# The absolute de Rham-Witt complex

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## Introduction

This note is a brief survey of the absolute de Rham-Witt complex. We explain the structure of this complex for a smooth scheme over a complete discrete valuation ring of mixed characteristic $(0, p)$ and its relation to the sheaf of $p$-adic vanishing cycles and refer to the papers [**3, 8, 9, 5, 7, 6**] for details. We do not discuss the relative de Rham-Witt complex of Langer-Zink which is closely related to crystalline cohomology. The reader is refered to the papers [**15, 16, 7**] for this material and for a comparison of the two complexes.

In more detail, let $K$ be a field that is complete with respect to a discrete valuation, let $V$ be the valuation ring, and let $k$ be the residue field. Let $\mathfrak{m}$ and $e$ be the maximal ideal and the ramification index of $V$ such that $pV = \mathfrak{m}^e$. We assume that $K$ has characteristic 0 and that $k$ is perfect of odd characteristic $p$, but we allow the ramification index $e$ to be any positive integer. Let $X$ be a smooth $V$-scheme, and let $U$ and $Y$ be the generic and special fibers.

$$\begin{array}{ccccc} Y & \xhookrightarrow{\;i\;} & X & \xhookleftarrow{\;j\;} & U \\ \big\downarrow & & \big\downarrow{\scriptstyle f} & & \big\downarrow \\ \operatorname{Spec} k & \hookrightarrow & \operatorname{Spec} V & \hookleftarrow & \operatorname{Spec} K. \end{array}$$

We recall that the canonical log-structure on $X$ is the map of sheaves of monoids on the small étale site of $X$ given by the canonical inclusion

$$\alpha\colon M_X = j_*\mathcal{O}_U^* \times_{j_*\mathcal{O}_U} \mathcal{O}_X \hookrightarrow \mathcal{O}_X$$

where $\mathcal{O}_X$ is considered a sheaf of monoids under multiplication. A basic global object associated with the $V$-scheme $X$ is given by the $p$-adic étale cohomology groups $H^*(U, \mu_{p^v}^{\otimes q})$ which may be studied by means of the Leray spectral sequence

$$E_2^{s,t} = H^s(Y, i^*R^tj_*\mu_{p^v}^{\otimes q}) \Rightarrow H^{s+t}(U, \mu_{p^v}^{\otimes q}).$$

We focus in particular on the cohomology sheaves

$$M_v^q = i^*R^qj_*\mu_{p^v}^{\otimes q}.$$

If the field $K$ contains the $p^v$th roots of unity, this is no restriction. The $\mathfrak{m}$-adic filtration of $V$ gives rise to a natural filtration of this sheaf. For $v = 1$, the graded pieces $\operatorname{gr}_U^m M_1^q$ were determined by Bloch-Kato [**3**, Cor. 1.4.1] in terms of sheaves of

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differentials on $Y$. The main result of [**5**], Thm. A, may be seen as an improvement on the result of Bloch-Kato in that it determines the whole sheaf $M_1^q$ in terms of the absolute de Rham-Witt complex of $(X, M_X)$ of [**8, 9**].

## 1. The sheaf of $p$-adic vanishing cycles

We briefly recall the results of Bloch-Kato [**3**] and Hyodo [**10**] on the structure of the sheaf $M_v^q$ of $p$-adic vanishing cycles. The sheaf $M_v^q$ is studied by means of symbols. The symbol map is defined to be the map of sheaves of graded rings

$$i^*T(M_X^{\mathrm{gp}}) \to M_v^*$$

induced from the composite map of sheaves of abelian groups

$$M_X^{\mathrm{gp}} \xrightarrow{\sim} j_*\mathcal{O}_U^* \xrightarrow{\partial_v} R^1 j_*\mu_{p^v}$$

where the map $\partial_v$ is the boundary map in the cohomology long-exact sequence associated with the Kummer sequence

