Lecture notes on the big de Rham-Witt complex

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Introduction

The purpose of these notes is to give a comprehensive and self-contained treatment big Witt vectors and the big de Rham-Witt complex.

We begin with an introduction to Witt vectors and cover both the classical p-typical Witt vectors of Teichmüller and Witt [**13**] and the generalized or big Witt vectors of Cartier [**3**]. The latter associates to every ring A and every set S of positive integers stable under division a ring $\mathbb{W}_S(A)$, and the former corresponds to the case where all elements of S are powers of a single prime number p.

We continue with the definition of the big de Rham-Witt complex. It generalizes the classical p-typical de Rham-Witt complex of Bloch, Deligne, and Illusie [**8, 7, 1**] and was first defined by the author and Madsen [**6**]. For rings in which the prime number 2 is not either invertible or zero, the original definitions of these complexes are not quite correct. The correct definition of the 2-typical de Rham-Witt complex was given by Costeanu [**4**] while the correct definition of the big de Rham-Witt complex is given first in these notes. The big de Rham-Witt complex associates to every ring A and every set S of integers stable under division an anti-symmetric differential graded ring $\mathbb{W}_S\Omega_A^{\cdot}$ together with maps

$$\begin{aligned} R_T^S &\colon \mathbb{W}_S\Omega_A^q \to \mathbb{W}_T\Omega_A^q && \text{(restriction)} \\ F_n &\colon \mathbb{W}_S\Omega_A^q \to \mathbb{W}_{S/n}\Omega_A^q && \text{(Frobenius)} \\ V_n &\colon \mathbb{W}_{S/n}\Omega_A^q \to \mathbb{W}_S\Omega_A^q && \text{(Verschiebung)} \\ \lambda &\colon \mathbb{W}_S(A) \to \mathbb{W}_S\Omega_A^0 \end{aligned}$$

where R_T^S is a map of differential graded rings, F_n is a map of graded rings, V_n is a map of graded abelian groups, and λ is a map of rings. Here S/n denotes the set of positive integers m such that $mn \in S$. The maps R_T^S, F_n, V_n, and λ are required to satisfy a number of additional relations listed in Def. 2.1 below. By definition, the big de Rham-Witt complex is the initial example of this algebraic structure. It is not difficult to show that the ring homomorphism λ is an isomorphism. We also prove in Cor. 5.7 below that the canonical map

$$\Omega_A^q \to \mathbb{W}_{\{1\}}\Omega_A^q$$

is an isomorphism, but this requires quite a bit of preparation. Thus, the big de Rham-Witt complex combines de Rham differentials and big Witt vectors. The

p-typical de Rham-Witt complex is obtained by restricting to the subsets S that consist of powers of a single prime number p.

Beyond the definition of the big de Rham-Witt complex, the topics treated are as follows. We show, in Prop. 3.6, that for a $\mathbb{Z}_{(p)}$-algebra A, the big de Rham-Witt groups decompose as products of p-typical de Rham-Witt groups. We then give, in Thm. 4.6, an explicit formula for the p-typical de Rham-Witt complex of the polynomial algebra $A[X]$ over the $\mathbb{Z}_{(p)}$-algebra A in terms of the p-typical de Rham-Witt complex of A. Finally, in Thm 6.1, we prove that if $f\colon A \to B$ is an étale morphism, then the induced map

$$\mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega^q_A \to \mathbb{W}_S\Omega^q_B$$

is an isomorphism.

1. Witt vectors

As promised, we begin these notes with a self-contained introduction to Witt vectors. In the approach taken here, all necessary congruences are isolated in the lemma of Dwork. A slightly different but very readable account may be found in Bergman [**11**, Appendix]. We conclude with a brief treatment of special λ-rings and Adams operations and a recent theorem of Borger [**2**, Thm. A] on the ring of Witt vectors and étale morphisms.

Let $\mathbb{N}$ be the set of positive integers, and let $S \subset \mathbb{N}$ be a subset with the property that, if $n \in S$, and if d is a divisor in n, then $d \in S$. We then say that S is a truncation set. The big Witt ring $\mathbb{W}_S(A)$ is defined to be the set A^S equipped with a ring structure such that the ghost map

$$w\colon \mathbb{W}_S(A) \to A^S$$

that takes the vector $(a_n \mid n \in S)$ to the sequence $(w_n \mid n \in S)$, where

$$w_n = \sum_{d|n} d a_d^{n/d},$$

is a natural transformation of functors from the category of rings to itself. Here, on the right-hand side, A^S is considered a ring with componentwise addition and multiplication. To prove that there exists a unique ring structure on $\mathbb{W}_S(A)$ that is characterized in this way, we first prove the following result.

Lemma 1.1 (Dwork). *Suppose that, for every prime number p, there exists a ring homomorphism $\phi_p\colon A \to A$ with the property that $\phi_p(a) \equiv a^p$ modulo pA. Then a sequence $(x_n \mid n \in S)$ is in the image of the ghost map*

$$w\colon \mathbb{W}_S(A) \to A^S$$

if and only if $x_n \equiv \phi_p(x_{n/p})$ modulo $p^{v_p(n)}A$, for every prime number p, and for every $n \in S$ with $v_p(n) \geqslant 1$. Here $v_p(n)$ denotes the p-adic valuation of n.

Proof. We first show that, if $a \equiv b$ modulo pA, then $a^{p^{v-1}} \equiv b^{p^{v-1}}$ modulo $p^v A$. If we write $a = b + p\epsilon$, then

$$a^{p^{v-1}} = b^{p^{v-1}} + \sum_{1 \leqslant i \leqslant p^{v-1}} \binom{p^{v-1}}{i} b^{p^{v-1}-i} p^i \epsilon^i.$$

In general, the p-adic valuation of the binomial coefficient $\binom{m+n}{n}$ is equal to the number of carriers in the addition of m and n in base p. So

$$v_p\left(\binom{p^{v-1}}{i}\right) = v - 1 - v_p(i),$$

and hence,

$$v_p\left(\binom{p^{v-1}}{i} p^i\right) = v - 1 + i - v_p(i) \geqslant v.$$

This proves the claim. Now, since ϕ_p is a ring-homomorphism,

$$\phi_p(w_{n/p}(a)) = \sum_{d|(n/p)} d\phi_p(a_d^{n/pd})$$

which is congruent to $\sum_{d|(n/p)} d a_d^{n/d}$ modulo $p^{v_p(n)}A$. If d divides n but not n/p, then $v_p(d) = v_p(n)$, and hence this sum is congruent to $\sum_{d|n} d a_d^{n/d} = w_n(a)$ modulo

$p^{v_p(n)}A$ as stated. Conversely, if $(x_n \mid n \in S)$ is a sequence such that $x_n \equiv \phi_p(x_{n/p})$ modulo $p^{v_p(n)}A$, we find a vector $a = (a_n \mid n \in S)$ with $w_n(a) = x_n$ as follows. We let $a_1 = x_1$ and assume, inductively, that a_d has been chosen, for all d that divides n, such that $w_d(a) = x_d$. The calculation above shows that the difference

$$x_n - \sum_{d|n, d\neq n} da_d^{n/d}$$

is congruent to zero modulo $p^{v_p(n)}A$. Hence, we can find $a_n \in A$ such that na_n is equal to this difference. □

Proposition 1.2. *There exists a unique ring structure such that the ghost map*

$$w\colon \mathbb{W}_S(A) \to A^S$$

is a natural transformation of functors from rings to rings.

Proof. Let A be the polynomial ring $\mathbb{Z}[a_n, b_n \mid n \in S]$. Then the unique ring homomorphism

$$\phi_p\colon A \to A$$

that maps a_n to a_n^p and b_n to b_n^p satisfies that $\phi_p(f) = f^p$ modulo pA. Let a and b be the sequences $(a_n \mid n \in S)$ and $(b_n \mid n \in S)$. Since ϕ_p is a ring homomorphism, Lemma 1.1 shows immediately that the sequences $w(a) + w(b)$, $w(a) \cdot w(b)$, and $-w(a)$ are in the image of the ghost map. It follows that there are sequences of polynomials $s = (s_n \mid n \in S)$, $p = (p_n \mid n \in S)$, and $\iota = (\iota_n \mid n \in S)$ such that $w(s) = w(a) + w(b)$, $w(p) = w(a) \cdot w(b)$, and $w(\iota) = -w(a)$. Moreover, since A is torsion free, the ghost map is injective, and hence, these polynomials are unique.

Let now A' be any ring, and let $a' = (a'_n \mid n \in S)$ and $b' = (b'_n \mid n \in S)$ be two vectors in $\mathbb{W}_S(A')$. Then there is a unique ring homomorphism $f\colon A \to A'$ such that $\mathbb{W}_S(f)(a) = a'$ and $\mathbb{W}_S(f)(b) = b'$. We define $a' + b' = \mathbb{W}_S(f)(s)$, $a \cdot b = \mathbb{W}_S(f)(p)$, and $-a = \mathbb{W}_S(f)(\iota)$. It remains to prove that the ring axioms are verified. Suppose first that A' is torsion free. Then the ghost map is injective, and hence, the ring axioms are satisfied in this case. In general, we choose a surjective ring homomorphism $g\colon A'' \to A'$ from a torsion free ring A''. Then

$$\mathbb{W}_S(g)\colon \mathbb{W}_S(A'') \to \mathbb{W}_S(A')$$

is again surjective, and since the ring axioms are satisfied on the left-hand side, they are satisfied on the right-hand side. □

If $T \subset S$ are two truncation sets, then the forgetful map

$$R^S_T\colon \mathbb{W}_S(A) \to \mathbb{W}_T(A)$$

is a natural ring homomorphism called the restriction from S to T. If $n \in \mathbb{N}$, and if $S \subset \mathbb{N}$ is a truncation set, then

$$S/n = \{d \in \mathbb{N} \mid nd \in S\}$$

is again a truncation set. We define the nth Verschiebung map

$$V_n\colon \mathbb{W}_{S/n}(A) \to \mathbb{W}_S(A)$$

by

$$V_n((a_d \mid d \in S/n))_m = \begin{cases} a_d, & \text{if } m = nd, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 1.3. *The Verschiebung map V_n is additive.*

Proof. There is a commutative diagram

$$\begin{array}{ccc} \mathbb{W}_{S/n}(A) & \xrightarrow{w} & A^{S/n} \\ \downarrow{\scriptstyle V_n} & & \downarrow{\scriptstyle V_n^w} \\ \mathbb{W}_S(A) & \xrightarrow{w} & A^S \end{array}$$

where the map V_n^w is given by

$$V_n^w((x_d \mid d \in S/n))_m = \begin{cases} nx_d, & \text{if } m = nd, \\ 0, & \text{otherwise.} \end{cases}$$

Since the map V_n^w is additive, so is the map V_n. Indeed, if A is torsion free, the horizontal maps are both injective, and hence, V_n is additive in this case. In the general case, we choose a surjective ring homomorphism $g\colon A' \to A$ and argue as in the proof of Prop. 1.2 above. □

Lemma 1.4. *There exists a unique natural ring homomorphism*

$$F_n\colon \mathbb{W}_S(A) \to \mathbb{W}_{S/n}(A)$$

such the diagram

$$\begin{array}{ccc} \mathbb{W}_S(A) & \xrightarrow{w} & A^S \\ \downarrow{\scriptstyle F_n} & & \downarrow{\scriptstyle F_n^w} \\ \mathbb{W}_{S/n}(A) & \xrightarrow{w} & A^{S/n}, \end{array}$$

where $F_n^w((x_m \mid m \in S))_d = x_{nd}$, commutes.

Proof. We construct the Frobenius map F_n in a manner similar to the construction of the ring operations on $\mathbb{W}_S(A)$ in Prop. 1.2. We let A be the polynomial ring $\mathbb{Z}[a_n \mid n \in S]$, and let a be the vector $(a_n \mid n \in S)$. Then Lemma 1.1 shows that the sequence $F_n^w(w(a)) \in A^{S/n}$ is the image of a (unique) element

$$F_n(a) = (f_{n,d} \mid d \in S/n) \in \mathbb{W}_{S/n}(A)$$

by the ghost map. If A' is any ring, and if $a' = (a'_n \mid n \in S)$ is a vector in $\mathbb{W}_S(A')$, then we define $F_n(a') = \mathbb{W}_{S/n}(g)(F_n(a))$, where $g\colon A \to A'$ is the unique ring homomorphism that maps a to a'. Finally, since F_n^w is a ring homomorphism, an argument similar to the proof of Lemma 1.3 shows that also F_n is a ring homomorphism. □

The Teichmüller representative is the map

$$[-]_S\colon A \to \mathbb{W}_S(A)$$

defined by

$$([a]_S)_n = \begin{cases} a, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

It is a multiplicative map. Indeed, there is a commutative diagram

$$\begin{array}{ccc} A & = & A \\ \Big\downarrow{\scriptstyle [-]_S} & & \Big\downarrow{\scriptstyle [-]_S^w} \\ \mathbb{W}_S(A) & \xrightarrow{w} & A^S, \end{array}$$

where $([a]_S^w)_n = a^n$, and $[-]_S^w$ is a multiplicative map.

LEMMA 1.5. *The following relations holds.*
(i) $a = \sum_{n \in S} V_n([a_n]_{S/n})$.
(ii) $F_n V_n(a) = na$.
(iii) $a V_n(a') = V_n(F_n(a) a')$.
(iv) $F_m V_n = V_n F_m$, *if* $(m, n) = 1$.

PROOF. One easily verifies that both sides of each equation have the same image by the ghost map. This shows that the relations hold, if A is torsion free, and hence, in general. □

PROPOSITION 1.6. *The ring* $\mathbb{W}_S(\mathbb{Z})$ *of big Witt vectors in the ring of rational integers is equal to the product*

$$\mathbb{W}_S(\mathbb{Z}) = \prod_{n \in S} \mathbb{Z} \cdot V_n([1]_{S/n})$$

with the multiplication given by

$$V_m([1]_{S/m}) \cdot V_n([1]_{S/n}) = c \cdot V_d([1]_{S/d}),$$

where $c = (m, n)$ *and* $d = mn/(m, n)$ *are the greatest common divisor and the least common multiple of* m *and* n.

PROOF. The formula for the multiplication follows from Lemma 1.5 (ii)-(iv). Suppose first that S is finite. If S is empty, the statement is trivial, so assume that S is non-empty. We let $m \in S$ be maximal, and let $T = S \smallsetminus \{m\}$. Then the sequence of abelian groups

$$0 \to \mathbb{W}_{\{1\}}(\mathbb{Z}) \xrightarrow{V_m} \mathbb{W}_S(\mathbb{Z}) \xrightarrow{R_T^S} \mathbb{W}_T(\mathbb{Z}) \to 0$$

is exact, and we wish to show that it is equal to the sequence

$$0 \to \mathbb{Z} \cdot [1]_{\{1\}} \xrightarrow{V_m} \prod_{n \in S} \mathbb{Z} \cdot V_n([1]_{S/n}) \xrightarrow{R_T^S} \prod_{n \in T} \mathbb{Z} \cdot V_n([1]_{T/n}) \to 0.$$

The latter sequence is a sub-sequence of the former sequence, and, inductively, the left-hand terms (resp. the right-hand terms) of the two sequences are equal. Hence, middle terms are equal, too. The statement for S finite follows. Finally, a general truncation set S is the union of the finite sub-truncation sets $S_\alpha \subset S$, and hence,

$$\mathbb{W}_S(\mathbb{Z}) = \lim_{\alpha} \mathbb{W}_{S_\alpha}(\mathbb{Z}).$$

This proves the stated formula in general. □

The action of the restriction, Frobenius, and Verschiebung operators on the generators $V_n([1]_{S/n})$ is easily derived from the relations Lemma 1.5 (ii)–(iv). To give a formula for the Teichmüller representative, we recall the Möbius inversion formula. Let $g\colon \mathbb{N} \to \mathbb{Z}$ be a function, and let $f\colon \mathbb{N} \to \mathbb{Z}$ be the function given by

$$f(n) = \sum_{d\mid n} g(d).$$

Then the function g is given by f by means of the formula

$$g(n) = \sum_{d\mid n} \mu(d) f(n/d),$$

where $\mu\colon \mathbb{N} \to \{-1,0,1\}$ is the Möbius function. Here $\mu(d) = (-1)^r$, if d is a product of $r \geqslant 0$ distinct prime numbers, and $\mu(d) = 0$, otherwise.

ADDENDUM 1.7. *Let m be an integer. Then*

$$[m]_S = \sum_{n\in S} \frac{1}{n}\big(\sum_{d\mid n} \mu(d) m^{n/d}\big) V_n([1]_{S/n}),$$

where $\mu\colon \mathbb{N} \to \{-1,0,1\}$ is the Möbius function.

PROOF. It suffices to prove that the formula holds in $\mathbb{W}_S(\mathbb{Z})$. We know from Prop. 1.6 that there are unique integers r_d, $d \in S$, such that

$$[m]_S = \sum_{d\in S} r_d V_d([1]_{S/d}).$$

Evaluating the nth ghost component of this equation, we get

$$m^n = \sum_{d\mid n} d r_d,$$

and the stated formula now follows from the Möbius inversion formula. □

LEMMA 1.8. *Suppose that A is an $\mathbb{F}_p$-algebra, and let $\varphi\colon A \to A$ be the Frobenius endomorphism. Then*

$$F_p = R^S_{S/p} \circ \mathbb{W}_S(\varphi)\colon \mathbb{W}_S(A) \to \mathbb{W}_{S/p}(A).$$

PROOF. We recall from the proof of Prop. 1.4 that

$$F_p(a) = (f_{p,d}(a) \mid d \in S/p),$$

where $f_{p,d}$ are the integral polynomials defined by the equations

$$\sum_{d\mid n} d f_{p,d}^{n/d} = \sum_{d\mid pn} d a_d^{pn/d}$$

for all $n \in S$. Let $A = \mathbb{Z}[a_n \mid n \in S]$. We shall prove that for all $n \in S/p$,

$$f_{p,n} \equiv a_n^p$$

modulo pA. This is equivalent to the statement of the lemma. If $n = 1$, we have $f_{p,1} = a_1^p + p a_p$, and we are done in this case. So let $n > 1$ and assume, inductively, that the stated congruence has been proved for all proper divisors in n. Then, if d is a proper divisor in n, $f_{p,d} \equiv a_d^p$ modulo pA, so

$$d f_{p,d}^{n/d} \equiv d a_d^{pn/d}$$

modulo $p^{v_p(n)+1}A$; compare the proof of Lemma 1.1. Rewriting the defining equations

$$\sum_{d\mid n} df_{p,d}^{n/d} = \sum_{d\mid n} da_d^{pn/d} + \sum_{d\mid pn, d\nmid n} da_d^{pn/d}$$

and noting that if $d \mid pn$ and $d \nmid n$, then $v_p(d) = v_p(n) + 1$, we find

$$nf_{p,n} \equiv na_n^p$$

modulo $p^{v_p(n)+1}A$. Since A is torsion free, we conclude that $f_{p,n} \equiv a_n^p$ modulo pA as desired. $\square$

We consider the truncation set $P = \{1, p, p^2, \dots\} \subset \mathbb{N}$ that consists of all powers of a fixed prime number p. The proper non-empty sub-truncation sets of P all are of the form $\{1, p, \dots, p^{n-1}\}$, for some positive integer n. The rings

$$W(A) = \mathbb{W}_P(A)$$
$$W_n(A) = \mathbb{W}_{\{1,p,\dots,p^{n-1}\}}(A)$$

are called the ring of p-typical Witt vectors in A and p-typical Witt vectors of length n in A, respectively. We shall now show that, if A is a $\mathbb{Z}_{(p)}$-algebra, the rings of big Witt vectors $\mathbb{W}_S(A)$ decompose canonically as a product of rings of p-typical Witt vectors. We begin with the following result.

LEMMA 1.9. *Let m be an integer and suppose that m is invertible (resp. a non-zero-divisor) in A. Then m is invertible (resp. a non-zero-divisor) in $\mathbb{W}_S(A)$.*

PROOF. It suffices to prove the lemma, for S finite. Indeed, in general, $\mathbb{W}_S(A)$ is the limit of $\mathbb{W}_T(A)$, where T ranges over the finite sub-truncation sets of S. So assume that S is finite and non-empty. Let $n \in S$ be maximal, and let $T = S \smallsetminus \{n\}$. Then $S/n = \{1\}$ and we have an exact sequence

$$0 \to A \xrightarrow{V_n} \mathbb{W}_S(A) \xrightarrow{R_T^S} \mathbb{W}_T(A) \to 0$$

from which the lemma follows by easy induction. $\square$

We next show that rings of big Witt vectors of a $\mathbb{Z}_{(p)}$-algebra admit a natural idempotent decomposition. In fact, we prove the following more precise result.

