Real algebraic K-theory

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Introduction

This account of our work on real algebraic K-theory is in a preliminary form, but we have chosen to make the manuscript available, since the definitions and results herein have already been used. At present, the manuscript contains a complete proof of Theorem A, while our proofs of Theorems B and C only exist as hand-written documents.

The central construction in the work present here is a variant of the definition of algebraic K-theory given in Waldhausen's seminal paper [25]. To briefly recall Waldhausen's construction, let Δ be the simplicial index category, and let

$$\Delta \xrightarrow{\ i\ } \mathsf{Cat}$$

be the fully faithful functor that to a non-empty finite ordinal $[n]$ assigns the category $i([n])$ with object set $[n]$ and with a unique morphism from i to j if and only if $i \leqslant j$. Now, to every exact category with weak equivalences $\mathscr{C}$, Waldhausen's construction associates a simplicial exact category with weak equivalences $S^{1,1}\mathscr{C}[-]$ defined as follows. Writing $[-,-]$ to for the category with objects the functors and morphisms the natural transformations, the category $S^{1,1}\mathscr{C}[n]$ is the full subcategory

$$S^{1,1}\mathscr{C}[n] \subset [[i([1]), i([n])], \mathscr{C}]$$

whose objects are the functors $A\colon [i([1]), i([n])] \to \mathscr{C}$ such that

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(i) for every functor $\mu\colon i([0]) \to i([n])$,

$$A(s_0\mu) = 0,$$

a chosen null-object in $\mathscr{C}$;

(ii) and for every functor $\sigma\colon i([2]) \to i([n])$, the sequence

$$A(d_2\sigma) \longrightarrow A(d_1\sigma) \longrightarrow A(d_0\sigma)$$

in $\mathscr{C}$ is exact.

Here $s_0\mu = \mu \circ i(s^0)$ and $d_v\sigma = \sigma \circ i(d^v)$; and the morphisms in (ii) are induced by the unique natural transformations $d^v \Rightarrow d^{v-1}$. A sequence $A' \to A \to A''$ in $S^{1,1}\mathscr{C}[n]$ is defined to be exact if for every functor $\theta\colon i([1]) \to i([n])$, the induced sequence $A'(\theta) \to A(\theta) \to A''(\theta)$ in $\mathscr{C}$ is exact; and a morphism $A \to A'$ in $S^{1,1}\mathscr{C}[n]$ is defined to be a weak equivalence if for every functor $\theta\colon i([1]) \to i([n])$, the induced morphism $A(\theta) \to A'(\theta)$ in $\mathscr{C}$ is a weak equivalence. The Waldhausen construction may be iterated, and we write $S^{r,r}\mathscr{C}[-]$ for the r-simplicial exact category with weak equivalences obtained by applying the construction r times. The classifying space of the r-simplicial subcategory of weak equivalences,

$$K(\mathscr{C})_r = B(wS^{r,r}\mathscr{C}[-]),$$

is a pointed space with a left action by the symmetric group Σ_r on r letters induced from permutation of the r simplicial directions. This family of pointed left Σ_r-spaces together with the pointed maps

$$K(\mathscr{C})_r \wedge S^1 \xrightarrow{\sigma_{r,1}} K(\mathscr{C})_{r+1}$$

induced from the inclusion of the 1-skeleton in the last simplicial direction forms a symmetric spectrum $K(\mathscr{C})$; this is Waldhausen's algebraic K-theory spectrum.

We pause to introduce some terminology to be used throughout. We write

$$G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$$

and say that a pointed space with a continuous left G-action is a pointed real space. We define the real circles $S^{1,0} = S^{\mathbb{R}}$ and $S^{1,1} = S^{i\mathbb{R}}$ to be the pointed real spaces given by the one-point compactifications of the 1-dimensional trivial representation and sign representation, respectively. More generally, for integers $p \geqslant q \geqslant 0$, we set

$$S^{p,q} = (S^{1,0})^{\wedge(p-q)} \wedge (S^{1,1})^{\wedge q},$$

following the indexing in motivic homotopy theory. It is a p-dimensional sphere and its subspace of points fixed by the G-action is a $(p-q)$-dimensional sphere.

Suppose now that (D,η) is a duality structure on the exact category with weak equivalences $\mathscr{C}$, that is, $D\colon \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$ is an exact functor and $\eta\colon \mathrm{id}_{\mathscr{C}} \Rightarrow D \circ D^{\mathrm{op}}$ is a natural weak equivalence such that the composite natural transformation

$$D \xRightarrow{\eta \circ D} D \circ D^{\mathrm{op}} \circ D \xRightarrow{D \circ \eta^{\mathrm{op}}} D$$

is equal to the identity natural transfomation of D. The duality structure (D,η) gives rise to an action of the group $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on the pointed left Σ_r-spaces $K(\mathscr{C})_r$. Let us here write $K(\mathscr{C},D,\eta)_r$ for this pointed real left Σ_r-space. Moreover, the spectrum structure maps in the algebraic K-theory spectrum $K(\mathscr{C})$ define real pointed maps

$$K(\mathscr{C},D,\eta)_r \wedge S^{1,1} \xrightarrow{\sigma_{r,1}} K(\mathscr{C},D,\eta)_{r+1}.$$

Hence, we obtain a symmetric spectrum $K(\mathscr{C},D,\eta)$ in the category of real pointed spaces with respect to the sphere $S^{1,1}$. From the point of view of stable equivariant homotopy theory, this is not a very reasonable object. For instance, the family of subspaces of points fixed by the G-action form a symmetric spectrum in the category of pointed spaces with respect to the sphere S^0 rather than the sphere S^1.

To address this, we introduce the following variant of Waldhausen's construction, which we call the real Waldhausen construction. First, to an exact category with weak equivalences $\mathscr{C}$, the real Waldhausen assigns the simplicial exact category with weak equivalences $S^{2,1}\mathscr{C}[-]$, where $S^{2,1}\mathscr{C}[n]$ is the full subcategory

$$S^{2,1}\mathscr{C}[n] \subset [[i([2]),i([n])],\mathscr{C}]$$

whose objects are the functors $A\colon [i([2]),i([n])] \to \mathscr{C}$ such that

(i) for every functor $\mu\colon i([1]) \to i([n])$,

$$A(s_0\mu) = A(s_1\mu) = 0,$$

a chosen null-object in $\mathscr{C}$;

(ii) for every functor $\sigma\colon i([3]) \to i([n])$, the sequence

$$A(d_3\sigma) \longrightarrow A(d_2\sigma) \longrightarrow A(d_1\sigma) \longrightarrow A(d_0\sigma)$$

in $\mathscr{C}$ is exact.

Here again $s_v\mu = \mu \circ i(s^v)$; $d_v\sigma = \sigma \circ i(d^v)$; the morphisms in (ii) are induced by the unique natural transformations $d^v \Rightarrow d^{v-1}$; and the exactness of the sequence in (ii) means that the left-hand morphism is an admissible monomorphism, that the right-hand morphism is an admissible epimorphism, and that the middle morphism induces an isomorphism of the cokernel of the left-hand morphism onto the kernel of the right-hand morphism. A sequence $A' \to A \to A''$ in $S^{2,1}\mathscr{C}[n]$ is defined to be exact if for every functor $\theta\colon i([2]) \to i([n])$, the induced sequence $A'(\theta) \to A(\theta) \to A''(\theta)$ in $\mathscr{C}$ is exact; and a morphism $A \to A'$ in $S^{2,1}\mathscr{C}[n]$ is defined to be a weak equivalence if for every functor $\theta\colon i([2]) \to i([n])$, the induced morphism $A(\theta) \to A'(\theta)$ in $\mathscr{C}$ is a weak equivalence. The real Waldhausen construction may be iterated, and we write $S^{2r,r}\mathscr{C}[-]$ for the r-simplicial exact category with weak equivalences obtained by applying the construction a total of r times. The classifying space of the r-simplicial subcategory of weak equivalences,

$$KR(\mathscr{C})_r = B(wS^{2r,r}\mathscr{C}[-]),$$

is a pointed space with a left action by the symmetric group Σ_r on r letters induced from permutation of the r simplicial directions. This family of pointed left Σ_r-spaces together with the pointed maps

$$KR(\mathscr{C})_r \wedge S^2 \xrightarrow{\sigma_{r,1}} KR(\mathscr{C})_{r+1}$$

induced from the inclusion of the 2-skeleton in the last simplicial direction constitute a symmetric spectrum $KR(\mathscr{C})$ with respect to the 2-sphere.

Suppose now that $(\mathscr{C},D,\eta)$ is an exact category with weak equivalences and duality. The duality structure (D,η) gives rise to a left G-action on the pointed left Σ_r-space $KR(\mathscr{C})_r$, and we denote the resulting pointed real left Σ_r-space by $KR(\mathscr{C},D,\eta)_r$. With respect to this real structure, the spectrum structure maps in the symmetric spectrum $KR(\mathscr{C})$ are real pointed maps

$$KR(\mathscr{C},D,\eta)_r \wedge S^{2,1} \xrightarrow{\sigma_{r,1}} KR(\mathscr{C},D,\eta)_{r+1}.$$

Hence, this defines a symmetric spectrum $KR(\mathscr{C},D,\eta)$ in the symmetric monoidal category of pointed real spaces under smash product with respect to the sphere $S^{2,1}$. We call a symmetric spectrum of this form a real symmetric spectrum, and we call the real symmetric spectrum $KR(\mathscr{C},D,\eta)$ the real algebraic K-theory of the exact category with weak equivalences and duality $(\mathscr{C},D,\eta)$. A real symmetric spectrum is precisely a G-equivariant symmetric spectrum in the sense of Mandell [12]. We recall that the category $\mathrm{Sp}^{\Sigma}(\mathsf{RealTop}_*,S^{2,1})$ of real symmetric spectra with the model structure defined in op. cit. is a model for the G-equivariant stable homotopy category, and we call the equivariant homotopy groups

$$KR_{p,q}(\mathscr{C},D,\eta) = \mathrm{Ho}(\mathrm{Sp}^{\Sigma}(\mathsf{RealTop}_*,S^{2,1}))(S^{p,q},KR(\mathscr{C},D,\eta))$$

the real algebraic K-groups of $(\mathscr{C},D,\eta)$. Here, by abuse of notation, we write $S^{p,q}$ for a choice of smash product of q copies of the real suspension spectrum of circle $S^{i\mathbb{R}}$ and $p-q$ copies of the real suspension spectrum of circle $S^{\mathbb{R}}$. We note that (p,q) is allowed to be any pair of integer.

Our two main theorems on real algebraic K-theory are the real additivity theorem and the real group completion theorem. To state them, we first discuss Segal's direct sum K-theory construction and our real version thereof. To a finite pointed set (X,x), we associate a category $P(X,x)$ defined as follows. The objects are all pointed subsets of $x \in U \subset X$; the set of morphisms from the object U to the object V is the set of pointed subsets $x \in F \subset U \cap V$; the composition of $G\colon V \to W$ and $F\colon U \to V$ is the subset $G \circ F = G \cap F\colon U \to W$; and the identity morphism of $x \in U \subset X$ is $x \in U \subset U$. Hence, the morphism $F\colon U \to V$ is the composition of $F\colon U \to F$ and $F\colon F \to V$, which may be thought of as the map that collapses the complement of $F \subset U$ to the basepoint and the inclusion of F into V, respectively. In addition, to a pointed map $f\colon (X_0,x_0) \to (X_1,x_1)$, we assign the functor $f^*\colon P(X_1,x_1) \to P(X_0,x_0)$ defined on both objects and morphisms by $f^*(T) = f^{-1}(T \smallsetminus \{x_1\}) \cup \{x_0\}$. The category $P(X,x)$ admits a Grothendieck topology J in which a sieve S on the object U is a covering sieve if and only if, for every $u \in U$, the morphisms $\{x,u\}\colon \{x,u\} \to U$ is in S. Now, if $\mathscr{C}$ is an additive category, then we define $\mathscr{C}(X,x)$ to be the category of pointed

$\mathscr{C}$-valued sheaves on $P(X,x)$ with respect to the topology J, where a $\mathscr{C}$-valued sheaf A is said to be pointed if its value at the object $x \in \{x\} \subset X$ is a chosen null-object 0. Moreover, to a pointed map $f\colon (X_0,x_0) \to (X_1,x_1)$, we assign the direct image functor $f_*\colon \mathscr{C}(X_0,x_0) \to \mathscr{C}(X_1,x_1)$ defined by $f_*(A)(V) = A(f^*(V))$. The functor

$$\mathscr{C}(X,x) \xrightarrow{i^*} \mathscr{C}^{(X,x)}$$

that to a sheaf A associates the family of stalks $(A(\{x,u\})_{u\in X}$ is an equivalence of categories onto the category of pointed functors from the discrete pointed category (X,x) to the pointed category $(\mathscr{C},0)$. However, while the target category depends contravariantly on (X,x), the domain category depends covariantly on (X,x). The latter functor $\mathscr{C}(-)$ is a variant of Segal's construction of a Γ-category associated with an additive category $\mathscr{C}$.

Now, given an additive category with weak equivalences $\mathscr{C}$, its direct sum K-theory is the symmetric spectrum $K^{\oplus}(\mathscr{C})$ defined as follows. We define the Segal construction of an additive category with weak equivalences $\mathscr{C}$ to be the simplicial additive category with weak equivalences $S^{1,1}_{\oplus}\mathscr{C}[-]$ obtained by applying $\mathscr{C}(-)$ to the simplicial circle $S^{1,1}[-] = \Delta[1][-]/\partial\Delta[1][-]$ and declaring a morphism $A \to A'$ in $S^{1,1}_{\oplus}\mathscr{C}[n]$ be a weak equivalence if for every object U of $P(S^{1,1}[n])$, the morphism $A(U) \to A'(U)$ is a weak equivalence in $\mathscr{C}$. We apply the Segal construction r times to get the r-simplicial additive category with weak equivalences $S^{r,r}_{\oplus}\mathscr{C}[-]$; define

$$K^{\oplus}(\mathscr{C})_r = B(wS^{r,r}_{\oplus}\mathscr{C}[-])$$

to be the pointed left Σ_r-space given by the classifying space of the r-simplicial subcategory of weak equivalences; and define

$$K^{\oplus}(\mathscr{C})_r \wedge S^1 \xrightarrow{\sigma_{r,1}} K^{\oplus}(\mathscr{C})_{r+1}.$$

to be the pointed map induced by the inclusion of the 1-skeleton in the last simplicial direction. This defines Segal's the direct sum K-theory symmetric $K^{\oplus}(\mathscr{C})$.