$$0 \to \mu_{p^v} \to \mathcal{O}_U^* \xrightarrow{p^v} \mathcal{O}_U^* \to 0.$$

It is well-known that the symbol map annihilates the two-sided ideal of the tensor algebra that is generated by the local sections $x \otimes y$ for which the local section $\alpha(x) + \alpha(y)$ is constant equal to 1. A choice of uniformizer $\pi$ of $V$ determines an isomorphism $\mathbb{Z} \times i^*\mathcal{O}_X^* \xrightarrow{\sim} i^*M_X^{\mathrm{gp}}$ that to $(n, u)$ assigns $\pi^n u$. We define a descending filtration of the tensor algebra by graded ideals

$$\mathrm{Fil}_U^m T(M_X^{\mathrm{gp}}) \subset T(M_X^{\mathrm{gp}})$$

where $\mathrm{Fil}_U^0 T(M_X^{\mathrm{gp}}) = T(M_X^{\mathrm{gp}})$, where $\mathrm{Fil}_U^1 T(M_X^{\mathrm{gp}})$ is the graded ideal generated by the subsheaf $\mathbb{Z} \times (1 + \mathfrak{m}\mathcal{O}_X)^*$, and where, for $j > 0$, $\mathrm{Fil}_U^{2j} T(M_X^{\mathrm{gp}})$ is the graded ideal generated by $\{0\} \times (1 + \mathfrak{m}^j\mathcal{O}_X)^*$ and $\mathrm{Fil}_U^{2j+1} T(M_X^{\mathrm{gp}})$ is the graded ideal generated by $(\{0\} \times (1 + \mathfrak{m}^j\mathcal{O}_X)^*) \otimes (\mathbb{Z} \times \{1\})$ and $\{0\} \times (1 + \mathfrak{m}^{j+1}\mathcal{O}_X)^*$. Let

$$\mathrm{Fil}_U^m M_v^q \subset M_v^q$$

be the image of $i^* \mathrm{Fil}_U^m T(M_X^{\mathrm{gp}})$ by the symbol map. If $y$ is a local section of $\mathcal{O}_Y$, we write $\tilde{y}$ for any lifting to a local section of $\mathcal{O}_X$. We define $e' = pe/(p-1)$. The following result is the improved version of [**3**, Cor. 1.4.1] proved in [**10**, Cor. 1.7].

THEOREM 1.1 (Bloch-Kato, Hyodo). *The sheaf of abelian groups*

$$M_1^q = i^*R^q j_*\mu_p^{\otimes q}$$

*on the small étale site of $Y$ in the étale topology has the following structure.*

(i) $\mathrm{Fil}_U^0 M_1^q = M_1^q$.

(ii) *There is a natural isomorphism*

$$\rho_0 \colon \Omega^q_{Y,\log} \xrightarrow{\sim} \mathrm{gr}_U^0 M_1^q \qquad (\textit{resp. } \rho_1 \colon \Omega^{q-1}_{Y,\log} \xrightarrow{\sim} \mathrm{gr}_U^1 M_1^q)$$

*that takes $d\log y_1 \dots d\log y_q$ (resp. $d\log y_1 \dots d\log y_{q-1}$) to the symbol $\{\tilde{y}_1, \dots, \tilde{y}_q\}$ (resp. the symbol $\{\tilde{y}_1, \dots, \tilde{y}_{q-1}, \pi\}$).*

(iii) *If $0 < j < e'$, and if $p$ does not divide $j$ (resp. if $p$ divides $j$), there is a natural isomorphism*

$$\rho_{2j} \colon \Omega_Y^{q-1}/B\Omega_Y^{q-1} \xrightarrow{\sim} \mathrm{gr}_U^{2j} M_1^q \qquad (\textit{resp. } \rho_{2j} \colon \Omega_Y^{q-1}/Z\Omega_Y^{q-1} \xrightarrow{\sim} \mathrm{gr}_U^{2j} M_1^q)$$

