie algebra of G with the adjoint action [29, section 8.1]. If $G = \mathbb{T}$, the adjoint representation \mathfrak{t} is trivial, and we fix an isomorphism $\mathbb{R} \xrightarrow{\sim} \mathfrak{t}$.

Let E be a free contractible \mathbb{T} -CW-complex; any two such \mathbb{T} -CW-complexes are canonically \mathbb{T} -homotopy equivalent. We use the unit sphere $E = S(\mathbb{C}^\infty)$ with the standard \mathbb{T} -CW-structure with one cell in every even non-negative dimension. Then the cyclic homology of A is defined by

$$\text{HC}_q(A) = [S^{q+1}, H(A) \wedge E_+]_{\mathbb{T}}. \tag{3.13}$$

There is a canonical isomorphism

$$[S^{q+1}, H(A) \wedge E_+]_{\mathbb{T}} \xleftarrow{\sim} [S^{q+1}, (H(A) \wedge E_+)^{\mathbb{T}}] ,$$

and hence the group $\mathrm{HC}_q(A)$ is canonically isomorphic to the q th homotopy group of the \mathbb{T} -group homology spectrum

$$\mathbb{H}.(\mathbb{T}, H(A)) = (H(A) \wedge E_+)^{\mathbb{T}} [+1]$$

(compare [40, theorem 7.1] for the dimension-shift). There is a canonical exact triangle of pointed \mathbb{T} -CW-complexes

$$\mathbb{T}_+ = \mathrm{sk}_1(E_+) \twoheadrightarrow \mathrm{sk}_2(E_+) \rightarrow \mathbb{T}_+[-2] \xrightarrow{\partial} \mathbb{T}_+[-1] ,$$

which induces an exact triangle of \mathbb{T} -spectra

$$H(A) \wedge \mathbb{T}_+ \rightarrow H(A) \wedge \mathrm{sk}_2(E_+) \rightarrow H(A) \wedge \mathbb{T}_+[-2] \xrightarrow{\partial} H(A) \wedge \mathbb{T}_+[-1] .$$

The boundary map induces Connes' (B -)operator

$$d: \mathrm{HH}_q(A) \rightarrow \mathrm{HH}_{q+1}(A) , \tag{3.14}$$

where we use the duality isomorphism (3.12) to identify

$$\mathrm{HH}_q(A) \xrightarrow{\sim} [S^{q+1} \wedge \mathbb{T}_+, H(A) \wedge S^t]_{\mathbb{T}} \xrightarrow{\sim} [S^{q+1}, H(A) \wedge \mathbb{T}_+]_{\mathbb{T}} .$$

The operator d is the d^2 -differential in the spectral sequence induced from the skeleton filtration of E . The spectral sequence is a first quadrant homology type spectral sequence with $E_{s,t}^2 = \mathrm{HH}_t(A)$, for s even, and zero, for s odd. Moreover, the groups $\mathrm{HH}_*(A)$ together with the operator d form a differential graded ring.

The definition (3.14) of the Connes' operator d makes sense for every \mathbb{T} -spectrum. The operator d is a derivation, for every ring \mathbb{T} -spectrum, but, in general, it is not a differential. Instead, one has

$$d \circ d = d \circ \iota = \iota \circ d ,$$

where ι is the map induced by the Hopf map $\eta: S^{q+1} \rightarrow S^q$.

We now explain the definition of topological cyclic homology, and refer to [20, 27, 29, 31] for details. Let p be a fixed prime and consider the groups

$$\mathrm{TR}_q^n(A; p) = [S^q \wedge \mathbb{T}/C_{p^{n-1}}, T(A)]_{\mathbb{T}} . \tag{3.15}$$

There is a canonical isomorphism

$$[S^q \wedge \mathbb{T}/C_{p^{n-1}}, T(A)]_{\mathbb{T}} \xleftarrow{\sim} [S^q, T(A)^{C_{p^{n-1}}}] ,$$

and hence the group $\mathrm{TR}_q^n(A; p)$ is canonically isomorphic to the q th homotopy group of the $C_{p^{n-1}}$ -fixed point spectrum

$$\mathrm{TR}^n(A; p) = T(A)^{C_{p^{n-1}}} .$$

By using the duality isomorphism (3.12), we can define a derivation

$$d: \mathrm{TR}_q^n(A; p) \rightarrow \mathrm{TR}_{q+1}^n(A; p)$$

in a manner similar to (3.14). The canonical projection from $\mathbb{T}/C_{p^{n-1}}$ to $\mathbb{T}/C_{p^{n-2}}$ induces a natural map

$$F: \mathrm{TR}_q^n(A; p) \rightarrow \mathrm{TR}_q^{n-1}(A; p),$$

called the Frobenius, and there is an associated transfer map

$$V: \mathrm{TR}_q^{n-1}(A; p) \rightarrow \mathrm{TR}_q^n(A; p),$$

called the Verschiebung. The two composites FV and VF are given by multiplication by the integer p and by the element $V(1)$, respectively. Moreover,

$$FdV = d + (p - 1)\iota.$$

There is an additional map

$$R: \mathrm{TR}_q^n(A; p) \rightarrow \mathrm{TR}_q^{n-1}(A; p),$$

called the restriction, whose definition we now explain.

In general, an isomorphism of compact Lie groups $f: G \xrightarrow{\sim} G'$ induces an equivalence of categories f^* from the G' -stable category to the G -stable category [40, II.1.7]. In particular, the isomorphism given by the r th root

$$\varrho_r: \mathbb{T} \xrightarrow{\sim} \mathbb{T}/C_r$$

induces an equivalence ϱ_r^* from the \mathbb{T}/C_r -stable category to the \mathbb{T} -stable category. We consider the following exact triangle of pointed \mathbb{T} -CW-complexes.

$$E_+ \rightarrow S^0 \rightarrow \tilde{E} \xrightarrow{\partial} E_+[-1].$$

Here \tilde{E} is defined as the mapping cone of the left-hand map, which collapses E to the non-base point of S^0 . It induces an exact triangle of \mathbb{T} -spectra

$$T(A) \wedge E_+ \rightarrow T(A) \rightarrow T(A) \wedge \tilde{E} \xrightarrow{\partial} T(A) \wedge E_+[-1].$$

The \mathbb{T} -spectrum $T(A)$ has the additional property that there is a natural \mathcal{F} -equivalence of \mathbb{T} -spectra

$$r: \varrho_p^*(T(A) \wedge \tilde{E})^{C_p} \xrightarrow{\sim} T(A),$$

and hence the exact triangle above induces an exact triangle of \mathbb{T} -spectra

$$\varrho_p^*(T(A) \wedge E_+)^{C_p} \rightarrow \varrho_p^*T(A)^{C_p} \xrightarrow{R} T(A) \xrightarrow{\partial} \varrho_p^*(T(A) \wedge E_+)^{C_p}[-1].$$

This induces an exact triangle of $C_{p^{n-2}}$ -fixed point spectra

$$\mathbb{H}. (C_{p^{n-1}}, T(A)) \rightarrow \mathrm{TR}^n(A; p) \xrightarrow{R} \mathrm{TR}^{n-1}(A; p) \xrightarrow{\partial} \mathbb{H}. (C_{p^{n-1}}, T(A)) [-1],$$

which, in turn, gives rise to a long-exact sequence of homotopy groups

$$\dots \rightarrow \mathbb{H}_q (C_{p^{n-1}}, T(A)) \rightarrow \mathrm{TR}_q^n(A; p) \xrightarrow{R} \mathrm{TR}_q^{n-1}(A; p) \xrightarrow{\partial} \dots$$

Here the left-hand term is the $C_{p^{n-1}}$ -group homology spectrum

$$\mathbb{H}. (C_{p^{n-1}}, T(A)) = (T(A) \wedge E_+)^{C_{p^{n-1}}},$$

whose homotopy groups are the abutment of a first quadrant homology type spectral sequence

$$E_{s,t}^2 = H_s (C_{p^{n-1}}, \mathrm{THH}_t(A)) \Rightarrow \mathbb{H}_{s+t} (C_{p^{n-1}}, T(A)).$$

The spectral sequence is obtained from the skeleton filtration of E considered as a $C_{p^{n-1}}$ -CW-complex, and the identification the E^2 -term with the group homology of $C_{p^{n-1}}$ acting on $\mathrm{THH}_*(A)$ uses the duality isomorphism (3.12). We refer to [31, §4] for a detailed discussion.