Similarly, we define the real Segal construction of $\mathscr{C}$ to be the simplicial additive category with weak equivalences $S^{2,1}_{\oplus}\mathscr{C}[-]$ obtained by applying $\mathscr{C}(-)$ to the simplicial 2-sphere $S^{2,1}[-] = \Delta[2][-]/\partial\Delta[2][-]$ and declaring a morphism $A \to A'$ in $S^{2,1}_{\oplus}\mathscr{C}[n]$ is a weak equivalence if for every object U of $P(S^{2,1}[n])$, the morphism $A(U) \to A'(U)$ is a weak equivalence in $\mathscr{C}$. We write $S^{2r,r}_{\oplus}\mathscr{C}[-]$ for the r-fold iterate of the real Segal construction; define

$$KR^{\oplus}(\mathscr{C})_r = B(wS^{2r,r}_{\oplus}\mathscr{C}[-])$$

to be the pointed left Σ_r-space given by the classifying space of the r-simplicial subcategory of weak equivalences; and define

$$KR^{\oplus}(\mathscr{C})_r \wedge S^2 \xrightarrow{\sigma_{r,1}} KR^{\oplus}(\mathscr{C})_{r+1}$$

to be the pointed map induced by the inclusion of the 2-skeleton in the last simplicial direction. This defines a symmetric spectrum $KR^{\oplus}(\mathscr{C})$ in the category of pointed spaces with respect to the 2-sphere.

Suppose now that $(\mathscr{C}, D, \eta)$ is an additive category with weak equivalences and duality. The duality structure (D, η) gives rise to a left G-action on the pointed left Σ_r-space $KR^{\oplus}(\mathscr{C})_r$, and we denote the resulting pointed real left Σ_r-space by $KR^{\oplus}(\mathscr{C}, D, \eta)_r$. With respect to this real structure, the spectrum structure maps in the symmetric spectrum $KR^{\oplus}(\mathscr{C})$ are real pointed maps

$$KR^{\oplus}(\mathscr{C}, D, \eta)_r \wedge S^{2,1} \xrightarrow{\sigma_{r,1}} KR^{\oplus}(\mathscr{C}, D, \eta)_{r+1}.$$

This defines a real symmetric spectrum $KR^{\oplus}(\mathscr{C}, D, \eta)$ that we call the real direct sum K-theory spectrum of $(\mathscr{C}, D, \eta)$.

We next consider an exact category with weak equivalences $\mathscr{C}$ and the real Segal construction $S_{\oplus}^{2,1}\mathscr{C}[-]$ of its underlying additive category with weak equivalences. It inherits a structure of simplicial exact category with weak equivalences by declaring a sequence $A' \to A \to A''$ in $S_{\oplus}^{2,1}\mathscr{C}[n]$ to be exact if for every object U of $P(S^{2,1}[n])$, $A'(U) \to A(U) \to A''(U)$ is exact in $\mathscr{C}$. There is an exact forgetful functor

$$S_{\oplus}^{2,1}\mathscr{C}[n] \xrightarrow{\phi^*} S^{2,1}\mathscr{C}[n]$$

defined to be the restriction along the functor

$$[i([2]), i([n])] \xrightarrow{\phi} P(S^{2,1}[n], \infty[n])$$

that to an object $i(\theta)\colon i([2]) \to i([n])$ associates the object

$$\phi(i(\theta)) = \{\rho\colon [n] \to [2] \mid \theta \circ \rho = \mathrm{id}_{[2]}\} \cup \{\infty[n]\}$$

and that to a morphism $i(\theta_0) \Rightarrow i(\theta_1)$, which is unique if it exists, associates the morphism $\phi(i(\theta_0)) \cap \phi(i(\theta_1))\colon \phi(i(\theta_0)) \Rightarrow \phi(i(\theta_1))$. The functor ϕ^* induces a map of symmetric spectra $\phi^*\colon KR^{\oplus}(\mathscr{C}) \to KR(\mathscr{C})$, and if $(\mathscr{C}, D, \eta)$ is an exact category with weak equivalences and dualty, then this map is one of real symmetric spectra,

$$KR^{\oplus}(\mathscr{C}, D, \eta) \xrightarrow{\phi^*} KR(\mathscr{C}, D, \eta).$$

Our first main result is the following comparison theorem, which we prove by a rather elaborate real version of Quillen's proof in [18] of the corresponding comparison theorem for direct sum K-theory and algebraic K-theory.

Theorem A. *Let $(\mathscr{C}, D, \eta)$ be an exact category with weak equivalences duality such that the exact sequences in $\mathscr{C}$ are the split-exact sequences and such that the weak equivalences in $\mathscr{C}$ are the isomorphisms. In this situation, the forgetful map*

$$KR^{\oplus}(\mathscr{C}, D, \eta) \xrightarrow{\phi^*} KR(\mathscr{C}, D, \eta)$$

is a level weak equivalence of real symmetric spectra.

The importance of this theorem lies in that the real group-completion theorem describes the equivariant homology of the underlying equivariant infinite loop space the real direct sum K-theory spectrum in terms of that of its zeroth space.

To state the second main result, let $(\mathscr{C}, D, \eta)$ be an exact category with weak equivalences and strict duality. The duality structure (D, η) on $\mathscr{C}$ gives rise to duality structures $(D[n], \eta[n])$ on $S^{1,1}\mathscr{C}[n]$ and $S^{2,1}\mathscr{C}[n]$ defined as follows. The unique isomorphism of categories $v_n \colon i([n])^{\mathrm{op}} \to i([n])$, given on objects by $v_n(s) = n - s$, defines a strict duality structure on the category $i([n])$, and this, in turn, induces a strict duality structure on the functor category $[i([m]), i([n])]$ with the duality functor given by $[v_m, v_n](-) = v_n \circ (-)^{\mathrm{op}} \circ v_m^{\mathrm{op}}$. Now, this duality structure and the duality structure on $\mathscr{C}$ gives the functor category $[[i([m]), i([n])], \mathscr{C}]$ a duality structure with the duality functor $[[v_m, v_n], D](-) = D \circ (-)^{\mathrm{op}} \circ [v_m, v_n]^{\mathrm{op}}$ and with the morphism to the double dual induced from that on $\mathscr{C}$. For $m = 1$ and $m = 2$, this duality structure restrict to the desired duality structures $(D[n], \eta[n])$ on $S^{1,1}\mathscr{C}[n]$ and $S^{2,1}\mathscr{C}[n]$, respectively.

A left G-action on a finite pointed set (X, x) gives rise to a strict duality structure on the category $P(X, x)$. The duality functor $v_{(X,x)}$ takes the object U to the object $\tau \cdot U$, where $\tau \in G$ is complex conjugation, and takes the morphism $F \colon U \to V$ to the morphism $\tau \cdot F \colon \tau \cdot V \to \tau \cdot U$. This duality structure and the given duality structure on $\mathscr{C}$, in turn, gives rise to a duality structure $(D(X,x), \eta(X,x))$ on $\mathscr{C}(X,x)$ with the duality functor $D(X,x)(-) = D \circ (-)^{\mathrm{op}} \circ v_{(X,x)}$ and with $\eta(X,x) = \eta \circ (-) \circ \mathrm{id}_{P(X,x)^{\mathrm{op}}}$. Taking (X, x) to be $S^{1,1}[n]$ and $S^{2,1}[n]$, respectively, with complex conjugation $\tau \in G$ acting by $(\tau \cdot \theta)(s) = n - \theta(1 - s)$ and $(\tau \cdot \theta)(s) = n - \theta(2 - s)$, this defines structures on $S^{1,1}_{\oplus}\mathscr{C}[n]$ and $S^{2,1}_{\oplus}\mathscr{C}[n]$, both of which we also write $(D[n], \eta[n])$. Moreover, the functor ϕ^* is a duality-preserving exact functor

$$(S^{2,1}_{\oplus}\mathscr{C}[n], D[n], \eta[n]) \xrightarrow{\phi^*} (S^{2,1}\mathscr{C}[n], D[n], \eta[n]),$$

and so is the analogously defined forgetful functor

$$(S^{1,1}_{\oplus}\mathscr{C}[n], D[n], \eta[n]) \xrightarrow{\phi^*} (S^{1,1}\mathscr{C}[n], D[n], \eta[n]).$$

Our second main result is the following real version of the additivity theorem, which we prove by an adaption of McCarthy's proof in [14] of the additivity theorem to the real setting. That this statement is the appropriate real version of the additivity theorem was first recognized by Schlichting, who proved the corresponding statement for his higher Grothendieck-Witt theory [21, Theorem 4].

Theorem B. *Let $(\mathscr{C}, D, \eta)$ be an exact category with weak equivalences and duality. For every positive integer r, the map of pointed real spaces*

$$KR(S^{1,1}_{\oplus}\mathscr{C}[3], D[3], \eta[3])_r \xrightarrow{\phi^*} KR(S^{1,1}\mathscr{C}[3], D[3], \eta[3])_r$$

induced by the forgetful functor is a weak equivalence.

The proof of the real additivity theorem presented here works in great generality, including the case of the real version of topological Hochschild homology that we define below. We remark that, by contrast, the real additivity theorem fails for real direct sum K-theory, which, accordingly, should be considered mainly a calculational devise. The domain of the map in the statement of the real additivitiy theorem may be understood as follows. First, for every finite real pointed set (X,x), the stalks functor induces a level weak equivalence of real symmetric spectra

$$KR(\mathscr{C}(X,x),D(X,x),\eta(X,x)) \xrightarrow{i^*} KR(\mathscr{C}^{(X,x)},D^{(X,x)},\eta^{(X,x)}),$$

since, in general, real algebraic K-theory takes equivalences of categories with weak equivalences and duality to homotopy equivalences of real symmetric spectra. Next, the canonical map

$$KR(\mathscr{C}^{(X,x)},D^{(X,x)},\eta^{(X,x)}) \longrightarrow KR(\mathscr{C},D,\eta)^{(X,x)}$$

from the real algebraic K-theory of the power of category with weak equivalences and duality by a finite pointed real set to the power of the real algebraic K-theory of the given category with weak equivalences and duality by the same finite real pointed set is a level weak equivalence of real symmetric spectra. In the situation of Theorem B, the finite real pointed set $(X,x) = S^{1,1}[3]$ has three elements 0001, 0011, and 0111 in addition to the basepoint $\infty = \{0000, 1111\}$, and the action by G permutes 0001 and 0111 and fixes 0011. Hence, the pointed real set $S^{1,1}[3]$ consists of one free G-orbit and one fixed G-orbit, which is the reason that Theorem B is the appropriate real version of the additivity theorem.

Based on the real additivity theorem, we show that, more generally, for every non-negative integer n and positive integer r, the map of pointed real spaces

$$KR(S^{1,1}_{\oplus}\mathscr{C}[n],D[n],\eta[n])_r \xrightarrow{\phi^*} KR(S^{1,1}\mathscr{C}[n],D[n],\eta[n])_r$$

is a weak equivalence. Similarly, for every non-negative integer n and positive integer r, the forgetful functor induces a weak equivalence

$$KR(S^{2,1}_{\oplus}\mathscr{C}[n],D[n],\eta[n])_r \xrightarrow{\phi^*} KR(S^{2,1}\mathscr{C}[n],D[n],\eta[n])_r$$

of real pointed spaces. Together with the real group completion theorem, this implies the following result.

Theorem C. *Let $(\mathscr{C},D,\eta)$ be an exact category with weak equivalences and duality. For every positive integer r, the adjoint structure map*

$$KR(\mathscr{C},D,\eta)_r \xrightarrow{\tilde{\sigma}_{r,1}} \Omega^{2,1}(KR(\mathscr{C},D,\eta)_{r+1})$$

is a weak equivalence of pointed real spaces.

By Theorem C, the real symmetric spectrum $KR(\mathscr{C},D,\eta)$ is positively fibrant. It follows that, for $p \geqslant q \geqslant 0$, there is a canonical isomorphism

$$\mathsf{Ho}(\mathsf{RealTop}_*)(S^{p,q},\Omega^{2,1}(KR(\mathscr{C},D,\eta)_1)) \longrightarrow KR_{p,q}(\mathscr{C},D,\eta))$$

onto the real algebraic K-groups from the equivariant homotopy groups of the real pointed space $\Omega^{2,1}(KR(\mathscr{C},D,\eta)_1)$. Concerning this real pointed space, we further use the real additivity theorem together with a new group-completion theorem of Moi [16, Theorem 5.14] and a key observation of Schlichting [21, Proposition 3] to prove the following rather surprising result.

Theorem D. *There is a canonical chain of natural weak equivalences between the two functors $\Omega^{1,1}(K(\mathscr{C},D,\eta)_1)$ and $\Omega^{2,1}(KR(\mathscr{C},D,\eta)_1)$ from the category of exact categories with weak equivalences and duality to the category of pointed real spaces.*

This result, in particular, identifies the underlying non-equivariant homotopy type of real algebraic K-theory canonically with that of algebraic K-theory, as one would expect. However, the weak equivalence between the subspaces of points fixed by the respective G-actions, implies that Schlichting's higher Grothendieck-Witt groups [21], which, using the terminology introduced above, may be defined by

$$GW_p(\mathscr{C},D,\eta) = \pi_p((\Omega^{1,1}(K(\mathscr{C},D,\eta)_1))^G),$$

are canonically naturally isomorphic to the groups $KR_{p,0}(\mathscr{C},D,\eta)$, for all $p \geqslant 0$. This is surprising, since the real circle $S^{1,1}$ is not a co-group object in the homotopy category of pointed real spaces, and, indeed, Moi has showed that Theorem D fails, if the pointed real spaces $K(\mathscr{C},D,\eta)_1$ and $KR(\mathscr{C},D,\eta)_1$ are replaced by their respective direct sum counterparts.