*that to $a d\log y_1 \dots d\log y_{q-1}$ assigns the symbol $\{1 + \pi^j\tilde{a}, \tilde{y}_1, \dots, \tilde{y}_{q-1}\}$.*

(iv) *If* $0 < j < e'$, *there is a natural isomorphism*

$$\rho_{2j+1} \colon \Omega_Y^{q-2}/Z\Omega_Y^{q-2} \xrightarrow{\sim} \operatorname{gr}_U^{2j+1} M_1^q$$

*that takes* $a d\log y_1 \dots d\log y_{q-2}$ *to the symbol* $\{1+\pi^j\tilde{a}, \tilde{y}_1, \dots, \tilde{y}_{q-2}, \pi\}$.

(iv) *If* $e' \leqslant j$, *then* $\operatorname{Fil}_U^{2j} M_1^q$ *is equal to zero.* □

More generally, Bloch-Kato and Hyodo determine the structure of the graded pieces $\operatorname{gr}_U^m M_v^q$, for all $v \geqslant 1$ and $0 \leqslant m < 2e'$ [**3**, Thm. 1.4] and [**10**, Thm. 1.6]. But if $v > 1$, the higher filtration quotients $\operatorname{gr}_U^m M_v^q$ with $m \geqslant 2e'$ are not zero and their structure is very complicated. It depends on the finer structure of the field $K$ and not only on $Y$, $q$, $v$, and $e$ [**3**, Rem. 6.8].

## 2. The absolute de Rham-Witt complex

We recall from [**14**] that the absolute de Rham complex $\Omega^*_{(X,M_X)}$ is a sheaf of log-differential graded rings on the small étale site of $X$ in the étale topology, that there is map of sheaves of log-rings

$$\lambda \colon (\mathcal{O}_X, M_X) \to (\Omega^0_{(X,M_X)}, M_X),$$

and that the absolute de Rham complex is universal with these properties. We now characterize the absolute de Rham-Witt complex $W_n\Omega^*_{(X,M_X)}$ in a similar way.

For every positive integer $n$, the de Rham-Witt complex $W_n\Omega^*_{(X,M_X)}$ is a sheaf log-differential graded rings on the small étale site of $X$ in the étale topology. This means that there is an anti-commutative product

$$W_n\Omega^q_{(X,M_X)} \otimes W_n\Omega^{q'}_{(X,M_X)} \to W_n\Omega^{q+q'}_{(X,M_X)}$$

and a differential $d \colon W_n\Omega^q_{(X,M_X)} \to W_n\Omega^{q+1}_{(X,M_X)}$ that is a derivation for the product along with maps of sheaves of monoids

$$\alpha_n \colon M_X \to (W_n\Omega^0_{(X,M_X)}, \cdot\ )$$
$$d\log_n \colon M_X \to (W_n\Omega^1_{(X,M_X)}, +)$$

such that, for every local section $x$ of $M_X$,

$$\alpha_n(x) \cdot d\log_n x = d\alpha_n(x).$$

In addition, there is a map of sheaves of log-rings

$$\lambda_n \colon (W_n(\mathcal{O}_X), M_X) \to (W_n\Omega^0_{(X,M_X)}, M_X)$$

where the pre-log structure on the left-hand side is given by the composite map

$$M_X \xrightarrow{\alpha} \mathcal{O}_X \xrightarrow{[-]_n} W_n(\mathcal{O}_X)$$

where $[-]_n$ is the multiplicative representative. As $n$ varies, the sheaves of log-differential graded rings $W_n\Omega^*_{(X,M_X)}$ are related by maps