PROPOSITION 1.10. *Let p be a prime number, S a truncation set, and $I(S)$ the set of $k \in S$ not divisible by p. Let A be a ring and assume that every $k \in I(S)$ is invertible in A. Then there is a natural idempotent decomposition*

$$\mathbb{W}_S(A) = \prod_{k \in I(S)} \mathbb{W}_S(A)e_k$$

where

$$e_k = \prod_{l \in I(S) \smallsetminus \{1\}} \left(\frac{1}{k} V_k([1]_{S/k}) - \frac{1}{kl} V_{kl}([1]_{S/kl}) \right).$$

Moreover, the composite map

$$\mathbb{W}_S(A)e_k \hookrightarrow \mathbb{W}_S(A) \xrightarrow{F_k} \mathbb{W}_{S/k}(A) \xrightarrow{R^{S/k}_{S/k \cap P}} \mathbb{W}_{S/k \cap P}(A)$$

is an isomorphism.

Proof. We calculate

$$w_n(\frac{1}{k}V_k([1]_{S/k})) = \begin{cases} 1, & \text{if } k \in S \cap k\mathbb{N}, \\ 0, & \text{otherwise,} \end{cases}$$

and hence,

$$w_n(e_k) = \begin{cases} 1, & \text{if } k \in S \cap kP, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that the elements e_k, $k \in I(S)$, are orthogonal idempotents in $\mathbb{W}_S(A)$. This proves the former part of the statement. To prove the latter part, we note that multiplication by k defines a bijection

$$S/k \cap P = (S \cap kP)/k \xrightarrow{\sim} S \cap kP$$

and that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{W}_S(A)e_k & \xrightarrow{w} & A^{S\cap kP} \\ \downarrow{\scriptstyle R^{S/k}_{S/k\cap P}F_k} & & \sim \downarrow{\scriptstyle k^*} \\ \mathbb{W}_{S/k\cap k}(A) & \xrightarrow{w} & A^{S/k\cap P}. \end{array}$$

We first assume that A is torsion free and has an endomorphism $\phi_p \colon A \to A$ such that $\phi_p(a) \equiv a^p$ modulo pA. Then the horizontal maps w are both injective. Moreover, Lemma 1.1 identifies the image of the top horizontal map w with the set of sequences $(x_d \mid d \in S \cap kP)$ such that $x_d \equiv \phi_p(x_{d/p})$ modulo $p^{v_p(d)}A$. Similarly, the image of the lower horizontal map w is the set of sequences $(y_d \mid d \in S/k \cap P)$ such that $y_d \equiv \phi_p(y_{d/p})$ modulo $p^{v_p(d)}A$. Since the right-hand vertical map k^* induces an isomorphism of these subrings, the left-hand vertical map $R^{S/k}_{S/k\cap P}F_k$ is an isomorphism in this case. □

Example 1.11. Let $S = \{1, 2, \dots, n\}$ such that $\mathbb{W}_S(A)$ is the ring $\mathbb{W}_n(A)$ of big Witt vectors of length n in A. Then $S/k \cap P = \{1, p, \dots, p^{s-1}\}$ where $s = s(n, k)$ is the unique integer with $p^{s-1}k \leqslant n < p^s k$. Hence, if every integer $1 \leqslant k \leqslant n$ not divisible by p is invertible in A, then Prop. 1.10 shows that

$$\mathbb{W}_n(A) \xrightarrow{\sim} \prod W_s(A)$$

where the product ranges over integers $1 \leqslant k \leqslant n$ not divisible by p and where $s = s(n, k)$ is given as above.

We now consider the ring $W_n(A)$ of p-typical Witt vectors of length n in A in more detail. The ghost map

$$w \colon W_n(A) \to A^n$$

takes the vector $(a_0, \dots, a_{n-1})$ to the sequence $(w_0, \dots, w_{n-1})$ where

$$w_i = a_0^{p^i} + pa_1^{p^{i-1}} + \dots + p^i a_i.$$

If $\phi \colon A \to A$ is a ring homomorphism with $\phi(a) \equiv a^p$ modulo pA, then Lemma 1.1 identifies the image of the ghost map with the subring of sequences $(x_0, \dots x_{n-1})$ such that $x_i \equiv \phi(x_{i-1})$ modulo $p^i A$, for all$1 \leqslant i \leqslant n-1$. We write

$$[-]_n \colon A \to W_n(A)$$

for the Teichmüller representative and

$$F: W_n(A) \to W_{n-1}(A)$$
$$V: W_{n-1}(A) \to W_n(A)$$

for the pth Frobenius and pth Verschiebung.

LEMMA 1.12. *If A is an $\mathbb{F}_p$-algebra, then $VF = p$.*

PROOF. For any ring A, the composite VF is given by multiplication by the element $V([1]_{n-1})$. Suppose that A is an $\mathbb{F}_p$-algebra. The exact sequences

$$0 \to A \xrightarrow{V^{n-1}} W_n(A) \xrightarrow{R} W_{n-1}(A) \to 0$$

show, inductively, that $W_n(A)$ is annihilated by p^n. Hence, $V([1]_{n-1})$ is annihilated by p^{n-1}. We show by induction on n that $V([1]_{n-1}) = p[1]_n$, the case $n = 1$ being trivial. The formula from Addendum 1.7 gives that

$$[p]_n = p[1]_n + \sum_{0<s<n} \frac{p^{p^s} - p^{p^{s-1}}}{p^s} V^s([1]_{n-s}).$$

Since $[p]_n = 0$, and since, inductively, $V^s([1]_{n-s}) = p^{s-1}V([1]_{n-1})$, for $0 < s < n$, we can rewrite this formula as

$$0 = p[1]_n + (p^{p^{n-1}-1} - 1)V([1]_{n-1}).$$

But $p^{n-1} - 1 \geqslant n - 1$, so we get $p[1]_n = V([1]_{n-1})$ as stated. □

We now suppose that A is a p-torsion free ring and that there exists a ring homomorphism $\phi\colon A \to A$ such that $\phi(a) \equiv a^p$ modulo pA. It follows from Lemma 1.1 that there is a unique ring homomorphism

$$s_\phi\colon A \to W(A)$$

such that the composite

$$A \xrightarrow{s_\phi} W(A) \xrightarrow{w} A^{\mathbb{N}_0}$$

maps a to $(a, \phi(a), \phi^2(a), \dots)$. We then define

$$t_\phi\colon A \to W(A/pA)$$

to be the composite of s_ϕ and the map induced by the canonical projection of A onto A/pA. We recall that the $\mathbb{F}_p$-algebra A/pA is said to be perfect, if the Frobenius endomorphism $\varphi\colon A/pA \to A/pA$ is an automorphism.

PROPOSITION 1.13. *Let A be a p-torsion free ring, and let $\phi\colon A \to A$ be a ring homomorphism such that $\phi(a) \equiv a^p$ modulo pA. Suppose that A/pA is a perfect $\mathbb{F}_p$-algebra. Then the map t_ϕ induces an isomorphism*

$$t_\phi\colon A/p^nA \xrightarrow{\sim} W_n(A),$$

for all $n \geqslant 1$.

PROOF. The map t_ϕ factors as in the statement since

$$V^nW(A/pA) = V^nW(\phi^n(A/pA)) = V^nF^nW(A/pA) = p^nW(A/pA).$$

The proof is now completed by an induction argument based on the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A/pA & \xrightarrow{p^{n-1}} & A/p^nA & \xrightarrow{\mathrm{pr}} & A/p^{n-1}A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi^{n-1}} & & \downarrow{\scriptstyle t_\phi} & & \downarrow{\scriptstyle t_\phi} & & \\ 0 & \longrightarrow & A/pA & \xrightarrow{V^{n-1}} & W_n(A/pA) & \xrightarrow{R} & W_{n-1}(A/pA) & \longrightarrow & 0. \end{array}$$

The top horizontal sequence is exact, since A is p-torsion free, and the left-hand vertical map is an isomorphism, since A/pA is perfect. The statement follows by induction on $n \geqslant 1$. □

We return to the ring of big Witt vectors. We write $(1 + tA[\![t]\!])^*$ for the multiplicative group of power series over A with constant term 1.

Proposition 1.14. *There is a natural commutative diagram*

$$\begin{array}{ccc} \mathbb{W}(A) & \xrightarrow{\gamma} & (1 + tA[\![t]\!])^* \\ \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle t\frac{d}{dt}\log} \\ A^{\mathbb{N}} & \xrightarrow{\gamma^w} & tA[\![t]\!] \end{array}$$

where

$$\gamma(a_1, a_2, \dots) = \prod_{n \geqslant 1} (1 - a_n t^n)^{-1},$$
$$\gamma^w(x_1, x_2, \dots) = \sum_{n \geqslant 1} x_n t^n,$$

and the horizontal maps are isomorphisms of abelian groups.

Proof. It is clear that γ^w is an isomorphism of additive abelian groups. We show that γ is a bijection. We have

$$\prod_{n \geqslant 1} (1 - a_n t^n)^{-1} = (1 + b_1 t + b_2 t^2 + \dots)^{-1}$$

where the coefficient b_n is given by the sum

$$b_n = \sum (-1)^r a_{i_1} \dots a_{i_r}$$

that runs over all $1 \leqslant i_1 < \dots < i_r \leqslant n$ such that $i_1 + 2i_2 + \dots + ri_r = n$. This formula shows that the coefficients a_n, $n \geqslant 1$, are determined uniquely by the coefficients b_n, $n \geqslant 1$. Indeed, we have the recursive formula

$$a_n = b_n - \sum (-1)^r a_{i_1} \dots a_{i_r},$$

where the sum on the right-hand side ranges over $1 \leqslant i_1 < \dots < i_r < n$ such that $i_1 + 2i_2 + \dots + ri_r = n$. To prove that the map γ is a homomorphism from the additive group $\mathbb{W}(A)$ to the multiplicative group $(1 + tA[\![t]\!])^*$, it suffices as usual to consider the case where A is torsion free. In this case the vertical maps in the diagram of the statement are both injective, and hence, it suffices to show that the diagram of the statement commutes. We calculate:

$$t\frac{d}{dt}\log(\prod_{d\geqslant 1}(1-a_dt^d)^{-1}) = -\sum_{d\geqslant 1} t\frac{d}{dt}\log(1-a_dt^d) = \sum_{d\geqslant 1}\frac{ta_dt^d}{1-a_dt^d}$$
$$= \sum_{d\geqslant 1}\sum_{s\geqslant 0} da_dt^d\cdot a_d^st^{sd} = \sum_{d\geqslant 1}\sum_{q\geqslant 1} da_d^qt^{qd} = \sum_{n\geqslant 1}\big(\sum_{d\mid n} da_d^{n/d}\big)t^n.$$

This completes the proof. □

ADDENDUM 1.15. *The map γ induces an isomorphism of abelian groups*

$$\gamma_S\colon \mathbb{W}_S(A) \xrightarrow{\sim} \Gamma_S(A)$$

where $\Gamma_S(A)$ is the quotient of the multiplicative group $\Gamma(A) = (1+tA[\![t]\!])^*$ by the subgroup $I_S(A)$ of all power series of the form $\prod_{n\in\mathbb{N}\smallsetminus S}(1-a_nt^n)^{-1}$.

PROOF. The kernel of the restriction map

$$R_S^{\mathbb{N}}\colon \mathbb{W}(A) \to \mathbb{W}_S(A)$$

is equal to the subset of all vectors $a = (a_n \mid n \in \mathbb{N})$ such that $a_n = 0$, if $n \in S$. The image of this subset by the map γ is the subset $I_S(A) \subset \Gamma$. □

EXAMPLE 1.16. If $S = \{1, 2, \dots, m\}$, then $I_S(A) = (1 + t^{m+1}A[\![t]\!])^*$. Hence, in this case, Addendum 1.15 gives an isomorphism of abelian groups

$$\gamma_S\colon \mathbb{W}_m(A) \xrightarrow{\sim} \Gamma_S(A) = (1+tA[\![t]\!])^*/(1+t^{m+1}A[\![t]\!])^*.$$

The structure of this group, for A a $\mathbb{Z}_{(p)}$-algebra, was examined in Example 1.11.

LEMMA 1.17. *Let p be a prime number, and let A be any ring. Then the ring homomorphism $F_p\colon \mathbb{W}(A) \to \mathbb{W}(A)$ satisfies that $F_p(a) \equiv a^p$ modulo $p\mathbb{W}(A)$.*

PROOF. We first let $A = \mathbb{Z}[a_1, a_2, \dots]$ and $a = (a_1, a_2, \dots)$. If suffices to show that there exists $b \in \mathbb{W}(A)$ such that $F_p(a) - a^p = pb$. By Lemma 1.9, the element is necessarily unique; we use Lemma 1.1 to prove that it exists. We have

$$w_n(F_p(a) - a^p) = \sum_{d\mid pn} da_d^{pn/d} - \big(\sum_{d\mid n} da_d^{n/d}\big)^p$$

which is clearly congruent to zero modulo pA. So let $x = (x_n \mid n \in \mathbb{N})$ with

$$x_n = \frac{1}{p}(F_p(a) - a^p).$$

We wish to show that $x = w(b)$, for some $b \in \mathbb{W}(A)$. The unique ring homomorphism $\phi_\ell\colon A \to A$ that maps a_n to a_n^ℓ satisfies that $\phi_\ell(f) = f^\ell$ modulo ℓA, and hence, Lemma 1.1 shows that x is in the image of the ghost map if and only if

$$x_n \equiv \phi_\ell(x_{n/\ell})$$

modulo $\ell^{v_\ell(n)}A$, for all primes ℓ and all $n \in \ell\mathbb{N}$. This is equivalent to showing that

$$w_n(F_p(a) - a^p) \equiv \phi_\ell(w_{n/p}(F_p(a) - a^p))$$

modulo $\ell^{v_\ell(n)}A$, if $\ell \neq p$ and $n \in \ell\mathbb{N}$, and modulo $\ell^{v_\ell(n)+1}A$, if $\ell = p$ and $n \in \ell\mathbb{N}$. If $\ell \neq p$, the statement follows from Lemma 1.1, and if $\ell = p$ and $n \in \ell\mathbb{N}$, we calculate

$$\begin{aligned} & w_n(F_p(a) - a^p) - \phi_p(w_{n/p}(F_p(a) - a^p)) \\ &= \sum_{d \mid pn, d \nmid n} d a_d^{pn/d} - (\sum_{d \mid n} d a_d^{n/d})^p + (\sum_{d \mid (n/p)} d a_d^{n/d})^p. \end{aligned}$$

If $d \mid pn$ and $d \nmid n$, then $v_p(d) = v_p(n) + 1$, so the first summand is congruent to zero modulo $p^{v_p(n)+1}A$. Similarly, if $d \mid n$ and $d \nmid (n/p)$, then $v_p(d) = v_p(n)$, and hence,

$$\sum_{d \mid n} d a_d^{n/d} \equiv \sum_{d \mid (n/p)} d a_d^{n/d}$$

modulo $p^{v_p(n)}A$. But then

$$(\sum_{d \mid n} d a_d^{n/d})^p \equiv (\sum_{d \mid (n/p)} d a_d^{n/d})^p$$

modulo $p^{v_p(n)+1}A$; compare the proof of Lemma 1.1. This completes the proof. □

Let $\epsilon\colon \mathbb{W}(A) \to A$ be the ring homomorphism that takes $a = (a_n \mid n \in \mathbb{N})$ to a_1.

Proposition 1.18. *There exists a unique natural ring homomorphism*

$$\Delta\colon \mathbb{W}(A) \to \mathbb{W}(\mathbb{W}(A))$$

such that $w_n(\Delta(a)) = F_n(a)$, for all $n \in \mathbb{N}$. Moreover, the functor $\mathbb{W}(-)$ and the ring homomorphisms Δ and ϵ form a comonad on the category of rings.

Proof. By naturality, we may assume that A is torsion free. Then Lemma 1.9 shows that also $\mathbb{W}(A)$ is torsion free, and hence, the ghost map

$$w\colon \mathbb{W}(\mathbb{W}(A)) \to \mathbb{W}(A)^{\mathbb{N}}$$

is injective. Lemma 1.17 and Lemma 1.1 show that the sequence $(F_n(a) \mid a \in \mathbb{N})$ is in the image of the ghost map. Hence, the natural ring homomorphism Δ exists. The second part of the statement means that

$$\mathbb{W}(\Delta_A) \circ \Delta_A = \Delta_{\mathbb{W}(A)} \circ \Delta_A \colon \mathbb{W}(A) \to \mathbb{W}(\mathbb{W}(\mathbb{W}(A)))$$

and

$$\mathbb{W}(\epsilon_A) \circ \Delta_A = \epsilon_{\mathbb{W}(A)} \circ \Delta_A \colon \mathbb{W}(A) \to \mathbb{W}(A).$$

Both equalities are readily verified by evaluating the ghost coordinates. □

Definition 1.19. A *special λ-ring* is a ring A and a ring homomorphism

$$\lambda\colon A \to \mathbb{W}(A)$$

that makes A a coalgebra over the comonad $(\mathbb{W}(-), \Delta, \epsilon)$.

Let $(A, \lambda\colon A \to \mathbb{W}(A))$ be a special λ-ring. Then the associated nth Adams operation is the ring homomorphism defined to be the composition

$$\psi^n\colon A \xrightarrow{\lambda} \mathbb{W}(A) \xrightarrow{w_n} A$$

of the structure map and the nth ghost map.

The record the following theorem proved independently by Borger [**2**, Thm. A, Cor. 9.5] and van der Kallen [**12**, Thm. 2.4]

Theorem 1.20. *Let $f\colon A \to B$ be an étale morphism, let S be a finite truncation set, and let n be a positive integer. Then the morphism*

$$\mathbb{W}_S(f)\colon \mathbb{W}_S(A) \to \mathbb{W}_S(B)$$

is étale and the diagram

$$\begin{array}{ccc} \mathbb{W}_S(A) & \xrightarrow{\mathbb{W}_S(f)} & \mathbb{W}_S(B) \\ \downarrow{\scriptstyle F_n} & & \downarrow{\scriptstyle F_n} \\ \mathbb{W}_{S/n}(A) & \xrightarrow{\mathbb{W}_{S/n}(f)} & \mathbb{W}_{S/n}(B) \end{array}$$

is cartesian. □

We remark that in loc. cit., the theorem above is stated only for those finite truncation sets that consist of all divisors of a given positive integer. These truncation sets, however, include all finite p-typical truncation sets, and therefore, the general case follows immediately by applying Prop. 1.10.

2. The de Rham-Witt complex

In this section, we introduce the big de Rham-Witt complex. More precisely, we fix a truncation set U and define a de Rham-Witt complex $\mathbb{W}_S^U\Omega_A^q$ for every sub-truncation set $S \subset U$. The big de Rham-Witt complex and the p-typical de Rham-Witt complex correspond to the cases where U is the set of all positive integers and the set of all powers of the single prime number p, respectively. We prove in Cor. 5.7 below that, up to canonical isomorphism, $\mathbb{W}_S^U\Omega_A^q$ depends only on S and not on the truncation set $U \supset S$.

We recall that a truncation set is a subset $S \subset \mathbb{N}$ of the set of positive integers with the property that if $n \in S$, and if $d \in \mathbb{N}$ divides n, then also $d \in S$. The set J of truncation sets is partially ordered under inclusion. We consider J as a category with one morphism from T to S if $T \subset S$. Then, if A is a ring, the assignment $S \mapsto \mathbb{W}_S(A)$ defines a contravariant functor from the category J to the category of rings, and this functor takes colimits in the category J to limits in the category of rings. Moreover, the assignment $S \mapsto S/n$ is an endo-functor on the category J, and the ring homomorphism

$$F_n \colon \mathbb{W}_S(A) \to \mathbb{W}_{S/n}(A)$$

and the abelian group homomorphism

$$V_n \colon \mathbb{W}_{S/n}(A) \to \mathbb{W}_S(A)$$

are natural transformations with respect to the variable S.