There is a cofibration sequence of pointed real spaces

$$S^{0,0} \wedge G_+ \xrightarrow{f} S^{0,0} \xrightarrow{i} S^{1,1} \xrightarrow{h} S^{1,0} \wedge G_+$$

in which the map i is the unique inclusion of the sub-pointed real space of points fixed by the G-action. It induces a cofibration sequence of real suspension spectra, which, in turn, gives rise to, for every integer q, a long exact sequences

$$\cdots \xrightarrow{F_q} K_p(\mathscr{C}) \xrightarrow{H_q} KR_{p,q}(\mathscr{C},D,\eta) \xrightarrow{I_q} KR_{p-1,q-1}(\mathscr{C},D,\eta) \xrightarrow{F_q} \cdots.$$

The map F_q is called the forgetful map; H_q is called the hyperbolic map; and I_q may be identified with multiplication by the Hopf map $\eta\colon S^{1,1} \to S^{0,0}$. Here we have used the identification of the equivariant homotopy group

$$\mathrm{Ho}(\mathsf{Sp}^{\Sigma}(\mathsf{RealTop}_*,S^{2,1}))(S^{p,q} \wedge G_+, KR(\mathscr{C},D,\eta))$$

with $K_p(\mathscr{C})$. This, in turn, uses the isomorphism of $S^{1,0} \wedge G_+$ onto $S^{1,1} \wedge G_+$ that to the class of (z,g) assigns the class of $(iz \cdot g, g)$, where $i \in \mathbb{C}$ is a choice of square root of -1. The action by G on itself by right multiplication gives to a left G-action on the equivariant homotopy group above, and hence, to a left G-action on $K_p(\mathscr{C})$. The exact sequences define an exact couple with $E^1_{s,t} = K_t(\mathscr{C})$ and $D^1_{s,t} = KR_{t,-s}(\mathscr{C},D,\eta)$, the associated spectral sequence of which is the Tate spectral sequence

$$E^2_{s,t} = \hat{H}^{-s}(G, K_t(\mathscr{C})) \Rightarrow \hat{\mathbb{H}}^{-s-t}(G, KR(\mathscr{C},D,\eta)).$$

The d^1-differential $d^1 = F_{2-s} \circ H_{1-s}\colon E^1_{s,t} \to E^1_{s-1,t}$ is equal to $\mathrm{id} + (-1)^{1-s}\tau$, where we write $\tau\colon K_t(\mathscr{C}) \to K_t(\mathscr{C})$ for the action by complex conjugation $\tau \in G$.

1 The classifying space of a category with strict duality

A category with strict duality structure has a real classifying space. To analyze the equivariant homotopy type of this real space, we first define the following diagram of closed symmetric monoidal categories and strong symmetric monoidal functors and show that it commutes, up to the indicated monoidal natural isomorphism.

$$
\begin{tikzcd}
\mathrm{Cat\,Dual} \arrow[r, "B"] \arrow[d, "\mathrm{Sym}"'] & \mathrm{Real\,Top} \arrow[d, "(-)^{G_{\mathbb{R}}}"] \\
\mathrm{Cat} \arrow[r, "B"'] \arrow[ur, Rightarrow, "\alpha"] & \mathrm{Top}
\end{tikzcd}
$$

We then prove a number of results based on this diagram. Here $G_{\mathbb{R}} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. We write $\sigma \in G_{\mathbb{R}}$ for complex conjugation.

Definition 1.1. The closed symmetric monoidal structure on the category Set_0 of all (κ-small) sets and all maps between these is defined as follows. The monoidal product of the sets X_1 and X_2 is the set of ordered pairs

$$X_1 \times X_2 = \{(x_1,x_2) \mid x_1 \in X_1, x_2 \in X_2\}.$$

The set $X_1 \times X_2$ together with the two projection maps $\mathrm{pr}_i \colon X_1 \times X_2 \to X_i$ defined by $\mathrm{pr}_i(x_1,x_2) = x_i$ is an explicit choice of product of the objects X and Y in Set_0. The monoidal product of the maps f_1 and f_2 is the map $f_1 \times f_2 = (f_1 \circ \mathrm{pr}_1, f_2 \circ \mathrm{pr}_2)$. The unit for the monoidal product is the set $\{1\}$; it is a terminal object in Set_0. The associativity natural isomorphism

$$a_{X_1,X_2,X_3} \colon (X_1 \times X_2) \times X_3 \to X_1 \times (X_2 \times X_3)$$

is the unique natural isomorphism determined by the domain and target both being products of the same three objects in Set_0, and the identity and symmetry natural isomorphisms are defined similarly. The internal Hom-object is the set $\mathrm{Set}_0(X_1,X_2)$ of all maps from X_1 to X_2, and the unit and counit maps

$$X_1 \xrightarrow{\eta_{X_1}} \mathrm{Set}_0(X_2, X_1 \times X_2), \qquad \mathrm{Set}_0(X_2,X_3) \times X_2 \xrightarrow{\varepsilon_{X_3}} X_3$$

are defined by $\eta_{X_1}(x_1)(x_2) = (x_1,x_2)$ and $\varepsilon_{X_3}(f)(x_2) = f(x_2)$.

Definition 1.2. The closed symmetric monoidal structure on the category Cat_0 of all (κ-small) categories and all functors between these is defined as follows. The monoidal product of $\mathscr{C}_1$ and $\mathscr{C}_2$ is the category $\mathscr{C}_1 \times \mathscr{C}_2$ defined by

$$\begin{aligned} \mathrm{ob}(\mathscr{C}_1 \times \mathscr{C}_2) &= \mathrm{ob}(\mathscr{C}_1) \times \mathrm{ob}(\mathscr{C}_2) \\ (\mathscr{C}_1 \times \mathscr{C})((c_1,c_2),(c_1',c_2')) &= \mathscr{C}_1(c_1,c_1') \times \mathscr{C}_2(c_2,c_2') \end{aligned}$$

with identity morphisms and compositions defined componentwise. The category $\mathscr{C}_1 \times \mathscr{C}_2$ together with the functors $\mathrm{pr}_i \colon \mathscr{C}_1 \times \mathscr{C}_2 \to \mathscr{C}_i$ defined on objects and morphisms by the canonical projections is an explicit choice of product of the objects

$\mathscr{C}_1$ and $\mathscr{C}_2$ in Cat_0. If F_1 and F_2 are morphisms in Cat_0, we define $F_1 \times F_2$ to be the morphism $(F_1 \circ \mathrm{pr}_1, F_2 \circ \mathrm{pr}_2)$. The unit for the monoidal product is defined to be the discrete category $\mathbf{1}$ with $\mathrm{ob}(\mathbf{1}) = \{1\}$; it is a terminal object in Cat_0. The associativity, identity, and symmetry natural isomorphisms are defined to be the unique isomorphisms between different choices of products of the same objects in Cat_0. The internal Hom-object is defined to be the category $\mathrm{Cat}(\mathscr{C}_1, \mathscr{C}_2)$, where the objects are all functors $F\colon \mathscr{C}_1 \to \mathscr{C}_2$, where the morphisms are all natural transformations between such functors, where the identity morphisms are the identity natural transformations, and where the composition of morphisms is given by vertical composition of natural transformations. Finally, the unit and counit natural transformations

$$\mathscr{C}_1 \xrightarrow{\eta_{\mathscr{C}_1}} \mathrm{Cat}(\mathscr{C}_2, \mathscr{C}_1 \times \mathscr{C}_2), \qquad \mathrm{Cat}(\mathscr{C}_2, \mathscr{C}_3) \times \mathscr{C}_2 \xrightarrow{\varepsilon_{\mathscr{C}_3}} \mathscr{C}_3$$

are defined, on objects, by $\eta_{\mathscr{C}_1}(c_1)(c_2) = (c_1, c_2)$ and $\varepsilon_{\mathscr{C}_3}(F, c_2) = F(c_2)$ and, on morphisms, by $\eta_{\mathscr{C}_1}(f)_{c_2} = (f, 1_{c_2})$ and $\varepsilon_{\mathscr{C}_3}(\phi\colon F \Rightarrow F', f\colon c_2 \to c_2') = F(f) \circ \phi_{c_2}$.

Definition 1.3. Let Δ be the full subcategory of Cat_0 whose set of objects consists of the categories $[n]$ generated by the oriented graphs $0 \leftarrow 1 \leftarrow \cdots \leftarrow n$ with n a non-negative integer. The category of (κ-small) simplicial sets is the category

$$\mathrm{Simpl\,Set}_0 = \mathrm{Cat}(\Delta^{\mathrm{op}}, \mathrm{Set}_0)$$

which is given the following closed symmetric monoidal structure. The monoidal product $(X_1 \times X_2)[-]$ of the objects $X_1[-]$ and $X_2[-]$ is the composition

$$\Delta^{\mathrm{op}} \xrightarrow{\Delta} \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \xrightarrow{X_1[-] \times X_2[-]} \mathrm{Set}_0 \times \mathrm{Set}_0 \xrightarrow{\times} \mathrm{Set}_0$$

where Δ is the diagonal functor and $\times$ is the monoidal product functor defined in Definition 1.1. The monoidal product $(f_1 \times f_2)[-]$ of the morphisms $f_1[-]$ and $f_2[-]$ is the horizontal composite $\times \circ (f_1[-] \times f_2[-]) \circ \Delta$. The object $(X_1 \times X_2)[-]$ together with the morphisms $\mathrm{pr}_i\colon (X_1 \times X_2)[-] \to X_i[-]$ defined by the horizontal compositions $\mathrm{pr}_i \circ (X_1[-] \times X_2[-]) \circ \Delta$ is an explicit choice of product of the objects $X_1[-]$ and $X_2[-]$ in $\mathrm{Simpl\,Set}_0$. The unit for the monoidal product is the constant simplicial set $\{1\}[-]$ with value $\{1\}$; it is a terminal object in $\mathrm{Simpl\,Set}_0$. The associativity, identity, and symmetry isomorphisms are defined to be the unique isomorphisms between different choices of products of the same objects of $\mathrm{Simpl\,Set}_0$. The internal Hom-object is defined to be the simplicial set

$$\mathrm{Simpl\,Set}(X_1, X_2)[-] = \mathrm{Simpl\,Set}_0((\Delta[-] \times X_1)[-], X_2[-]),$$

where $\Delta[n][-] = \mathrm{Cat}_0([-], [n])$ is the simplicial standard n-simplex. The unit map

$$X_1[-] \xrightarrow{\eta_{X_1}[-]} \mathrm{Simpl\,Set}(X_2, X_1 \times X_2)[-],$$

is defined by $\eta_{X_1}[n](x_1)(\theta, x_2) = (\theta^*(x_1), x_2)$, and the counit map

$$(\mathrm{Simpl\,Set}(X_2, X_3) \times X_2)[-] \xrightarrow{\varepsilon_{X_3}[-]} X_3[-]$$

is defined by $\varepsilon_{X_3}[n](f, x_2) = f(\mathrm{id}_{[n]}, x_2)$.

We note that the definition of the monoidal product on $\mathrm{Simpl\,Set}$ is independent of the choice of the product categories $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ and $\mathrm{Set}_0 \times \mathrm{Set}_0$.

Definition 1.4. The closed symmetric monoidal structure on the category Top_0 of all (κ-small) compactly generated topological spaces and all continuous maps is defined as follows. The monoidal product of X_1 and X_2 is the set $X_1 \times X_2$ with the compactly generated topology associated with the product topology. The space $X_1 \times X_2$ together with the two projections $\mathrm{pr}_i \colon X_1 \times X_2 \to X_i$ is an explicit choice of product of the objects X_1 and X_2 in Top_0. The unit for the monoidal product is defined to be the discrete space $\{1\}$; it is a terminal object in Top_0. The associativity, identity, and symmetry isomorphisms are defined to be the associativity, identity, and symmetry isomorphisms of the underlying sets. The internal Hom-object $\mathrm{Top}(X_1, X_2)$ is defined to be the set $\mathrm{Top}_0(X_1, X_2)$ equipped with the compactly generated topology associated with the topology for which a subbasis is given by the subsets

$$N(h,U) = \{f \colon X_1 \to X_2 \mid f(h(K)) \subset U\} \subset \mathrm{Top}_0(X_1, X_2),$$

where h ranges over all continuous maps $h \colon K \to X_1$ from a compact Hausdorff space, and where U ranges over all open subsets $U \subset X_2$. The unit and counit maps

$$X_1 \xrightarrow{\eta_{X_1}} \mathrm{Top}(X_2, X_1 \times X_2), \qquad \mathrm{Top}(X_2, X_3) \times X_2 \xrightarrow{\varepsilon_{X_3}} X_3$$

are defined by $\eta_{X_1}(x_1)(x_2) = (x_1, x_2)$ and $\varepsilon_{X_3}(f, x_2) = f(x_2)$.

A detailed proof that Definition 1.4 indeed defines a closed symmetric monoidal structure on Top_0 is given in the appendix of the thesis of Gaunce Lewis [10].

Definition 1.5. The nerve is the strong symmetric monoidal functor

$$N = (N, \phi, \psi) \colon \mathrm{Cat} \to \mathrm{Simpl\,Set},$$

where $N \colon \mathrm{Cat}_0 \to \mathrm{Simpl\,Set}_0$ is the nerve functor defined by

$$N(\mathscr{C})[-] = \mathrm{ob}\,\mathrm{Cat}([-], \mathscr{C}),$$

and where $\phi \colon (N(\mathscr{C}_1) \times N(\mathscr{C}_2))[-] \to N(\mathscr{C}_1 \times \mathscr{C}_2)[-]$ and $\psi \colon \{1\}[-] \to N(\mathbf{1})[-]$ are the inverses of the canonical isomorphisms.