$$R \colon W_n\Omega^q_{(X,M_X)} \to W_{n-1}\Omega^q_{(X,M_X)}$$

that preserve all of the structure above, by maps of sheaves of graded rings

$$F \colon W_n\Omega^q_{(X,M_X)} \to W_{n-1}\Omega^q_{(X,M_X)}$$

such that $F\lambda_n = \lambda_{n-1}F$, $F\alpha_n = \alpha_{n-1}$, $Fd\log_n = d\log_{n-1}$, $FR = RF$, and such that, for every local section $a$ of $\mathcal{O}_X$,

$$Fd\lambda_n([a]_n) = \lambda_{n-1}([a]_{n-1}^{p-1})d\lambda_{n-1}([a]_{n-1}),$$

and by maps of sheaves of graded abelian groups

$$V\colon W_{n-1}\Omega^q_{(X,M_X)} \to W_n\Omega^q_{(X,M_X)}$$

such that $\lambda_n V = V\lambda_{n-1}$, $V d\log_{n-1} = d\log_n$, $VR = RV$, and such that for every pair of local sections $\omega$ and $\omega'$ of $W_n\Omega^q_{(X,M_X)}$ and $W_{n-1}\Omega^{q'}_{(X,M_X)}$,

$$\omega \cdot V(\omega') = V(F(\omega)\cdot\omega').$$

The absolute de Rham-Witt complex is defined to be the initial example of the algebraic structure described above. It is proved in [**9**, Thm. A] that it exists and that the map of sheaves of log-differential graded rings

$$\lambda_n^q\colon \Omega^q_{(W_n(\mathcal{O}_X),M_X)} \to W_n\Omega^q_{(\mathcal{O}_X,M_X)}$$

induced from the map $\lambda_n$ is surjective. It follows from *op. cit.*, Thm. D, that the maps $\lambda_n^0$ and $\lambda_1^q$ are isomorphisms, and therefore, we shall often suppress these maps in the notation. Moreover, [**15**, Cor. A.18] and the proof of [**9**, Prop. 6.2.3] show that the sheaves $W_n\Omega^q_{(X,M_X)}$ are quasi-coherent $W_n(\mathcal{O}_X)$-modules on the small étale site of $X$. It follows, in particular, that if $X' \to X$ is an étale morphism with $X'$ affine, then the abelian group of sections over $X'$ is given by

$$\Gamma(X', W_n\Omega^q_{(X,M_X)}) = W_n\Omega^q_{(\Gamma(X',\mathcal{O}_X),\Gamma(X',M_X))};$$

compare [**7**, Lemma 2.1].

The $U$-filtration of the absolute de Rham-Witt complex is defined to be the natural descending filtration by the sheaves of differential graded ideals

$$\operatorname{Fil}^m_U W_n\Omega^*_{(X,M_X)} \subset W_n\Omega^*_{(X,M_X)}$$

generated by $W_n(\mathfrak{m}^j\mathcal{O}_X)$, if $m = 2j$ is even, and by $W_n(\mathfrak{m}^j\mathcal{O}_X)\cdot d\log_n M_X$ and by $W_n(\mathfrak{m}^{j+1})$, if $m = 2j+1$ is odd. There are canonical isomorphisms

$$i^*W_n\Omega^*_{(X,M_X)}/i^*\operatorname{Fil}^1_U W_n\Omega^*_{(X,M_X)} \xrightarrow{\sim} W_n\Omega^*_Y$$
$$i^*W_n\Omega^*_{(X,M_X)}/i^*\operatorname{Fil}^2_U W_n\Omega^*_{(X,M_X)} \xrightarrow{\sim} W_n\Omega^*_{(Y,M_Y)}$$

onto the de Rham-Witt complex of Bloch-Deligne-Illusie [**12**] of the special fiber and the de Rham-Witt complex of Hyodo-Kato [**11**] of the special fiber with the induced log-structure, respectively. The latter complex is denoted by $W_n\tilde{\omega}^*_Y$ in *op. cit.* The quotients by the higher terms of the $U$-filtration, however, are new and were first considered in [**8**].