One defines $\mathrm{TC}^n(A; p)$ as the homotopy equalizer of the maps

$$R, F: \mathrm{TR}^n(A; p) \rightarrow \mathrm{TR}^{n-1}(A; p)$$

and $\mathrm{TC}(A; p)$ as the homotopy limit of the spectra $\mathrm{TC}^n(A; p)$. Hence, there is a natural long-exact sequence of pro-abelian groups

$$\dots \rightarrow \mathrm{TC}_q^*(A; p) \rightarrow \mathrm{TR}_q^*(A; p) \xrightarrow{1-F} \mathrm{TR}_q^*(A; p) \xrightarrow{\partial} \mathrm{TC}_{q-1}^*(A; p) \rightarrow \dots$$

and a natural short-exact sequence

$$0 \rightarrow R^1 \lim \mathrm{TC}_{q+1}^*(A; p) \rightarrow \mathrm{TC}_q(A; p) \rightarrow \lim \mathrm{TC}_q^*(A; p) \rightarrow 0.$$

Here we use the restriction map as the structure map for the pro-systems, but we could just as well have used the Frobenius map. We shall use the notation $\mathrm{TF}_q^*(A; p)$ to denote the pro-abelian group consisting of the groups $\mathrm{TR}_q^n(A; p)$, for $n \geq 1$, with the Frobenius as the structure map.

3.8

The de Rham–Witt Complex

The Hochschild homology groups $\mathrm{HH}_*(A)$ form a differential graded ring, with the differential given by Connes’ operator, and there is a canonical ring homomorphism $\lambda: A \rightarrow \mathrm{HH}_0(A)$. The de Rham complex Ω_A^* is the initial example of this algebraic structure, and hence there is a canonical map

$$\lambda: \Omega_A^q \rightarrow \mathrm{HH}_q(A).$$

Analogously, the groups $\mathrm{TR}_*^*(A; p)$ form a more complex algebraic structure called a Witt complex [27]. The de Rham–Witt complex $W.\Omega_A^*$ is the initial example of this algebraic structure, and hence there is a canonical map

$$\lambda: W_n.\Omega_A^q \rightarrow \mathrm{TR}_q^n(A; p).$$

The construction of $W.\Omega_A^*$ was given first for \mathbb{F}_p -algebras by Bloch–Deligne–Illusie [5, 35], who also showed that for k a perfect field of characteristic p and $X \rightarrow \mathrm{Spec} k$ smooth, there is a canonical isomorphism

$$H^q(X, W.\Omega_X^*) \xrightarrow{\sim} H_{\mathrm{crys}}^q(X/W(k))$$

of the (hyper-)cohomology of X with coefficients in $W.\Omega_X^*$ and the crystalline cohomology of X over $\mathrm{Spec} W(k)$ defined by Berthelot–Grothendieck [3]. Recent work by Madsen and the author [27, 31] has shown that the construction can be naturally extended to $\mathbb{Z}_{(p)}$ -algebras (where p is odd) and that the extended construction is strongly related to the p -adic K -theory of local number fields. See also [19, 26].

Suppose either that A is a $\mathbb{Z}_{(p)}$ -algebra with p odd or an \mathbb{F}_2 -algebra. Then we define a Witt complex over A to be the following (i)–(iii).

(i) A pro-differential graded ring E^* and a strict map of pro-rings

$$\lambda: W.(A) \rightarrow E^0$$

from the pro-ring of Witt vectors in A .

(ii) A strict map of pro-graded rings

$$F: E_i^* \rightarrow E_{i-1}^*$$

such that $\lambda F = F\lambda$ and such that for all $a \in A$,

$$Fd\lambda([a]_n) = \lambda([a]_{n-1})^{p-1} d\lambda([a]_{n-1}),$$

where $[a]_n = (a, 0, \dots, 0) \in W_n(A)$ is the multiplicative representative.

(iii) A strict map of graded E_i^* -modules

$$V: F_*E_{i-1}^* \rightarrow E_i^*$$

such that $\lambda V = V\lambda$ and such that $FdV = d$ and $FV = p$.

A map of Witt complexes over A is a strict map $f: E^* \rightarrow E'^*$ of pro-differential graded rings such that $\lambda' = f\lambda$, $F'f = fF$ and $V'f = fV$. We call F the Frobenius, V the Verschiebung, and the structure map of the pro-differential graded ring the restriction.

It is proved in [27, theorem A] that there exists an initial Witt complex over A , $W.\Omega_A^*$, and that the canonical map $\Omega_{W_n(A)}^q \rightarrow W_n.\Omega_A^q$ is surjective. The following result, which we will be need below, is [27, theorem B].

6 **Theorem 6** Suppose either that A is a $\mathbb{Z}_{(p)}$ -algebra, with p an odd prime, or an \mathbb{F}_2 -algebra. Then every element $\omega^{(n)} \in W_n \Omega_{A[t]}^q$ can be written uniquely as a (direct) sum

$$\begin{aligned} \omega^{(n)} = & \sum_{j \in \mathbb{N}_0} a_{0,j}^{(n)} [t]_n^j + \sum_{j \in \mathbb{N}} b_{0,j}^{(n)} [t]_n^{j-1} d[t]_n \\ & + \sum_{s=1}^{n-1} \sum_{j \in I_p} \left(V^s \left(a_{s,j}^{(n-s)} [t]_{n-s}^j \right) + dV^s \left(b_{s,j}^{(n-s)} [t]_{n-s}^j \right) \right) \end{aligned}$$

with $a_{s,j}^{(n-s)} \in W_{n-s} \Omega_A^q$ and $b_{s,j}^{(n-s)} \in W_{n-s} \Omega_A^{q-1}$.

If A is a regular \mathbb{F}_p -algebra, then the structure of the groups $W_n \Omega_A^q$ is well understood [35]. There is a multiplicative descending filtration by the differential graded ideals given by

$$\text{Fil}^s W_n \Omega_A^q = V^s W_{n-s} \Omega^q + dV^s W_{n-s} \Omega_A^{q-1}.$$

The filtration has length n and the filtration quotients can be expressed in terms of the de Rham complex of A and the Cartier operator [35, I.3.9].

3.9 Cyclic Homology of $A[x]/(x^e)$

As we recalled in the introduction, there is a natural isomorphism

$$K_q(A[x]/(x^e), (x)) \otimes \mathbb{Q} \approx \text{HC}_{q-1}(A[x]/(x^e), (x)) \otimes \mathbb{Q}.$$

We use the calculation of the \mathbb{T} -equivariant homotopy type of $N^{\text{cy}}(II_e)$ to derive a formula that expresses the left-hand groups in terms of the rational Hochschild homology groups of the ring A .

7 **Proposition 7** There is a natural isomorphism, valid for all rings A ,

$$\bigoplus_{i \in \mathbb{N} \setminus e\mathbb{N}} \text{HH}_{q-2d}(A) \otimes \mathbb{Q} \xrightarrow{\sim} \text{HC}_q(A[x]/(x^e), (x)) \otimes \mathbb{Q},$$

where $d = [(i-1)/e]$.