We define $\Delta[-] \colon \Delta \to \mathrm{Top}_0$ to be the functor that to the object $[n]$ associates the convex hull $\Delta[n]$ of the set $\mathrm{ob}([n])$ in the real vector space that it spans and that to the morphism $\theta \colon [m] \to [n]$ associates the affine map $\theta_* \colon \Delta[m] \to \Delta[n]$ induced by the set map $\theta \colon \mathrm{ob}([m]) \to \mathrm{ob}([n])$. The geometric realization $|X[-]|$ of the simplicial set $X[-]$, we recall, is defined by the following coequalizer diagram in Top_0.

$$\coprod X[n] \times \Delta([m],[n]) \times \Delta[m] \overset{f}{\underset{g}{\rightrightarrows}} \coprod X[p] \times \Delta[p] \longrightarrow |X[-]|$$

Here the middle and left-hand coproducts are indexed by the sets of objects and ordered pairs of objects in Δ, respectively, and the maps f and g are defined by

$$\begin{aligned} f \circ \mathrm{in}_{([m],[n])} &= \mathrm{in}_{[m]} \circ (f_{m,n} \times \mathrm{id}) \\ g \circ \mathrm{in}_{([m],[n])} &= \mathrm{in}_{[n]} \circ (\mathrm{id} \times g_{m,n}) \end{aligned}$$

where the maps $f_{m,n}$ and $g_{m,n}$ are the composite maps

$$X[n] \times \Delta([m],[n]) \xrightarrow{\mathrm{id} \times X} X[n] \times \mathrm{Set}(X[n], X[m]) \xrightarrow{\varepsilon \circ t} X[m]$$
$$\Delta([m],[n]) \times \Delta[m] \xrightarrow{\Delta \times \mathrm{id}} \mathrm{Top}(\Delta[m], \Delta[n]) \times \Delta[m] \xrightarrow{\varepsilon} \Delta[n].$$

We also recall that the canonical map

$$(|\mathrm{pr}_1|, |\mathrm{pr}_2|)\colon |(X_1 \times X_2)[-]| \to |X_1[-]| \times |X_2[-]|$$

is a homeomorphism; see [4] for an elegant proof.

Definition 1.6. The geometric realization functor is the strong symmetric monoidal functor

$$|-| = (|-|, \phi, \psi)\colon \mathrm{Simpl\,Set} \to \mathrm{Top},$$

where $|-|\colon \mathrm{Simpl\,Set}_0 \to \mathrm{Top}_0$ is the geometric realization functor defined above, and where $\phi\colon |X_1[-]| \times |X_2[-]| \to |(X_1 \times X_2)[-]|$ and $\psi\colon \{1\} \to |\{1\}[-]|$ are the inverses of the canonical homeomorphisms.

Let $(F, \phi_F, \psi_F)\colon \mathscr{U} \to \mathscr{V}$ and $(G, \phi_G, \psi_G)\colon \mathscr{V} \to \mathscr{W}$ be composable symmetric monoidal functors between symmetric monoidal categories. The composition is the symmetric monoidal functor defined by

$$(G \circ F, G(\phi_F) \circ \phi_G, G(\psi_F) \circ \psi_G)\colon \mathscr{U} \to \mathscr{W}.$$

If both F and G are strong monoidal, then also $G \circ F$ is strong monoidal.

Definition 1.7. The classifying space functor is the strong symmetric monoidal functor

$$B = (B, \phi, \psi)\colon \mathrm{Cat} \to \mathrm{Top}$$

given by the composition of the nerve and geometric realization symmetric monoidal functors.

Construction 1.8. A closed symmetric monoidal category

$$\mathscr{V} = (\mathscr{V}_0, \otimes, I, a, l, t, [-,-], \eta, \varepsilon)$$

has an associated $\mathscr{V}$-category, also denoted by $\mathscr{V}$, that is defined as follows. The set of objects is $\mathrm{ob}(\mathscr{V}) = \mathrm{ob}(\mathscr{V}_0)$, the morphism object $\mathscr{V}(x,y)$ is the internal Hom-object $[x,y]$, the composition morphism

$$\circ\colon \mathscr{V}(y,z) \otimes \mathscr{V}(x,y) \to \mathscr{V}(x,z)$$

is the adjoint of the composition

$$(\mathscr{V}(y,z) \otimes \mathscr{V}(x,y)) \otimes x \xrightarrow{a} \mathscr{V}(y,z) \otimes (\mathscr{V}(x,y) \otimes x) \xrightarrow{\mathrm{id} \otimes \varepsilon} \mathscr{V}(y,z) \otimes z \xrightarrow{\varepsilon} z,$$

and the identity morphism $1_x\colon I \to \mathscr{V}(x,x)$ is the adjoint of the identity morphism of the object x in $\mathscr{V}_0$.

Construction 1.9. Let $(F,\phi,\psi)\colon \mathscr{V} \to \mathscr{V}'$ be a symmetric monoidal functor between symmetric monoidal categories, and let $\mathscr{C}$ be a $\mathscr{V}$-category. In this situation, we define the $\mathscr{V}'$-category $\mathscr{C}_F$ as follows. The set of objects is $\mathrm{ob}(\mathscr{C}_F) = \mathrm{ob}(\mathscr{C})$, the morphism object $\mathscr{C}_F(c_1,c_2)$ is $F(\mathscr{C}(c_1,c_2))$, the composition morphism

$$\circ_F\colon \mathscr{C}_F(c_2,c_3) \otimes' \mathscr{C}_F(c_1,c_2) \to \mathscr{C}_F(c_1,c_3)$$

is the composite morphism

$$F(\mathscr{C}(c_2,c_3)) \otimes' F(\mathscr{C}(c_1,c_2)) \xrightarrow{\ \phi\ } F(\mathscr{C}(c_2,c_3) \otimes \mathscr{C}(c_1,c_2)) \xrightarrow{F(\circ)} F(\mathscr{C}(c_1,c_3))$$

and the identity morphism $1_{x,F}\colon I' \to \mathscr{C}_F(x,x)$ is the composite morphism

$$I' \xrightarrow{\ \psi\ } F(I) \xrightarrow{F(1_x)} F(\mathscr{C}(x,x)).$$

In the case where $\mathscr{V}$ and $\mathscr{V}'$ are closed symmetric monoidal categories, the symmetric monoidal functor $(F,\phi,\psi)\colon \mathscr{V} \to \mathscr{V}'$ gives rise to a $\mathscr{V}'$-functor $F\colon \mathscr{V}_F \to \mathscr{V}'$ that is given, on objects, by the map $F\colon \mathrm{ob}\,\mathscr{V} \to \mathrm{ob}\,\mathscr{V}'$ and, on morphism objects, by the morphism

$$F\colon F(\mathscr{V}(x,y)) \longrightarrow \mathscr{V}'(F(x),F(y))$$

that is adjoint to the composite morphism

$$F(\mathscr{V}(x,y)) \otimes' F(x) \xrightarrow{\ \phi\ } F(\mathscr{V}(x,y) \otimes F(x)) \xrightarrow{F(\varepsilon)} F(y)$$

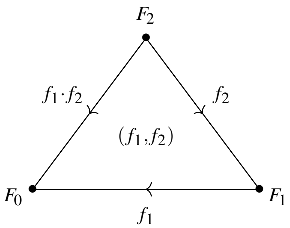
Remark 1.10. Since the classifying space functor in Definition 1.7 is symmetric monoidal, we obtain by Constructions 1.8 and 1.9 a map of spaces

$$B\,\mathrm{Cat}(\mathscr{C}_1,\mathscr{C}_2) \xrightarrow{\ B\ } \mathrm{Top}(B\mathscr{C}_1,B\mathscr{C}_2)$$

as part of the Top-functor $B\colon \mathrm{Cat}_B \to \mathrm{Top}$. Hence, for every n-tuple

$$F_0 \xLeftarrow{\ f_1\ } F_1 \xLeftarrow{\ f_2\ } \cdots \xLeftarrow{\ f_n\ } F_n$$

of composable natural transformations between functors $F_i\colon \mathscr{C}_1 \to \mathscr{C}_2$, we obtain a continuous map $\Delta[n] \to \mathrm{Top}(B\mathscr{C}_1,B\mathscr{C}_2)$ from the topological standard n-simplex to the space of maps from $B\mathscr{C}_1$ to $B\mathscr{C}_2$. In particular, a functor $F\colon \mathscr{C}_1 \to \mathscr{C}_2$ determines a point in the mapping space; a natural transformation $f_1\colon F_1 \Rightarrow F_0$ determines a path in the mapping space between the points determined by F_0 and F_1; and two composable natural transformations $f_1\colon F_1 \Rightarrow F_2$ and $f_2\colon F_2 \Rightarrow F_1$ determine a map from a 2-simplex to the mapping space as indicated by the figure below.



This is the extent to which the classifying space functor preserves composition.

We have now defined the lower row in the diagram at the beginning of the section and proceed to define the rest of the diagram. We recall that $G_{\mathbb{R}} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and that $\sigma \in G_{\mathbb{R}}$ is the generator.

Definition 1.11. The category $\mathrm{Real\,Set}_0$ of all (κ-small) left $G_{\mathbb{R}}$-sets and all equivariant maps has a closed symmetric monoidal structure defined as follows. The monoidal product of X_1 and X_2 is the set $X_1 \times X_2$ with the diagonal $G_{\mathbb{R}}$-action, the unit for the monoidal product is the set $\{1\}$ with the trivial $G_{\mathbb{R}}$-action, and the associativity, identity, and symmetry natural isomorphisms are the corresponding isomorphisms of the underlying sets. The internal Hom-object is the real set $\mathrm{Real\,Set}(X_1, X_2)$ given by the set $\mathrm{Set}(X_1, X_2)$ with the left $G_{\mathbb{R}}$-action defined by $(\sigma f)(x_1) = \sigma f(\sigma^{-1} x_1)$, and the unit and counit maps are defined to be the unit and counit maps of the underlying sets.

We remark that the set $\mathrm{Real\,Set}_0(X_1, X_2)$ is equal to the subset of $G_{\mathbb{R}}$-fixed points in the real set $\mathrm{Real\,Set}(X_1, X_2)$.

Definition 1.12. A real category is a category enriched in $\mathrm{Real\,Set}$; a real functor between real categories is a $\mathrm{Real\,Set}$-functor; and a real natural transformation between real functors is a $\mathrm{Real\,Set}$-natural transformation.

We recall that a category with strict duality is a pair $(\mathscr{C}, D)$ of a category $\mathscr{C}$ and a functor $D \colon \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$ such that $D \circ D^{\mathrm{op}}$ and $D^{\mathrm{op}} \circ D$ are equal to the respective identity functors, and that a duality preserving functor $F \colon (\mathscr{C}_1, D_1) \to (\mathscr{C}_2, D_2)$ between categories with strict duality is a functor $F \colon \mathscr{C}_1 \to \mathscr{C}_2$ such that $F \circ D_1 = D_2 \circ F^{\mathrm{op}}$.

Definition 1.13. The category $\mathrm{Cat\,Dual}_0$ of all (κ-small) categories with strict duality and duality preserving functors has the following closed symmetric monoidal structure. The monoidal product of the objects $(\mathscr{C}_1, D_1)$ and $(\mathscr{C}_2, D_2)$ is the pair

$$(\mathscr{C}_1, D_1) \times (\mathscr{C}_2, D_2) = (\mathscr{C}_1 \times \mathscr{C}_2, D_1 \times D_2),$$

where $\mathscr{C}_1 \times \mathscr{C}_2$ and $D_1 \times D_2$ are the monoidal products in Cat, and the monoidal product of the morphisms F_1 and F_2 is the monoidal product $F_1 \times F_2$ in Cat. The unit for the monoidal product is $(\mathbf{1}, \mathrm{id})$, and the associativity, identity, and symmetry isomorphisms are corresponding isomorphisms of the underlying categories. The internal Hom-object is the pair

$$\mathrm{Cat\,Dual}((\mathscr{C}_1, D_1), (\mathscr{C}_2, D_2)) = (\mathrm{Cat}(\mathscr{C}_1, \mathscr{C}_2), (D_1, D_2)),$$

where $(D_1, D_2) \colon \mathrm{Cat}(\mathscr{C}_1, \mathscr{C}_2)^{\mathrm{op}} \to \mathrm{Cat}(\mathscr{C}_1, \mathscr{C}_2)$ is the functor that to

$$\begin{tikzcd} \mathscr{C}_1 \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow f"] & \mathscr{C}_2 \end{tikzcd}$$

assigns the horizontal composition

$$\begin{tikzcd} \mathscr{C}_1 \arrow[r, "D_1^{\mathrm{op}}"] & \mathscr{C}_1^{\mathrm{op}} \arrow[r, bend left, "F^{\mathrm{op}}"] \arrow[r, bend right, "G^{\mathrm{op}}"'] \arrow[r, phantom, "\Uparrow f^{\mathrm{op}}"] & \mathscr{C}_2^{\mathrm{op}} \arrow[r, "D_2"] & \mathscr{C}_2, \end{tikzcd}$$

and the unit and counit functors are defined to be the unit and counit functors that are part of the closed symmetric monoidal structure on Cat.

Example 1.14. The set of objects in a category with strict duality is naturally a real set. More precisely, there is a strong symmetric monoidal functor

$$\mathrm{ob}_R = (\mathrm{ob}_R, \mathrm{id}, \mathrm{id})\colon \ \mathrm{Cat\,Dual} \to \mathrm{Real\,Set},$$

where $\mathrm{ob}_R\colon \ \mathrm{Cat\,Dual}_0 \to \mathrm{Real\,Set}_0$ is the functor that to $(\mathscr{C}, D)$ associates the real set $\mathrm{ob}_R(\mathscr{C})$ defined to be the set $\mathrm{ob}(\mathscr{C})$ with the left $G_{\mathbb{R}}$-action where the generator $\sigma \in G_{\mathbb{R}}$ acts by the map $D\colon \ \mathrm{ob}(\mathscr{C}) = \mathrm{ob}(\mathscr{C}^{\mathrm{op}}) \to \mathrm{ob}(\mathscr{C})$. In particular, a category enriched in Cat Dual gives rise to an underlying real category upon applying the functor $\mathrm{ob}_R(-)$ to the Hom-objects. For example, the category $[n]$ has a unique strict duality structure given by the functor $D\colon \ [n]^{\mathrm{op}} \to [n]$ defined by $D(i) = n - i$ and the real index category ΔR may be identified with the underlying real category of the full sub-Cat Dual-category of Cat Dual whose objects are the categories with strict duality $([n], D)$ with n a non-negative integer.

We remark that the set $\mathrm{Cat\,Dual}_0((\mathscr{C}_1, D_1), (\mathscr{C}_2, D_2))$ is equal to the subset of $G_{\mathbb{R}}$-fixed set points in the real set $\mathrm{ob}_R \mathrm{Cat\,Dual}((\mathscr{C}_1, D_1), (\mathscr{C}_2, D_2))$.