Definition 2.1. Let U be a fixed truncation set. A *U-Witt complex* over A is a contravariant functor

$$S \mapsto E_S^{\cdot}$$

that to every sub-truncation set $S \subset U$ assigns an anti-symmetric graded ring $E_S^{\cdot}$ and that takes colimits to limits together with a natural ring homomorphism

$$\lambda \colon \mathbb{W}_S(A) \to E_S^0$$

and natural maps of graded abelian groups

$$d \colon E_S^q \to E_S^{q+1}$$
$$F_n \colon E_S^q \to E_{S/n}^q$$
$$V_n \colon E_{S/n}^q \to E_S^q$$

such that the following (i)–(v) hold:

(i) For all $x \in E_S^q$, $y \in E_S^{q'}$, and $a \in A$,

$$d(x \cdot y) = d(x) \cdot y + (-1)^q x \cdot d(y)$$
$$d(d(x)) = d\log\lambda([-1]_S) \cdot d(x)$$
$$d\lambda([a]_S) \cdot d\lambda([a]_S) = d\log\lambda([-1]_S)\lambda([a]_S)d\lambda([a]_S).$$

(ii) For all positive integers m and n,

$$F_1 = V_1 = \mathrm{id}, \quad F_mF_n = F_{mn}, \quad V_nV_m = V_{mn},$$
$$F_nV_n = n \cdot \mathrm{id}, \quad F_mV_n = V_nF_m \quad \text{if } (m,n) = 1.$$
$$F_n\lambda = \lambda F_n, \qquad V_n\lambda = \lambda V_n,$$

(iii) For all positive integers n, the map F_n is a ring homomorphism, and the maps F_n and V_n satisfy the projection formula that for all $x \in E_S^q$ and $y \in E_{S/n}^{q'}$,

$$x \cdot V_n(y) = V_n(F_n(x) \cdot y).$$

(iv) For all positive integers n and all $y \in E_{S/n}^q$,

$$F_n dV_n(y) = d(y) + (n-1) d\log \lambda([-1]_{S/n}) \cdot y.$$

(v) For all positive integers n and all $a \in A$,

$$F_n d\lambda([a]_S) = \lambda([a]_{S/n}^{n-1}) d\lambda([a]_{S/n}).$$

A map of U-Witt complexes is a natural map of graded rings

$$f \colon E_S^{\cdot} \to E_S'^{\cdot}$$

such that $f\lambda = \lambda' f$, $fd = d'f$, $fF_n = F_n' f$, and $fV_n = V_n' f$.

Remark 2.2. We abbreviate $d\log[-1]_S = d\log\lambda([-1]_S)$. This element is not always zero, but it is zero in many interesting cases. We mention the following:

(a) If 2 is invertible in A, then $d\log[-1]_S = 0$. Indeed, since d is a derivation,

$$2d\log[-1]_S = d\log[1]_S = 0.$$

(b) If $2 = 0$ in A, then $d\log[-1]_S = d\log[1]_S = 0$.

(c) If all $n \in S$ are odd, then $d\log[-1]_S = 0$. Indeed, in general,

$$[-1]_S = -[1]_S + V_2([1]_{S/2}).$$

If $d\log[-1]_S$ is zero, then d is a differential and makes $E_S^{\cdot}$ an anti-symmetric differential graded ring. We also note that every U-Witt functor is determined by its value on finite sub-truncation sets $S \subset U$.

Lemma 2.3. *Let m and n be positive integers, let $c = (m,n)$ be the greatest common divisor, and let k and l be any pair of integers such that $km + ln = c$. The following relations hold in every U-Witt complex:*

$dF_n = nF_n d, \quad V_n d = ndV_n,$

$F_m dV_n = kdF_{m/c}V_{n/c} + lF_{m/c}V_{n/c}d + (c-1)d\log[-1]_{S/m} \cdot F_{m/c}V_{n/c},$

$d\log[-1]_S = \sum_{r \geqslant 1} 2^{r-1} dV_{2^r}([1]_{S/2^r}),$

$d\log[-1]_S \cdot d\log[-1]_S = 0, \quad dd\log[-1]_S = 0,$

$F_n(d\log[-1]_S) = d\log[-1]_{S/n}, \quad V_n(d\log[-1]_{S/n}) = V_n([1]_{S/n}) d\log[-1]_S.$

Proof. The calculation

$$\begin{aligned}
dF_n(x) &= F_n dV_n F_n(x) - (n-1)d\log[-1] \cdot F_n(x) \\
&= F_n d(V_n([1]) \cdot x) - (n-1)d\log[-1] \cdot F_n(x) \\
&= F_n(dV_n([1]) \cdot x + V_n([1]) \cdot dx) - (n-1)d\log[-1] \cdot F_n(x) \\
&= F_n dV_n([1]) \cdot F_n(x) + F_n V_n([1]) \cdot F_n d(x) - (n-1)d\log[-1] \cdot F_n(x) \\
&= (n-1)d\log[-1] \cdot F_n(x) + nF_n d(x) - (n-1)d\log[-1] \cdot F_n(x) \\
&= nF_n d(x)
\end{aligned}$$

verifies the first relation, and the second relation is verified similarly. Next,

$$\begin{aligned} F_m(d\log[-1]_S) &= F_m([-1]_S^{-1}d[-1]_S) = F_m([-1]_S^{-1})F_m d([-1]_S) \\ &= [-1]_{S/m}^{-m}[-1]_{S/m}^{m-1}d[-1]_{S/m} = [-1]_{S/m}^{-1}d[-1]_{S/m} \\ &= d\log[-1]_{S/m}, \end{aligned}$$

which shows the next to last formula, and the last formula then follows from the projection formula. Using the four relations proved thus far, we find

$$\begin{aligned} F_m dV_n(x) &= F_{m/c}F_c dV_c V_{n/c}(x) \\ &= F_{m/c}dV_{n/c}(x) + (c-1)d\log[-1]\cdot F_{m/c}V_{n/c}(x) \\ &= (k(m/c)+l(n/c))F_{m/c}dV_{n/c}(x) + (c-1)d\log[-1]\cdot F_{m/c}V_{n/c}(x) \\ &= kdF_{m/c}V_{n/c}(x) + lF_{m/c}V_{n/c}d(x) + (c-1)d\log[-1]\cdot F_{m/c}V_{n/c}(x). \end{aligned}$$

To prove the formula for $d\log[-1]_S$, we recall from Addendum 1.7 that

$$[-1]_S = -[1]_S + V_2([1]_{S/2}).$$

This is also easily verified by evaluating ghost coordinates. Hence,

$$\begin{aligned} d\log[-1]_S &= [-1]_S d[-1]_S = (-[1]_S + V_2([1]_{S/2})d(-[1]_S + V_2([1]_{S/2}) \\ &= -dV_2([1]_{S/2}) + V_2(F_2 dV_2([1]_{S/2})) \\ &= -dV_2([1]_{S/2}) + V_2(d\log[-1]_{S/2}) \\ &= dV_2([1]_{S/2}) + V_2(d\log[-1]_{S/2}) \end{aligned}$$

where the last equality uses that $2dV_2([1]_{S/2}) = Vd([1]_{S/2}) = 0$. The stated formula now follows by easy induction. Using this formula, we find

$$\begin{aligned} dV_2 d\log[-1]_{S/2} &= \sum_{r\geqslant 1} 2^r ddV_{2^{r+1}}([1]_{S/2^{r+1}}) \\ &= \sum_{r\geqslant 1} 2^r d\log[-1]_S \cdot dV_{2^{r+1}}([1]_{S/2^{r+1}}) \end{aligned}$$

which is zero, since $2d\log[-1]_S = 0$. The calculation

$$\begin{aligned} (d\log[-1]_S)^2 &= (d[-1]_S)^2 = (dV_2([1]_{S/2}))^2 \\ &= d(V_2([1]_{S/2})dV_2([1]_{S/2})) - V_2([1]_{S/2})ddV_2([1]_{S/2}) \\ &= dV_2 d\log[-1]_{S/2} - V_2([1]_{S/2})dV_2 d\log[-1]_{S/2} \\ &= dV_2 d\log[-1]_{S/2}([1]_S - V_2([1]_{S/2})) \end{aligned}$$

then shows that $(d\log[-1]_S)^2$ is zero as stated. This, in turn, shows that

$$\begin{aligned} dd\log[-1]_S &= d([-1]_S d[-1]_S) = d[-1]_S d[-1]_S + [-1]_S dd[-1]_S \\ &= [-1]_S dd[-1]_S = [-1]_S d\log[-1]_S d[-1]_S = d[-1]_S d[-1]_S = 0. \end{aligned}$$

This completes the proof. □

Corollary 2.4. *Let $E_S^{\cdot}$ be a U-Witt complex and define $\hat{d}\colon E_S^q \to E_S^{q+1}$ by*

$$\hat{d}(x) = \begin{cases} d(x), & \text{if } q \text{ is even,} \\ d(x) + d\log[-1]_S \cdot x, & \text{if } q \text{ is odd.} \end{cases}$$

Then $\hat{d}$ is a differential and a derivation for the product on $E_S^{\cdot}$.

Proof. To see that $\hat{d}$ is a differential, we calculate:

$$\begin{aligned} d(d(x) + d\log[-1]_S \cdot x) &= dd(x) - d\log[-1]_S \cdot d(x) \\ &= d(x)\cdot d\log[-1]_S - d\log[-1]_S \cdot d(x) = 0, \\ d(d(x)) + d\log[-1]_S \cdot d(x) &= d\log[-1]_S \cdot d(x) + d\log[-1]_S \cdot d(x) = 0, \end{aligned}$$

where we have used that $dd\log[-1]_S = 0$. We leave it to the reader to verify that $\hat{d}$ is a derivation. □

We recall that a graded ring $E^\cdot$ is said to be anti-symmetric if, for every pair of homogeneous elements x and y,

$$x \cdot y = (-1)^{|x||y|} y \cdot x,$$

and alternating if, in addition, the square of every homogeneous element of odd degree is equal to zero. The two notions agree, if 2 is invertible in E^0. If A is a ring and M an A-module, then the graded ring

$$\hat{\Lambda}_A(M) = T_A(M)/(x \otimes y - y \otimes x \mid x, y \in M)$$

is the free anti-symmetric graded A-algebra generated by M. We let $d\colon A \to \Omega^1_A$ be the universal derivation from A to an A-module and consider the graded A-algebra

$$\hat{\Omega}^\cdot_A = \hat{\Lambda}_A \Omega^1_A.$$

The derivation $d\colon A \to \Omega^1_A$ gives rise to a differential on $\hat{\Omega}^\cdot_A$ defined by

$$d(a_0 da_1 \dots da_q) = da_0 da_1 \dots da_q,$$

where $a_0 da_1 \dots da_q$ denotes the class of $a_0 da_1 \otimes \dots \otimes da_q$ in $\hat{\Omega}^q_A$. The differential d makes $\hat{\Omega}^\cdot_A$ an anti-symmetric differential graded ring. Suppose that $E^\cdot$ is an anti-symmetric differential graded ring and that $\lambda\colon A \to E^0$ is a ring homomorphism. Then there exists a unique map of differential graded rings

$$\hat{\Omega}^\cdot_A \to E^\cdot$$

that extends the given map $\lambda\colon A \to E^0$. We let $E^\cdot_S$ be a U-Witt complex over A and consider $E^\cdot_S$ as an anti-symmetric differential graded ring with respect to the differential $\hat{d}$ of Cor. 2.4. Then there is a unique map of differential rings

$$\hat{\Omega}^\cdot_{\mathbb{W}_S(A)} \to E^\cdot_S$$

that extends the given map $\lambda\colon \mathbb{W}_S(A) \to E^0_S$.

Let U be a truncation set. We now prove that the category of U-Witt complexes has an initial object $\mathbb{W}^U_S\Omega^\cdot_A$. If $U \subset U'$ are two truncation sets, then for every sub-truncation set $S \subset U$, there is a canonical map

$$\mathbb{W}^U_S\Omega^\cdot_A \to \mathbb{W}^{U'}_S\Omega^\cdot_A.$$

We prove in Cor. 5.7 below that this map is an isomorphism, and thereafter, we willl write $\mathbb{W}_S\Omega^\cdot_A$ instead of $\mathbb{W}^U_S\Omega^\cdot_A$.

Proposition 2.5. *Let U be a truncation set. Then the category $\mathscr{W}^U_A$ of U-Witt complexes over A has an initial object $\mathbb{W}^U_S\Omega^\cdot_A$. Moreover, the canonical map*

$$\hat{\Omega}^\cdot_{\mathbb{W}_S(A)} \to \mathbb{W}^U_S\Omega^\cdot_A$$

is a surjection.

Proof. We use Freyd's adjoint functor theorem [**10**, Chap. V, §6, Thm. 1] to show that $\mathscr{W}_A^U$ has an initial object. Clearly, $\mathscr{W}_A^U$ has all small limits, so we must verify the solution set condition: There exists a *set* $\{E(\alpha)_S^\cdot \mid \alpha \in \mathscr{A}\}$ of objects in $\mathscr{W}_A^U$, such that for every object $E_S^\cdot$ of $\mathscr{W}_A^U$, there exists a morphism in $\mathscr{W}_A^U$

$$f\colon E(\alpha)_S^\cdot \to E_S^\cdot,$$

for some $\alpha \in \mathscr{A}$. To this end, we show that the image $I_S^\cdot$ of the canonical map

$$\hat{\Omega}_{\mathbb{W}_S(A)}^\cdot \to E_S^\cdot$$

is a sub-U-Witt complex of $E_S^\cdot$. Since the image is canonically isomorphic to a quotient of $\hat{\Omega}_{\mathbb{W}_S(A)}^\cdot$ and since there are only a set of such quotients, it will follow that the solution set condition is verified. By definition, $I_S^\cdot \subset E_S^\cdot$ is a sub-differential graded ring, and since $d\log[-1]_S \in I_S^1$, the operator $d\colon E_S^q \to E_S^{q+1}$ restricts to an operator $d\colon I_S^q \to I_S^{q+1}$. Moreover, the restriction $R_T^S\colon E_S^\cdot \to E_T^\cdot$ restricts to a map of graded rings $R_T^S\colon I_S^\cdot \to I_T^\cdot$. It remains only to show that the Frobenius and Verschiebung satisfy $F_m(I_S^\cdot) \subset I_{S/m}^\cdot$ and $V_m(I_{S/m}^\cdot) \subset I_S^\cdot$. Every element of $I_{S/m}^q$ can be written, non-uniquely, as a sum of elements of the form

$$\omega = \lambda(a_0)d\lambda(a_1)\dots d\lambda(a_q),$$

where $a_0, \dots, a_q \in \mathbb{W}_{S/m}(A)$. We have

$$\begin{aligned} V_m(\lambda(a)d\lambda(a')) &= V_m(\lambda(a)F_m dV_m(\lambda(a')) - (m-1)d\log[-1]_{S/m}\lambda(a')) \\ &= \lambda(V_m(a))d\lambda(V_m(a')) - (m-1)d\log[-1]_S\lambda(V_m(aa')). \end{aligned}$$

Since $d\log[-1]_S d\log[-1]_S = 0$, this shows that

$$\begin{aligned} &V_m(\omega) = \lambda(V_m(a_0))d\lambda(V_m(a_1))\dots d\lambda(V_m(a_q)) \\ &+ (m-1)d\log[-1]_S \sum_{1\leqslant i\leqslant q} \lambda(V_m(a_0a_i))d\lambda(V_m(a_1))\dots d\widehat{\lambda(V_m(a_i))}\dots d\lambda(V_m(a_q)) \end{aligned}$$

which is an element of I_S^m as desired. Since F_m is multiplicative, it suffices to show that it maps each factor on the right-hand side of the formula for ω to an element of $I_{S/m}^q$. We have $F_m(\lambda(a_0)) = \lambda(F_m(a_0))$ which proves the statement for the first factor. For the remaining factors, we write out $a_i \in \mathbb{W}_S(A)$ as a sum

$$a_i = \sum_{n\in S} V_n([a_{i,n}]_{S/n}).$$

Let $c = (m,n)$ be the greatest common divisor of m and n. We then have

$$\begin{aligned} &F_m d\lambda(V_n([a_{i,n}]_{S/n})) = F_{m/c}F_c dV_c V_{n/c}\lambda([a_{i,n}]_{S/n})) \\ &= F_{m/c}dV_{n/c}\lambda([a_{i,n}]_{S/n}) - d\log[-1]_{S/m}\lambda(F_{m/c}V_{n/c}([a_{i,n}])). \end{aligned}$$

The second summand in the lower line is in $I_{S/m}^1$. To prove that this holds also for the first summand, we choose k and l such that $km + ln = c$. Then

$$\begin{aligned} &F_{m/c}dV_{n/c}\lambda([a_{i,n}]_{S/n}) = (km/c + ln/c)F_{m/c}dV_{n/c}\lambda([a_{i,n}]_{S/n}) \\ &= kd\lambda(F_{m/c}V_{n/c}([a_{i,n}]_{S/n})) + lF_{m/c}V_{n/c}d\lambda([a_{i,n}]_{S/n}). \end{aligned}$$

The left-hand term on the bottom line is in $I_{S/m}^1$, and to see that the same is true for the right-hand term, we write

$$\begin{aligned} &lF_{m/c}V_{n/c}d\lambda([a_{i,n}]_{S/n}) = lV_{n/c}F_{m/c}d\lambda([a_{i,n}]_{S/n}) \\ &= lV_{n/c}(\lambda([a_{i,n}]_{S/e}^{m/c-1})d\lambda([a_{i,n}]_{S/e})), \end{aligned}$$

where $e = mn/c$. Since $V_{n/c}(I^1_{S/e}) \subset I^1_{S/m}$, we find that $F_m(I^q_S) \subset I^q_{S/m}$ as desired. We conclude that $\mathscr{W}^U_A$ has an initial object $\mathbb{W}^U_S\Omega^{\cdot}_A$ as stated.

Finally, the argument above shows that the image $I^{\cdot}_S$ of the canonical map

$$\hat{\Omega}^{\cdot}_{\mathbb{W}_S(A)} \to \mathbb{W}^U_S\Omega^{\cdot}_A$$

is a sub-U-Witt complex of $\mathbb{W}^U_S\Omega^{\cdot}_A$. Hence, there exists a map $r\colon \mathbb{W}^U_S\Omega^{\cdot}_A \to I^{\cdot}_S$ of U-Witt complexes over A. The composite map

$$\mathbb{W}^U_S\Omega^{\cdot}_A \xrightarrow{r} I^{\cdot}_S \hookrightarrow \mathbb{W}^U_S\Omega^{\cdot}_A$$

is an endomorphism of the initial object of $\mathscr{W}_A$. But the only endomorphism of an initial object is the identity map. It follows that the inclusion of $I^{\cdot}_S$ into $\mathbb{W}^U_S\Omega^{\cdot}_A$ is surjective, and hence, an isomorphism. □

Remark 2.6. Suppose that A is an algebra over a ring k. We define a U-Witt complex over A relative to k to be U-Witt complex E over A such that the derivation $d\colon E^q_S \to E^{q+1}_S$ is $\mathbb{W}_S(k)$-linear. The proof of Prop. 2.5 shows that the category of U-Witt complexes over A relative to k has an initial object $\mathbb{W}^U_S\Omega^{\cdot}_{A/k}$, the relative de Rham-Witt complex. This complex was constructed and studied by Langer and Zink [**9**], at least for p-typical U. There is a canonical surjection

$$\mathbb{W}^U_S\Omega^q_A \to \mathbb{W}^U_S\Omega^q_{A/k}$$

from the absolute de Rham-Witt complex onto the relative de Rham-Witt complex. In general, it is *not* an isomorphism for $k = \mathbb{Z}$; compare [**5**].