Proof We recall that the composite

$$H(A) \wedge N^{\text{cy}}(II_e) \rightarrow H(A[x]/(x^e)) \wedge N^{\text{cy}}(A[x]/(x^e)) \xrightarrow{\mu} H(A[x]/(x^e))$$

is an equivalence of \mathbb{T} -spectra. Hence, theorem 5 gives rise to an exact triangle of \mathbb{T} -spectra

$$\begin{aligned} \bigvee_{i \in e\mathbb{N}} H(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i|e+} &\xrightarrow{\text{id} \wedge \text{pr}} \bigvee_{i \in \mathbb{N}} H(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i+} \\ &\rightarrow H(A[x]/(x^e), (x)) \xrightarrow{\partial} \bigvee_{i \in e\mathbb{N}} H(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i|e+}[-1], \end{aligned}$$

which induces an exact triangle of \mathbb{T} -group homology spectra. The associated long-exact sequence of homotopy groups takes the form

$$\begin{aligned} \cdots \rightarrow \bigoplus_{i \in e\mathbb{N}} \mathbb{H}_q(\mathbb{T}, H(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i|e+}) &\rightarrow \bigoplus_{i \in \mathbb{N}} \mathbb{H}_q(\mathbb{T}, H(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i+}) \\ &\rightarrow \text{HC}_q(A[x]/(x^e), (x)) \xrightarrow{\partial} \bigoplus_{i \in e\mathbb{N}} \mathbb{H}_{q-1}(\mathbb{T}, H(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i|e+}) \rightarrow \cdots \end{aligned}$$

There is a natural isomorphism

$$\mathbb{H}_q(\mathbb{T}, H(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i+}) \xleftarrow{\sim} \mathbb{H}_q(C_i, H(A) \wedge S^{\lambda_d}) .$$

Moreover, the edge-homomorphism

$$H_0(C_i, \pi_q(H(A) \wedge S^{\lambda_d})) \rightarrow \mathbb{H}_q(C_i, T(A) \wedge S^{\lambda_d})$$

of the spectral sequence

$$E_{s,t}^2 = H_s(C_i, \pi_t(H(A) \wedge S^{\lambda_d}) \otimes \mathbb{Q}) \Rightarrow \mathbb{H}_{s+t}(C_i, H(A) \wedge S^{\lambda_d}) \otimes \mathbb{Q}$$

is an isomorphism. Indeed, for every C_i -module M , the composition

$$H_s(C_i, M) \xrightarrow{f^*} H_s(\{1\}, M) \xrightarrow{f_*} H_s(C_i, M)$$

is equal to multiplication by $i = [C_i : \{1\}]$. In the case at hand, this map is an isomorphism. But $H_s(\{1\}, M)$ is zero, for $s > 0$, and therefore also $H_s(C_i, M)$ is zero, for $s > 0$. Next, the action by C_i on $H(A) \wedge S^{\lambda_d}$ extends to an action by \mathbb{T} , and therefore, it induces the trivial action on homotopy groups. Hence, we further have an isomorphism

$$\text{HH}_{t-2d}(A) = \pi_t(H(A) \wedge S^{\lambda_d}) \xrightarrow{\sim} H_0(C_i, \pi_t(H(A) \wedge S^{\lambda_d})) .$$

Finally, one sees in a similar manner that, after tensoring with \mathbb{Q} , the map

$$\text{pr}_* : \mathbb{H}_q(\mathbb{T}, H(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i|e+}) \rightarrow \mathbb{H}_q(\mathbb{T}, H(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i+})$$

becomes an isomorphism. Hence, after tensoring with \mathbb{Q} , the top left-hand term of the long-exact sequence above maps isomorphically onto the direct summands of the top right-hand term indexed by $i \in e\mathbb{N}$. The proposition follows.

Topological Cyclic Homology of $A[x]/(x^e)$

We recall that for every ring A and every prime p , the cyclotomic trace induces an isomorphism

$$K_q(A[x]/(x^e), (x), \mathbb{Z}_p) \xrightarrow{\sim} \mathrm{TC}_q(A[x]/(x^e), (x), \mathbb{Z}_p) .$$

In this paragraph, we evaluate the topological cyclic homology groups on the right in terms of the groups

$$\mathrm{TR}_{q-\lambda}^n(A; p) = [S^q \wedge \mathbb{T}/C_{p^{n-1+}}, T(A) \wedge S^\lambda]_{\mathbb{T}} .$$

Indeed, we shall prove the following formula.

Proposition 8 Let $e = p^v e'$ with e' prime to p , and let A be an \mathbb{F}_p -algebra. Then there is a natural long-exact sequence of abelian groups

$$\begin{aligned} \cdots \rightarrow \prod_{j \in e' I_p} \lim_R \mathrm{TR}_{q-1-\lambda_d}^{r-v}(A; p) &\xrightarrow{e' V^v} \prod_{j \in I_p} \lim_R \mathrm{TR}_{q-1-\lambda_d}^r(A; p) \\ \rightarrow \mathrm{TC}_q(A[x]/(x^e), (x); p) &\xrightarrow{\partial} \prod_{j \in e' I_p} \lim_R \mathrm{TR}_{q-2-\lambda_d}^{r-v}(A; p) \rightarrow \cdots \end{aligned}$$

where $d = [(p^{r-1}j - 1)/e]$. The analogous sequence for the homotopy groups with \mathbb{Z}/p^v -coefficients is valid for every ring A .

Let λ be a finite dimensional orthogonal \mathbb{T} -representation, and let λ' denote the \mathbb{T} -representation $\varphi_p^* \lambda^{C_p}$. Then the restriction map induces a map

$$R: \mathrm{TR}_{q-\lambda}^n(A; p) \rightarrow \mathrm{TR}_{q-\lambda'}^{n-1}(A; p) .$$

This is the structure map in the limits of proposition 8. We note that, if $j \in I_p$ and $r \in \mathbb{N}$, and if we let $d = [(p^{r-1}j - 1)/e]$ and $d' = [(p^{r-2}j - 1)/e]$, then $(\lambda_d)' = \lambda_{d'}$ as required. Moreover, by [29, theorem 2.2], there is a natural long-exact sequence

$$\cdots \rightarrow \mathbb{H}_q(C_{p^{n-1}}, T(A) \wedge S^\lambda) \rightarrow \mathrm{TR}_{q-\lambda}^n(A; p) \xrightarrow{R} \mathrm{TR}_{q-\lambda'}^{n-1}(A; p) \rightarrow \cdots$$

Since the left-hand term is zero, for $q < \dim_{\mathbb{R}}(\lambda)$, it follows that the map R in this sequence is an epimorphism, for $q \leq \dim_{\mathbb{R}}(\lambda)$, and an isomorphism, for $q < \dim_{\mathbb{R}}(\lambda)$. Hence, the limits of proposition 8 are attained. In addition, we see that the j th factor of the upper right-hand term of the long-exact sequence of proposition 8 is non-zero if and only if $q - 1 \geq 2[(j - 1)/e]$ and that the j th factor of the upper left-hand term is non-zero if and only if $q - 1 \geq 2[(p^v j - 1)/e]$. Hence, the products are finite in each degree q .

Proof (*Proof of proposition 8*) From proposition 3 and theorem 5, we get an exact triangle of \mathbb{T} -spectra

$$\begin{aligned} \bigvee_{i \in \mathbb{N}} T(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i|e^+} &\xrightarrow{\text{id} \wedge \text{pr}} \bigvee_{i \in \mathbb{N}} T(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i^+} \\ \rightarrow T(A[x]/(x^e), (x)) &\xrightarrow{\partial} \bigvee_{i \in \mathbb{N}} T(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i|e^+}[-1], \end{aligned} \tag{3.16}$$

where $d = [(i - 1)/e]$. We wish to evaluate the map of homotopy groups of $C_{p^{n-1}}$ -fixed points induced by the map of \mathbb{T} -spectra in the top line. We first consider the top right-hand term in (3.16). By re-indexing after the p -adic valuation of $i \in \mathbb{N}$, this term can be rewritten as

$$\bigvee_{j \in \mathbb{N}} T(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{p^{n-1}j^+} \vee \bigvee_{r=1}^{n-1} \bigvee_{j \in I_p} T(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{p^{r-1}j^+},$$

and hence the $C_{p^{n-1}}$ -fixed point are expressed as a wedge sum

$$\begin{aligned} &\bigvee_{j \in \mathbb{N}} \varrho_{p^{n-1}}^* (T(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{p^{n-1}j^+})^{C_{p^{n-1}}} \\ &\vee \bigvee_{r=1}^{n-1} \bigvee_{j \in I_p} \varrho_{p^{n-r}}^* \left(\varrho_{p^{r-1}}^* (T(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{p^{r-1}j^+})^{C_{p^{r-1}}} \right)^{C_{p^{n-r}}}. \end{aligned}$$