Definition 1.15. The category $\mathrm{Real\,Simpl\,Set}_0$ of all (κ-small) real simplicial sets and real simplicial maps has a closed symmetric monoidal structure defined as follows. The monoidal product $(X_1 \times X_2)[-]$ of the real simplicial sets $X_1[-]$ and $X_2[-]$ is given by the monoidal product of the underlying simplicial sets with the diagonal left $G_{\mathbb{R}}$-action on $(X_1 \times X_2)[n] = X_1[n] \times X_2[n]$; the unit for the monoidal product is the constant real simplicial set $\{1\}[-]$ with value $\{1\}$; and the internal Hom-object $\mathrm{Real\,Simpl\,Set}(X_1, X_2)[-]$ is given by the simplicial mapping space of the underlying simplicial sets with the conjugation left $G_{\mathbb{R}}$-action on $\mathrm{Simpl}_0((\Delta[n] \times X_1)[-], X_2[-])$. The associativity, identity, and symmetry isomorphisms and the unit and counit maps are the corresponding maps of the underlying simplicial sets.

We note that the set $\mathrm{Real\,Simpl\,Set}_0(X_1[-], X_2[-])$ is canonically isomorphic to the subset of $G_{\mathbb{R}}$-fixed point in the real set $\mathrm{Real\,Simpl\,Set}(X_1, X_2)[0]$.

Definition 1.16. The real nerve is the strong symmetric monoidal functor

$$N = (N, \phi, \psi)\colon \ \mathrm{Cat\,Dual} \to \mathrm{Real\,Simpl\,Set},$$

where $N\colon \ \mathrm{Cat\,Dual}_0 \to \mathrm{Real\,Simpl\,Set}_0$ is the functor defined by

$$N(\mathscr{C}, D)[-] = \mathrm{ob}_R \mathrm{Cat\,Dual}(([-], D), (\mathscr{C}, D)),$$

and where the maps $\phi\colon \ (N(\mathscr{C}_1, D_1) \times N(\mathscr{C}_2, D_2))[-] \to N(\mathscr{C}_1 \times \mathscr{C}_2, D_1 \times D_2)[-]$ and $\psi\colon \ \{1\}[-] \to N(\mathbf{1}, \mathrm{id})[-]$ are the inverses of the canonical isomorphisms.

Definition 1.17. The category $\mathrm{Real\,Top}_0$ of all (κ-small) compactly generated left $G_{\mathbb{R}}$-spaces and all continuous $G_{\mathbb{R}}$-equivariant maps has the following closed symmetric monoidal structure. The monoidal product of X_1 and X_2 is the product space $X_1 \times X_2$ with the diagonal left $G_{\mathbb{R}}$-action, the unit is the discrete space $\{1\}$ with the trivial $G_{\mathbb{R}}$-action, and the associativity, identity, and symmetry natural isomorphisms are the corresponding isomorphisms of the underlying spaces. The internal Hom-objects is the mapping space $\mathrm{Top}(X_1,X_2)$ equipped with the conjugation left $G_{\mathbb{R}}$-action given by $(\sigma f)(x_1) = \sigma f(\sigma^{-1}x_1)$, and the unit and counit maps are the unit and counit maps of the underlying spaces.

We note that the set $\mathrm{Real\,Top}_0(X,Y)$ is equal to the underlying set of the subspace of $G_{\mathbb{R}}$-fixed points in the mapping space $\mathrm{Real\,Top}(X,Y)$.

Let $\Delta R[n]$ be the real space defined by the space $\Delta[n]$ with the left $G_{\mathbb{R}}$-action where the generator $\sigma \in G_{\mathbb{R}}$ acts through the affine map $\sigma\colon \Delta R[n] \to \Delta R[n]$ that maps the vertex i to then vertex $n-i$. There is a real functor

$$\Delta R[-]\colon \Delta R \to \mathrm{Real\,Top}$$

that takes $[n]$ to $\Delta R[n]$ and that on morphism real sets is given by the real map

$$\Delta R([m],[n]) \xrightarrow{\Delta R[-]} \mathrm{Real\,Top}(\Delta R[m],\Delta R[n])$$

that to the functor $\theta\colon\ [m] \to [n]$ associates the affine map $\theta_*\colon \Delta R[m] \to \Delta R[n]$ induced by $\mathrm{ob}\,\theta\colon\ \mathrm{ob}[m] \to \mathrm{ob}[n]$. Here we view $\mathrm{Real\,Top}$ as a real category by forgetting the topology on the mapping spaces.

We now define the geometric realization of the real simplicial set $X[-]$ to be the real space $|X[-]|_R$ given by the following coequalizer diagram in $\mathrm{Real\,Top}_0$.

$$\coprod X[n] \times \Delta R([m],[n]) \times \Delta R[m] \underset{g}{\overset{f}{\rightrightarrows}} \coprod X[p] \times \Delta R[p] \longrightarrow |X[-]|_R$$

The middle and left-hand coproducts range over the sets of objects and ordered pairs of objects in ΔR, respectively, and the real maps f and g are defined by

$$\begin{aligned} f \circ \mathrm{in}_{([m],[n])} &= \mathrm{in}_{[m]} \circ (f_{m,n} \times \mathrm{id}) \\ g \circ \mathrm{in}_{([m],[n])} &= \mathrm{in}_{[n]} \circ (\mathrm{id} \times g_{m,n}) \end{aligned}$$

where $f_{m,n}$ and $g_{m,n}$ are the composite real maps

$$X[n] \times \Delta R([m],[n]) \xrightarrow{\mathrm{id}\times\Delta R} X[n] \times \mathrm{Real\,Set}(X[n],X[m]) \xrightarrow{\varepsilon\circ t} X[m]$$

$$\Delta R([m],[n]) \times \Delta R[m] \xrightarrow{\mathrm{id}\times X} \mathrm{Real\,Top}(\Delta R[m],\Delta R[n]) \times \Delta R[m] \xrightarrow{\varepsilon} \Delta R[n].$$

Here the real sets $X[m]$, $X[n]$, $\Delta R([m],[n])$, and $\mathrm{Real\,Set}(X[n],X[m])$ are considered as real spaces with the discrete topology.

Definition 1.18. The geometric realization is the strong symmetric monoidal functor

$$|-|_R = (|-|_R, \phi, \psi)\colon\ \mathrm{Real\,Simpl\,Set} \to \mathrm{Real\,Top},$$

where $|-|_R\colon\ \mathrm{Real\,Simpl\,Set}_0 \to \mathrm{Real\,Top}_0$ is the geometric realization functor defined above, and where the real maps $\phi\colon |X_1[-]|_R \times |X_2[-]|_R \to |(X_1 \times X_2)[-]|_R$ and $\{1\} \to |\{1\}[-]|_R$ are the inverses of the canonical isomorphisms.

Definition 1.19. The real classifying space functor is the strong symmetric monoidal functor

$$B = (B, \phi, \psi)\colon \mathrm{Cat\,Dual} \to \mathrm{Real\,Top}$$

defined by the composition of the real nerve and the real geometric realization symmetric monoidal functors.

We next recall Segal's subdivision [22]. We view the simplicial index category Δ as a real category with trivial $G_{\mathbb{R}}$-action on the morphism sets $\Delta([m],[n])$ and define

$$\Delta \xrightarrow{\mathrm{sd}} \Delta R$$

to be the real functor that takes the object $[n]$ to the object $[2n+1]$ and that, on morphism real sets, is given by the real maps

$$\Delta([m],[n]) \xrightarrow{\mathrm{sd}} \Delta R(\mathrm{sd}[m], \mathrm{sd}[n])$$

defined by

$$\mathrm{sd}(\theta)(i) = \begin{cases} n - \theta(m-i) & (0 \leqslant i \leqslant m) \\ n+1+\theta(i-(m+1)) & (m+1 \leqslant i \leqslant 2m+1). \end{cases}$$

The induced symmetric monoidal functor

$$\mathrm{sd} = (\mathrm{sd}^*, \phi, \psi)\colon \mathrm{Real\,Simpl\,Set} \to \mathrm{Simpl\,Real\,Set}$$

is the Segal subdivision. Here, the target is the category of simplicial objects in $\mathrm{Real\,Set}_0$ with the closed symmetric monoidal structure defined as in Definition 1.3, and the maps ϕ and ψ are the inverses of the canonical isomorphisms. Moreover, the composition of the maps

$$\coprod X[2p+1] \times \Delta[p] \longrightarrow \coprod X[2p+1] \times \Delta R[2p+1] \longrightarrow \coprod X[p] \times \Delta R[p],$$

where the coproducts range over non-negative integers p, where the left-hand map is induced by the affine maps $d_p\colon \Delta[p] \to \Delta R[2p+1]$ that take the vertex i to the barycenter of the 1-simplex that connects the vertices $p-i$ and $p+1+i$, and where the right-hand map is the canonical inclusion induces a natural transformation

$$|\,\mathrm{sd}X[-]| \xrightarrow{d} |X[-]|_R.$$

It follows immediately from the definitions that d is a monoidal natural transformation, which means that the following diagrams commute.

$$\begin{tikzcd}
|\,\mathrm{sd}X_1[-]| \times |\,\mathrm{sd}X_2[-]| \arrow[r, "d\times d"] \arrow[d, "\phi"] & |X_1[-]|_R \times |X_2[-]|_R \arrow[d, "\phi"] \\
|\,\mathrm{sd}(X_1 \times X_2)[-]| \arrow[r, "d"] & |(X_1 \times X_2)[-]|_R
\end{tikzcd}$$

$$\begin{tikzcd}
 & \{1\} \arrow[dl, "\psi"'] \arrow[dr, "\psi"] & \\
{|\,\mathrm{sd}\{1\}[-]|} \arrow[rr, "d"] & & {|\{1\}[-]|_R}
\end{tikzcd}$$

The following result is proved in [22, Proposition A.1], but see also [19].

Lemma 1.20. *The monoidal natural transformation*

$$|\,\mathrm{sd}X[-]| \xrightarrow{d} |X[-]|_R$$

is a real homeomorphism.

The lemma has the following corollary which provides a simplicial model for the subspace of $G_{\mathbb{R}}$-fixed points in the real space $|X[-]|^{G_{\mathbb{R}}}$.

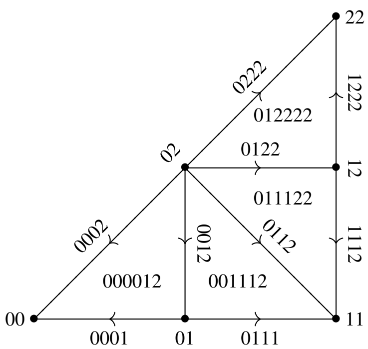
Corollary 1.21. *The composition*

$$|(\mathrm{sd}X[-])^{G_{\mathbb{R}}}| \xrightarrow{\gamma} |X[-]|_R^{G_{\mathbb{R}}}$$

of the canonical map $|(\mathrm{sd}X[-])^{G_{\mathbb{R}}}| \to |\,\mathrm{sd}X[-]|^{G_{\mathbb{R}}}$ and the map $d^{G_{\mathbb{R}}}$ of $G_{\mathbb{R}}$-fixed sets induced by d is a monoidal natural homeomorphism.

Proof. The canonical map $|Y[-]^{G_{\mathbb{R}}}| \to |Y[-]|^{G_{\mathbb{R}}}$ is a monoidal natural transformation of symmetric monoidal functors from Real Simpl Set to Top, and it is a homeomorphism, since geometric realization preserves finite limits [4]. The map $d^{G_{\mathbb{R}}}$ is a monoidal natural homeomorphism by Lemma 1.20, and finally, the composition of monoidal natural homeomorphisms is a monoidal natural homeomorphism. ☐

Example 1.22. The following figure indicates the non-degenerate simplices in the subdivision $\mathrm{sd}\,\Delta R[2][-]$ of the real simplicial set $\Delta R[2][-]$.



The generator $\sigma \in G_{\mathbb{R}}$ acts by reflection in the line through the 0-simplices 11 and 02.

A symmetric space in the category with strict duality $(\mathscr{C}, D)$ is defined to be a pair (c, f) of an object c and a morphism $f\colon c \to Dc^{\mathrm{op}}$ in $\mathscr{C}$ such that $f = Df^{\mathrm{op}}$. A map of symmetric spaces $g\colon (c_1, f_1) \to (c_2, f_2)$ is a morphism $g\colon c_1 \to c_2$ in $\mathscr{C}$ such that the following diagram commutes.

$$\begin{tikzcd} c_1 \arrow[r, "g"] \arrow[d, "f_1"] & c_2 \arrow[d, "f_2"] \\ Dc_1^{\mathrm{op}} & Dc_2^{\mathrm{op}} \arrow[l, "Dg^{\mathrm{op}}"'] \end{tikzcd}$$

We view the morphism $f\colon c \to Dc^{\mathrm{op}}$ with $f = Df^{\mathrm{op}}$ as a symmetric form on the object c with respect to the strict duality structure $D\colon \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$. The symmetric form is said to be non-singular if, in addition, the morphism f is an isomorphism.

Definition 1.23. The symmetric space functor is the strong symmetric monoidal functor

$$\mathrm{Sym} = (\mathrm{Sym}, \phi, \psi)\colon \mathrm{Cat\,Dual} \to \mathrm{Cat},$$

where $\mathrm{Sym}\colon \mathrm{Cat\,Dual}_0 \to \mathrm{Cat}_0$ is the functor defined above, and where the natural transformations $\phi\colon \mathrm{Sym}(\mathscr{C}_1, D_1) \times \mathrm{Sym}(\mathscr{C}_2, D_2) \to \mathrm{Sym}(\mathscr{C}_1 \times \mathscr{C}_2, D_1 \times D_2)$ and $\psi\colon \mathbf{1} \to \mathrm{Sym}(\mathbf{1}, \mathrm{id})$ are defined to be the inverses of the canonical isomorphisms.

We now prove the following result promised at the beginning of the section.