Lemma 2.7. *The map* $\lambda\colon \mathbb{W}_S(A) \to \mathbb{W}^U_S\Omega^0_A$ *is an isomorphism.*

Proof. We may define a U-Witt complex with E^q_S equal to $\mathbb{W}_S(A)$, for $q = 0$, equal to zero, for $q > 0$, and with $\lambda\colon \mathbb{W}_S(A) \to E^0_S$ the identity map. Hence, there is a unique map of UWitt complexes $\mathbb{W}^U_S\Omega^q \to E^q_S$. The composite map

$$\hat{\Omega}^q_{\mathbb{W}_S(A)} \to \mathbb{W}^U_S\Omega^q_A \to E^q_S$$

is an isomorphism, for $q = 0$, and the left-hand map is surjective. But then the left-hand map is an isomorphism, for $q = 0$, as desired. □

Remark 2.8. If $U = \mathbb{N}$ or $U = P$, the axiom that

$$d\lambda([a]_S) \cdot d\lambda([a]_S) = d\log\lambda([-1]_S)\lambda([a]_S)d\lambda([a]_S)$$

is a consequence of the remaining axioms. Indeed, given a sub-truncation set $S \subset \mathbb{N}$, we can find a sub-truncation set $R \subset \mathbb{N}$ such that $S = R/2$, and then

$$\begin{aligned} d\lambda([a]_S) \cdot d\lambda([a]_S) &= d(\lambda([a]_S)d\lambda([a]_S)) - \lambda([a]_S)dd\lambda([a]_S) \\ &= dF_2 d\lambda([a]_R) - d\log\lambda([-1]_S)\lambda([a]_S)d\lambda([a]_S) \\ &= 2F_2 dd\lambda([a]_R) - d\log\lambda([-1]_S)\lambda([a]_S)d\lambda([a]_S) \\ &= d\log\lambda([-1]_S)\lambda([a]_S)d\lambda([a]_S) \end{aligned}$$

since $2d\log\lambda([-1]_S)$ is zero. The case $U = P$ is similar.

3. p-typical decomposition

We recall from Prop. 1.10 that the ring of big Witt vectors of a $\mathbb{Z}_{(p)}$-algebra decomposes as a product of rings of p-typical Witt vectors. We show in Prop. 3.6 that the big de Rham-Witt complex of a $\mathbb{Z}_{(p)}$-algebra decomposes similarly. Let again $P = \{1, p, p^2, \dots\}$ be the set of all powers of the single prime number p.

Let A be a $\mathbb{Z}_{(p)}$-algebra, and let $i^*\colon \mathscr{W}_A^U \to \mathscr{W}_A^{U\cap P}$ be the forgetful functor from the category of U-Witt complexes over A to the category of $U \cap P$-Witt complexes over A. We define a functor

$$i_!\colon \mathscr{W}_A^{U\cap P} \to \mathscr{W}_A^U \tag{3.1}$$

as follows. Let D be a $U \cap P$-Witt complex over A. We define

$$(i_!D)_S^q = \prod_{k\in I(S)} D^q_{S/k\cap P}$$

with componentwise multiplication. Here, we recall, $I(S)$ denotes the set of $k \in S$ such that p does not divide k. We define the ring homomorphism

$$\lambda\colon \mathbb{W}_S(A) \to \prod_{k\in I(S)} \mathbb{W}_{S/k\cap P}(A) \to \prod_{k\in I(S)} D^0_{S/k\cap P} = (i_!D)^0_S$$

to be the composition of the ring isomorphism of Prop. 1.10 and the product of the given ring homomorphisms $\lambda\colon \mathbb{W}_{S/k\cap P}(A) \to D^0_{S/k\cap P}$. If $S' \subset S$ is a sub-truncation set, then $I(S') \subset I(S)$, and we define the restriction map

$$R^S_{S'}\colon (i_!D)^q_S = \prod_{k\in I(S)} D^q_{S/k\cap P} \to \prod_{k\in I(S')} D^q_{S/k\cap P} \to \prod_{k\in I(S')} D^q_{S'/k\cap P} = (i_!D)^q_{S'}$$

to be the composition of the canonical projection and the product of the given restriction maps $R^{S/k\cap P}_{S'/k\cap P}\colon D^q_{S/k\cap P} \to D^q_{S'/k\cap P}$. We define the derivation

$$d\colon (i_!D)^q_S = \prod_{k\in I(S)} D^q_{S/k\cap P} \to \prod_{k\in I(S)} D^{q+1}_{S/k\cap P} = (i_!D)^{q+1}_S$$

to be the product of the maps $\frac{1}{k}d\colon D^q_{S/k\cap P} \to D^{q+1}_{S/k\cap P}$. If n is a positive integer, and if $n = p^s h$ with h prime to p, then we define the Frobenius operator

$$F_n\colon (i_!D)^q_S = \prod_{k\in I(S)} D^q_{S/k\cap P} \to \prod_{hl\in I(S)} D^q_{S/hl\cap P} \to \prod_{l\in I(S/n)} D^q_{S/p^s hl\cap P} = (i_!D)^q_{S/n}$$

to be the composition of the canonical projection and the product of the given Frobenius maps $F_{p^s}\colon D^q_{S/hl\cap P} \to D^q_{S/p^s hl\cap P}$. We define the Verschiebung map

$$V_n\colon (i_!D)^q_{S/n} = \prod_{l\in I(S/n)} D^q_{S/p^s hl\cap P} \to \prod_{hl\in I(S)} D^q_{S/hl\cap P} \to \prod_{k\in I(S)} D^q_{S/k\cap P} = (i_!D)^q_S$$

to be the composition of the product of the maps $hV_{p^s}\colon D^q_{S/p^s hl\cap P} \to D^q_{S/hl\cap P}$ and the canonical inclusion.

LEMMA 3.2. *Let A be a $\mathbb{Z}_{(p)}$-algebra. Then $i_!\colon \mathscr{W}_A^{U\cap P} \to \mathscr{W}_A^U$ is a well-defined functor and $(i^*, i_!)$ forms an adjoint pair of functors.*

PROOF. We leave it to the reader to verify the first statement and prove the second. Let E be a U-Witt complex over A, and let D be a $U \cap P$-Witt complex over A. We define a the unit of the adjunction to be the map

$$\eta\colon E^q_S \to (i_! i^* E)^q_S = \prod_{k \in I(S)} E^q_{S/k \cap P}$$

whose kth component is $R^{S/k}_{S/k\cap P} \circ F_k$, and the counit to be the identity map

$$\epsilon\colon (i^* i_! D)^q_T = D^q_T \to D^q_T.$$

Now, let $f\colon i^*E \to D$ be a map of $U \cap P$-Witt complexes over A. It follows immediately from the definitions that the composition

$$i^*E \xrightarrow{i^*\eta} i^* i_! i^* E \xrightarrow{i^* i_! f} i^* i_! D \xrightarrow{\epsilon} D$$

is equal to f as required. Finally, let $g\colon E \to i_! D$ be a map of U-Witt complexes over A. Then the following diagram commutes.

$$\begin{array}{ccc} E^q_S & \xrightarrow{g^q_S} & \prod_{d \in I(S)} D^q_{S/d\cap P} \\ \downarrow{\scriptstyle F_k} & & \downarrow{\scriptstyle \mathrm{pr}} \\ E^q_{S/k} & \xrightarrow{g^q_{S/k}} & \prod_{l \in I(S/k)} D^q_{S/kl\cap P} \\ \downarrow{\scriptstyle R^{S/k}_{S/k\cap P}} & & \downarrow{\scriptstyle \mathrm{pr}} \\ E^q_{S/k\cap P} & \xrightarrow{g^q_{S/k\cap P}} & D^q_{S/k\cap P}. \end{array}$$

The composition of the top horizontal map and the right-hand vertical maps is the kth component of the map $g\colon E \to i_! D$, and the composition of the left-hand vertical maps and the lower horizontal map is the kth component of the composition

$$E \xrightarrow{\eta} i_! i^* E \xrightarrow{i_! i^* g} i_! i^* i_! D \xrightarrow{i_! \epsilon} i_! D.$$

Hence, the two maps are equal as required. □

COROLLARY 3.3. *Let A be a $\mathbb{Z}_{(p)}$-algebra, and let U be a truncation set. Then for every sub-truncation set $T \subset U \cap P$, the canonical map*

$$\mathbb{W}^{U\cap P}_T \Omega^{\cdot}_A \to \mathbb{W}^U_T \Omega^{\cdot}_A$$

is an isomorphism.

PROOF. The map of the statement is the unique map from the initial object in the category $\mathscr{W}^{U\cap P}_A$ to the image by the functor i^* of the initial object in the category $\mathscr{W}^U_A$. Since i^* admits the right adjoint $i_!$, it preserves initial objects. □

We again assume that A is a $\mathbb{Z}_{(p)}$-algebra and define a new functor

$$i^! \colon \mathscr{W}^U_A \to \mathscr{W}^{U\cap P}_A \tag{3.4}$$

as follows. Given a sub-truncation set $T \subset U \cap P$, we define $[T] \subset U$ to be the largest sub-truncation set with the property that $[T] \cap P = T$. Let $e_1 \in \mathbb{W}_{[T]}(A)$ be the idempotent defined in the statement of Prop. 1.10 and recall that the composition

$$\alpha_T \colon \mathbb{W}_{[T]}(A) \cdot e_1 \hookrightarrow \mathbb{W}_{[T]}(A) \xrightarrow{R^{[T]}_T} \mathbb{W}_T(A)$$

is an isomorphism. Then, given a U-Witt complex E over A and a sub-truncation set $T \subset U \cap P$, we define

$$(i^! E)^q_T = E^q_{[T]} \cdot \lambda(e_1) \subset E^q_{[T]}$$

and define the ring homomorphism

$$\lambda\colon \mathbb{W}_T(A) \to \mathbb{W}_{[T]}(A) \cdot e_1 \to E^0_{[T]} \cdot \lambda(e_1) = (i^! E)^0_T$$

to be the composition of the inverse of the isomorphism α_T and the map induced from the given map $\lambda\colon \mathbb{W}_{[T]}(A) \to E^0_{[T]}$. The derivation $d\colon E^q_{[T]} \to E^{q+1}_{[T]}$ annihilates $\lambda(e_1)$, and hence, restricts to a derivation

$$d\colon (i^! E)^q_T \to (i^! E)^{q+1}_T.$$

The restriction map $R^{[T]}_{[T]/p}\colon E^*_{[T]} \to E^*_{[T]/p}$ maps $R^{[T]}_{[T]/p}(\lambda(e_1)) = \lambda(e_1)$, and hence, defines a restriction map

$$R^T_{T/p}\colon (i^! E)^q_T \to (i^! E)^q_{T/p}.$$

Finally, the Frobenius map $F_p\colon E^*_{[T]} \to E^*_{[T]/p}$ maps $F_p(\lambda(e_1)) = \lambda(e_1)$, and hence, F_p and V_p define Frobenius and Verschiebung maps

$$F_p\colon (i^! E)^q_T \to (i^! E)^q_{T/p}$$
$$V_p\colon (i^! E)^q_{T/p} \to (i^! E)^q_T.$$

LEMMA 3.5. *Let A be a $\mathbb{Z}_{(p)}$-algebra. Then $i^!\colon \mathscr{W}^U_A \to \mathscr{W}^{U\cap P}_A$ is a well-defined functor and $(i_!, i^!)$ forms an adjoint pair of functors.*

PROOF. Only the second statement needs proof. The composition

$$(i^! i_! D)^q_T = \Big(\prod_{k\in I([T])} D^q_{[T]/p\cap P}\Big) \cdot \lambda(e_1) \hookrightarrow \prod_{k\in I([T])} D^q_{[T]/p\cap P} \xrightarrow{\mathrm{pr}_1} D^q_T$$

is an isomorphism, and we define the unit map $\eta\colon D^q_T \to (i^! i_! D)^q_T$ to be the inverse map. Let E be a big Witt complex over A, and let k be a positive integer that is not divisible by p. We claim that the map $\frac{1}{k}V_k\colon E^q_{S/k} \to E^q_S$ restricts to a map

$$\tfrac{1}{k}V_k\colon E^q_{S/k} \cdot \lambda(e_1) \to E^q_S \cdot \lambda(e_k).$$

Indeed, we have $\frac{1}{k}V_k(\lambda(e_1)) = \lambda(e_k)$, and therefore, that

$$\tfrac{1}{k}V_k(x\lambda(e_1)) = \tfrac{1}{k}V_k(F_k(\tfrac{1}{k}V_k(x)) \cdot \lambda(e_1)) = \tfrac{1}{k}V_k(x)\tfrac{1}{k}V_k(\lambda(e_1)) = \tfrac{1}{k}V_k(x)\lambda(e_k).$$

We now define the counit $\epsilon\colon (i_! i^! E)^q_S \to E^q_S$ to be the composition

$$\prod_{k\in I(S)} E^q_{[S/k\cap P]} \cdot \lambda(e_1) \to \prod_{k\in I(S)} E^q_{[S/k\cap P]} \cdot \lambda(e_k) \to \prod_{k\in I(S)} E^q_S \cdot \lambda(e_k)$$

of the product of the maps $\frac{1}{k}V_k$ and the product of the maps $R^{[S/k\cap P]}_S$. One readily verifies that this makes $(i_!, i^!)$ an adjoint pair. □

PROPOSITION 3.6. *Let A be a $\mathbb{Z}_{(p)}$-algebra, let S a truncation set, and let $I(S)$ be the set of $k \in S$ with k not divisible by p. Then there is a canonical isomorphism*

$$\mathbb{W}^U_S \Omega^q_A \xrightarrow{\sim} \prod_{k\in I(S)} \mathbb{W}^{U\cap P}_{S/k\cap P} \Omega^q_A$$

whose kth component is the composite map

$$\mathbb{W}_S^U\Omega_A^q \xrightarrow{F_k} \mathbb{W}_{S/k}^U\Omega_A^q \xrightarrow{R^{S/k}_{S/k\cap P}} \mathbb{W}_{S/k\cap P}^U\Omega_A^q \xleftarrow{\sim} \mathbb{W}_{S/k\cap P}^{U\cap P}\Omega_A^q.$$

PROOF. The map of the statement as the unique map from the initial object in the category $\mathscr{W}_A^U$ to the image by $i_!$ of the initial object in the category $\mathscr{W}_A^{U\cap P}$. Since $i_!$ admits the right adjoint functor $i^!$, it preserves initial objects. □

LEMMA 3.7. *Let A be a ring, let U be a truncation set, and let p be prime number. Then for every finite sub-truncation set $S \subset U$, the canonical map*

$$(\mathbb{W}_S^U\Omega_A^q) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \to \mathbb{W}_S^U\Omega^q_{A\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}}$$

is an isomorphism.

PROOF. The statement for $q = 0$ is proved by an induction argument similar to the proof of Lemma 1.9. It follows that the groups $(\mathbb{W}_S^U\Omega_A^q) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ for $S \subset U$ finite define a big Witt complex over $A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. The resulting map

$$\mathbb{W}_S^U\Omega^q_{A\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}} \to (\mathbb{W}_S^U\Omega_A^q) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$$

is inverse to the map in the statement. □

Let $U \subset P$ be a p-typical truncation set, and let E be a U-Witt complex. Every finite sub-truncation set $S \subset U$ is of the form $S = \{1, p, \dots, p^{n-1}\}$ for some non-negative integer n. In this case, we will often abbreviate $E_n^q = E_S^q$, $R = R^S_{S/p}$, $F = F_p$, and $V = V_p$. In particular, we write

$$W_n^U\Omega_A^q = \mathbb{W}_S^U\Omega_A^q$$

for the U-de Rham-Witt complex. The following result is very helpful in verifying axiom (v) of Def. 2.1. The case where p is odd was proved by Langer-Zink [**9**, proof of Prop. 1.3], and the case where $p = 2$ was proved by Costeanu [**4**, Lemma 3.6].

LEMMA 3.8. *Let p be a prime number, let $U \subset P$ be a p-typical truncation set, and let $S = \{1, p, \dots, p^{n-1}\}$ be a finite sub-truncation set. Suppose that, in the situation of Def. 2.1, the axioms (i)–(iv) hold in general and the axiom (v) holds for all $m < n$. Then the difference*

$$Fd\lambda([a]_n) - \lambda([a]_{n-1}^{p-1})d\lambda([a]_{n-1})$$

is an additive function of $a \in A$.

PROOF. We suppress the ring homomorphism λ and use p-typical notation. It suffices, by naturality, to consider $A = \mathbb{Z}[x, y]$ and show that

$$\begin{aligned} &Fd[x+y]_n - Fd[x]_n - Fd[y]_n \\ &= [x+y]_{n-1}^{p-1}d[x+y]_{n-1} - [x]_{n-1}^{p-1}d[x]_{n-1} - [y]_{n-1}^{p-1}d[y]_{n-1}. \end{aligned}$$

If we define $\tau \in W_{n-1}(\mathbb{Z}[x, y])$ by the equation

$$[x + y]_n = [x]_n + [y]_n + V\tau,$$

then the left-hand side of the equality in question becomes

$$Fd[x + y]_n - Fd[x]_n - Fd[y]_n = FdV\tau = d\tau + (p - 1)d\log[-1]_{n-1}\tau.$$

Applying F and R, respectively, to the equation defining τ, we get

$$[x+y]_{n-1}^p = [x]_{n-1}^p + [y]_{n-1}^p + p\tau,$$
$$[x+y]_{n-1} = [x]_{n-1} + [y]_{n-1} + V\bar{\tau},$$

where $\bar{\tau} = R\tau \in W_{n-2}(\mathbb{Z}[x,y])$. Hence,

$$p\tau = ([x]_{n-1} + [y]_{n-1} + V\bar{\tau})^p - [x]_{n-1}^p - [y]_{n-1}^p.$$

Since $W_{n-1}(\mathbb{Z}[x,y])$ is p-torsion free, and since

$$(V\bar{\tau})^p = p^{p-1}V(\bar{\tau}^p),$$

we may divide this equation by p to obtain the following formula.

$$\tau = \sum_{\substack{0\leqslant i,j,k<p\\ i+j+k=p}} \frac{(p-1)!}{i!j!k!}[x]_{n-1}^i[y]_{n-1}^j(V\bar{\tau})^k + p^{p-2}V(\bar{\tau}^p).$$

For the right-hand side of the equation in question, we calculate:

$$\begin{aligned}
&[x+y]_{n-1}^{p-1}d[x+y]_{n-1} - [x]_{n-1}^{p-1}d[x]_{n-1} - [y]_{n-1}^{p-1}d[y]_{n-1} \\
&= \quad ([x]_{n-1} + [y]_{n-1} + V\bar{\tau})^{p-1}d[x]_{n-1} - [x]_{n-1}^{p-1}d[x]_{n-1} \\
&\quad + ([x]_{n-1} + [y]_{n-1} + V\bar{\tau})^{p-1}d[y]_{n-1} - [y]_{n-1}^{p-1}d[y]_{n-1} \\
&\quad + ([x]_{n-1} + [y]_{n-1} + V\bar{\tau})^{p-1}dV\bar{\tau} \\
&= \sum_{\substack{0\leqslant i,j,k<p\\ i+j+k=p-1\\ i\neq p-1}} \frac{(p-1)!}{i!j!k!}[x]_{n-1}^i[y]_{n-1}^j(V\bar{\tau})^k d[x]_{n-1} \\
&\quad + \sum_{\substack{0\leqslant i,j,k<p\\ i+j+k=p-1\\ j\neq p-1}} \frac{(p-1)!}{i!j!k!}[x]_{n-1}^i[y]_{n-1}^j(V\bar{\tau})^k d[y]_{n-1} \\
&\quad + \sum_{\substack{0\leqslant i,j,k<p\\ i+j+k=p-1\\ i\neq p-1}} \frac{(p-1)!}{i!j!k!}[x]_{n-1}^i[y]_{n-1}^j(V\bar{\tau})^k dV\bar{\tau} \\
&\quad + \quad (V\bar{\tau})^{p-1}dV\bar{\tau} \\
&= d\tau + (V\bar{\tau})^{p-1}dV\bar{\tau} - p^{p-2}dV(\bar{\tau}^p).
\end{aligned}$$

Hence, to complete the proof, it suffices to prove the formula

$$(V\bar{\tau})^{p-1}dV\bar{\tau} - p^{p-2}dV(\bar{\tau}^p) = (p-1)d\log[-1]_{n-1}\tau.$$