Moreover, for every \mathbb{T} -spectrum T , there is a natural equivalence of \mathbb{T} -spectra

$$\varrho_{p^m}^* T^{C_{p^m}} \wedge \varrho_{p^m}^* (\mathbb{T}/C_{p^m j^+})^{C_{p^m}} \xrightarrow{\sim} \varrho_{p^m}^* (T \wedge \mathbb{T}/C_{p^m j^+})^{C_{p^m}},$$

and the p^m th root defines a \mathbb{T} -equivariant homeomorphism

$$\mathbb{T}/C_{j^+} \xrightarrow{\sim} \varrho_{p^m}^* (\mathbb{T}/C_{p^m j^+})^{C_{p^m}}.$$

Hence, the wedge sum above is canonically equivalent to the following wedge sum of \mathbb{T} -spectra.

$$\begin{aligned} &\bigvee_{j \in \mathbb{N}} \varrho_{p^{n-1}}^* (T(A) \wedge S^{\lambda_d})^{C_{p^{n-1}}} \wedge \mathbb{T}/C_{j^+} \\ &\vee \bigvee_{r=1}^{n-1} \bigvee_{j \in I_p} \varrho_{p^{n-r}}^* \left(\varrho_{p^{r-1}}^* (T(A) \wedge S^{\lambda_d})^{C_{p^{r-1}}} \wedge \mathbb{T}/C_{j^+} \right)^{C_{p^{n-r}}}. \end{aligned} \tag{3.17}$$

Here, in the top line, $i = p^{n-1}j$, and in the bottom line, $i = p^{r-1}j$, and in both cases, $d = [(i - 1)/e]$. The following easy result is [27, lemma 3.4.1].

Lemma 9 Let T be a \mathbb{T} -spectrum, let $j \in I_p$, and let $\iota: C_j/C_j \rightarrow \mathbb{T}/C_j$ be the canonical inclusion. Then the map

$$V^m \iota_* + dV^m \iota_*: \pi_q(T) \oplus \pi_{q-1}(T) \rightarrow \pi_q\left(\mathcal{C}_{p^m}^*(T \wedge \mathbb{T}/C_{j+})^{C_{p^m}}\right)$$

is an isomorphism.

The wedge decomposition (3.17) and lemma 9 gives rise to an isomorphism of the following direct sum onto the q th homotopy group of the top right-hand term of (3.16).

$$\begin{aligned} & \bigoplus_{j \in \mathbb{N}} \left(\mathrm{TR}_{q-\lambda_d}^n(A; p) \oplus \mathrm{TR}_{q-1-\lambda_d}^n(A; p) \right) \\ & \oplus \bigoplus_{j \in I_p} \bigoplus_{r=1}^{n-1} \left(\mathrm{TR}_{q-\lambda_d}^r(A; p) \oplus \mathrm{TR}_{q-1-\lambda_d}^r(A; p) \right). \end{aligned} \tag{3.18}$$

The same argument gives an isomorphism of the following direct sum onto the q th homotopy group of the top left-hand term of (3.16).

$$\begin{aligned} & \bigoplus_{j \in e'\mathbb{N}} \left(\mathrm{TR}_{q-\lambda_d}^m(A; p) \oplus \mathrm{TR}_{q-1-\lambda_d}^m(A; p) \right) \\ & \oplus \bigoplus_{j \in e'I_p} \bigoplus_{r=v+1}^{n-1} \left(\mathrm{TR}_{q-\lambda_d}^{r-v}(A; p) \oplus \mathrm{TR}_{q-1-\lambda_d}^{r-v}(A; p) \right). \end{aligned} \tag{3.19}$$

Here in the top line $m = \min\{n, n - v + v_p(j)\}$. Moreover, the map of q th homotopy groups induced by the map $\mathrm{id} \wedge \mathrm{pr}$ in (3.16) preserves the indices of the direct sum decompositions (3.18) and (3.19) of the q th homotopy groups of the target and domain. It is given on the summands in the bottom lines of (3.18) and (3.19) by the following maps [29, lemma 8.1].

$$\begin{aligned} V^v: \mathrm{TR}_{q-\lambda_d}^{r-v}(A; p) &\rightarrow \mathrm{TR}_{q-\lambda_d}^r(A; p), \\ e'V^v: \mathrm{TR}_{q-1-\lambda_d}^{r-v}(A; p) &\rightarrow \mathrm{TR}_{q-1-\lambda_d}^r(A; p). \end{aligned}$$

We now assume that A is an \mathbb{F}_p -algebra and consider the groups in (3.18) for varying $n \geq 1$ as a pro-abelian group with structure map given by the Frobenius map. The Frobenius map takes the summand with index $j \in \mathbb{N}$ in the top line of (3.18) for n to the summand with index $pj \in \mathbb{N}$ in the top line of (3.18) for $n - 1$. It takes the summand with index $j \in I_p$ and $1 \leq r < n - 1$ in the bottom line of (3.18) for n to the summand with the same indices in the bottom line of (3.18) for $n - 1$. Finally, it takes the summand with index $j \in I_p$ and $r = n - 1$ in the bottom line of (3.18) for n to the summand with index $j \in \mathbb{N}$ in the top line of (3.18) for $n - 1$. It follows that the projection onto the quotient-pro-abelian group given by the bottom line of (3.18) is an isomorphism of pro-abelian groups. Indeed, the sub-pro-abelian group given by the upper line of *loc.cit.* is Mittag-Leffler zero since

the sum is finite. The value of the Frobenius map on the summands in the bottom line of (3.18) follows immediately from lemma 9 and the relations $FV = p$ and $FdV = d$. We find that

$$F = p: \text{TR}_{q-\lambda_d}^r(A; p) \rightarrow \text{TR}_{q-\lambda_d}^r(A; p),$$

$$F = \text{id}: \text{TR}_{q-1-\lambda_d}^r(A; p) \rightarrow \text{TR}_{q-1-\lambda_d}^r(A; p).$$

It follows that the pro-abelian group with degree n term given by the direct sum (3.18) and with structure map given by the Frobenius is canonically isomorphic to the pro-abelian group with degree n term the direct sum

$$\bigoplus_{r=1}^{n-1} \bigoplus_{j \in I_p} \text{TR}_{q-1-\lambda_d}^r(A; p)$$

and with structure map the canonical projection. A similar argument shows that the pro-abelian group with degree n term given by the direct sum (3.19) and with structure map given by the Frobenius is canonically isomorphic to the pro-abelian group with degree n term the direct sum

$$\bigoplus_{r=v+1}^{n-1} \bigoplus_{j \in e'I_p} \text{TR}_{q-1-\lambda_d}^{r-v}(A; p)$$

and with structure map the canonical projection. Hence, we have a long-exact sequence of pro-abelian groups with degree n terms

$$\begin{aligned} \cdots \rightarrow \bigoplus_{r=v+1}^{n-1} \bigoplus_{j \in e'I_p} \text{TR}_{q-1-\lambda_d}^{r-v}(A; p) &\xrightarrow{e'V^v} \bigoplus_{r=1}^{n-1} \bigoplus_{j \in I_p} \text{TR}_{q-1-\lambda_d}^r(A; p) \\ &\rightarrow \text{TF}_q^n(A[x]/(x^e), (x); p) \xrightarrow{\partial} \bigoplus_{r=v+1}^{n-1} \bigoplus_{j \in e'I_p} \text{TR}_{q-2-\lambda_d}^{r-v}(A; p) \rightarrow \cdots \end{aligned} \tag{3.20}$$