Proposition 1.24. *There is a canonical monoidal natural homeomorphism α that makes the following diagram of symmetric monoidal functors commute.*

$$\begin{tikzcd} \mathrm{Cat\,Dual} \arrow[r, "B"] \arrow[d, "\mathrm{Sym}"'] & \mathrm{Real\,Top} \arrow[d, "(-)^{G_{\mathbb{R}}}"] \\ \mathrm{Cat} \arrow[r, "B"'] \arrow[ur, phantom, "\overset{\alpha}{\Nearrow}"] & \mathrm{Top} \end{tikzcd}$$

Proof. There is a natural isomorphism of simplicial sets

$$\beta\colon N\,\mathrm{Sym}(\mathscr{C}, D)[-] \to (\mathrm{sd}\,N(\mathscr{C}, D)[-])^{G_{\mathbb{R}}}$$

that to the n-simplex $(c_0, f_0) \xleftarrow{g_1} \cdots \xleftarrow{g_n} (c_n, f_n)$ assigns the n-simplex

$$Dc_n^{\mathrm{op}} \xleftarrow{Dg_n^{\mathrm{op}}} \cdots \xleftarrow{Dg_1^{\mathrm{op}}} Dc_0^{\mathrm{op}} \xleftarrow{f_0} c_0 \xleftarrow{g_1} \cdots \xleftarrow{g_n} c_n.$$

It follows immediately from the definitions that β is monoidal, and the desired monoidal natural homeomorphism α now is the composition

$$|N\,\mathrm{Sym}(\mathscr{C}, D)[-]| \xrightarrow{|\beta|} |(\mathrm{sd}\,N(\mathscr{C}, D)[-])^{G_{\mathbb{R}}}| \xrightarrow{\gamma} |N(\mathscr{C}, D)[-]|^{G_{\mathbb{R}}}$$

of the monoidal natural homeomorphisms $|\beta|$ and γ. $\square$

Remark 1.25. We explain how Proposition 1.24 is used to produce equivariant maps and equivariant homotopies between the real classifying spaces of categories with strict duality. From Constructions 1.8 and 1.9, we have the Top-functor

$$\mathrm{Cat\,Dual}_{B(-)^{G_{\mathbb{R}}}} \xrightarrow{B(-)^{G_{\mathbb{R}}}} \mathrm{Top}$$

which we precompose with the Top-functor

$$\mathrm{Cat\,Dual}_{B\,\mathrm{Sym}} \xrightarrow{F_\alpha} \mathrm{Cat\,Dual}_{B(-)^{G_{\mathbb{R}}}}$$

that, on objects, is the identity and, on morphism spaces, is the homeomorphism

$$B\,\mathrm{Sym\,Cat\,Dual}((\mathscr{C}_1,D_1),(\mathscr{C}_2,D_2)) \xrightarrow{F_\alpha} B\,\mathrm{Cat\,Dual}((\mathscr{C}_1,D_1),(\mathscr{C}_2,D_2))^{G_{\mathbb{R}}}$$

given by the monoidal natural homeomorphism α from Proposition 1.24. Let

$$\mathrm{Cat\,Dual}_{B\,\mathrm{Sym}} \xrightarrow{H} \mathrm{Top}$$

be the composite Top-functor; it gives a map of spaces

$$B\,\mathrm{Sym\,Cat\,Dual}((\mathscr{C}_1,D_1),(\mathscr{C}_2,D_2)) \xrightarrow{H} \mathrm{Real\,Top}(B(\mathscr{C}_1,D_1),B(\mathscr{C}_2,D_2))^{G_{\mathbb{R}}}.$$

Hence, every n-tuple of morphisms of symmetric spaces

$$(F_0,f_0) \xLeftarrow{g_0} (F_1,f_1) \xLeftarrow{g_1} \cdots \xLeftarrow{g_n} (F_n,f_n)$$

in $\mathrm{Cat\,Dual}((\mathscr{C}_1,D_1),(\mathscr{C}_2,D_2))$ determines a continuous map

$$\Delta[n] \to \mathrm{Real\,Top}(B(\mathscr{C}_1,D_1),B(\mathscr{C}_2,D_2))^{G_{\mathbb{R}}}$$

from the topological standard n-simplex $\Delta[n]$ to the space of real maps from $B(\mathscr{C}_1,D_1)$ to $B(\mathscr{C}_2,D_2)$. In particular, every symmetric space (F,f) determines a real map

$$H(F,f)\colon B(\mathscr{C}_1,D_1) \to B(\mathscr{C}_2,D_2),$$

and every map of symmetric spaces $g_1\colon (F_1,f_1) \Rightarrow (F_0,f_0)$ determines a real homotopy $H(g_1)$ from the real map $H(F_0,f_0)$ to the real map $H(F_1,f_1)$.

Corollary 1.26. *If the category with strict duality $(\mathscr{C},D)$ has an initial object 0, then the real classifying space $B(\mathscr{C},D)$ is equivariantly contractible.*

Proof. The image of the map

$$B\,\mathrm{Sym\,Cat\,Dual}((\mathscr{C},D),(\mathscr{C},D)) \xrightarrow{H} \mathrm{Real\,Top}(B(\mathscr{C},D),B(\mathscr{C},D))^{G_{\mathbb{R}}},$$

which is part of the Top-functor $H\colon \mathrm{Cat\,Dual}_{B\,\mathrm{Sym}} \to \mathrm{Top}$, contains both the identity map and the constant map $B0$. Hence, it will suffice to show that the domain, and hence also the image, of this map is a contractible space. The object $1 = D(0^{\mathrm{op}})$ is a terminal object of $\mathscr{C}$ and $0 = D(1^{\mathrm{op}})$. Now, the constant functors $0, 1\colon \mathscr{C} \to \mathscr{C}$ with values 0 and 1 are an initial object and a terminal object, respectively, of the functor category $\mathrm{Cat}(\mathscr{C},\mathscr{C})$. Let $u\colon 0 \Rightarrow 1$ be the unique morphism. We claim that $(0,u)$ is an initial object of $\mathrm{Sym\,Cat\,Dual}((\mathscr{C},D),(\mathscr{C},D))$. Indeed, the pair $(0,u)$ is an object of said category, since u and Du^{op} both are morphisms from 0 to 1 and therefore necessarily equal. Similarly, if (F,f) is another object, then the unique morphism $g\colon 0 \Rightarrow F$ necessarily makes the following diagram commute.

$$\begin{tikzcd} 0 \arrow[r, Rightarrow, "g"] \arrow[d, Rightarrow, "u"] & F \arrow[d, Rightarrow, "f"] \\ D0^{\mathrm{op}} & DF^{\mathrm{op}} \arrow[l, Rightarrow, "Dg^{\mathrm{op}}"'] \end{tikzcd}$$

This shows that g is a morphism of symmetric spaces $g\colon (0,u) \to (F,f)$, and since this morphism is unique, we conclude that $(0,u)$ is an initial object as claimed. It follows that the domain of the map H is contractible as desired. $\square$

Definition 1.27. An adjunction from the category with strict duality $(\mathscr{C}_1,D_1)$ to the category with strict duality $(\mathscr{C}_2,D_2)$ is a sextuple $(F,G,\eta,\varepsilon,f,g)$ in which the quadruple (F,G,η,ε) is an adjunction from the category $\mathscr{C}_1$ to the category $\mathscr{C}_2$ and in which $f\colon F \Rightarrow (D_1,D_2)F^{\mathrm{op}}$ and $g\colon G \Rightarrow (D_2,D_2)G^{\mathrm{op}}$ are natural transformations such that $f = (D_1,D_2)f^{\mathrm{op}}$ and $g = (D_2,D_1)g^{\mathrm{op}}$ and such that the diagrams

$$\begin{tikzcd} \mathrm{id}_{\mathscr{C}_1} \arrow[r, Rightarrow, "\eta"] \arrow[d, Rightarrow, "1"] & G \circ F \arrow[d, Rightarrow, "g \circ f"] \\ (D_1,D_1)\,\mathrm{id}_{\mathscr{C}_1}^{\mathrm{op}} & (D_1,D_1)(G \circ F)^{\mathrm{op}} \arrow[l, Rightarrow, "(D_1,D_1)\eta^{\mathrm{op}}"'] \end{tikzcd}$$

$$\begin{tikzcd} F \circ G \arrow[r, Rightarrow, "\varepsilon"] \arrow[d, Rightarrow, "f \circ g"] & \mathrm{id}_{\mathscr{C}_2} \arrow[d, Rightarrow, "1"] \\ (D_2,D_2)(F \circ G)^{\mathrm{op}} & (D_2,D_2)\,\mathrm{id}_{\mathscr{C}_2}^{\mathrm{op}} \arrow[l, Rightarrow, "(D_2,D_2)\varepsilon^{\mathrm{op}}"'] \end{tikzcd}$$

commute.

Corollary 1.28. *If $(F,G,\eta,\varepsilon,f,g)$ is an adjunction from the category with strict duality $(\mathscr{C}_1,D_1)$ to the category with strict duality $(\mathscr{C}_2,D_2)$, then the real maps*

$$\begin{tikzcd} B(\mathscr{C}_1,D_1) \arrow[r, shift left, "H(F{,}f)"] & B(\mathscr{C}_2,D_2) \arrow[l, shift left, "H(G{,}g)"] \end{tikzcd}$$

are inverse equivariant homotopy equivalences.

Proof. Since $H\colon \mathrm{Cat\,Dual}_{B\mathrm{Sym}} \to \mathrm{Top}$ is a Top-functor, we have

$$H(G,g) \circ H(F,f) = H(G \circ F, g \circ f)$$
$$H(F,f) \circ H(G,f) = H(F \circ G, f \circ g).$$

Moreover, the unit and counit of the adjunction define maps of symmetric spaces

$$\eta\colon (\mathrm{id}_{\mathscr{C}_1},1) \to (G\circ F, g\circ f)$$
$$\varepsilon\colon (F\circ G, f\circ f) \to (\mathrm{id}_{\mathscr{C}_2},1).$$

It follows that $H(\eta)$ is an equivariant homotopy from $H(G,g)\circ H(F,f)$ to the identity map of $B(\mathscr{C}_1,D_1)$ and that $H(\varepsilon)$ is a equivariant homotopy from the identity map of $B(\mathscr{C}_2,D_2)$ to $H(F,f)\circ H(G,g)$. □

We finally discuss pointed analogues of the above. In general, if $\mathscr{C}$ is a category and if $1\in\mathrm{ob}\mathscr{C}$ is a terminal object, then the associated category of pointed objects is defined to be the under-category $1/\mathscr{C}$. If also $1'\in\mathrm{ob}\mathscr{C}$ is a terminal object, then the categories $1/\mathscr{C}$ and $1'/\mathscr{C}$ are canonically isomorphic. We will often suppress the choice of terminal object and write $\mathscr{C}_*$ for the category $1/\mathscr{C}$ and (X,x) for the pointed object $x\colon 1\to X$. If $\mathscr{C}$ has finite coproducts, then the forgetful functor $1/\mathscr{C}\to\mathscr{C}$ that to (X,x) assigns X admits the left adjoint functor $(-)_+\colon \mathscr{C}\to 1/\mathscr{C}$ that to X associates a choice of coproduct $X_+ = (1\sqcup X, \mathrm{in}_1)$. The unit and counit maps are defined to be $\eta_X = \mathrm{in}_2\colon X\to 1\sqcup X$ and $\varepsilon_{(X,x)} = x+\mathrm{id}_X\colon (1\sqcup X,\mathrm{in}_1)\to(X,x)$, respectively.

Now, if $\mathscr{V} = (\mathscr{V}_0,\otimes,I,a,l,t,[-,-],\eta,\varepsilon)$ is a closed symmetric monoidal category that admits finite limits and colimits, then the category $\mathscr{V}_{0*} = 1/\mathscr{V}_0$ of pointed objects in $\mathscr{V}_0$ inherits a closed symmetric monoidal structure defined as follows. The monoidal product $(X,x)\wedge(Y,y)$ is defined to be a choice of push-out

$$\begin{tikzcd}
(X\otimes 1)\sqcup(1\otimes Y) \arrow[r, "\mathrm{id}\otimes y + x\otimes\mathrm{id}"] \arrow[d] & X\otimes Y \arrow[d, "p"] \\
1 \arrow[r] & (X,x)\wedge(Y,y)
\end{tikzcd}$$

with the basepoint given by the lower horizontal morphism and is called the smash product. The unit object for the smash product is I_+, and the left identity isomorphism $l_{(X,x)}\colon I_+\wedge(X,x)\to(X,x)$ is defined to be the unique map determined by the given map $x\colon 1\to X$ and by the composition

$$(1\sqcup I)\otimes X \xleftarrow{\sim} (1\otimes X)\sqcup(I\otimes X) \longrightarrow 1\sqcup(I\otimes X) \xrightarrow{x+l_X} X$$

of the inverse of the canonical map, the map induced by the unique map $1\otimes X\to 1$, and the sum of the maps $x\colon 1\to X$ and $l_X\colon I\otimes X\to X$. The associativity and symmetry isomorphisms in $\mathscr{V}_*$ are induced by those in $\mathscr{V}$ using that the functor $-\otimes X$ preserves colimits. Dually, since the functor $[X,-]$ preserves limits, the object $[X,1]$ is terminal, and we define the internal Hom-object $[(X,x),(Y,y)]_*$ to be a choice of pull-back

$$\begin{tikzcd}
{[X,Y]_*} \arrow[r, "i"] \arrow[d] & {[X,Y]} \arrow[d, "{[x,\mathrm{id}]}"] \\
{[1,1]} \arrow[r, "{[1,y]}"] & {[1,Y]}
\end{tikzcd}$$

with the basepoint $1 \to [X,Y]_*$ defined to be the map determined by the unique map $1 \to [1,1]$ and by the composite map

$$1 \longrightarrow [X,1] \xrightarrow{[\mathrm{id},y]} [X,Y].$$

The unit and counit maps

$$(X,x) \xrightarrow{\eta_{(X,x)}} [(Y,y),(X,x)\wedge(Y,y)]_*, \qquad [(Y,y),(Z,z)]_* \wedge (Y,y) \xrightarrow{\varepsilon_{(Z,z)}} (Z,z)$$

are induced by the following composite maps, respectively.

$$X \xrightarrow{\eta_X} [Y, X\otimes Y] \xrightarrow{[\mathrm{id},p]} [Y,(X,x)\wedge(Y,y)]$$

$$[(Y,y),(Z,z)]_* \otimes Y \xrightarrow{i\otimes \mathrm{id}} [Y,Z]\otimes Y \xrightarrow{\varepsilon_Z} Z$$

This defines the induced closed symmetric monoidal structure on $\mathscr{V}_*$. We obtain the following pointed version of Proposition 1.24.