We show that, in fact, this formula holds for every element $\tau \in W_{n-1}(A)$. Suppose first that $p > 2$. Then, using the axioms (i)–(iv) only, we have

$$\begin{aligned}
&p^{p-2}dV(\bar{\tau}^p) = p^{p-3}Vd(\bar{\tau}^p) = p^{p-2}V(\bar{\tau}^{p-1}d\bar{\tau}) \\
&= p^{p-2}V(\bar{\tau}^{p-1})dV\bar{\tau} = (V\bar{\tau})^{p-1}dV\bar{\tau},
\end{aligned}$$

where we have used that $(p-1)d\log[-1]_{n-1}$ is zero. This completes the proof for $p > 2$. So assume that $p = 2$. We must show that

$$V(\bar{\tau})dV(\bar{\tau}) - dV(\bar{\tau}^2) = d\log[-1]_{n-1}\tau.$$

The proof is by induction on n and has three parts. We first show that the formula holds if $\tau = [a]_{n-1}$, $a \in A$. We then assume the formula for $\eta \in W_{n-2}(A)$ and show that it holds for $\tau = V\eta$. Finally, we show that if the formula holds for τ and τ', it holds for $\tau + \tau'$. So let $a \in A$, and recall from the proof of Lemma 2.3 that

$$d\log[-1]_{n-1} = V([1]_{n-2})dV([1]_{n-2}) - dV([1]_{n-2}).$$

Hence,

$$\begin{aligned} d\log[-1]_{n-1}[a]_{n-1} &= V([a]_{n-2}^2)dV([1]_{n-2}) - dV([1]_{n-2})[a]_{n-1} \\ &= V([a]_{n-2}^2)dV([1]_{n-2}) - dV([a]_{n-2}^2) + V([1]_{n-2})d[a]_{n-1}, \end{aligned}$$

and the assumption that axiom (v) holds for $m = n-1$ shows that

$$\begin{aligned} V([1]_{n-2})d[a]_{n-1} &= V(Fd[a]_{n-1}) = V([a]_{n-2}d[a]_{n-2}) \\ &= V([a]_{n-2})dV([a]_{n-2}) - V([a]_{n-2}^2)d\log[-1]_{n-1}. \end{aligned}$$

Hence, to complete the proof for $\tau = [a]_{n-1}$, it remains to show that

$$V([a]_{n-2}^2)d\log[-1]_{n-1} = V([a]_{n-2}^2)dV([1]_{n-2}).$$

But the formula for $d\log[-1]_{n-1}$ that we recalled above shows that the difference between the two sides is equal to

$$V([a]_{n-2}^2)V([1]_{n-2})dV([1]_{n-2}) = 2V([a]_{n-2}^2)dV([1])_{n-2} = V([a]_{n-2}^2)Vd([1]_{n-2})$$

which is zero as desired. Next, let $\eta \in W_{n-2}(A)$ and assume, inductively, that

$$V(\bar{\eta})dV(\bar{\eta}) - dV(\bar{\eta}^2) = d\log[-1]_{n-2}\eta.$$

We claim that by applying the Verschiebung to this equation, we get

$$V^2(\bar{\eta})dV^2(\bar{\eta}) - dV((V\bar{\eta})^2) = d\log[-1]_{n-1}V(\eta)$$

as desired. Indeed,

$$\begin{aligned} V(V(\bar{\eta})dV(\bar{\eta})) &= V(V(\bar{\eta})dV(\bar{\eta}) + V(\bar{\eta})^2 d\log[-1]_{n-2}) \\ &= V(V(\bar{\eta})FdV^2(\bar{\eta})) = V^2(\bar{\eta})dV^2(\bar{\eta}), \\ V(dV(\bar{\eta}^2)) &= 2dV^2(\bar{\eta}^2) = dV^2(\bar{\eta}F\bar{\eta}) = dV((V\bar{\eta})^2), \\ V(d\log[-1]_{n-2}\eta) &= d\log[-1]_{n-1}V(\eta). \end{aligned}$$

Finally, suppose that $\tau, \tau' \in W_{n-1}(A)$ both satisfy the formula in question. Then

$$\begin{aligned} V(\bar{\tau} + \bar{\tau}')dV(\bar{\tau} + \bar{\tau}') &= V(\bar{\tau})dV\bar{\tau} + V(\bar{\tau}')dV\bar{\tau}' + V(\bar{\tau})dV\bar{\tau}' + V(\bar{\tau}')dV\bar{\tau}, \\ dV((\bar{\tau} + \bar{\tau}')^2) &= dV(\bar{\tau}^2) + dV(\bar{\tau}'^2) + 2dV(\bar{\tau}\bar{\tau}') \\ &= dV(\bar{\tau}^2) + dV(\bar{\tau}'^2) + d(V(\bar{\tau})V(\bar{\tau}')) \\ &= dV(\bar{\tau}^2) + dV(\bar{\tau}'^2) + V(\bar{\tau})dV\bar{\tau}' + V(\bar{\tau}')dV\bar{\tau}, \\ d\log[-1]_{n-1}(\tau + \tau') &= d\log[-1]_{n-1}\tau + d\log[-1]_{n-1}\tau'. \end{aligned}$$

It follows that

$$V(\bar{\tau} + \bar{\tau}')dV(\bar{\tau} + \bar{\tau}') - dV((\bar{\tau} + \bar{\tau}')) = d\log[-1]_{n-1}(\tau + \tau')$$

as desired. This completes the proof of the induction step. □

We apply Lemma 3.8 to evaluate the de Rham-Witt complex of the ring of integers. It turns out that, up to canonical isomorphism, the groups $\mathbb{W}_S^U\Omega_{\mathbb{Z}}^q$ are independent of the truncation set U, so we omit this in the statement.

Proposition 3.9. *The big de Rham-Witt complex of $\mathbb{Z}$ is given as follows:*

$$\mathbb{W}_S\Omega^0_{\mathbb{Z}} = \prod_{n\in S} \mathbb{Z}\cdot V_n\lambda([1]_{S/n}),$$
$$\mathbb{W}_S\Omega^1_{\mathbb{Z}} = \prod_{n\in S} \mathbb{Z}/n\mathbb{Z}\cdot dV_n\lambda([1]_{S/n}),$$

and the groups in degrees $q \geqslant 2$ are zero. The multiplication is given by

$$\begin{aligned} V_m\lambda([1]_{S/m})\cdot V_n\lambda([1]_{S/n}) &= c\cdot V_e\lambda([1]_{S/e}) \\ V_m\lambda([1]_{S/m})\cdot dV_n\lambda([1]_{S/n}) &= km\cdot dV_e\lambda([1]_{S/e}) \\ &\qquad + (c-1)\sum_{r\geqslant 1} 2^{r-1}e\cdot dV_{2^re}\lambda([1]_{S/2^re}), \end{aligned}$$

where $c=(m,n)$ is the greatest common divisor of m and n, $e = mn/c$ is the least common multiple of m and n, and k and l are any integers such that $km+ln = c$. The Frobenius and Verschiebung operators are given by

$$\begin{aligned} F_mV_n\lambda([1]_{S/n}) &= c\cdot V_{n/c}\lambda([1]_{S/e}), \\ F_m dV_n\lambda([1]_{S/n}) &= k\cdot dV_{n/c}\lambda([1]_{S/e}) \\ &\qquad + (c-1)\sum_{r\geqslant 1}(2^{r-1}n/c)\cdot dV_{2^rn/c}\lambda([1]_{S/2^re}), \\ V_m(V_n\lambda([1]_{S/mn})) &= V_{mn}\lambda([1]_{S/mn}), \\ V_m(dV_n\lambda([1]_{S/mn})) &= m\cdot dV_{mn}\lambda([1]_{S/mn}). \end{aligned}$$

Proof. We may assume that $S\subset U$ is finite. We will also omit the map λ from the notation. The statement for $q=0$ follows from Lemma 2.7 and Prop. 1.6. We first show that $\mathbb{W}^U_S\Omega^1_{\mathbb{Z}}$ is a quotient of the stated group and that $\mathbb{W}^U_S\Omega^2_{\mathbb{Z}}$ is zero. By Prop. 2.5, the group $\mathbb{W}^U_S\Omega^1_{\mathbb{Z}}$ is generated by the elements $V_m([1]_{S/m})dV_n([1]_{S/n})$ with $m,n\in S$, and to show that the elements $dV_n([1]_{S/n})$ with $n\in S$ suffice as generators, we verify the stated formula for the multiplication. To this end, we first verify the stated formula for the Frobenius. By Lemma 2.3,

$$\begin{aligned} F_mdV_n([1]_{S/n}) &= kdF_{m/c}V_{n/c}([1]_{S/n}) + d\log[-1]_{S/m}\cdot F_{m/c}V_{n/c}([1]_{S/n}) \\ &= kdV_{n/c}([1]_{S/e}) + d\log[-1]_{S/m}V_{n/c}([1]_{S/e}) \\ &= kdV_{n/c}([1]_{S/e}) + V_{n/c}(d\log[-1]_{S/e}). \end{aligned}$$

and since

$$d\log[-1]_{S/m} = \sum_{r\geqslant 1} 2^{r-1}dV_{2^r}([1]_{S/2^rm})$$

we obtain the stated formula. Now, since

$$V_m([1]_{S/m})dV_n([1]_{S/n}) = V_m(F_mdV_n([1]_{S/n}))$$

and since $V_md = mdV_m$, we obtain the stated formula for the multiplication. Hence, the elements $dV_n([1]_{S/n})$ with $n\in S$ generate the group $\mathbb{W}^U_S\Omega^1_{\mathbb{Z}}$. It follows that this group is a quotient of the stated group. Indeed,

$$ndV_n([1]_{S/n}) = V_nd([1]_{S/n}) = 0$$

since d is a derivation. To show that $\mathbb{W}^U_S\Omega^2_{\mathbb{Z}}$ is zero, it will suffice to show that $ddV_n([1]_{S/n})$ is zero for all $n\in S$. Indeed, by the formula for the multiplication, this implies that also $dV_m([1]_{S/m}) \cdot dV_n([1]_{S/n})$ is zero for all $m, n \in S$. Now,

$$\begin{aligned} ddV_n([1]_{S/n}) &= d\log([-1]_S) \cdot dV_n([1]_{S/n}) = d(d\log([-1]_S) \cdot V_n([1]_{S/n})) \\ &= dV_n(d\log([-1]_{S/n})) = \sum_{r \geqslant 1} 2^{r-1} dV_n dV_{2^r}([1]_{S/2^r n}) \\ &= nddV_{2n}([1]_{S/2n}), \end{aligned}$$

and since S is finite, we conclude that $ddV_n([1]_{S/n})$ is zero by easy induction. This shows that $\mathbb{W}_S^U \Omega_{\mathbb{Z}}^q$ is zero for $q \geqslant 2$ as stated.

It remains to prove that the canonical quotient map

$$\prod_{n \in S} \mathbb{Z}/n\mathbb{Z} \cdot dV_n([1]_{S/n}) \to \mathbb{W}_S^U \Omega_{\mathbb{Z}}^1$$

is the identity map. Since the groups in question are finite, it will suffice to show that, for every prime number p, the p-primary subgroups of the two groups have the same order. Now, by Cor. 3.6 and Lemma 3.7, the induced map of p-primary subgroups takes the form

$$\prod_{k \in I(S)} \prod_{p^s \in S/k \cap P} \mathbb{Z}/p^s\mathbb{Z} \cdot dV_{p^s k}([1]_{S/p^s k}) \to \prod_{k \in I(S)} \mathbb{W}_{S/k \cap P}^{U \cap P} \Omega_{\mathbb{Z}}^1.$$

Therefore, it will suffice to prove the stated formula for the group $\mathbb{W}_S^U \Omega_A^1$ in the case where $U \subset P$ is a p-typical truncation set. So we assume that $U \subset P$ is p-typical and proceed to show that there exists a U-Witt complex E over $\mathbb{Z}$ with

$$\begin{aligned} E_n^0 &= \prod_{0 \leqslant s < n} \mathbb{Z} \cdot V^s([1]_{n-s}) \\ E_n^1 &= \prod_{1 \leqslant s < n} \mathbb{Z}/p^s\mathbb{Z} \cdot dV^s([1]_{n-s}) \end{aligned}$$

and with $E_n^q = 0$ for $q \geqslant 2$. Here, and in the rest of the proof, we use p-typical notation. As we have seen above, the multiplication and the Frobenius and Verschiebung operators must necessarily be given by the stated formulas. Conversely, one readily verifies that the multiplication and Frobenius and Verschiebung operators so defined satisfy the axioms (i)–(iv) of Def. 2.1. Hence, to show that E is a U-Witt complex over $\mathbb{Z}$, it remains only to prove axiom (v): For all integers a and all positive integers n such that $p^n \in U$,

$$Fd[a]_n = [a]_{n-1}^{p-1} d[a]_{n-1}.$$

By Lemma 3.8, it suffice to consider $a = 1$, and since d is a derivation, both sides are zero. This completes the proof. □

Remark 3.10. We invite the reader to attempt to prove the equality

$$F_n d\lambda([a]_S) = \lambda([a]_S^{n-1}) d\lambda([a]_S)$$

directly from the formula for $[a]_S$ given in Addendum 1.7 and the formulas for the multiplication and the Frobenius given in statement of Prop. 3.9.

4. The de Rham-Witt complex of a polynomial algebra

Let U be a truncation set, and let $f\colon A \to A'$ be a ring homomorphism. Given a U-Witt complex E over A', we may view E as a Witt complex over A by replacing the natural transformation λ by the composition $\lambda \circ W_S(f)$. This defines a direct image functor $f_*\colon \mathscr{W}^U_{A'} \to \mathscr{W}^U_A$. One may show, as in the proof of Prop. 2.5, that f_* has a left adjoint inverse image functor $f^*\colon \mathscr{W}^U_A \to \mathscr{W}^U_{A'}$, and since the left adjoint functor preserves colimits, the canonical map

$$\mathbb{W}^U_S\Omega^{\cdot}_{A'} \to f^*\mathbb{W}^U_S\Omega^{\cdot}_A$$

is an isomorphism. In this section, we let U be a p-typical truncation set, let A be a $\mathbb{Z}_{(p)}$-algebra, and derive a formula for the inverse image functor f^* associated with the map $f\colon A \to A[X]$ that includes the constant polynomials.

We fix a prime number p and a p-typical truncation set U. Let

$$\eta\colon E^q_n \to f_*f^*E^q_n$$

be the unit of the adjunction (f^*, f_*). We consider the maps of abelian groups

$$e(s,j,\epsilon)\colon E^{q-\epsilon}_{n-s} \to (f_*f^*E)^q_n = (f^*E)^q_n$$

that takes to ξ to the basic Witt differential $e(s,j,\epsilon)(\xi)$ of one of the following four types (I)–(IV). Let I_p be the set of positive integers not divisible by p.

(I) For $s = 0$, $j \in \mathbb{N}_0$, and $\epsilon = 0$,

$$e(s,j,\epsilon)(\xi) = \eta(\xi)[X]^j_n.$$

(II) For $s = 0$, $j \in \mathbb{N}$, and $\epsilon = 1$,

$$e(s,j,\epsilon)(\xi) = \eta(\xi)[X]^{j-1}_n d[X]_n.$$

(III) For $1 \leqslant s < n$, $j \in I_p$, and $\epsilon = 0$,

$$e(s,j,\epsilon)(\xi) = V^s(\eta(\xi)[X]^j_{n-s}).$$

(IV) For $1 \leqslant s < n$, $j \in I_p$, and $\epsilon = 1$,

$$e(s,j,\epsilon)(\xi) = dV^s(\eta(\xi)[X]^j_{n-s}).$$

The basic Witt differentials define a map of abelian groups

$$e\colon P(E)^q_n = \bigoplus E^{q-\epsilon}_{n-s} \to (f^*E)^q_n$$

where the indices s, j, and ϵ for the direct sum vary as in (I)–(IV) above. We proceed to show that this map is an isomorphism.

The requirement that the map e be a natural map of U-Witt complexes over $A[X]$ leaves but one way to define the structure of U-Witt complex on the collection of groups $P(E)^q_n$. We record the definition and begin with the product

$$\mu\colon P(E)^q_n \otimes P(E)^{q'}_n \to P(E)^{q+q'}_n. \tag{4.1}$$

We write (μ: A–B) to indicate the product of basic Witt differentials of type (A) and type (B). We omit the unit η and the subscript on the Teichmüller representatives from the notation. We also recall that, for p odd, the class $d\log[-1]$ is zero.

(μ: I–I). If $j, j' \in \mathbb{N}_0$, then

$$\xi[X]^j \cdot \xi'[X]^{j'} = \xi\xi'[X]^{j+j'}.$$

(μ: I–II). If $j \in \mathbb{N}_0$ and $j' \in \mathbb{N}$, then

$$\xi[X]^j \cdot \xi'[X]^{j'-1}d[X] = \xi\xi'[X]^{j+j'-1}d[X].$$

(μ: I–III). If $1 \leqslant s' < n$, $j \in \mathbb{N}_0$, and $j' \in I_p$, then

$$\xi[X]^j \cdot V^{s'}(\xi'[X]^{j'}) = V^{s'}(F^{s'}(\xi)\xi'[X]^{p^{s'}j+j'}).$$

(μ: I–IV). If $1 \leqslant s' < n$, $j \in \mathbb{N}_0$, and $j' \in I_p$, then

$$\xi[X]^j \cdot dV^{s'}(\xi'[X]^{j'}) = (-1)^{|\xi|}\frac{j'}{p^{s'}j+j'}dV^{s'}(F^{s'}(\xi)\xi'[X]^{p^{s'}j+j'})$$
$$-(-1)^{|\xi|}V^{s'}(F^{s'}d(\xi)\xi'[X]^{p^{s'}j+j'}) + (-1)^{|\xi|}\frac{j}{p^{s'}j+j'}V^{s'}(d(F^{s'}(\xi)\xi')[X]^{p^{s'}j+j'}).$$

(μ: II–II). If $j, j' \in \mathbb{N}$, then

$$\xi[X]^{j-1}d[X] \cdot \xi'[X]^{j'-1}d[X] = d\log[-1]\xi\xi'[X]^{j+j'-1}d[X].$$

(μ: II–III). If $1 \leqslant s' < n$, $j \in \mathbb{N}$, and $j' \in I_p$, then

$$\xi[X]^{j-1}d[X] \cdot V^{s'}(\xi'[X]^{j'}) = (-1)^{|\xi|}\frac{p^{s'}}{p^{s'}j+j'}dV^{s'}(F^{s'}(\xi)\xi'[X]^{p^{s'}j+j'})$$
$$-(-1)^{|\xi|}\frac{1}{p^{s'}j+j'}V^{s'}(d(F^{s'}(\xi)\xi')[X]^{p^{s'}j+j'}).$$

(μ: II–IV). If $1 \leqslant s' < n$, $j \in \mathbb{N}$, and $j' \in I_p$, then

$$\xi[X]^{j-1}d[X] \cdot dV^{s'}(\xi'[X]^{j'}) = (-1)^{|\xi|}\frac{1}{p^{s'}j+j'}dV^{s'}(F^{s'}(\xi)d(\xi')[X]^{p^{s'}j+j'})$$
$$-(-1)^{|\xi|}\frac{1}{p^{s'}j+j'}V^{s'}(F^{s'}d(\xi)d(\xi')[X]^{p^{s'}j+j'}) + jV^{s'}(d\log[-1]F^{s'}(\xi)d\xi'[X]^{p^{s'}j+j'})$$

(μ: III–III). If $1 \leqslant s < s' < n$ and $j, j' \in I_p$, then

$$V^s(\xi[X]^j) \cdot V^{s'}(\xi'[X]^{j'}) = p^sV^{s'}(F^{s'-s}(\xi)\xi'[X]^{p^{s'-s}j+j'});$$

if $1 \leqslant s = s' < n$, $j, j' \in I_p$, and $v = v_p(j+j') < s$, then

$$V^s(\xi[X]^j) \cdot V^{s'}(\xi'[X]^{j'}) = p^sV^{s-v}(V^v(\xi\xi')[X]^{p^{-v}(j+j')});$$

if $1 \leqslant s = s' < n$, $j, j' \in I_p$, and $v = v_p(j+j') \geqslant s$, then

$$V^s(\xi[X]^j) \cdot V^{s'}(\xi'[X]^{j'}) = p^sV^s(\xi\xi')[X]^{p^{-v}(j+j')}.$$