and with the structure map given by the canonical projection in the two terms of the upper line and in the right-hand term of the lower line and by the Frobenius map in the left-hand term of the lower line. In particular, the pro-abelian group $\text{TF}_q^n(A[x]/(x^e), (x); p)$ satisfies the Mittag-Leffler condition, so the derived limit vanishes. Hence, we have an isomorphism

$$\text{TF}_q(A[x]/(x^e), (x); p) \xrightarrow{\sim} \lim_F \text{TF}_q^n(A[x]/(x^e), (x); p) ,$$

and the long-exact sequence (3.20) induces a long-exact sequence

$$\begin{aligned} \cdots \rightarrow \prod_{r \geq v+1} \prod_{j \in e'I_p} \text{TR}_{q-1-\lambda_d}^{r-v}(A; p) &\xrightarrow{e'V^v} \prod_{r \geq 1} \prod_{j \in I_p} \text{TR}_{q-1-\lambda_d}^r(A; p) \\ &\rightarrow \text{TF}_q(A[x]/(x^e), (x); p) \xrightarrow{\partial} \prod_{r \geq v+1} \prod_{j \in e'I_p} \text{TR}_{q-2-\lambda_d}^{r-v}(A; p) \rightarrow \cdots \end{aligned} \tag{3.21}$$

of the limits. Finally, the restriction map induces a self-map of the sequence (3.21) which on the two terms of the upper line and the right-hand term of the lower line is given by the map

$$R: \mathrm{TR}_{q-1-\lambda_d}^r(A; p) \rightarrow \mathrm{TR}_{q-1-\lambda_d'}^{r-1}(A; p) .$$

As we remarked after the statement of proposition 8, this map is an isomorphism for all but finitely many r . It follows that the map

$$\mathrm{TF}_q(A[x]/(x^e), (x); p) \xrightarrow{R-\mathrm{id}} \mathrm{TF}_q(A[x]/(x^e), (x); p)$$

is surjective and identifies $\mathrm{TC}_q(A[x]/(x^e), (x); p)$ with the kernel. Indeed, the self-map $R - \mathrm{id}$ of the sequence (3.21) is a split surjection with compatible sections on the remaining terms. Finally, the long-exact sequence of the statement is obtained as the long-exact sequence of kernels of the self-map $R - \mathrm{id}$ of the sequence (3.21).

3.11

The Characteristic Zero Case

We use the description of the cyclic homology and topological cyclic homology from the paragraphs above to prove the following result.

Theorem 10 Suppose that A is a regular noetherian ring and a \mathbb{Q} -algebra. Then there is a natural isomorphism of abelian groups

$$K_{q-1}(A[x]/(x^e), (x)) \xleftarrow{\sim} \bigoplus_{m \geq 1} \left(\Omega_A^{q-2m} \right)^{e-1} ,$$

where the superscript $e - 1$ indicates product.

Proof We first show that the relative K -groups with \mathbb{Z}_p -coefficients are zero. By McCarthy [47], the cyclotomic trace induces an isomorphism

$$K_q(A[x]/(x^e), (x), \mathbb{Z}_p) \xrightarrow{\sim} \mathrm{TC}_q(A[x]/(x^e), (x); p, \mathbb{Z}_p)$$

and for every spectrum X , there is a natural short-exact sequence

$$0 \rightarrow R^1 \lim_{\nu} \pi_{q+1}(X, \mathbb{Z}/p^\nu) \rightarrow \pi_q(X, \mathbb{Z}_p) \rightarrow \lim_{\nu} \pi_q(X, \mathbb{Z}/p^\nu) \rightarrow 0 .$$

Hence, by proposition 8, it suffices to show that the groups $\mathrm{TR}_{q-\lambda}^n(A; p, \mathbb{Z}/p^\nu)$ are zero. Moreover, there is a natural long-exact sequence

$$\dots \rightarrow \mathbb{H}_q(C_{p^{n-1}}, T(A) \wedge S^\lambda) \rightarrow \mathrm{TR}_{q-\lambda}^n(A; p) \xrightarrow{R} \mathrm{TR}_{q-\lambda}^{n-1}(A; p) \rightarrow \dots$$

and a natural spectral sequence

$$E_{s,t}^2 = H_s(C_{p^{n-1}}, \pi_t(T(A) \wedge S^\lambda)) \Rightarrow \mathbb{H}_{s+t}(C_{p^{n-1}}, T(A) \wedge S^\lambda) .$$

The same is true for the homotopy groups with \mathbb{Z}/p^v -coefficients. Hence, it will be enough to show that the groups

$$\pi_r(T(A) \wedge S^\lambda, \mathbb{Z}/p^v) = \mathrm{THH}_{t-\dim_{\mathbb{R}}(\lambda)}(A, \mathbb{Z}/p^v)$$

are zero. But these groups are at the same time A -modules and annihilated by p^v . Therefore, they are zero, for every \mathbb{Q} -algebra A . It follows that in the arithmetic square (which is homotopy-cartesian [8])

$$\begin{array}{ccc} K(A[x]/(x^e), (x)) & \longrightarrow & \prod_p K(A[x]/(x^e), (x))_p^\wedge \\ \downarrow & & \downarrow \\ K(A[x]/(x^e), (x))_{\mathbb{Q}} & \longrightarrow & \left(\prod_p K(A[x]/(x^e), (x))_p^\wedge\right)_{\mathbb{Q}} \end{array} \tag{3.22}$$

the spectra on the right are trivial, and hence the left-hand vertical map is an equivalence. Hence, the canonical maps

$$K_q(A[x]/(x^e), (x)) \rightarrow \mathrm{HC}_q^-(A[x]/(x^e), (x)) \leftarrow \mathrm{HC}_{q-1}(A[x]/(x^e), (x))$$

are isomorphisms and the common group uniquely divisible. The statement now follows from proposition 7 and the fact that the canonical map

$$\Omega_A^* \rightarrow \mathrm{HH}_*(A)$$

is an isomorphism. The latter is true for A a smooth \mathbb{Q} -algebra by Hochschild–Kostant–Rosenberg [32], and the general case follows from Popescu [50]. Indeed, the domain and target both commute with filtered colimits.

The Groups $\mathrm{TR}_{q-\lambda}^n(A; p)$

In this paragraph, we evaluate the groups

$$\mathrm{TR}_{q-\lambda}^n(A; p) = [S^q \wedge \mathbb{T}/C_{p^{n-1+}}, T(A) \wedge S^\lambda]_{\mathbb{T}}$$

for A a regular \mathbb{F}_p -algebra. In the basic case of the field \mathbb{F}_p , or, more generally, a perfect field of characteristic p , the groups were evaluated in [29, proposition 9.1]. To state the result, we define

$$\ell_s = \dim_{\mathbb{C}}(\mathcal{A}^{C_{p^s}}),$$

for all $s \geq 0$, and $\ell_s = \infty$, for $s < 0$. Then there is a canonical isomorphism of graded abelian groups

$$\bigoplus_{\ell_{n-r} \leq m < \ell_{n-1-r}} W_r(k)[-2m] \xrightarrow{\sim} \mathrm{TR}_{*-\lambda}^n(k; p),$$

where the sum is over all integers m , and where r is the unique integer such that $\ell_{n-r} \leq m < \ell_{n-1-r}$. Equivalently, the group $\mathrm{TR}_{q-\lambda}^n(k; p)$ is isomorphic to $W_r(k)$, for $q = 2m$ and $\ell_{n-r} \leq m < \ell_{n-1-r}$, and is zero, for q is odd.