Addendum 1.29. *There is a canonical monoidal natural pointed homeomorphism α that makes the following diagram of symmetric monoidal functors commute.*

$$\begin{tikzcd}
\mathrm{Cat\,Dual}_* \arrow[r, "B"] \arrow[d, "\mathrm{Sym}"'] & \mathrm{Real\,Top}_* \arrow[d, "(-)^{G_{\mathbb{R}}}"] \\
\mathrm{Cat}_* \arrow[r, "B"'] \arrow[ur, Rightarrow, "\alpha"] & \mathrm{Top}_*
\end{tikzcd}$$

Proof. The natural homeomorphism α defined in the proof of Proposition 1.24 is basepoint preserving. $\square$

Remark 1.30. We call an object $(\mathscr{C},c)$ of the category $\mathrm{Cat}_{0*} = \mathbf{1}/\mathrm{Cat}_0$ a (κ-small) pointed category. Here the functor $c\colon \mathbf{1} \to \mathscr{C}$ determines and is determined by the object $c(1) \in \mathrm{ob}(\mathscr{C})$. We stress that $c(1) \in \mathrm{ob}(\mathscr{C})$ can be any object and is not required to be a nullobject. The functor $F\colon (\mathscr{C}_1,c_1) \to (\mathscr{C}_2,c_2)$ is pointed if $Fc_1 = c_2$, and the natural transformation $f\colon F_1 \Rightarrow F_2$ between pointed functors is pointed if $f_{c_1} = \mathrm{id}_{c_2}$. Similarly, the object $(\mathscr{C},D,c)$ of $\mathrm{Cat\,Dual}_{0*} = (\mathbf{1},\mathrm{id})/\mathrm{Cat\,Dual}_0$ is said to be a pointed category with strict duality. The morphism $c\colon (\mathbf{1},\mathrm{id}) \to (\mathscr{C},D)$ determines and is determined by the object $c(1) \in \mathrm{ob}(\mathscr{C})$ with $D(c(1)) = c(1)$.

Remark 1.31. We use Addendum 1.29 to produce pointed equivariant maps and pointed equivariant homotopies between the pointed real classifying spaces of pointed categories with strict duality as follows. We define the Top_*-functor

$$\mathrm{Cat\,Dual}_{*,B\,\mathrm{Sym}} \xrightarrow{H_*} \mathrm{Top}_*$$

to be the composition of the Top_*-functor

$$\mathrm{Cat\,Dual}_{*,B\,\mathrm{Sym}} \xrightarrow{F\alpha} \mathrm{Cat\,Dual}_{*,B(-)^{G_{\mathbb{R}}}}$$

that, on objects, is the identity and, on morphism pointed spaces, is given by the pointed homeomorphism

$$\begin{aligned}&B\,\mathrm{Sym\,Cat\,Dual}_*((\mathscr{C}_1,D_1,c_1),(\mathscr{C}_2,D_2,c_2))\\ &\xrightarrow{F\alpha} B\,\mathrm{Cat\,Dual}_*((\mathscr{C}_1,D_1,c_1),(\mathscr{C}_2,D_2,c_2))^{G_{\mathbb{R}}}\end{aligned}$$

induced from the pointed monoidal natural homeomorphism α from Addendum 1.29 and the Top_*-functor $B(-)^{G_{\mathbb{R}}}\colon\ \mathrm{Cat\,Dual}_{*,B(-)^{G_{\mathbb{R}}}} \to \mathrm{Top}_*$ given by Constructions 1.8 and 1.9. It gives a map of pointed spaces

$$\begin{aligned}&B\,\mathrm{Sym\,Cat\,Dual}_*((\mathscr{C}_1,D_1,c_1),(\mathscr{C}_2,D_2,c_2))\\ &\xrightarrow{H_*} \mathrm{Real\,Top}_*((B(\mathscr{C}_1,D_1),Bc_1),(B(\mathscr{C}_2,D_2),Bc_2))^{G_{\mathbb{R}}}.\end{aligned} \tag{1.32}$$

In particular, every symmetric space (F,f) in the pointed functor category on the left-hand side determines a pointed real map

$$H_*(F,f)\colon\ (B(\mathscr{C}_1,D_1),Bc_1) \to (B(\mathscr{C}_2,D_2),Bc_2),$$

and every map of symmetric spaces $g\colon\ (F_1,f_1) \to (F_0,f_0)$ determines a pointed real homotopy $H_*(g)$ from $H_*(F_0,f_0)$ to $H_*(F_1,f_1)$.

Corollary 1.33. *If $(\mathscr{C},D,c)$ is a pointed category with strict duality such that $c(1)$ is a nullobject of $\mathscr{C}$, then the pointed real classifying space $(B(\mathscr{C},D),Bc)$ is pointed equivariantly contractible.*

Proof. The image of the map (1.32) contains both the identity map and the constant map $Bc(1)$. Therefore, it will suffice to show that the domain, and hence also the image, is pointed contractible. But this follows from $(c(1),\mathrm{id})$ being a nullobject of the category of symmetric spaces in $\mathrm{Cat\,Dual}_*((\mathscr{C},D,c),(\mathscr{C},D,c))$. □

We say that the adjunction $(F,G,\eta,\varepsilon,f,g)$ from the pointed adjunction from the pointed category with strict duality $(\mathscr{C}_1,D_1,c_1)$ to the pointed category with strict duality $(\mathscr{C}_2,D_2,c_2)$ is pointed if the functors F and G and the natural transformations η, ε, f, and g all are pointed.

Corollary 1.34. *If $(F,G,\eta,\varepsilon,f,g)$ is a pointed adjunction from the pointed category with strict duality $(\mathscr{C}_1,D_1,c_1)$ to the pointed category with strict duality $(\mathscr{C}_2,D_2,c_2)$, then the two composites of the pointed real maps*

$$(B(\mathscr{C}_1,D_1),Bc_1) \underset{H_*(G,g)}{\overset{H_*(F,f)}{\rightleftarrows}} (B(\mathscr{C}_2,D_2),Bc_2)$$

are pointed equivariantly homotopic to the respective identity maps.

Proof. Since $H_*\colon \mathrm{Cat}\,\mathrm{Dual}_{*,B\mathrm{Sym}} \to \mathrm{Top}_*$ is a Top_*-functor,

$$\begin{aligned} H_*(G,g)\circ H_*(F,f) &= H_*(G\circ F, g\circ f) \\ H_*(F,f)\circ H_*(G,f) &= H_*(F\circ G, f\circ g). \end{aligned}$$

Moreover, the commutative diagrams in Definition 1.27 express that the unit and counit of the adjunction define maps of symmetric spaces

$$\begin{aligned} \eta\colon (\mathrm{id}_{\mathscr{C}_1}, 1) &\to (G\circ F, g\circ f) \\ \varepsilon\colon (F\circ G, f\circ f) &\to (\mathrm{id}_{\mathscr{C}_2}, 1). \end{aligned}$$

It follows that $H_*(\eta)$ is a pointed equivariant homotopy from $H_*(G,g)\circ H_*(F,f)$ to the identity map of $B(\mathscr{C}_1, D_1)$ and that $H_*(\varepsilon)$ is a pointed equivariant homotopy from the identity map of $B(\mathscr{C}_2, D_2)$ to $H_*(F,f)\circ H_*(G,g)$. $\square$

2 The real Waldhausen construction

In this section, we introduce a variant of Waldhausen's S-construction that we call the real Waldhausen construction. It associates to a pointed exact category with weak equivalences and strict duality $(\mathscr{C}, w\mathscr{C}, D, 0)$ a real simplicial pointed exact category with weak equivalences and strict duality $(S^{2,1}\mathscr{C}[-], wS^{2,1}\mathscr{C}[-], D[-], 0[-])$. We first recall the definition of a pointed exact category with weak equivalences and strict duality.

Let $\mathscr{C}$ be an additive category. The diagram

$$\begin{tikzcd} A \arrow[r, shift left, "i"] & B \arrow[l, shift left, "q"] \arrow[r, shift right, "p"'] & C \arrow[l, shift right, "j"'] \end{tikzcd}$$

in $\mathscr{C}$ is a biproduct diagram if $qi = \mathrm{id}_A$, $pj = \mathrm{id}_C$, and $iq + pj = \mathrm{id}_A$. In this case, it follows that p is a cokernel of i, that i is a kernel of p, that q is a cokernel of j, and that j is a kernel of q. We say the underlying sequence

$$A \xrightarrow{i} B \xrightarrow{p} C$$

is a split-exact sequence in $\mathscr{C}$.

We recall from [17, §2] that an exact category is a pair $(\mathscr{C}, \mathscr{E})$ of an additive category $\mathscr{C}$ and a set $\mathscr{E}$ of sequences

$$A \xrightarrow{i} B \xrightarrow{p} C$$

in $\mathscr{C}$ called the exact sequences for which the axioms (1)–(5) below are satisfied. A morphism in $\mathscr{C}$ that appears as the left-hand morphism i in a sequence in $\mathscr{E}$ is called an admissible monomorphisms, and a morphism in $\mathscr{C}$ that appears as the right-hand morphism p in a sequence in $\mathscr{E}$ is called an admissible epimorphism.

(1) For every sequence

$$A \xrightarrow{i} B \xrightarrow{p} C$$

in $\mathscr{E}$, p is a cokernel of i and i is a kernel of p.

(2) Every sequence isomorphic to a sequence in $\mathscr{E}$ is itself in $\mathscr{E}$.

(3) The composition of two admissible monomorphisms is an admissible monomorphism; the composition of two admissible epimorphisms is an admissible epimorphism.

(4) The cobase-change of an admissible monomorphism along any morphism exists and is an admissible monomorphism; the base-change of an admissible epimorphism along any morphism exists and is an admissible epimorphism.

(5) Every split-exact sequence in $\mathscr{C}$ is in $\mathscr{E}$.

It was proved by Keller [9, Appendix A] that the additional axiom c) in [17, §2] is a consequence of the axioms above. We will often abuse notation and write $\mathscr{C}$ for the exact category $(\mathscr{C}, \mathscr{E})$. If $(\mathscr{C}, \mathscr{E})$ is an exact category, then so is $(\mathscr{C}^{\mathrm{op}}, \mathscr{E}^{\mathrm{op}})$. We also abuse notation and simply write $\mathscr{C}^{\mathrm{op}}$ for the exact category $(\mathscr{C}^{\mathrm{op}}, \mathscr{E}^{\mathrm{op}})$.

Example 2.1. Let $\mathscr{E}_0$ be the set of split-exact sequences in the additive category $\mathscr{C}$. Then $(\mathscr{C},\mathscr{E}_0)$ is an exact category.

We define the subcategory $w\mathscr{C}$ of the exact category $\mathscr{C}$ to be a subcategory of weak equivalences, if it contains all isomorphisms, if for all diagrams

$$\begin{tikzcd} B \arrow[d, "\sim"] & A \arrow[l, tail, "i"'] \arrow[r] \arrow[d, "\sim"] & C \arrow[d, "\sim"] \\ B' & A' \arrow[l, tail, "i'"'] \arrow[r] & C' \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, two heads, "p"] \arrow[d, "\sim"] & X \arrow[d, "\sim"] & Z \arrow[l] \arrow[d, "\sim"] \\ Y' \arrow[r, two heads, "p'"] & X' & Z' \arrow[l] \end{tikzcd}$$

with i and i' admissible monomorphisms, with p and p' admissible epimorpisms, with and the vertical maps in $w\mathscr{C}$, the induced maps

$$B\textstyle\coprod_A C \longrightarrow B\coprod_{A'} C' \qquad\qquad Y\times_X Z \longrightarrow Y'\times_{X'} Z'$$

of pushouts and pullbacks, respectively, again are in $w\mathscr{C}$. We define an exact category with weak equivalences to be a pair $(\mathscr{C},w\mathscr{C})$ of an exact category $\mathscr{C}$ and a subcategory of weak equivalences $w\mathscr{C}\subset\mathscr{C}$, and define an exact functor

$$F\colon (\mathscr{C}_1,w\mathscr{C}_1)\to(\mathscr{C}_2,w\mathscr{C}_2)$$

between exact categories with weak equivalences to be a functor $F\colon \mathscr{C}_1\to\mathscr{C}_2$ that takes exact sequences in $\mathscr{C}_1$ to exact sequences in $\mathscr{C}_2$ and weak equivalences in $\mathscr{C}_1$ to weak equivalences in $\mathscr{C}_2$. We remark that if $(\mathscr{C},w\mathscr{C})$ is an exact category with weak equivalences, then so is $(\mathscr{C}^{\mathrm{op}},w\mathscr{C}^{\mathrm{op}})$.

Example 2.2. Let $\mathscr{C}$ be an exact category, and let $i\mathscr{C}$ be the subcategory of isomorphisms. Then $(\mathscr{C},i\mathscr{C})$ is an exact category with weak equivalences.

Finally, we define an exact category with weak equivalences and strict duality to be a triple $(\mathscr{C},w\mathscr{C},D)$, where $(\mathscr{C},w\mathscr{C})$ is an exact category with weak equivalences, and where $D\colon (\mathscr{C}^{\mathrm{op}},w\mathscr{C}^{\mathrm{op}})\to(\mathscr{C},w\mathscr{C})$ is an exact functor such that $D\circ D^{\mathrm{op}}=\mathrm{id}_{\mathscr{C}}$ and $D^{\mathrm{op}}\circ D=\mathrm{id}_{\mathscr{C}^{\mathrm{op}}}$. A duality preserving exact functor

$$F\colon (\mathscr{C}_1,w\mathscr{C}_1,D_1)\to(\mathscr{C}_2,w\mathscr{C}_2,D_2)$$

between exact categories with weak equivalences and strict duality is an exact functor $F\colon (\mathscr{C}_1,w\mathscr{C}_1)\to(\mathscr{C}_2,w\mathscr{C}_2)$ such that $F\circ D_1=D_2\circ F^{\mathrm{op}}$. The set of all ($\kappa$-small) exact categories with weak equivalences and strict duality is the set of objects in the real category $\mathrm{w\,Ex\,Cat\,Dual}$ whose real set of morphisms from $(\mathscr{C}_1,w\mathscr{C}_1,D_1)$ to $(\mathscr{C}_2,w\mathscr{C}_2,D_2)$ is defined to be the sub-real set

$$\mathrm{w\,Ex\,Cat\,Dual}((\mathscr{C}_1,w\mathscr{C}_1,D_1),(\mathscr{C}_2,w\mathscr{C}_2,D_2))\subset \mathrm{ob}_R\,\mathrm{Cat\,Dual}((\mathscr{C}_1,D_1),(\mathscr{C}_2,D_2))$$

that consists of all exact (but not necessarily duality preserving) functors.