(μ: III–IV): If $1 \leqslant s < s' < n$ and $j, j' \in I_p$, then

$$V^s(\xi[X]^j) \cdot dV^{s'}(\xi'[X]^{j'}) = (-1)^{|\xi|}\frac{p^sj'}{p^{s'-s}j+j'}dV^{s'}(F^{s'-s}(\xi)\xi'[X]^{p^{s'-s}j+j'})$$
$$-(-1)^{|\xi|}V^{s'}(F^{s'-s}d(\xi)\xi'[X]^{p^{s'-s}j+j'}) + V^{s'}(d\log[-1]F^{s'-s}(\xi)\xi'[X]^{p^{s'-s}j+j'})$$
$$+(-1)^{|\xi|}\frac{j}{p^{s'-s}j+j'}V^{s'}(d(F^{s'-s}(\xi)\xi')[X]^{p^{s'-s}j+j'});$$

if $1 \leqslant s' < s < n$ and $j, j' \in I_p$, then

$$V^s(\xi[X]^j) \cdot dV^{s'}(\xi'[X]^{j'}) = (-1)^{|\xi|} \frac{p^s j'}{j + p^{s-s'} j'} dV^s(\xi F^{s-s'}(\xi')[X]^{j+p^{s-s'}j'})$$
$$+ V^s(\xi F^{s-s'} d(\xi')[X]^{j+p^{s-s'}j'}) + V^s(d\log[-1]\xi F^{s-s'}(\xi')[X]^{j+p^{s-s'}j'})$$
$$- (-1)^{|\xi|} \frac{j'}{j + p^{s-s'} j'} V^s(d(\xi F^{s-s'}(\xi'))[X]^{j+p^{s-s'}j'});$$

if $1 \leqslant s = s' < n$, $j, j' \in I_p$, and $v = v_p(j + j') < s$, then

$$V^s(\xi[X]^j) \cdot dV^{s'}(\xi'[X]^{j'}) = (-1)^{|\xi|} \frac{p^s j'}{j + j'} dV^{s-v}(V^v(\xi\xi')[X]^{p^{-v}(j+j')})$$
$$+ V^{s-v}(V^v(\xi d\xi')[X]^{p^{-v}(j+j')}) + V^{s-v}(V^v(d\log[-1]\xi\xi')[X]^{p^{-v}(j+j')})$$
$$- (-1)^{|\xi|} \frac{p^v j'}{j + j'} V^{s-v}(dV^v(\xi\xi')[X]^{p^{-v}(j+j')});$$

if $1 \leqslant s = s' < n$, $j, j' \in I_p$, and $v = v_p(j + j') \geqslant s$, then

$$V^s(\xi[X]^j) \cdot dV^{s'}(\xi'[X]^{j'}) = (-1)^{|\xi'|} j' V^s(\xi\xi')[X]^{p^{-s}(j+j')-1} d[X]$$
$$V^s(\xi d\xi')[X]^{p^{-s}(j+j')} + V^s(d\log[-1]\xi\xi')[X]^{p^{-s}(j+j')}.$$

(μ: IV–IV). If $1 \leqslant s < s' < n$ and $j, j' \in I_p$, then

$$dV^s(\xi[X]^j) \cdot dV^{s'}(\xi'[X]^{j'}) = -(-1)^{|\xi|} dV^{s'}(F^{s'-s} d(\xi)\xi'[X]^{p^{s'-s}j+j'})$$
$$+ dV^{s'}(d\log[-1]F^{s'-s}(\xi)\xi'[X]^{p^{s'-s}j+j'})$$
$$+ (-1)^{|\xi|} \frac{j}{p^{s'-s} j + j'} dV^{s'}(d(F^{s'-s}(\xi)\xi')[X]^{p^{s'-s}j+j'})$$
$$+ V^{s'}(d\log[-1]F^{s'-s} d(\xi)\xi'[X]^{p^{s'-s}j+j'})$$
$$+ jV^{s'}(d\log[-1]d(F^{s'-s}(\xi)\xi')[X]^{p^{s'-s}j+j'});$$

if $1 \leqslant s = s' < n$, $j, j' \in I_p$, and $v = v_p(j + j') < s$, then

$$dV^s(\xi[X]^j) \cdot dV^{s'}(\xi'[X]^{j'}) = dV^{s-v}(V^v(\xi d\xi')[X]^{p^{-v}(j+j')})$$
$$+ dV^{s-v}(V^v(d\log[-1]\xi\xi')[X]^{p^{-v}(j+j')})$$
$$- (-1)^{|\xi|} \frac{p^v j'}{j + j'} dV^{s-v}(dV^v(\xi\xi')[X]^{p^{-v}(j+j')})$$
$$+ V^{s-v}(V^v(d\log[-1]\xi d\xi')[X]^{p^{-v}(j+j')})$$
$$+ p^v V^{s-v}(dV^v(d\log[-1]\xi\xi')[X]^{p^{-v}(j+j')});$$

if $1 \leqslant s = s' < n$, $j, j' \in I_p$, and $v = v_p(j + j') \geqslant s$, then

$$dV^s(\xi[X]^j) \cdot dV^{s'}(\xi'[X]^{j'}) = (-1)^{|\xi'|} j' dV^s(\xi\xi')[X]^{p^{-s}(j+j')-1} d[X]$$
$$- (-1)^{|\xi|+|\xi'|} p^{-s}(j + j') V^s(\xi d\xi')[X]^{p^{-s}(j+j')-1} d[X]$$
$$+ dV^s(\xi d\xi')[X]^{p^{-s}(j+j')} + dV^s(d\log[-1]\xi\xi')[X]^{p^{-s}(j+j')}$$
$$+ V^s(d\log[-1]\xi\xi')[X]^{p^{-s}(j+j')-1} d[X] + V^s(d\log[-1]\xi d\xi')[X]^{p^{-s}(j+j')}$$

We next define the Frobenius operator

$$F\colon P(E)^q_n \to P(E)^q_{n-1} \tag{4.2}$$

and write $(F\colon \text{A})$ to indicate the definition of the Frobenius of the basic Witt differential of type (A).

$(F\colon \text{I})$. If $j \in \mathbb{N}_0$, then

$$F(\xi[X]^j) = F(\xi)[X]^{pj}.$$

$(F\colon \text{II})$. If $j \in \mathbb{N}$, then

$$F(\xi[X]^{j-1}d[X]) = F(\xi)[X]^{pj-1}d[X].$$

$(F\colon \text{III})$ If $1 \leqslant s < n$ and $j \in I_p$, then

$$F(V^s(\xi[X]^j)) = pV^{s-1}(\xi[X]^j).$$

$(F\colon \text{IV})$. If $j \in I_p$, then

$$F(dV(\xi[X]^j)) = d(\xi)[X]^j + (-1)^{|\xi|}j\xi[X]^{j-1}d[X] + d\log[-1]\xi[X]^j;$$

if $2 \leqslant s < n$ and $j \in I_p$, then

$$F(dV^s(\xi[X]^j)) = dV^{s-1}(\xi[X]^j) + V^{s-1}(d\log[-1]\xi[X]^j).$$

We define the Verschiebung operator

$$V\colon P(E)^q_{n-1} \to P(E)^q_n \tag{4.3}$$

and write $(V\colon \text{A})$ to indicate the definition of the Verschiebung of the basic Witt differential of type (A).

$(V\colon \text{I})$. If $j \in \mathbb{N}_0$, then

$$V(\xi[X]^j) = V(\xi[X]^j).$$

$(V\colon \text{II})$. If $j \in p\mathbb{N}$, then

$$V(\xi[X]^{j-1}d[X]) = V(\xi)[X]^{j/p-1}d[X];$$

if $j \in I_p$, then

$$V(\xi[X]^{j-1}d[X]) = (-1)^{|\xi|}\frac{p}{j}dV(\xi[X]^j) - (-1)^{|\xi|}\frac{1}{j}V(d(\xi)[X]^j).$$

$(V\colon \text{III})$. If $1 \leqslant s < n-1$ and $j \in I_p$, then

$$V(V^s(\xi[X]^j)) = V^{s+1}(\xi[X]^j).$$

$(V\colon \text{IV})$. If $1 \leqslant s < n-1$ and $j \in I_p$, then

$$V(dV^s(\xi[X]^j)) = pdV^{s+1}(\xi[X]^j).$$

We define the derivation

$$d\colon P(E)^q_n \to P(E)^{q+1}_n \tag{4.4}$$

and write $(d\text{:A})$ to indicate the definition of the derivation of the basic Witt differential of type (A).

$(d\colon \text{I})$. If $j = 0$, then

$$d(\xi[X]^0) = d(\xi)[X]^0;$$

if $j \in \mathbb{N}$, then

$$d(\xi[X]^j) = d(\xi)[X]^j + (-1)^{|\xi|}j\xi[X]^{j-1}d[X].$$

(d: II). If $j \in \mathbb{N}$, then

$$d(\xi[X]^{j-1}d[X]) = d(\xi)[X]^{j-1}d[X] + jd\log[-1]\xi[X]^{j-1}d[X].$$

(d: III). If $1 \leqslant s < n$ and $j \in I_p$, then

$$d(V^s(\xi[X]^j)) = dV^s(\xi[X]^j).$$

(d: IV). If $1 \leqslant s < n$ and $j \in I_p$, then

$$d(dV^s(\xi[X]^j)) = dV^s(d\log[-1]\xi[X]^j).$$

Finally, we define the ring homomorphism

$$\lambda\colon W_n(A[X]) \to P(E)^0_n. \tag{4.5}$$

To this end, we recall from [**7**, Lemma 4.1.1] that every element of $W_n(A[X])$ may be written uniquely as a (finite) sum of elements of the form $\xi[X]^j$, where $\xi \in W_n(A)$ and $j \in \mathbb{N}_0$, and $V^s(\xi[X]^j)$, where $\xi \in W_{n-s}(A)$, $1 \leqslant s < n$, and $j \in I_p$. We write (λ: I) and (λ: III) to indicate the definition of the map λ on these two types of elements.

(λ: I). If $j \in \mathbb{N}_0$ and $\xi \in W_n(A)$, then

$$\lambda(\xi[X]^j) = \lambda(\xi)[X]^j.$$

(λ: III). If $1 \leqslant s < n$, $j \in I_p$, and $\xi \in W_{n-s}(A)$, then

$$\lambda(V^s(\xi[X]^j)) = V^s(\lambda(\xi)[X]^j).$$

The following result is [**7**, Thm. B] (p odd) and [**4**, Thm. 4.3] ($p = 2$).

THEOREM 4.6. *Let p be a prime number, let $U \subset P$ be a p-typical truncation set, and let E be a U-Witt complex over a $\mathbb{Z}_{(p)}$-algebra A. Then the collection of abelian groups $P(E)^q_n$ together with the operators defined by the formulas (4.1)–(4.5) form a U-Witt complex over $A[X]$. Moreover, the map U-Witt complexes*

$$e\colon P(E) \to f^*E$$

defined by the basic Witt differentials is an isomorphism.

PROOF. We first show that $P(E)$ is a U-Witt complex over $A[X]$. We must show that $P(E)^{\cdot}_n$ is an anti-commutative graded ring and verify the axioms (i)–(v) of Def. 2.1. This is largely a tedious task most of which we leave to the reader. However, we illustrate the verification of the associativity of the product in a particular case and verify the axiom (v).

We verify the associativity identity

$$(\xi[X]^j \cdot \xi'[X]^{j'-1}d[X]) \cdot dV^{s''}(\xi''[X]^{j''}) = \xi[X]^j \cdot (\xi'[X]^{j'-1}d[X]) \cdot dV^{s''}(\xi''[X]^{j''})).$$

By definition, the left-hand side is

$$\begin{aligned}
&(\xi[X]^j \cdot \xi'[X]^{j'-1}d[X]) \cdot dV^{s''}(\xi''[X]^{j''}) = \xi\xi'[X]^{j+j'-1}d[X] \cdot dV^{s''}(\xi''[X]^{j''}) \\
&= (-1)^{|\xi|+|\xi'|}\frac{1}{p^{s''}(j+j')+j''}dV^{s''}(F^{s''}(\xi\xi')d(\xi'')[X]^{p^{s''}(j+j')+j''}) \\
&\quad - (-1)^{|\xi|+|\xi'|}\frac{1}{p^{s''}(j+j')+j''}V^{s''}(F^{s''}d(\xi\xi')d(\xi'')[X]^{p^{s''}(j+j')+j''}) \\
&\quad + (j+j')V^{s''}(d\log[-1]F^{s''}(\xi\xi')d(\xi'')[X]^{p^{s''}(j+j')+j''}),
\end{aligned}$$

and using that F is multiplicative and d a derivation, we may rewrite the second summand as

$$-(-1)^{|\xi|+|\xi'|}\frac{1}{p^{s''}(j+j')+j''}V^{s''}(F^{s''}d(\xi)F^{s''}(\xi')d(\xi'')[X]^{p^{s''}(j+j')+j''})$$
$$-(-1)^{|\xi'|}\frac{1}{p^{s''}(j+j')+j''}V^{s''}(F^{s''}(\xi)F^{s''}d(\xi')d(\xi'')[X]^{p^{s''}(j+j')+j''}).$$

By definition, the right-hand side of the associativity identity in question is

$$\xi[X]^j\cdot(\xi'[X]^{j'-1}d[X]\cdot dV^{s''}(\xi''[X]^{j''}))$$
$$=\xi[X]^j\cdot\big((-1)^{|\xi'|}\frac{1}{p^{s''}j'+j''}dV^{s''}(F^{s''}(\xi')d(\xi'')[X]^{p^{s''}j'+j''})$$
$$-(-1)^{|\xi'|}\frac{1}{p^{s''}j'+j''}V^{s''}(F^{s''}d(\xi')d(\xi'')[X]^{p^{s''}j'+j''})$$
$$+jV^{s''}(d\log[-1]F^{s''}(\xi')d(\xi'')[X]^{p^{s''}j'+j''})\big)$$
$$=(-1)^{|\xi|+|\xi'|}\frac{1}{p^{s''}j'+j''}\frac{p^{s''}j'+j''}{p^{s''}(j+j')+j''}dV^{s''}(F^{s''}(\xi\xi')d(\xi'')[X]^{p^{s''}(j+j')+j''})$$
$$-(-1)^{|\xi|+|\xi'|}\frac{1}{p^{s''}j'+j''}V^{s''}(F^{s''}d(\xi)F^{s''}(\xi')d(\xi'')[X]^{p^{s''}(j+j')+j''})$$
$$+(-1)^{|\xi|+|\xi'|}\frac{1}{p^{s''}j'+j''}\frac{j}{p^{s''}(j+j')+j''}V^{s''}(d(F^{s''}(\xi\xi')d(\xi''))[X]^{p^{s''}(j+j')+j''})$$
$$-(-1)^{|\xi'|}\frac{1}{p^{s''}j'+j''}V^{s''}(F^{s''}(\xi)F^{s''}d(\xi')d(\xi'')[X]^{p^{s''}(j+j')+j''})$$
$$+j'V^{s''}(d\log[-1]F^{s''}(\xi\xi')d(\xi'')[X]^{p^{s''}(j+j')+j''}).$$

We see that the first summand of the left-side is equal to the first summand of the right-hand side. Moreover, by using that F is multiplicative, that d is a derivation, and that $dd(-)$ is equal to $d\log[-1]d(-)$, we rewrite the sum of the second, third, and fourth summands of the right-hand side as

$$-(-1)^{|\xi|+|\xi'|}\frac{1}{p^{s''}j'+j''}V^{s''}(F^{s''}d(\xi)F^{s''}(\xi')d(\xi'')[X]^{p^{s''}(j+j')+j''})$$
$$+(-1)^{|\xi|+|\xi'|}\frac{1}{p^{s''}j'+j''}\frac{p^{s''}j}{p^{s''}(j+j')+j''}V^{s''}(F^{s''}d(\xi)F^{s''}(\xi')d(\xi''))[X]^{p^{s''}(j+j')+j''})$$
$$+(-1)^{|\xi'|}\frac{1}{p^{s''}j'+j''}\frac{p^{s''}j}{p^{s''}(j+j')+j''}V^{s''}(F^{s''}(\xi)F^{s''}d(\xi')d(\xi''))[X]^{p^{s''}(j+j')+j''})$$
$$+jV^{s''}(d\log[-1]F^{s''}(\xi\xi')d(\xi'')[X]^{p^{s''}(j+j')+j''}).$$
$$-(-1)^{|\xi'|}\frac{1}{p^{s''}j'+j''}V^{s''}(F^{s''}(\xi)F^{s''}d(\xi')d(\xi'')[X]^{p^{s''}(j+j')+j''})$$

which by elementary arithmetic becomes

$$-(-1)^{|\xi|+|\xi'|}\frac{1}{p^{s''}(j+j')+j''}V^{s''}(F^{s''}d(\xi)F^{s''}(\xi')d(\xi'')[X]^{p^{s''}(j+j')+j''})$$
$$-(-1)^{|\xi'|}\frac{1}{p^{s''}(j+j')+j''}V^{s''}(F^{s''}(\xi)F^{s''}d(\xi')d(\xi'')[X]^{p^{s''}(j+j')+j''}).$$
$$+jV^{s''}(d\log[-1]F^{s''}(\xi\xi')d(\xi'')[X]^{p^{s''}(j+j')+j''}).$$

This proves the associativity identity in question. With the exception of axiom (v) of Def. 2.1, the remaining identities are verified in a similar manner.

Finally, we verify axiom (v). By Lemma 3.8, it suffices to consider monomials $aX^j \in A[X]$. Now, since the maps d, F, and λ defined by (4.4), (4.2), (4.5) are a derivation, a ring homomorphism, and a ring homomorphism, respectively, and since axiom (v) holds for E, we have

$$\begin{aligned} Fd\lambda([aX^j]) &= Fd(\lambda([a])[X]^j) = F(d\lambda([a]) \cdot [X]^j + \lambda([a]) \cdot j[X]^{j-1}d[X]) \\ &= \lambda([a])^{p-1}d\lambda([a]) \cdot [X]^{pj} + \lambda([a])^p \cdot j[X]^{pj-1}d[X] \\ &= \lambda([a])^{p-1}[X]^{pj-j}(d\lambda([a]) \cdot [X]^j + \lambda([a]) \cdot j[X]^{j-1}d[X]) \\ &= \lambda([aX^j])^{p-1}d\lambda([aX^j]). \end{aligned}$$

This proves that $P(E)$ is a U-Witt complex over $A[X]$ as stated.

Finally, let $\eta' \colon E \to f_*P(E)$ be the map of U-Witt complexes over A defined by

$$\eta'(\xi) = \xi[X]^0.$$

Then, given a map $\alpha \colon E \to f_*E'$ of U-Witt complexes over A, there is a unique map $\alpha' \colon P(E) \to E'$ of U-Witt complexes over $A[X]$ such that

$$\alpha = \alpha' \circ \eta' \colon E \to f_*E'.$$

This shows that $P(-)$ is left adjoint to f_* with η' the unit of the adjunction. □

Remark 4.7. In the case of the de Rham-Witt complex $W^U\Omega_A$, one may alternatively prove that $P(W^U\Omega_A)$ is a U-Witt complex over $A[X]$ by comparison with the topologically defined U-Witt complex $\mathrm{TR}^n_q(A;p)$. We refer to [**7**, Sects. 2–3] for the definition of the latter and for the proof that the canonical map

$$\lambda \colon W^U_n\Omega^q_{\mathbb{Z}_{(p)}} \to \mathrm{TR}^n_q(\mathbb{Z}_{(p)};p)$$

is injective, and the map given by the basic Witt differentials

$$e \colon P(\mathrm{TR}(A;p))^q_n \to \mathrm{TR}^n_q(A[X];p)$$

an isomorphism, for every $\mathbb{Z}_{(p)}$-algebra A. Since the map e is an isomorphism, it follows, in particular, that $P(\mathrm{TR}(A;p))$ is a U-Witt complex over $A[X]$ for every $\mathbb{Z}_{(p)}$-algebra A. Now, suppose that A is a polynomial algebra on $m \geqslant 0$ variables over $\mathbb{Z}_{(p)}$. Then we conclude by induction on m that

$$P(W^U\Omega_A) \to P(\mathrm{TR}(A;p))$$

is injective and that $P(W^U\Omega_A)$ is a U-Witt complex over $A[X]$. But this implies that for every $\mathbb{Z}_{(p)}$-algebra A, $P(W^U\Omega_A)$ is a U-Witt complex over $A[X]$. Indeed, each of the identities that must be verified in order to prove this statement involve only finitely many elements of A, and these finitely many elements are contained in the homorphic image of a polynomial algebra on finitely many variables over $\mathbb{Z}_{(p)}$.