We consider the reduced sheaves

$$E^q_{v,n} = i^*W_n\Omega^q_{(X,M_X)}/p^v i^*W_n\Omega^q_{(X,M_X)}$$

which are quasi-coherent $E^0_{v,n}$-modules on the small étale site of $Y$. We choose local coordinates $x_1,\dots,x_r$ of an open neighborhood of $X$ around a point of $Y$ and denote by $\bar{x}_1,\dots,\bar{x}_r$ the corresponding coordinates of the corresponding open neighborhood of $Y$. Then, in this neighborhood of $Y$, the choice of local coordinates determine ring homomorphisms

$$\delta_n\colon \mathcal{O}_Y \to E^0_{1,n}$$

such that $\delta_n(\bar{x}_i) = [x_i]_n$, $R\delta_n = \delta_{n-1}$, and such that $F\delta_n = \delta_{n-1}\varphi$. Here $\varphi$ is the absolute Frobenius endomorphism on $Y$. It follows that, in this neighborhood of

$Y$, we may consider $E_{1,n}^q$ as a quasi-coherent $\mathcal{O}_Y$-module. We note that

$$e' = \frac{pe}{p-1} = \lim_{v\to\infty} e\,\big(\frac{p^{-v}-1}{p^{-1}-1}\big).$$

The following result is proved in [**5**, Thm. 1.3.5]. We shall abbreviate $[a] = [a]_n$ and $d\log a = d\log_n a$.

THEOREM 2.1. *For $0 \leqslant j < e'$, let $v$ be the unique integer such that*

$$e\,\big(\frac{p^{-v}-1}{p^{-1}-1}\big) \leqslant j < e\,\big(\frac{p^{-(v+1)}-1}{p^{-1}-1}\big),$$

*and let $x_1,\dots,x_r$ be local coordinates of an open neighborhood of $X$ around a point of $Y$. Then, in the corresponding open neighborhood of $Y$, the sheaf*

$$E_{1,n}^q = i^*W_n\Omega^q_{(X,M_X)}/pi^*W_n\Omega^q_{(X,M_X)}$$

*has the following structure.*

(i) *If $0 \leqslant v < n$, then $\operatorname{gr}_U^{2j} E_{1,n}^q$ is a free $\mathcal{O}_Y$-module with a basis given as follows: If $p$ does not divide $j$ (resp. if $p$ divides $j$), let $0 < s < n-v$, and let $0 \leqslant i_1,\dots,i_r < p^s$ (resp. let $0 \leqslant i_1,\dots,i_r < p^s$ not all divisible by $p$); let $1 \leqslant m \leqslant r$ be maximal with $i_m$ not divisible by $p$. Then the local sections*

$$V^s([x_1]^{i_1}\dots[x_r]^{i_r}[\pi]^j d\log x_{m_1}\dots d\log x_{m_q})$$
$$dV^s([x_1]^{i_1}\dots[x_r]^{i_r}[\pi]^j d\log x_{m'_1}\dots d\log x_{m_{q-1}'}),$$

*where $1 \leqslant m_1 < \dots < m_q \leqslant r$ and $1 \leqslant m'_1 < \dots < m'_{q-1} \leqslant r$, and where all $m_i$ and $m'_i$ are different from $m$, together with the local sections*

$$[\pi]^j d\log x_{m_1}\dots d\log x_{m_q},$$

*where $1 \leqslant m_1 < \dots < m_q \leqslant r$, form a basis.*

(ii) *If $0 \leqslant v < n$, then $\operatorname{gr}_U^{2j+1} E_{1,n}^q$ is a free $\mathcal{O}_Y$-module with a basis given as follows: Let $0 < s < n-v$, and let $0 \leqslant i_1,\dots,i_r < p^s$ not all divisible by $p$; let $1 \leqslant m \leqslant r$ be maximal with $i_m$ not divisible by $p$. Then the local sections*

$$V^s([x_1]^{i_1}\dots[x_r]^{i_r}[\pi]^j d\log x_{m_1}\dots d\log x_{m_{q-1}} d\log\pi)$$
$$dV^s([x_1]^{i_1}\dots[x_r]^{i_r}[\pi]^j d\log x_{m'_1}\dots d\log x_{m_{q-2}'} d\log\pi),$$