The groups $\mathrm{TR}_{*-\lambda}^n(A; p)$ naturally form a differential graded module over the differential graded ring $\mathrm{TR}_*^n(A; p)$. Hence, we have a natural pairing

$$W_n \Omega_A^* \otimes_{W_n(k)} \mathrm{TR}_{*-\lambda}^n(k; p) \rightarrow \mathrm{TR}_{*-\lambda}^n(A; p) . \tag{3.23}$$

11 Theorem 11 Let A be a regular \mathbb{F}_p -algebra, and let λ be a finite dimensional complex \mathbb{T} -representation. Then the pairing (3.23) induces an isomorphism of graded abelian groups

$$\bigoplus_{\ell_{n-r} \leq m < \ell_{n-1-r}} W_r \Omega_A^*[-2m] \xrightarrow{\sim} \mathrm{TR}_{*-\lambda}^n(A; p) .$$

Proof The domain and target of the map of the statement both commute with filtered colimits. Hence, by Popescu [50], we can assume that A is a smooth k -algebra. Moreover, if $f: A \rightarrow A'$ is an étale map of k -algebras, then the canonical map

$$W_n(A') \otimes_{W_n(A)} \mathrm{TR}_{q-\lambda}^n(A; p) \rightarrow \mathrm{TR}_{q-\lambda}^n(A'; p)$$

is an isomorphism. The analogous statement holds for the domain of the map of the statement. By a covering argument, we are reduced to consider the case where A is a polynomial algebra over k in a finite number of variables, see [30, lemma 2.2.8] for details. Hence, it suffices to show that the statement for A implies the statement for $A' = A[t]$.

The following result follows by an argument similar to [27, theorem C].

12 Proposition 12 Let λ be a finite dimensional orthogonal \mathbb{T} -representation, and let A be a $\mathbb{Z}_{(p)}$ -algebra. Then every element $\omega^{(n)} \in \mathrm{TR}_{q-\lambda}^n(A[t]; p)$ can be written uniquely as a (direct) sum

$$\begin{aligned} \omega^{(n)} = & \sum_{j \in \mathbb{N}_0} a_{0,j}^{(n)} [t]_n^j + \sum_{j \in \mathbb{N}} b_{0,j}^{(n)} [t]_n^{j-1} d[t]_n \\ & + \sum_{s=1}^{n-1} \sum_{j \in I_p} \left(V^s \left(a_{s,j}^{(n-s)} [t]_{n-s}^j \right) + dV^s \left(b_{s,j}^{(n-s)} [t]_{n-s}^j \right) \right) \end{aligned}$$

with $a_{s,j}^{(n-s)} \in \mathrm{TR}_{q-\lambda}^{n-s}(A; p)$ and $b_{s,j}^{(n-s)} \in \mathrm{TR}_{q-1-\lambda}^{n-s}(A; p)$.

Proof We outline the proof. The group $\mathrm{TR}_{q-\lambda}^n(A[t]; p)$ is the q th homotopy group of the \mathbb{T} -spectrum

$$\mathfrak{e}_{p^{n-1}}^* (T(A[t]) \wedge S^\lambda)^{C_{p^{n-1}}} \tag{3.24}$$

Let $\Pi = \{0, 1, t, t^2, \dots\}$ be the sub-pointed monoid of $A[t]$ generated by the variable. We recall from proposition 3 that the composite

$$T(A) \wedge N^{\mathrm{cy}}(\Pi) \xrightarrow{f \wedge 1} T(A[t]) \wedge N^{\mathrm{cy}}(A[t]) \xrightarrow{\mu} T(A[t])$$

is an \mathcal{F} -equivalence of \mathbb{T} -spectra. Moreover, the \mathbb{T} -space $N^{\mathrm{cy}}(\Pi)$ decomposes as a wedge sum

$$\bigvee_{i \in \mathbb{N}_0} N^{\mathrm{cy}}(\Pi, i) \xrightarrow{\sim} N^{\mathrm{cy}}(\Pi),$$

where $N^{\mathrm{cy}}(\Pi, i)$ is the realization of the cyclic subset of $N^{\mathrm{cy}}(\Pi)$ generated by the 0-simplex 1, if $i = 0$, and by the $(i - 1)$ -simplex $t \wedge \dots \wedge t$, if $i > 0$. Hence, the \mathbb{T} -spectrum (3.24) can be expressed, up to \mathcal{F} -equivalence, as a wedge sum

$$\begin{aligned} & \bigvee_{j \in \mathbb{N}_0} \mathfrak{e}_{p^{n-1}}^* (T(A) \wedge S^\lambda \wedge N^{\mathrm{cy}}(\Pi, p^{n-1}j))^{C_{p^{n-1}}} \\ & \vee \bigvee_{s=1}^{n-1} \bigvee_{j \in I_p} \mathfrak{e}_{p^s}^* \left(\mathfrak{e}_{p^{n-1-s}}^* (T(A) \wedge S^\lambda \wedge N^{\mathrm{cy}}(\Pi, p^{n-1-s}j))^{C_{p^{n-1-s}}} \right)^{C_{p^s}}. \end{aligned}$$

In addition, there is a natural equivalence of \mathbb{T} -spectra

$$\begin{aligned} & \mathfrak{e}_{p^m}^* (T(A) \wedge S^\lambda)^{C_{p^m}} \wedge \mathfrak{e}_{p^m}^* N^{\mathrm{cy}}(\Pi, p^m j)^{C_{p^m}} \\ & \xrightarrow{\sim} \mathfrak{e}_{p^m}^* (T(A) \wedge S^\lambda \wedge N^{\mathrm{cy}}(\Pi, p^m j))^{C_{p^m}}, \end{aligned}$$

and a \mathbb{T} -equivariant homeomorphism

$$\Delta: N^{\mathrm{cy}}(\Pi, j) \xrightarrow{\sim} \mathfrak{e}_{p^m}^* N^{\mathrm{cy}}(\Pi, p^m j)^{C_{p^m}}.$$

Hence, we obtain the following wedge decomposition, up to \mathcal{F} -equivalence, of the \mathbb{T} -spectrum (3.24).

$$\begin{aligned} & \bigvee_{j \in \mathbb{N}_0} \mathfrak{e}_{p^{n-1}}^* (T(A) \wedge S^\lambda)^{C_{p^{n-1}}} \wedge N^{\mathrm{cy}}(\Pi, j) \\ & \vee \bigvee_{s=1}^{n-1} \bigvee_{j \in I_p} \mathfrak{e}_{p^s}^* \left(\mathfrak{e}_{p^{n-1-s}}^* (T(A) \wedge S^\lambda)^{C_{p^{n-1-s}}} \wedge N^{\mathrm{cy}}(\Pi, j) \right)^{C_{p^s}}. \end{aligned}$$

We claim that the induced direct sum decomposition of the q th homotopy group corresponds to the direct sum decomposition of the statement. To prove this, one first proves that the map

$$\mathrm{TR}_{*-\lambda}^n(A; p) \otimes \Omega_{\mathbb{Z}[t]}^* \rightarrow \mathrm{TR}_{*-\lambda}^n(A[t]; p)$$

that takes $a \otimes t^j$ to $f(a)[t]_n^j$ and $a \otimes t^{j-1} dt$ to $f(a)[t]_n^{j-1} d[t]_n$ is an isomorphism onto the direct summand $\pi_*(\mathcal{O}_{p^{n-1}}^*(T(A) \wedge S^\lambda)^{C_{p^{n-1}}} \wedge N^{\mathrm{cy}}(\Pi))$. We refer to [27, lemma 3.3.1] for the proof. Secondly, one shows that if T is any \mathbb{T} -spectrum, and if $j \in I_p$, then the map

$$V^s \iota + dV^s \iota: \pi_q(T) \oplus \pi_{q-1}(T) \rightarrow \pi_q(\mathcal{O}_{p^s}^*(T \wedge \mathbb{T}/C_{j^+})^{C_{p^s}})$$

is an isomorphism. Here $\iota: C_j/C_j \rightarrow \mathbb{T}/C_j$ is the canonical inclusion. This is lemma 9. This completes our outline of the proof of the proposition.