5. V-pro-complexes

The construction of Deligne and Illusie [**8**] of the p-typical de Rham-Witt complex for $\mathbb{F}_p$-algebras proceeds in two steps. First, a de Rham-Witt complex without a Frobenius operator is constructed, and second, a Frobenius operator is constructed on this complex. In this section, we show, using the polynomial formula of Thm. 4.6, that the p-typical de Rham-Witt complex may be constructed in this manner for all $\mathbb{Z}_{(p)}$-algebras. As a consequence, we finally prove that, up to canonical isomorphism, the groups $\mathbb{W}_S^U\Omega_A^q$ are independent of U. Following [**8**, Def. I.1.1], we introduce the auxiliary notion of a V-pro-complex.

Definition 5.1. Let p be a prime number, let $U \subset P$ be a p-typical truncation set, and let A be a ring. A U-V-pro-complex over A is a contravariant functor

$$S \mapsto E_S^{\cdot}$$

that to a sub-truncation set $S \subset U$ assigns an anti-symmetric graded ring $E_S^{\cdot}$ and that takes colimits to limits together with a natural ring homomorphism

$$\lambda \colon \mathbb{W}_S(A) \to E_S^0$$

and natural maps of graded abelian groups

$$d \colon E_S^q \to E_S^{q+1}$$
$$V \colon E_{S/p}^q \to E_S^q$$

such that $V\lambda = \lambda V$ and such that the following (i)–(iii) hold:

(i) For all $x \in E_S^q$, $y \in E_S^{q'}$, and $a \in A$,

$$d(x \cdot y) = d(x) \cdot y + (-1)^q x \cdot d(y)$$
$$d(d(x)) = d\log\lambda([-1]_S) \cdot d(x)$$
$$d\lambda([a]_S) \cdot d\lambda([a]_S) = d\log\lambda([-1]_S)\lambda([a]_S)d\lambda([a]_S).$$

(ii) For all $x \in E_{S/p}^q$ and $y \in E_{S/p}^{q'}$,

$$V(xdy) = V(x)dV(y) + d\log\lambda([-1]_S)V(xy).$$

(iii) For all $x \in E_{S/p}^q$, and all $a \in A$,

$$V(x)d\lambda([a]_S) = V(x\lambda([a]_{S/p}^{p-1}))dV\lambda([a]_{S/p})) + d\log\lambda([-1]_S)V(x\lambda([a]_{S/p}^p)).$$

A map of U-V-pro-complexes is a natural map of graded rings

$$f \colon E_S^{\cdot} \to E_S'^{\cdot}$$

such that $f\lambda = \lambda' f$, $fd = d'f$, and $fV = V'f$.

We remark that for a V-pro-complex E, the map $\hat{d} \colon E_S^q \to E_S^{q+1}$ defined by

$$\hat{d}(x) = \begin{cases} d(x) & \text{if } q \text{ is even} \\ d(x) + d\log\lambda([-1]_S) \cdot x & \text{if } q \text{ is odd} \end{cases}$$

is a derivation and a differential. For p odd, the class $d\log\lambda([-1]_S)$ is zero.

A U-Witt complex determines a U-V-pro-complex by forgetting the Frobenius operator $F = F_p$. Indeed, this follows from the calculation

$$\begin{aligned} V(x)dV(y) &= V(xFdV(y)) = V(xdy) + V(d\log\lambda([-1]_{S/p})xy) \\ &= V(xdy) + d\log\lambda([-1]_S)V(xy) \\ V(x)d\lambda([a]_S) &= V(xFd\lambda([a]_S)) = V(x\lambda([a]_{S/p}^{p-1})d\lambda([a]_{S/p})). \end{aligned}$$

Hence, we have a forgetful functor

$$u\colon \mathscr{W}_A^U \to \tilde{\mathscr{W}}_A^U$$

from the category of U-Witt complexes over A to the category of U-V-pro-complexes over A. We prove in Thm. 5.6 below that the functor u preserves initial objects.

PROPOSITION 5.2. *Let p be a prime number, let $U \subset P$ be a p-typical truncation set, and let A be a ring.*

(i) *There exists an initial object $\tilde{\mathbb{W}}_S^U\Omega_A^{\cdot}$ in the category $\tilde{\mathscr{W}}_A^U$.*

(ii) *The canonical map $\hat{\Omega}_{\mathbb{W}_S(A)}^q \to \tilde{\mathbb{W}}_S^U\Omega_A^q$ is surjective for all $S \subset U$.*

(iii) *The canonical map $\tilde{\mathbb{W}}_S^U\Omega_A^q \to \tilde{\mathbb{W}}_S^P\Omega_A^q$ is an isomorphism for all $S \subset U$.*

PROOF. It suffices to consider finite sub-truncation sets $S \subset U$. In this case, we inductively define anti-symmetric graded rings $\tilde{\mathbb{W}}_S^U\Omega_A^{\cdot}$ along with maps

$$R\colon \tilde{\mathbb{W}}_S^U\Omega_A^q \to \tilde{\mathbb{W}}_{S/p}^U\Omega_A^q, \quad V\colon \tilde{\mathbb{W}}_{S/p}^U\Omega_A^q \to \tilde{\mathbb{W}}_S^U\Omega_A^q, \quad d\colon \tilde{\mathbb{W}}_S^U\Omega_A^q \to \tilde{\mathbb{W}}_S^U\Omega_A^{q+1}$$

such that the axioms of Def. 5.1 are satisfied and such that the canonical maps

$$\hat{\Omega}_{\mathbb{W}_S(A)}^q \to \tilde{\mathbb{W}}_S^U\Omega_A^q$$

are surjective, beginning with $\tilde{\mathbb{W}}_\emptyset^U\Omega_A^{\cdot}$ which we define to be the zero graded ring. So we let $S \subset U$ be a finite sub-truncation set and assume, inductively, that for all proper sub-truncation sets $T \subset S$, the anti-symmetric graded ring $\tilde{\mathbb{W}}_T^U\Omega_A^{\cdot}$ and the maps R, V, d, and λ have been defined and satisfy the axioms of Def. 5.1 and that the canonical map $\hat{\Omega}_{\mathbb{W}_T(A)}^q \to \tilde{\mathbb{W}}_T^U\Omega_A^q$ is surjective. We then define

$$\tilde{\mathbb{W}}_S^U\Omega_A^{\cdot} = \hat{\Omega}_{\mathbb{W}_S(A)}^{\cdot}/N_S^{\cdot}$$

where $N_S^{\cdot}$ is the differential graded ideal generated by the elements

$$\begin{aligned} &\sum_\alpha \lambda V(x_\alpha)d\lambda V(y_{1,\alpha})\dots d\lambda V(y_{q,\alpha}) \\ &+ d\log\lambda([-1]_S)\sum_\alpha\sum_{1\leqslant i\leqslant q}\lambda V(x_\alpha y_{i,\alpha})d\lambda V(y_{1,\alpha})\dots \widehat{d\lambda V(y_{i,\alpha})}\dots d\lambda V(y_{q,\alpha}), \end{aligned}$$

for all $x_\alpha, y_{i,\alpha} \in \mathbb{W}_{S/n}(A)$ such that the class of the sum

$$\sum_\alpha \lambda(x_\alpha)d\lambda(y_{1,\alpha})\cdots d\lambda(y_{q,\alpha})$$

is zero in $\tilde{\mathbb{W}}_{S/p}^U\Omega_A^q$, and by the elements

$$\begin{aligned} &\lambda V(x)d\lambda([a]_S) - \lambda V(x[a]_{S/p}^{p-1})d\lambda V([a]_{S/p}) + d\log\lambda([-1]_S)\lambda V(x[a]_{S/p}^p) \\ &d\lambda([a]_S)d\lambda([a]_S) - d\log\lambda([-1]_S)\lambda([a]_S)d\lambda([a]_S) \end{aligned}$$

for all $x \in W_{S/n}(A)$ and all $a \in A$. We define the ring homomorphism

$$\lambda \colon \mathbb{W}_S(A) \to \tilde{\mathbb{W}}_S^U \Omega_A^0$$

to be the canonical projection. It is an isomorphism, since N_S^0 is zero. The unique map of differential graded rings

$$\hat{\Omega}^{\cdot}_{\mathbb{W}_S(A)} \to \tilde{\mathbb{W}}_{S/p}^U \Omega^{\cdot}_A$$

that extends λR factors through a map of differential graded rings

$$R \colon \tilde{\mathbb{W}}_S^U \Omega^{\cdot}_A \to \tilde{\mathbb{W}}_{S/p}^U \Omega^{\cdot}_A$$

and satisfies $\lambda R = R\lambda$. If $\hat{d}$ denotes the differential of the differential graded ring $\tilde{W}_S^U \Omega^{\cdot}_A$, then we define the derivation

$$d \colon \tilde{\mathbb{W}}_S^U \Omega_A^q \to \tilde{\mathbb{W}}_S^U \Omega_A^{q+1}$$

by the formula

$$d(x) = \begin{cases} \hat{d}(x) & \text{if } q \text{ is even} \\ \hat{d}(x) + d\log \lambda([-1]_S) \cdot x & \text{if } q \text{ is odd.} \end{cases}$$

Finally, the additive map

$$V \colon \tilde{\mathbb{W}}_{S/p}^U \Omega_A^q \to \tilde{\mathbb{W}}_S^U \Omega_A^q$$

defined by

$$\begin{aligned} V(\lambda(x) d\lambda(y_1) \cdots d\lambda(y_q)) &= \lambda V(x) d\lambda V(y_1) \cdots d\lambda V(y_q) \\ &+ \sum_{1 \leqslant i \leqslant q} d\log \lambda([-1]_S) \lambda V(xy_i) d\lambda V(y_1) \cdots \widehat{d\lambda V(y_i)} \cdots d\lambda V(y_q) \end{aligned}$$

is well-defined and satisfies $V\lambda = \lambda V$ along with axioms (ii)–(iii) of Def. 5.1. This completes the recursive definition of the V-pro-complex $\tilde{\mathbb{W}}_S^U \Omega^{\cdot}_A$. It clearly is an initial object of the category $\tilde{\mathscr{W}}_A^U$. It is also clear that the canonical map

$$\tilde{\mathbb{W}}_S^U \Omega_A^q \to \tilde{\mathbb{W}}_S^P \Omega_A^q$$

is the identity map for all $S \subset U$. $\square$

Since the group $\tilde{\mathbb{W}}_S^U \Omega_A^q$ is independent, up to canonical isomorphism, of the p-typical truncation set U, we will denote it by $\tilde{\mathbb{W}}_S \Omega_A^q$.

Lemma 5.3. *The following relations hold in* $\tilde{\mathbb{W}}_S \Omega_A$*:*

(i) *For all* $a \in \mathbb{W}_S(A)$ *and* $\xi \in \tilde{\mathbb{W}}_{S/p} \Omega_A^q$, $\lambda(a) V(\xi) = V(\lambda(F(a))\xi)$.
(ii) *For all* $\xi \in \tilde{\mathbb{W}}_{S/p} \Omega_A^q$, $V(\xi) d\log \lambda([-1]_S) = V(\xi d\log \lambda([-1]_{S/p}))$.
(iii) *For all* $\xi \in \tilde{\mathbb{W}}_{S/p} \Omega_A^q$, $Vd(\xi) = pdV(\xi)$.
(iv) *For* p *odd,* $d\log \lambda([-1]_S)$ *is zero, and for* $p = 2$*,*

$$d\log([-1]_S) = \sum_{r \geqslant 1} 2^{r-1} dV^r \lambda([1]_{S/2^r}).$$

PROOF. To prove (i), we may assume that $\xi = \lambda(x)d\lambda(y_1)\cdots d\lambda(y_q)$. Then

$$\begin{aligned}
\lambda(a)V(\xi) &= \lambda(aV(x))d\lambda V(y_1)\cdots d\lambda V(y_q) \\
&\quad + \sum_{1\leqslant i\leqslant q} d\log\lambda([-1]_S)\lambda(aV(xy_i))d\lambda V(y_1)\cdots \widehat{d\lambda V(y_i)}\cdots d\lambda V(y_q) \\
&= \lambda V(F(a)x)d\lambda V(y_1)\cdots d\lambda V(y_q) \\
&\quad + \sum_{1\leqslant i\leqslant q} d\log\lambda([-1]_S)\lambda V(F(a)xy_i)d\lambda V(y_1)\cdots \widehat{d\lambda V(y_i)}\cdots d\lambda V(y_q) \\
&= V(\lambda(F(a))\xi)
\end{aligned}$$

as stated. We next prove (ii). If p is odd, both sides are zero, and if $p=2$, we find

$$\begin{aligned}
V(\xi)d\log\lambda([-1]) &= V(\xi)\lambda([-1])d\lambda([-1]) = V(\xi\lambda(F([-1])))d\lambda([-1]) \\
&= V(\xi)d\lambda([-1]) = V(\xi\lambda([-1])d\lambda([-1])) = V(\xi d\log\lambda([-1])).
\end{aligned}$$

Next, the calculation

$$\begin{aligned}
Vd(\xi) &= V(\lambda([1])d\xi) = V\lambda([1])dV(\xi) + d\log\lambda([-1])V(\xi) \\
&= d(\lambda V([1])V(\xi)) - dV\lambda([1])V(\xi) + d\log\lambda([-1])V(\xi) \\
&= dV(\lambda(FV([1]))\xi) - V(FdV\lambda([1])\xi) + d\log\lambda([-1])V(\xi) \\
&= pdV(\xi) - V(d\log\lambda([-1])\xi) + d\log\lambda([-1])V(\xi)
\end{aligned}$$

shows that (ii) implies (iii). Finally, using the relations (i)–(iii), one proves (iv) as in Lemma 2.3. $\square$

We remark that, by constrast with Lemma 2.3, Lemma 5.3 is not valid for a general V-pro-complex.

LEMMA 5.4. *Let p be a prime number. Then the canonical map*

$$\tilde{\mathbb{W}}_S\Omega^q_{\mathbb{Z}} \to \mathbb{W}_S\Omega^q_{\mathbb{Z}}$$

is an isomorphism for every p-typical truncation set S.

PROOF. In the following composition of canonical maps, the left-hand map and the composite map are both isomorphisms for $q=0$, and surjections for $q\geqslant 1$.

$$\hat{\Omega}^q_{\mathbb{W}_S(\mathbb{Z})} \to \tilde{\mathbb{W}}_S\Omega^q_{\mathbb{Z}} \to \mathbb{W}_S\Omega^q_{\mathbb{Z}}$$

Hence, the right-hand map is an isomorphism for $q=0$, and a surjection for $q\geqslant 1$. To prove that the right-hand map is also injective for $q\geqslant 1$, we may assume that $S=\{1,p,\ldots,p^{n-1}\}$ is finite. Now, Prop. 3.9 shows that

$$\mathbb{W}_S\Omega^1_{\mathbb{Z}} = \prod_{1\leqslant s<n} \mathbb{Z}/p^s\mathbb{Z}\cdot dV^s\lambda([1]_{n-s})$$

and that $\mathbb{W}_S\Omega^2_{\mathbb{Z}}$ is zero. Therefore, it suffices to show that the elements $dV^s\lambda([1]_{n-s})$ with $1\leqslant s<n$ generate the group $\tilde{\mathbb{W}}_S\Omega^1_{\mathbb{Z}}$, that $dV^s\lambda([1]_{n-s})$ is annihilated by p^s, and that the group $\tilde{\mathbb{W}}_S\Omega^2_{\mathbb{Z}}$ is zero. Lemma 5.3 shows that for p odd,

$$V^s\lambda([1])dV^t\lambda([1]) = \begin{cases} p^s dV^t\lambda([1]) & \text{if } s<t \\ 0 & \text{if } s\geqslant t, \end{cases}$$

that for $p = 2$,

$$V^s\lambda([1])dV^t\lambda([1]) = \begin{cases} 2^s dV^t\lambda([1]) + \sum_{t<r<n} 2^{r-1}dV^r\lambda([1]) & \text{if } s < t \\ \sum_{s<r<n} 2^{r-1}dV^r\lambda([1]) & \text{if } s \geqslant t, \end{cases}$$

and that for any prime number p, $p^s dV^s\lambda([1]_{n-s}) = 0$. To show that $\tilde{\mathbb{W}}_S\Omega^2_{\mathbb{Z}}$ is zero, we must prove that $ddV^s\lambda([1]_{n-s}) = 0$ for all $1 \leqslant s < n$. For p odd, this is clear. For $p = 2$, we use the relations of Lemma 5.3 to show, as in the proof of Lemma 2.3, that $dd\log\lambda([-1]) = 0$, and hence, that

$$\begin{aligned} ddV^s\lambda([1]) &= d\log\lambda([-1]) \cdot dV^s\lambda([1]) = d(d\log\lambda([-1]) \cdot V^s\lambda([1]) \\ &= dV^s(d\log\lambda([-1])) = \sum_{1\leqslant r<n-s} 2^{r+s-1}ddV^{r+s}([1]). \end{aligned}$$

But this is zero, since $r + s - 1 \geqslant 1$. This completes the proof. $\square$

Lemma 5.5. *Let p be a prime number and let $U \subset P$ be a p-typical truncation set. Let A be a $\mathbb{Z}_{(p)}$-algebra and assume that the canonical map*

$$\tilde{\mathbb{W}}_S\Omega^q_A \to \mathbb{W}^U_S\Omega^q_A$$

is an isomorphism for all sub-truncation sets $S \subset U$. Then the same holds for the polynomial algebra $A[X]$.

Proof. We consider the commutative diagram of abelian groups

$$\begin{array}{ccc} \tilde{W}_n\Omega^q_{A[X]} & \longrightarrow & W^U_n\Omega^q_{A[X]} \\ \uparrow{\scriptstyle e} & & \uparrow{\scriptstyle e} \\ P(\tilde{W}\,\Omega_A)^q_n & \longrightarrow & P(W^U\Omega_A)^q_n \end{array}$$

where the vertical maps are given by the basic Witt differentials and where the horizontal maps are the canonical maps. The assumption on the $\mathbb{Z}_{(p)}$-algebra A implies that the lower horizontal map is an isomorphism for all n and q. Moreover, we proved in Thm. 4.6 that the right-hand vertical map is an isomorphism. In particular, the left-hand vertical map is a map of V-pro-complexes over $A[X]$. Since the target of this map is the initial V-pro-complex over $A[X]$, it is necessarily surjective, and therefore, an isomorphism. This shows that the upper horizontal map is an isomorphism as stated. $\square$

The following result was proved in [**7**, Thm. B] for p odd.

Theorem 5.6. *Let p be a prime number, let $U \subset P$ be a p-typical truncation set, and let A be a $\mathbb{Z}_{(p)}$-algebra. Then the canonical map*

$$\tilde{\mathbb{W}}_S\Omega^q_A \to \mathbb{W}^U_S\Omega^q_A$$

is an isomorphism for all sub-truncation sets $S \subset U$.

Proof. First, an induction argument based on Lemmas 5.4 and 5.5 shows that the theorem holds if A is a polynomial algebra over $\mathbb{Z}_{(p)}$ on a finite number of variables. In the general case, we define a Frobenius operator

$$\tilde{F} \colon \tilde{W}_n\Omega^q_A \to \tilde{W}_{n-1}\Omega^q_A$$

and show that this defines on the V-pro-complex $\tilde{W}_n\Omega^q_A$ the structure of a U-Witt complex over A. Here, and throughout the proof, we use p-typical notation.