*where $1 \leqslant m_1 < \dots < m_{q-1} \leqslant r$ and $1 \leqslant m'_1 < \dots < m'_{q-2} \leqslant r$, and where all $m_i$ and $m'_i$ are different from $m$, together with the local sections*

$$[\pi]^j d\log x_{m_1}\dots d\log x_{m_{q-1}} d\log\pi,$$

*where $1 \leqslant m_1 < \dots < m_{q-1} \leqslant r$, form a basis.*

(iii) *If $n \leqslant v$ or if $e' \leqslant j$, then $\operatorname{Fil}_U^{2j} E_{1,n}^q = 0$.* □

We remark that the rank of the free $\mathcal{O}_Y$-module $E_{1,n}^q$ is equal to

$$\operatorname{rk}_{\mathcal{O}_Y} E_{1,n}^q = \binom{r+1}{q}\cdot e \cdot \sum_{0\leqslant s<n} p^{rs}.$$

If $v > 1$, the structure of the sheaves $E_{v,n}^q$ is not presently known. In particular, these sheaves are generally not flat $\mathbb{Z}/p^v\mathbb{Z}$-modules. However, it is known that, if $\mu_{p^v} \subset K$, then, for $0 \leqslant m < v$, multiplication by $p^m$ induces an isomorphism

$$E_{1,\cdot}^q = \operatorname{gr}_p^0 E_{v,\cdot}^q \xrightarrow{\sim} \operatorname{gr}_p^m E_{v,\cdot}^q.$$

of sheaves of pro-abelian groups on the small étale site of $Y$. The proof, which is given in [**5**, Prop. 1.4.1], uses topology. We do not know an algebraic proof.

## 3. The kernel of $1 - F$

We define the subsheaf $M^q_{v,n} \subset E^q_{v,n}$ to be the kernel of the map

$$R - F\colon E^q_{v,n} \to E^q_{v,n-1}.$$

The $U$-filtration of the sheaf $E^q_{v,n}$ induces a filtration of the sheaf $M^q_{v,n}$. The following result, which follows from Thm. 2.1 above, is proved in [**5**, Thm. 2.1.2].

THEOREM 3.1. *The sheaf of pro-abelian groups*

$$M^q_{1,\cdot} = (i^*W.\Omega^q_{(X,M_X)}/pi^*W.\Omega^q_{(X,M_X)})^{F=1}$$

*on the small étale site of $Y$ in the étale topology has the following structure.*

(i) *There is a natural isomorphism*

$$\rho_0\colon \Omega^q_{Y,\log} \xrightarrow{\sim} \operatorname{gr}^0_U M^q_{1,\cdot} \qquad (resp.\ \rho_1\colon \Omega^{q-1}_{Y,\log} \xrightarrow{\sim} \operatorname{gr}^1_U M^q_{1,\cdot})$$

*that to $d\log y_1 \dots d\log y_q$ (resp. $d\log y_1 \dots d\log y_{q-1}$) assigns $d\log \tilde{y}_1 \dots d\log \tilde{y}_q$ (resp. $d\log \tilde{y}_1 \dots d\log \tilde{y}_{q-1} d\log \pi$).*

(ii) *If $0 < j < e'$, and if $p$ does not divide $j$ (resp. if $p$ divides $j$), there is a natural isomorphism*

$$\rho_{2j}\colon \Omega^{q-1}_Y/B\Omega^{q-1}_Y \xrightarrow{\sim} \operatorname{gr}^{2j}_U M^q_{1,\cdot} \qquad (resp.\ \rho_{2j}\colon \Omega^{q-1}_Y/Z\Omega^{q-1}_Y \xrightarrow{\sim} \operatorname{gr}^{2j}_U M^q_{1,\cdot})$$