As we recalled in theorem 6 above, the de Rham–Witt groups of $A[t]$ can be similarly expressed in terms of those of A . We can now complete the proof of theorem 11. Let $E_n^q(A)$ denote the left-hand side of the statement. We claim that every element $\omega^{(n)} \in E_n^q(A[t])$ can be written uniquely as a direct sum

$$\begin{aligned} \omega^{(n)} &= \sum_{j \in \mathbb{N}_0} a_{0,j}^{(n)} [t]_n^j + \sum_{j \in \mathbb{N}} b_{0,j}^{(n)} [t]_n^{j-1} d[t]_n \\ &\quad + \sum_{s=1}^{n-1} \sum_{j \in I_p} \left(V^s \left(a_{s,j}^{(n-s)} [t]_{n-s}^j \right) + dV^s \left(b_{s,j}^{(n-s)} [t]_{n-s}^j \right) \right) \end{aligned}$$

with $a_{s,j}^{(n-s)} \in E_{n-s}^q(A)$ and $b_{s,j}^{(n-s)} \in E_{n-s}^{q-1}(A)$, or equivalently, that the map

$$\bigoplus_{j \in \mathbb{N}_0} E_n^q(A) \oplus \bigoplus_{j \in \mathbb{N}} E_n^{q-1}(A) \oplus \bigoplus_{s=1}^{n-1} \bigoplus_{j \in I_p} (E_{n-s}^q(A) \oplus E_{n-s}^{q-1}(A)) \rightarrow E_n^q(A[t])$$

given by this formula is an isomorphism. On the one-hand, by theorem 6, the right-hand side is given by the direct sum

$$\begin{aligned} &\bigoplus_{\ell_{n-r} \leq m < \ell_{n-1-r}} \left(\bigoplus_{j \in \mathbb{N}_0} W_{r-s} \Omega_A^{q-2m} \oplus \bigoplus_{j \in \mathbb{N}} W_{r-s} \Omega_A^{q-1-2m} \right) \oplus \\ &\bigoplus_{\ell_{n-r} \leq m < \ell_{n-1-r}} \left(\bigoplus_{s=1}^{r-1} \bigoplus_{j \in I_p} \left(W_{r-s} \Omega_A^{q-2m} \oplus W_{r-s} \Omega_A^{q-1-2m} \right) \right), \end{aligned} \tag{3.25}$$

and on the other-hand, by the definition of $E_n^q(A)$, the left-hand side is given by the direct sum

$$\begin{aligned} & \bigoplus_{j \in \mathbb{N}_0} \bigoplus_{\ell_{n-r} \leq m < \ell_{n-1-r}} W_r \Omega_A^{q-2m} \oplus \bigoplus_{j \in \mathbb{N}} \bigoplus_{\ell_{n-r} \leq m < \ell_{n-1-r}} W_r \Omega_A^{q-1-2m} \oplus \\ & \bigoplus_{s=1}^{n-1} \bigoplus_{j \in I_p} \bigoplus_{\ell_{n-s-r} \leq m < \ell_{n-s-1-r}} (W_r \Omega_A^{q-2m} \oplus W_r \Omega_A^{q-1-2m}) . \end{aligned} \tag{3.26}$$

The top lines in (3.25) and (3.26) clearly are isomorphic, and the bottom lines both are seen to be isomorphic to the direct sum

$$\bigoplus_{r=1}^n \bigoplus_{\ell_{n-1-r} \leq m} \bigoplus_{j \in I_p} (W_r \Omega_A^{q-2m} \oplus W_r \Omega_A^{q-1-2m}) .$$

This proves the claim.

The Positive Characteristic Case

3.13

The following result was proved by Madsen and the author in [28, 30] but stated there in terms of big de Rham–Witt differential forms.

Theorem 13 Suppose that A is a regular noetherian ring and an \mathbb{F}_p -algebra, and write $e = p^v e'$ with e' not divisible by p . Then there is a natural long-exact sequence of abelian groups

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$$\begin{aligned} \cdots \rightarrow \bigoplus_{m \geq 1} \bigoplus_{j \in e' I_p} W_{s-v} \Omega_A^{q-2m} \xrightarrow{e' V^v} \bigoplus_{m \geq 1} \bigoplus_{j \in I_p} W_s \Omega_A^{q-2m} \\ \rightarrow K_{q-1}(A[x]/(x^e), (x)) \xrightarrow{\partial} \bigoplus_{m \geq 1} \bigoplus_{j \in e' I_p} W_{s-v} \Omega_A^{q-1-2m} \rightarrow \cdots , \end{aligned}$$

where $s = s(m, j)$ is the unique integer such that $p^{s-1} j \leq me < p^s j$.

Proof We recall that by Goodwillie [22], there is a canonical isomorphism

$$K_q(A[x]/(x^e), (x)) \otimes \mathbb{Q} \approx \mathrm{HC}_{q-1}(A[x]/(x^e), (x)) \otimes \mathbb{Q} .$$

But proposition 7 shows that the groups on the left are zero. Indeed, the groups $\mathrm{HH}_*(A)$ are A -modules, and therefore, annihilated by p . If ℓ is a prime, then by McCarthy [47], the cyclotomic trace induces an isomorphism

$$K_q(A[x]/(x^e), (x), \mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{TC}_q(A[x]/(x^e), (x); \ell, \mathbb{Z}_\ell) ,$$

and for every spectrum X , there is a natural short-exact sequence

$$0 \rightarrow R^1 \lim_{\nu} \pi_{q+1}(X, \mathbb{Z}/\ell^\nu) \rightarrow \pi_q(X, \mathbb{Z}/\ell) \rightarrow \lim_{\nu} \pi_q(X, \mathbb{Z}/\ell^\nu) \rightarrow 0.$$

The groups $\mathrm{TC}_q(A[x]/(x^\ell), (x); \ell, \mathbb{Z}/\ell^\nu)$ are given by proposition 8 in terms of the groups $\mathrm{TR}_{q-\lambda}^n(A; \ell, \mathbb{Z}/\ell^\nu)$. We claim that the latter are zero, for $\ell \neq p$. In effect, we claim the slightly stronger statement that for every prime ℓ , the groups $\mathrm{TR}_{q-\lambda}^n(A; \ell)$ are p -groups of a bounded exponent (which depends on ℓ, q, n , and λ). The proof is by induction on $n \geq 1$ and uses the natural long-exact sequence

$$\cdots \rightarrow \mathbb{H}_q(C_{\ell^{n-1}}, T(A) \wedge S^\lambda) \rightarrow \mathrm{TR}_{q-\lambda}^n(A; \ell) \xrightarrow{R} \mathrm{TR}_{q-\lambda}^{n-1}(A; \ell) \rightarrow \cdots$$

and the natural spectral sequence

$$E_{s,t}^2 = H_s(C_{\ell^{n-1}}, \pi_t(T(A) \wedge S^\lambda)) \Rightarrow \mathbb{H}_{s+t}(C_{\ell^{n-1}}, T(A) \wedge S^\lambda).$$

Since the groups

$$\pi_t(T(A) \wedge S^\lambda) = \mathrm{THH}_{t-\dim_{\mathbb{R}}(\lambda)}(A)$$

are A -modules, and hence annihilated by p , the claim follows. We conclude from the arithmetic square (3.22) that the cyclotomic trace

$$K_q(A[x]/(x^\ell), (x)) \rightarrow \mathrm{TC}_q(A[x]/(x^\ell), (x); p)$$

is an isomorphism. The right-hand side can be read off from proposition 8 and theorem 11. For notational reasons we evaluate the group in degree $q - 1$ rather than the one in degree q . We find that the j th factor of the upper right-hand term of the long-exact sequence of proposition 8 is given by

$$\lim_R \mathrm{TR}_{q-2-\lambda_d}^r(A; p) \xleftarrow{\sim} \bigoplus_{m \geq 0} W_s \Omega_A^{q-2(m+1)},$$

where s is the unique integer that satisfies

$$[(p^{s-1}j - 1)/e] \leq m < [(p^s j - 1)/e]$$

or equivalently,

$$p^{s-1}j \leq (m + 1)e < p^s j.$$

Writing m instead of $m + 1$, we obtain the upper right-hand term of the long-exact sequence of the statement. The upper left-hand term is evaluated similarly. Finally, because the isomorphism of theorem 11 is induced by the pairing (3.23), the map V^ν in the long-exact sequence of proposition 8 induces the iterated Verschiebung V^ν of the de Rham–Witt complex. The theorem follows.