Let $F_*\tilde{W}_{n-1}\Omega^q_A$ denote $\tilde{W}_{n-1}\Omega^q_A$ considered as a $W_n(A)$-module via the Frobenius $F\colon W_n(A) \to W_{n-1}(A)$. We first consider the map

$$\delta\colon W_n(A) \to F_*\tilde{W}_{n-1}\Omega^1_A$$

that to the Witt vector $a = [a_0]_n + V([a_1]_{n-1}) + \cdots + V^{n-1}([a_{n-1}]_1)$ assigns

$$\begin{aligned}\delta(a) = {}& \lambda([a_0]^{p-1}_{n-1})d\lambda([a_0]_{n-1}) + d\lambda([a_1]_{n-1}) + \cdots + dV^{n-2}\lambda([a_{n-1}]_1)\\ &+ d\log\lambda([-1]_{n-1})\big(\lambda([a_1]_{n-1}) + V\lambda([a_2]_{n-2}) + \cdots + V^{n-2}\lambda([a_{n-1}]_1)\big).\end{aligned}$$

We note that δ is defined so that the following diagram commutes.

$$\begin{array}{ccc} W_n(A) & \xrightarrow{\quad\delta\quad} & \tilde{W}_{n-1}\Omega^1_A \\ \Big\| & & \big\downarrow \\ W_n(A) & \xrightarrow{\quad Fd\lambda\quad} & W^U_{n-1}\Omega^1_A \end{array}$$

We claim that δ is a derivation. To prove this, it will suffice to consider the case where A is a polynomial algebra over $\mathbb{Z}_{(p)}$ in finitely many variables. For any finite number of elements of $W_n(A)$ are contained in the homomorphic image of the map of rings of Witt vectors induced by a ring homomorphism from a polynomial algebra over $\mathbb{Z}_{(p)}$ in finitely many variables. But in this case, the right-hand vertical map in the diagram above is an isomorphism, and since $Fd\lambda$ is a derivation, so is δ. It follows that there exists a unique map of graded $W_n(A)$-algebras

$$\tilde{F}\colon \hat{\Omega}^{\cdot}_{W_n(A)} \to F_*\tilde{W}_{n-1}\Omega^{\cdot}_A$$

such that $\tilde{F}d\lambda = \delta$. We claim that the map $\tilde{F}$ annihilates the kernel $N^{\cdot}_n$ of the canonical surjection $\hat{\Omega}^{\cdot}_{W_n(A)} \to \tilde{W}_n\Omega^{\cdot}_A$. Indeed, by definition, $\tilde{F}$ maps the sum

$$\begin{aligned}&\sum_\alpha \lambda V(x_\alpha)d\lambda V(y_{1,\alpha})\dots d\lambda V(y_{q,\alpha})\\ &+ d\log\lambda([-1]_S)\sum_\alpha \sum_{1\leqslant i\leqslant q} \lambda V(x_\alpha y_{i,\alpha})d\lambda V(y_{1,\alpha})\dots d\widehat{\lambda V(y_{i,\alpha})}\dots d\lambda V(y_{q,\alpha}),\end{aligned}$$

to the sum

$$p\sum_\alpha \lambda(x_\alpha)d\lambda(y_{1,\alpha})\cdots d\lambda(y_{q,\alpha})$$

and the elements

$$\begin{aligned}&\lambda V(x)d\lambda([a]_n) - \lambda V(x[a]^{p-1}_{n-1})d\lambda V([a]_{n-1}) - d\log\lambda([-1]_n)\lambda V(x[a]^p_{n-1})\\ &d\lambda([a]_n)d\lambda([a]_n) - d\log\lambda([-1]_n)\lambda([a]_n)d\lambda([a]_n)\end{aligned}$$

to the elements

$$\begin{aligned}&p\lambda(x)(\delta([a]_{n-1}) - \lambda([a]^{p-1}_{n-1})d\lambda([a]_{n-1}))\\ &\lambda([a]^{2p-2}_{n-1})(d\lambda([a]_{n-1})d\lambda([a]_{n-1}) - d\log\lambda([-1]_{n-1})\lambda([a]_{n-1})d\lambda([a]_{n-1})).\end{aligned}$$

We conclude that the map $\tilde{F}$ induces a map of graded rings

$$\tilde{F}\colon \tilde{W}_n\Omega^{\cdot}_A \to \tilde{W}_{n-1}\Omega^{\cdot}_A.$$

Moreover, the definition of the map $\tilde{F}$ implies that the following diagram, where the horizontal maps are the canonical maps, commutes.

$$\begin{array}{ccc} \tilde{W}_n\Omega^q_A & \longrightarrow & W^U_n\Omega^q_A \\ \big\downarrow{\scriptstyle \tilde{F}} & & \big\downarrow{\scriptstyle F} \\ \tilde{W}_{n-1}\Omega^q_A & \longrightarrow & W^U_{n-1}\Omega^q_A \end{array}$$

The horizontal maps in this diagram are isomorphisms, if A is a polynomial algebra over $\mathbb{Z}_{(p)}$ in finitely many variables. Hence, in this case, the map $\tilde{F}$ defines on the V-pro-complex $\tilde{W}_n\Omega^q_A$ the structure of a U-Witt complex over A. But then the same is true for every $\mathbb{Z}_{(p)}$-algebra A. Indeed, every relation in the definition of U-Witt complex involves only finitely many elements of $\tilde{W}_n\Omega^q_A$. Therefore, these elements are contained in the homomorphic image of the map induced by a ring homomorphism from a polynomial algebra over $\mathbb{Z}_{(p)}$ on a finite number of variables. It follows that we have maps

$$\tilde{W}_n\Omega^q_A \rightleftarrows W^U_n\Omega^q_A$$

where the top map is the unique map from the initial V-pro-complex over A and where the bottom map is the unique map from the initial U-Witt complex over A. The two compositions are selfmaps of initial objects, and therefore, are identity maps. This completes the proof. $\square$

COROLLARY 5.7. *Let $U \subset U'$ be arbitrary truncation sets, and let A be a ring. Then for every sub-truncation set $S \subset U$, the canonical map*

$$\mathbb{W}^U_S\Omega^q_A \to \mathbb{W}^{U'}_S\Omega^q_A$$

is an isomorphism.

PROOF. It suffices to prove that for every prime number p, the map

$$\mathbb{W}^U_S\Omega^q_A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \to \mathbb{W}^{U'}_S\Omega^q_A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$$

induced by the map of the statement is an isomorphism. Therefore, by Prop. 3.6 and Lemma 3.7, we may assume that U and U' are p-typical and that A is a $\mathbb{Z}_{(p)}$-algebra. But in this case, the statement follows from Thm. 5.6. $\square$

Since the group $\mathbb{W}^U_S\Omega^q_A$ is independent, up to canonical isomorphism, of the truncation set U, we will denote it by $\mathbb{W}_S\Omega^q_A$.

REMARK 5.8. It follows, in particular, from Cor. 5.7 that the canonical map

$$\Omega^q_A = \mathbb{W}^{\{1\}}_{\{1\}}\Omega^q_A \to \mathbb{W}_{\{1\}}\Omega^q_A$$

is an isomorphism.

6. Étale morphisms

The functor that to the ring A associates the $\mathbb{W}_S(A)$-module $\mathbb{W}_S\Omega^q_A$ defines a presheaf of $\mathbb{W}_S(\mathscr{O})$-modules on the category of affine schemes. In this section, we show that for S finite, this presheaf is a quasi-coherent sheaf of $\mathbb{W}_S(\mathscr{O})$-modules for the étale topology. In other words, we prove the following result.

THEOREM 6.1. *Let S be a finite truncation set and let $f\colon A \to B$ be an étale map. Then the canonical map*

$$\mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega^q_A \to \mathbb{W}_S\Omega^q_B$$

is an isomorphism.

PROOF. By Cor. 5.7, it suffices to prove the following statement: For every finite truncation set U and every sub-truncation set $S \subset U$, the canonical map

$$\alpha\colon \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}^U_S\Omega^q_A \to \mathbb{W}^U_S\Omega^q_B$$

is an isomorphism. We prove the statement by induction on U, beginning with the trivial case $U = \emptyset$. So we let U be a finite truncation set and assume that the statement has been proved for all proper sub-truncation sets of U. To prove the induction step, we define the structure of a U-Witt complex over B on the domains of the canonical map α. By Thm. 1.20, the map

$$\mathbb{W}_S(f)\colon \mathbb{W}_S(A) \to \mathbb{W}_S(B)$$

is étale. Therefore, we may define a derivation

$$d\colon \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}^U_S\Omega^q_A \to \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}^U_S\Omega^{q+1}_A$$

which uniquely extends the derivation $d\colon \mathbb{W}^U_S\Omega^q_A \to \mathbb{W}^U_S\Omega^{q+1}_A$ by

$$d(b \otimes x) = (db)x + b \otimes dx$$

where db is the image of b by the composition

$$\mathbb{W}_S(B) \xrightarrow{d} \Omega^1_{\mathbb{W}_S(B)} \xleftarrow{\sim} \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \Omega^1_{\mathbb{W}_S(A)} \to \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}^U_S\Omega^1_A.$$

Here the middle map and the right-hand map are the canonical isomorphism and the canonical projection, respectively. We further define $R^S_T = R^S_T \otimes R^S_T$ and $F_n = F_n \otimes F_n$. To define the map

$$V_n\colon \mathbb{W}_{S/n}(B) \otimes_{\mathbb{W}_{S/n}(A)} \mathbb{W}^U_{S/n}\Omega^q_A \to\colon \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}^U_S\Omega^q_A$$

we use that, by Thm. 1.20, the following square of rings is cocartesian.

$$\begin{array}{ccc} \mathbb{W}_S(A) & \xrightarrow{\mathbb{W}_S(f)} & \mathbb{W}_S(B) \\ \downarrow{\scriptstyle F_n} & & \downarrow{\scriptstyle F_n} \\ \mathbb{W}_{S/n}(A) & \xrightarrow{\mathbb{W}_{S/n}(f)} & \mathbb{W}_{S/n}(B) \end{array}$$

It follows that the map

$$F_n \otimes \mathrm{id}\colon \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_{S/n}\Omega^q_A \to \mathbb{W}_{S/n}(B) \otimes_{\mathbb{W}_{S/n}(A)} \mathbb{W}_{S/n}\Omega^q_A$$

is an isomorphism, and we then define V_n to be the composition of the inverse of this isomorphism and the map

$$\mathrm{id} \otimes V_n\colon \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_{S/n}\Omega^q_A \to \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega^q_A.$$

Finally, we define the map λ to be the composition

$$\mathbb{W}_S(B) \to \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S(A) \xrightarrow{\mathrm{id} \otimes \lambda} \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S^U \Omega_A^0$$

of the canonical isomorphism and the map $\mathrm{id} \otimes \lambda$. We proceed to show that this defines a U-Witt complex over B. The axioms (i)–(iii) of Def. 2.1 follow immediately from the definitions. For example, the identity $dd(x) = d\log \lambda([-1]_S)d(x)$ holds because the two sides are derivations which agree on $\mathbb{W}_S^U \Omega_A^q$. It remains to prove the axioms (iv)–(v) of Def. 2.1.

To prove axiom (iv), we first show that for all $b \in \mathbb{W}_{S/n}(B)$,

$$F_n dV_n(b) + (n-1)d\log[-1]_{S/n} \cdot b = d(b).$$

The right-hand side is the unique extension to $\mathbb{W}_{S/n}(B)$ of the derivation

$$d\colon \mathbb{W}_{S/n}(A) \to \mathbb{W}_{S/n}^U \Omega_A^1.$$

The left-hand side, $D_n(b)$, is also an extension of this map since $\mathbb{W}^U \Omega_A$ is a U-Witt complex over A. Hence, it will suffice to show that D_n is a derivation. Moreover, since D_n is an additive function of b, and since the square of rings in Thm. 1.20, which we recalled above, is cocartesion, it is enough to consider elements of the form $F_n(b)a$ with $a \in \mathbb{W}_{S/n}(A)$ and $b \in \mathbb{W}_S(B)$. Now,

$$\begin{aligned}
&D_n(F_n(b)aF_n(b')a') = D_n(F_n(bb')aa') \\
&= F_n d(bb'V_n(aa')) + (n-1)d\log[-1]F_n(bb')aa' \\
&= F_n d(bb')F_nV_n(aa') + F_n(bb')F_n dV_n(aa') + (n-1)d\log[-1]F_n(bb')aa' \\
&= F_n d(bb')paa' + F_n(bb')d(aa') \\
&= (F_n d(b)F_n(b') + F_n(b)F_n d(b'))paa' + F_n(bb')(d(a)a' + ad(a')) \\
&= (F_n d(b)pa + F_n(b)d(a))F_n(b')a' + F_n(b)a(F_n d(b')pa + F_n(b)d(a')) \\
&= D_n(F_n(b)a)F_n(b')a' + F_n(b)aD_n(F_n(b')a')
\end{aligned}$$

which shows that D_n is a derivation, and hence, equal to the derivation d. We note that the special case of axiom (iv) proved thus far implies that

$$dF_n = nF_n d\colon \mathbb{W}_S(B) \to \mathbb{W}_{S/n}(B) \otimes_{\mathbb{W}_{S/n}(A)} \mathbb{W}_{S/n}^U \Omega_A^1.$$

To prove axiom (iv) in general, we must show that

$$F_n dV_n(b \otimes x) + (n-1)d\log[-1]_{S/n} \cdot b \otimes x = d(b \otimes x)$$

for all $b \in \mathbb{W}_{S/n}(B)$ and $x \in \mathbb{W}_{S/n}^U \Omega_A^q$. As before, this follows once we prove that the left-hand side, $D_n(b \otimes x)$, is a derivation. Moreover, since D_n is additive and since the square of rings in Thm. 1.20 is cocartesian, it will suffice to consider elements of the form $F_n(b) \otimes x$ with $b \in \mathbb{W}_S(B)$ and $x \in \mathbb{W}_{S/n}^U \Omega_A^q$. Finally, to prove that D_n is a derivation on elements of this form, it suffices to show that

$$D_n(F_n(b) \otimes x) = D_n(F_n(b))x + F_n(b) \otimes D_n(x).$$

Now, by the definition of V_n and by the case of axiom (iv) proved above, we find

$$\begin{aligned} &D_n(F_n(b) \otimes x) = F_n d V_n(F_n(b) \otimes x) + (n-1) d\log[-1] \cdot F_n(b) \otimes x \\ &= F_n d(b \otimes V_n(x)) + (n-1) d\log[-1] \cdot F_n(b) \otimes x \\ &= F_n d(b) \cdot F_n V_n(x) + F_n(b) \otimes F_n d V_n(x) + (n-1) d\log[-1] \cdot F_n(b) \otimes x \\ &= n F_n d(b) \cdot x + F_n(b) \otimes D_n(x) = d F_n(b) \cdot x + F_n(b) \otimes D_n(x) \\ &= D_n(F_n(b)) \cdot x + F_n(b) \otimes D_n(x) \end{aligned}$$

as desired. This completes the proof of axiom (iv).

To prove axiom (v), we must show that for every sub-truncation set $S \subset U$ and every positive integer n, the derivation

$$F_n d\lambda \colon \mathbb{W}_S(B) \to \mathbb{W}_{S/n}(B) \otimes_{\mathbb{W}_{S/n}(A)} \mathbb{W}^U_{S/n}\Omega^1_A$$

maps $[b]_S$ to $\lambda([b]^{n-1}_{S/n}) d\lambda([b]_{S/n})$. To this end, we consider the following diagram where the unmarked arrows are the canonical maps.

$$\begin{array}{ccccc} \mathbb{W}_S(B) & \xrightarrow{F_n d\lambda} & \mathbb{W}_{S/n}(B) \otimes_{\mathbb{W}_{S/n}(A)} \mathbb{W}^U_{S/n}\Omega^1_A & \longleftarrow & \mathbb{W}_{S/n}(B) \otimes_{\mathbb{W}_{S/n}(A)} \mathbb{W}^{S/n}_{S/n}\Omega^1_A \\ \Vert & & \downarrow & & \downarrow \\ \mathbb{W}_S(B) & \xrightarrow{F_n d\lambda} & \mathbb{W}^U_{S/n}\Omega^1_B & \longleftarrow & \mathbb{W}^{S/n}_{S/n}\Omega^1_B \end{array}$$

The right-hand horizontal maps are isomorphisms by Cor. 5.7, and the right-hand vertical map is an isomorphism by the inductive hypothesis. Therefore, also the middle vertical map is an isomorphism. But the lower left-hand horizontal map takes $[b]_S$ to $\lambda([b]^{n-1}_{S/n}) d\lambda([b]_{S/n})$, since $\mathbb{W}^U\Omega_B$ is a U-Witt complex over B, and hence, so does the upper left-hand horizontal map. This proves axiom (v).

We have proved that the domains of the canonical map α at the beginning of the proof form a U-Witt complex over B. Therefore, there exists a unique map

$$\beta \colon \mathbb{W}^U_S\Omega^q_B \to \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}^U_S\Omega^q_A$$

of U-Witt complexes over B. The composition $\alpha \circ \beta$ is a selfmap of the initial object $\mathbb{W}^U_S\Omega^q_B$, and therefore, is the identity map. The composition $\beta \circ \alpha$ is a map of U-Witt complexes over B. In particular, it is a map of $\mathbb{W}_S(B)$-modules, and therefore, is determined by the composition with the map

$$\iota \colon \mathbb{W}^U_S\Omega^q_A \to \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}^U_S\Omega^q_A$$

that takes x to $[1]_S \otimes x$. But ι and $\beta \circ \alpha \circ \iota$ both are maps of U-Witt complexes over A with domain the initial U-Witt complex over A. Therefore, the two maps are equal, and hence, also $\beta \circ \alpha$ is the identity map. This completes the proof of the induction step and the theorem. $\square$

7. Frobenius invariants

In this section, we consider the invariants of Frobenius operators the de Rham-Witt complex. We note that the definition of the ring of Witt vectors $\mathbb{W}_S(A)$ makes perfect sense also if A does not have a unit element.

LEMMA 7.1. *Let S be a truncation set, let A be a ring, and let $I \subset A$ be an ideal. Then $\mathbb{W}_S(I) \subset \mathbb{W}_S(A)$ is an ideal and the following sequence is exact.*

$$0 \longrightarrow \mathbb{W}_S(I) \longrightarrow \mathbb{W}_S(A) \longrightarrow \mathbb{W}_S(A/I) \longrightarrow 0$$

PROOF. We may assume that S is finite. The proof, for S finite, is by induction starting from the case $S = \emptyset$ which is trivial. In the induction step, we let S be non-empty and assume that the statement has been proved for all proper sub-truncation sets of S. We let $m \in S$ be maximal and let $T = S \smallsetminus \{m\}$. Then the induction step is proved by using the following natural exact sequence.

$$0 \longrightarrow A \xrightarrow{V_m} \mathbb{W}_S(A) \xrightarrow{R^S_T} \mathbb{W}_T(A) \longrightarrow 0$$

This completes the proof. □

The canonical projection from A onto A/I^j induces a canonical surjection

$$\mathbb{W}_S\Omega^q_A \to \mathbb{W}_S\Omega^q_{A/I^j}$$

from the de Rham-Witt complex of A onto that of A/I^j. We define

$$\mathrm{Fil}^j_I \, \mathbb{W}_S\Omega^q_A \subset \mathbb{W}_S\Omega^q_A$$

to be the kernel of this map.

LEMMA 7.2. *Let S be a truncation set, let A be a ring, and let $I \subset A$ be an ideal. Let $x \in I^j$ and suppose that $1 + x$ is a unit in A. Then*

$$d\log[1+x]_S \equiv \sum_{n \in S} dV_n([x]_{S/n})$$

modulo $\mathrm{Fil}^{2j}_I \, \mathbb{W}_S\Omega^1_A$.

PROOF. We may assume that S is finite and that $j = 1$. We claim that

$$[1+x]_S - [1]_S \equiv \sum_{n \in S} V_n([x]_{S/n})$$

modulo $\mathbb{W}_S(I^2)$. It will suffice to consider $A = \mathbb{Z}[x]$ and $I = (x)$. We write

$$[1+x]_S - [1]_S = \sum_{n \in S} V_n([a_n]_{S/n})$$

and must show that $a_n \equiv x$ modulo I^2. We proceed by induction on S beginning with the trivial $S = \emptyset$. So let S be non-empty and assume that the claim has been proved for all sub-truncation sets $T \subset S$. We let $m \in S$ be maximal and consider the mth ghost coordinate of the equation in question.

$$(1+x)^m - 1 = \sum_{d \mid m} d a_d^{m/d}$$

The left-hand side is congruent to mx modulo (x^2) and, by the inductive hypothesis, the right-hand side is congruent to ma_m module (x^2). Since $\mathbb{Z}[x]$ is torsion-free, the claim follows.

We return to the general case and differentiate to find that

$$d([1+x]_S) \equiv \sum_{n \in S} dV_n([x]_S)$$

modulo $\mathrm{Fil}_I^2 \mathbb{W}_S\Omega_A^1$. By the assumption that $1+x$ is a unit in A, $d\log[1+x]_S$ is defined and belongs to $\mathrm{Fil}_I^1 \mathbb{W}_S\Omega_A^1$. But $[1+x]_S - [1]_S$ belongs to $\mathbb{W}_S(I)$, so

$$([1+x]_S - [1]_S)d\log[1+x]_S = d([1+x]_S) - d\log[1+x]_S$$

belongs to $\mathrm{Fil}_I^2 \mathbb{W}_S\Omega_A^1$. This completes the proof. □

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