*that to $a d\log y_1 \dots d\log y_{q-1}$ assigns $d\log(1+\pi^j\tilde{a}) d\log \tilde{y}_1 \dots d\log \tilde{y}_{q-1}$.*

(iii) *If $0 < j < e'$, there is a natural isomorphism*

$$\rho_{2j+1}\colon \Omega^{q-2}_Y/Z\Omega^{q-2}_Y \xrightarrow{\sim} \operatorname{gr}^{2j+1}_U M^q_{1,\cdot}$$

*that takes $a d\log y_1 \dots d\log y_{q-2}$ to $d\log(1+\pi^j\tilde{a}) d\log \tilde{y}_1 \dots d\log \tilde{y}_{q-2} d\log \pi$.*

(iv) *If $e' \leqslant j$, then $\operatorname{Fil}^{2j}_U M^q_{1,\cdot}$ is zero.* □

One observes that the results of Thms. 1.1 and 3.1 agree. In order to relate the sheaves $M^q_1$ and $M^q_{1,\cdot}$ we consider the symbol map

$$c^*_n\colon T(M^{\mathrm{gp}}_X) \to W_n\Omega^*_{(X,M_X)}$$

defined to be the map of sheaves of graded rings induced from the map

$$d\log_n\colon M^{\mathrm{gp}}_X \to W_n\Omega^1_{(X,M_X)}.$$

It was proved by Costeanu that this map annihilates the two-sided ideal of the tensor algebra generated by the local sections $x \otimes y$ for which $\alpha(x) + \alpha(y)$ is the constant local section 1. The proof is given in [**5**, Appendix B]. Moreover, the symbol map is compatible with the $U$-filtrations in the sense that

$$c^*_n(\operatorname{Fil}^m_U T(M^{\mathrm{gp}}_X)) \subset \operatorname{Fil}^m_U W_n\Omega^*_{(X,M_X)}.$$

It follows that we have maps of sheaves of pro-abelian groups

$$i^*R^qj_*\mu_p^{\otimes q} \xleftarrow{c^q} (i^*M^{\mathrm{gp}}_X)^{\otimes q} \xrightarrow{c^q_\cdot} i^*W.\Omega^q_{(X,M_X)}/pi^*W.\Omega^q_{(X,M_X)}$$

which are compatible with the $U$-filtrations of the three terms. Moreover, the two maps both annihilate the two-sided ideal of the tensor algebra generated by the local sections $x \otimes y$ such that $\alpha(x) + \alpha(y)$ is equal to the constant local section 1. It then follows from [**3**, Prop. 6.1] and from Thms. 1.1 and 3.1 that the kernels of the two maps agree and that there is a natural short-exact sequence of sheaves of pro-abelian groups on the small étale site of $Y$ in the étale topology

$$0 \to i^*R^qj_*\mu_p^{\otimes q} \to i^*W.\Omega^q_{(X,M_X)}/p \xrightarrow{1-F} i^*W.\Omega^q_{(X,M_X)}/p \to 0.$$

The argument is given in [**5**, Thm. 2.1.4]. This gives the promised description of the sheaf $M_1^q$ of $p$-adic vanishing cycles in terms of the absolute de Rham-Witt complex. More generally, the following result is [**5**, Thm. A]. We remark that the ramification index $e$ and the degree $q$ may take all possible values.