We first proved theorem 13 in [28] in the case of a perfect field k of characteristic p , and the extension to all regular noetherian \mathbb{F}_p -algebras was given in [30]. Our main motivation was that the more general case makes it possible to also evaluate the groups $NK_q(R) = K_q(R[t], (t))$, which occur in the fundamental theorem for non-regular rings, in the following situation.

Corollary 14 Let A be a regular noetherian ring that is also an \mathbb{F}_p -algebra, and write $e = p^v e'$ with e' prime to p . Then there is a natural long-exact sequence of abelian groups

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$$\begin{aligned} \cdots \rightarrow \bigoplus_{m \geq 1} \bigoplus_{j \in I_p} W_{s-v} \Omega_{(A[t], (t))}^{q-2m} &\xrightarrow{e' V^v} \bigoplus_{m \geq 1} \bigoplus_{j \in I_p} W_s \Omega_{(A[t], (t))}^{q-2m} \\ \rightarrow NK_{q-1}(A[x]/(x^e)) &\xrightarrow{\partial} \bigoplus_{m \geq 1} \bigoplus_{j \in I_p} W_{s-v} \Omega_{(A[t], (t))}^{q-1-2m} \rightarrow \cdots, \end{aligned}$$

where $s = s(m, j)$ is the unique integers such that $p^{s-1}j \leq me < p^s j$.

Proof We have a natural split-exact sequence

$$0 \rightarrow NK_q(A[x]/(x^e), (x)) \rightarrow NK_q(A[x]/(x^e)) \rightarrow NK_q(A) \rightarrow 0.$$

Since A is regular, the fundamental theorem shows that the right-hand term is zero. Hence, the left-hand map is an isomorphism, and the left-hand group is given by theorem 13.

We remark that Weibel [57] has shown that the groups $NK_*(A)$ are naturally modules over the big ring of Witt vectors $\mathbf{W}(A)$. We do not know if this module structure is compatible with the obvious $\mathbf{W}(A)$ -module structure on the remaining terms of the long-exact sequence of corollary 14.

Miscellaneous

3.14

As e varies, the spectra $K(A[x]/(x^e))$ are related by several maps. Let $e' = de$ and let $\pi_d: A[x]/(x^{e'}) \rightarrow A[x]/(x^e)$ and $\iota_d: A[x]/(x^e) \rightarrow A[x]/(x^{e'})$ be the maps of A -algebras that take x to x and x^d , respectively. Then there are maps of K -theory spectra

$$\begin{aligned} \pi_d^*, \iota_{d*}: K(A[x]/(x^{e'})) &\rightarrow K(A[x]/(x^e)), \\ \iota_d^*: K(A[x]/(x^e)) &\rightarrow K(A[x]/(x^{e'})), \end{aligned} \tag{3.27}$$

which are related in a manner similar to that of the restriction, Frobenius, and Verschiebung, respectively. It remains an unsolved problem to determine the value

of these maps under the isomorphisms of theorems 10 and 13 above. Another very interesting and open problem is to determine the multiplicative structure of the groups $K_*(A[x]/(x^e), \mathbb{Z}/p)$.

There are two limit cases, however, where the maps (3.27) are well understood. One case is a recent result by Betley–Schlichtkrull [4] which expresses topological cyclic homology in terms of K -theory of truncated polynomial algebras. To state it, let \mathbb{I} be the category with objects the set of positive integers and with morphisms generated by morphisms r_d and f_d from $e' = de$ to e , for all positive integers e and d , subject to the following relations.

$$r_1 = f_1 = \text{id}, \quad r_d r_{d'} = r_{dd'}, \quad f_d f_{d'} = f_{dd'}, \quad r_d f_{d'} = f_{d'} r_d.$$

Then there is a functor from the category \mathbb{I} to the category of symmetric spectra that to the object e assigns $K(A[x]/(x^e))$ and that to the morphisms r_d and f_d from $e' = de$ to e assign the maps π_d^* and ι_{d*} , respectively. The following result is [4, theorem 1.1].

15 Theorem 15 There is a natural weak equivalence of symmetric spectra

$$\text{TC}(A; p, \mathbb{Z}/p^v) \simeq \text{holim}_{\mathbb{I}} K(A[x]/(x^e), \mathbb{Z}/p^v) [1].$$

Another case concerns the limit over the maps π_d^* . This limit was considered first by Bloch [5] who used it to give a K -theoretical construction of the de Rham–Witt complex. This, in turn, led to the purely algebraic construction of the de Rham–Witt complex which we recalled in Sect. 3.8 above. The K -theoretical construction begins with the spectrum of curves on $K(A)$ defined to be the following homotopy limit with respect to the maps π_d^* .

$$C(A) = \text{holim}_e K(A[x]/(x^e), (x)) [1].$$

This is a ring spectrum in such a way that the ring of components is canonically isomorphic to the ring of big Witt vectors $W(A)$. If A is a $\mathbb{Z}_{(p)}$ -algebra, the ring $W(A)$ has a canonical idempotent decomposition as a product of copies of the ring of p -typical Witt vectors $W(A)$. These idempotents give rise to an analogous decomposition of the ring spectrum $C(A)$ as a product of copies of a ring spectrum $C(A; p)$ which is called the p -typical curves on $K(A)$. The following result was proved by the author in [25, theorem C]. The corresponding result for the symbolic part of K -theory was proved by Bloch [5] under the additional assumption that A be local of dimension less than p . The restriction on the dimension of A was removed by Kato [39].

16 Theorem 16 Let A be a smooth algebra over a perfect field of positive characteristic p . Then there are natural isomorphisms of abelian groups

$$C_*(A; p) \approx \text{TR}_*(A; p) \approx W\Omega_A^*.$$

Finally, we briefly discuss the K -theory of \mathbb{Z}/p^e relative to the ideal generated by p . The cyclotomic trace also induces an isomorphism

$$K_q(\mathbb{Z}/p^e, (p)) \xrightarrow{\sim} \mathrm{TC}_q(\mathbb{Z}/p^e, (p); p) ,$$

and hence one may attempt to evaluate the relative K -groups by evaluating the relative topological cyclic homology groups on the right. The following result of Brun [11, 12] was proved by these methods. It generalizes earlier results by Evens and Friedlander [15] and by Aisbett, Lluís-Puebla and Snaith [2].

Theorem 17 Suppose that $0 \leq q \leq p - 3$. Then $K_q(\mathbb{Z}/p^e, (p))$ is a cyclic group of order $p^{j(e-1)}$, if $q = 2j - 1$ is odd, and is zero, if q is even.

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We remark that the groups $K_q(\mathbb{F}_p[x]/(x^e), (x))$ and $K_q(\mathbb{Z}/p^e, (p))$ have the same order, for $0 \leq q \leq p - 3$. The order of the former group is given by the formula of theorem 17, for all non-negative integers q , but this cannot be true for the latter group. Indeed, by Suslin–Panin [48, 53], the canonical map

$$K_q(\mathbb{Z}_p, (p), \mathbb{Z}_p) \rightarrow \lim_e K_q(\mathbb{Z}/p^e, (p))$$

is an isomorphism, for all integers q . The left-hand group is non-zero, if q is even and divisible by $2(p - 1)$. Hence the groups $K_q(\mathbb{Z}/p^e, (p))$ cannot be zero for every even integer q and positive integer e . This comparison also shows that $K_q(\mathbb{Z}/p^e, (p))$ cannot be a cyclic group for every odd integer q and every positive integer e . But this also follows from the result of Geisser [18] that the group $K_3(\mathbb{Z}/9, (3))$ is isomorphic to $\mathbb{Z}/3 \oplus \mathbb{Z}/3$.

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