Theorem 3.2. *Suppose that the field $K$ contains the $p^v$th roots of unity. Then there is a natural short-exact sequence*

$$0 \to i^*R^qj_*\mu_{p^v}^{\otimes q} \to i^*W.\Omega^q_{(X,M_X)}/p^v \xrightarrow{1-F} i^*W.\Omega^q_{(X,M_X)}/p^v \to 0$$

*of sheaves of pro-abelian groups on the small étale site of $Y$ in the étale topology.* □

We conclude this paper with some conjectures generalizing Thm. 3.2. Firstly, we conjecture that the absolute de Rham-Witt complex satisfies Galois descent in the following sense: Let $K' = K(\mu_{p^v})$, let $V' \subset K'$ be the integral closure of $V$ in $K'$, and let $X' = X \times_{\operatorname{Spec} V} \operatorname{Spec} V'$. Then we conjecture that the canonical map

$$W.\Omega^q_{(X,M_X)}/p^vW.\Omega^q_{(X,M_X)} \to (W.\Omega^q_{(X',M_{X'})}/p^vW.\Omega^q_{(X',M_{X'})})^{\operatorname{Gal}(K'/K)}$$

is an isomorphism of sheaves of pro-abelian groups. This implies that the statement of Thm. 3.2 holds also if the field $K$ does not contain the $p^v$th roots of unity. We further conjecture that Thm. 3.2 holds if $f\colon (X, M_X) \to (\operatorname{Spec} V, M_V)$ is a smooth map of log-schemes with the special fiber of Cartier type.

Secondly, one would like to determine the structure of the complex of sheaves

$$M_v = \tau_{\leqslant q} i^*Rj_*\mu_{p^v}^{\otimes q}$$

and not only its cohomology sheaves. The cohomology sheaves in degrees greater than $q$ are also of considerable interest, but they are of a rather different nature [**13**]. The following conjectures are similar to the Beilinson-Lichtenbaum conjectures [**1, 2, 17**]. We conjecture that there exist complexes $W_n(q)$ of quasi-coherent $W_n(\mathcal{O}_X)$-modules on the small étale site of $X$ related by restriction, Frobenius, and Verschiebung maps that have the following properties (i)–(iii).

(i) The cohomology sheaves $H^i(W_n(q))$ are located in degrees $0 \leqslant i \leqslant q$ and there is a canonical isomorphism of quasi-coherent $W_n(\mathcal{O}_X)$-modules

$$W_n\Omega^q_{(X,M_X)} \xrightarrow{\sim} H^q(W_n(q)).$$

(ii) There is a distinguished triangle

$$\tau_{\leqslant q} i^*Rj_*\mu_{p^v}^{\otimes q} \to i^*W.(q)/p^v \xrightarrow{1-F} i^*W.(q)/p^v \xrightarrow{\partial} \tau_{\leqslant q} i^*Rj_*\mu_{p^v}^{\otimes q}[+1]$$

in the derived category of the category of sheaves of pro-abelian groups on the small étale site of $Y$.

(iii) There is a spectral sequence of quasi-coherent $W_n(\mathcal{O}_X)$-modules

$$E^2_{s,t} = H^{t-s}(W_n(t)) \Rightarrow \operatorname{TR}^n_{s+t}(X|U;p).$$

The conjectural spectral sequence (iii) is analogous to the Bloch-Lichtenbaum spectral sequence [**4**]. To explain the abutment, we recall that the absolute de Rham-Witt complex is defined to be the universal example of the algebraic structure that was described in Sect. 2 above. The sheaves $\operatorname{TR}^n_q(X|U;p)$, which are obtained from topological Hochschild homology, form another important example of this algebraic structure. They have no algebraic definition but are defined as the equivariant homotopy groups of a presheaf of $\mathbb{T}$-spectra; see [**8**, Sect. 1] for a detailed account. The universal property of the absolute de Rham-Witt complex implies that there is a unique map of sheaves

$$I\colon W_n\Omega^q_{(X,M_X)} \to \mathrm{TR}^n_q(X|U;p)$$

which commutes with all the structure maps. This map should be the composition of the isomorphism (i) and the edge homomorphism of the spectral sequence (iii). Moreover, the exact sequence of Thm. 3.2 should be the sequence of cohomology sheaves in degree $q$ associated with the distinguished triangle (ii).

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