Let $(\mathbf{1},\mathbf{1},\mathrm{id})$ be the unique exact category with weak equivalences and strict duality whose underlying category is $\mathbf{1}$; it is a terminal object in the category of all (κ-small) exact categories with weak equivalences and strict duality and duality preserving exact functors. A duality preserving exact functor

$$0\colon (\mathbf{1},\mathbf{1},\mathrm{id})\to(\mathscr{C},w\mathscr{C},D)$$

determines and is determined by the null-object $0(1) \in \mathrm{ob}(\mathscr{C})$ with $D(0(1)) = 0(1)$. We define $\mathrm{w\,Ex\,Cat\,Dual}_*$ to be the real category of pointed exact categories with weak equivalences and strict duality.

We define the sequence

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \xrightarrow{\ h\ } D$$

in the exact category $\mathscr{C}$ to be 4-term exact if f is an admissible monomorphism, if h is an admissible epimorphism, and if g factors as the composition of a cokernel $p\colon B \to E$ of f and a kernel $i\colon E \to C$ of h. We remind the reader that the definition of the internal Hom-object in the category (κ-small) categories with strict duality is given in Definition 1.13 and that the categories with strict duality $([n], D)$ are defined in Example 1.14.

Definition 2.3. The real Waldhausen construction of the pointed exact category with weak equivalences and strict duality $(\mathscr{C}, w\mathscr{C}, D, 0)$ is the real simplicial pointed exact category with weak equivalences and strict duality

$$(S^{2,1}\mathscr{C}[-], wS^{2,1}\mathscr{C}[-], D[-], 0[-]),$$

where

$$(S^{2,1}\mathscr{C}[n], D[n]) \subset \mathrm{Cat\,Dual}(\mathrm{Cat\,Dual}(([2], D), ([n], D)), (\mathscr{C}, D))$$

is the full subcategory with strict duality of all functors

$$A\colon\ \mathrm{Cat}([2],[n]) \longrightarrow \mathscr{C}$$

such that

(i) for all functors $\mu\colon\ [1] \to [n]$,

$$A(s_0\mu) = A(s_1\mu) = 0(1),$$

(ii) for all functors $\sigma\colon\ [3] \to [n]$, the sequence

$$A(d_0\sigma) \longrightarrow A(d_1\sigma) \longrightarrow A(d_2\sigma) \longrightarrow A(d_3\sigma)$$

is 4-term exact;

where the sequence $A \to B \to C$ in $S^{2,1}\mathscr{C}[n]$ is exact if, for all functors $\theta\colon\ [2] \to [n]$, the sequence $A(\theta) \to B(\theta) \to C(\theta)$ in $\mathscr{C}$ is exact; where the morphism $A \to B$ is in $wS^{2,1}\mathscr{C}[n]$ if, for all functors $\theta\colon\ [2] \to [n]$, the morphism $A(\theta) \to B(\theta)$ is in $w\mathscr{C}$, and where the basepoint $0[n]$ is the constant diagram $0[n](1)(\theta) = 0(1)$.

We postpone discussion of the functoriality of the real Waldhausen construction to Proposition 3.6 below.

Example 2.4. For $n = 0$ and $n = 1$, the category $S^{2,1}\mathscr{C}[n]$ is equal to the discrete category with the single object $0[n](1)$, and for $n = 2$, the forgetful functor

$$(S^{2,1}\mathscr{C}[2], wS^{2,1}\mathscr{C}[2], D[2], 0[2]) \to (\mathscr{C}, w\mathscr{C}, D, 0)$$

that to A associates $A(\mathrm{id}_{[2]})$ is an isomorphism of pointed exact categories with weak equivalences and strict duality.

If A is an object of $S^{2,1}\mathscr{C}[n]$, then for every functor $\sigma\colon [3] \to [n]$, the sequence

$$A(d_0\sigma) \longrightarrow A(d_1\sigma) \longrightarrow A(d_2\sigma) \longrightarrow A(d_3\sigma)$$

is 4-term exact, and hence, the middle map induces an isomorphism of a cokernel of the left-hand map onto a kernel of the right-hand map. It is sometimes convenient to have an explicit choice of kernel and cokernel be part of the structure. The purpose of the following definition is to include this choice.

Definition 2.5. The extended real Waldhausen construction of the pointed exact category with weak equivalence and strict duality $(\mathscr{C}, w\mathscr{C}, D, 0)$ is the real simplicial pointed exact category with weak equivalences and strict duality

$$(\tilde{S}^{2,1}\mathscr{C}[-], w\tilde{S}^{2,1}\mathscr{C}[-], \tilde{D}[-], \tilde{0}[-]),$$

where

$$(\tilde{S}^{2,1}\mathscr{C}[n], \tilde{D}[n]) \subset \mathrm{Cat\,Dual}(\mathrm{Cat\,Dual}(([3], D), ([n], D)), (\mathscr{C}, D))$$

is the full subcategory with strict duality of all functors

$$A\colon\ \mathrm{Cat}([3],[n]) \longrightarrow \mathscr{C}$$

such that

(i) for every functor $\theta\colon [2] \to [n]$,

$$A(s_0\theta) = A(s_2\theta) = 0(1),$$

(ii) for every functor $\tau\colon [4] \to [n]$, the sequences

$$A(d_0\tau) \longrightarrow A(d_1\tau) \longrightarrow A(d_2\tau)$$

$$A(d_2\tau) \longrightarrow A(d_3\tau) \longrightarrow A(d_4\tau)$$

are exact;

where the sequence $A \to B \to C$ in $\tilde{S}^{2,1}\mathscr{C}[n]$ is exact if, for all functors $\sigma\colon [3] \to [n]$, the sequence $A(\sigma) \to B(\sigma) \to C(\sigma)$ in $\mathscr{C}$ is exact; where the morphism $A \to B$ is in $w\tilde{S}^{2,1}\mathscr{C}[n]$ if, for all functors $\sigma\colon [3] \to [n]$, the morphism $A(\sigma) \to B(\sigma)$ is in $w\mathscr{C}$, and where the base point $\tilde{0}[n]$ is the constant diagram $\tilde{0}[n](1)(\sigma) = 0(1)$.

Example 2.6. For $n = 0$ and $n = 1$, the category $\tilde{S}^{2,1}\mathscr{C}[n]$ is equal to the discrete category on the single object $\tilde{0}[n](1)$; and for $n = 2$, the forgetful functor

$$(\tilde{S}^{2,1}\mathscr{C}[2], w\tilde{S}^{2,1}\mathscr{C}[2], \tilde{D}[2], \tilde{0}[2]) \to (\mathscr{C}, w\mathscr{C}, D, 0)$$

that to A associates $A(s^1)$ is an isomorphism of pointed exact categories with weak equivalences and strict duality.

We will say that a pointed adjunction $(F, G, \eta, \varepsilon, f, g)$ from the pointed exact category with weak equivalences and strict duality $(\mathscr{C}_1, w\mathscr{C}_1, D_1, 0_1)$ to the pointed exact category with weak equivalences and strict duality $(\mathscr{C}_2, w\mathscr{C}_2, D_2, 0_2)$ is exact if the pointed functors F and G are exact.

Lemma 2.7. *Let $(\mathscr{C}, w\mathscr{C}, D, 0)$ be a pointed exact category with weak equivalences and strict duality. For every integer $n \geqslant 0$, there exists a pointed exact adjunction*

$$(s_1^*, h, \eta, \varepsilon, 1, g)$$

from $(\tilde{S}^{2,1}\mathscr{C}[n], w\tilde{S}^{2,1}\mathscr{C}[n], \tilde{D}[n], \tilde{0}[n])$ to $(S^{2,1}\mathscr{C}[n], wS^{2,1}\mathscr{C}[n], D[n], 0[n])$. Moreover, the natural transformations η, ε, and g all are isomorphisms.

Proof. We first show that $s^1\colon [3] \to [2]$ induces a pointed exact duality preserving functor $s_1^*\colon \tilde{S}^{2,1}\mathscr{C}[n] \to S^{2,1}\mathscr{C}[n]$. To this end, we verify that if $A \in \operatorname{ob} \tilde{S}^{2,1}\mathscr{C}[n]$ then $s_1^*(A) \in \operatorname{ob} S^{2,1}\mathscr{C}[n]$. Given $\mu\colon [1] \to [n]$,

$$\begin{aligned} s_1^*(A)(s_0\mu) &= A(s_1 s_0 \mu) = A(s_0 s_0 \mu) = 0 \\ s_1^*(A)(s_1\mu) &= A(s_1 s_1 \mu) = A(s_2 s_1 \mu) = 0 \end{aligned}$$

which shows that $s_1^*(A)$ satisfies (i) of Definition 2.3, and given $\sigma\colon [3] \to [n]$, we have

$$\begin{tikzcd}
s_1^*(A)(d_0\sigma) \arrow[r] \arrow[d, equal] & s_1^*(A)(d_1\sigma) \arrow[r] \arrow[d, equal] & s_1^*(A)(d_2\sigma) \arrow[r] \arrow[d, equal] & s_1^*(A)(d_3\sigma) \arrow[d, equal] \\
A(s_1 d_0\sigma) \arrow[r] \arrow[d, equal] & A(s_1 d_1\sigma) \arrow[r] \arrow[d, equal] & A(s_1 d_2\sigma) \arrow[r] \arrow[d, equal] & A(s_1 d_3\sigma) \arrow[d, equal] \\
A(d_0 s_2\sigma) \arrow[r] & A(d_2 s_2\sigma) \arrow[r] \arrow[d] & A(d_3 s_1\sigma) \arrow[r] & A(d_4 s_1\sigma) \\
& A(d_2 s_2\sigma) \arrow[r, equal] & A(d_2 s_1\sigma) \arrow[u] &
\end{tikzcd}$$

which shows that $s_1^*(A)$ satisfies (ii) of Definition 2.3. Finally, $s_1^* \circ \tilde{0}[n] = 0[n]$, so the functor s_1^* is pointed; it is clear that it is duality preserving and exact.

We next define the pointed exact functor

$$h\colon S^{2,1}\mathscr{C}[n] \to \tilde{S}^{2,1}\mathscr{C}[n].$$

Given $A \in \operatorname{ob} S^{2,1}\mathscr{C}[n]$, we define the functor $h(A)\colon \operatorname{Cat}([3],[n]) \to \mathscr{C}$ as follows. For every functor $\sigma\colon [3] \to [n]$, we have the 4-term exact sequence

$$A(d_0\sigma) \xrightarrow{a} A(d_1\sigma) \xrightarrow{b} A(d_2\sigma) \xrightarrow{c} A(d_3\sigma).$$

If b is the zero morphism, then we define $h(A)(\sigma)$ to be the given null-object $0(1)$, and otherwise, we choose any factorization of b as the composition

$$A(d_1\sigma) \overset{p}{\twoheadrightarrow} h(A)(\sigma) \overset{i}{\rightarrowtail} A(d_2\sigma)$$

of a cokernel p of the admissible monomorphism a and a kernel i of the admissible epimorphism c. This defines $h(A)$ on objects, and the universal property of kernels and cokernels determines the value of $h(A)$ on morphisms. We proceed to show that $h(A)$ satisfies (i) and (ii) of Definition 2.5.

We first show that $h(A)$ satisfies (i). Given $\theta\colon [2] \to [n]$, the 4-term exact sequence

$$A(d_0s_0\theta) = A(d_1s_0\theta) \xrightarrow{b} A(d_2s_0\theta) = A(d_3s_0\theta)$$

shows that $h(A)(s_0\theta) = 0(1)$, and the 4-term exact sequence

$$A(d_0s_2\theta) = A(d_1s_2\theta) \xrightarrow{b} A(d_2s_2\theta) = A(d_3s_2\theta),$$

shows that $h(A)(s_2\theta) = 0(1)$. This proves that $h(A)$ satisfies (i). We next show that $h(A)$ satisfies (ii). Given $\tau\colon [4] \to [n]$, we consider the following diagram.

$$\begin{tikzcd}
A(d_0d_0\tau) \arrow[r, tail] \arrow[d, equal] & A(d_1d_0\tau) \arrow[r, two heads, "p"] \arrow[d, tail] & h(A)(d_0\tau) \arrow[d] \\
A(d_0d_1\tau) \arrow[r, tail] & A(d_1d_1\tau) \arrow[r, two heads, "p"] \arrow[d, two heads, "p"] & h(A)(d_1\tau) \arrow[d] \\
 & h(A)(d_2\tau) \arrow[r, equal] & h(A)(d_2\tau)
\end{tikzcd}$$

The upper right-hand square is a push-out. Therefore, the upper right-hand vertical map is an admissible monomorphism, and the composition of a cokernel of this map and the middle right-hand horizontal map is a cokernel of the upper middle vertical map. Now, since $d_1d_1\tau = d_1d_2\tau$ and $d_1d_0\tau = d_0d_2\tau$, the lower middle vertical map is such a cokernel. This proves that the right-hand column is an exact sequence in $\mathscr{C}$. Similarly, in the diagram

$$\begin{tikzcd}
h(A)(d_2\tau) \arrow[r, equal] \arrow[d] & h(A)(d_2\tau) \arrow[d, tail, "i"] & \\
h(A)(d_3\tau) \arrow[r, tail, "i"] \arrow[d] & A(d_2d_3\tau) \arrow[r, two heads] \arrow[d, two heads] & A(d_3d_3\tau) \arrow[d, equal] \\
h(A)(d_4\tau) \arrow[r, tail, "i"] & A(d_2d_4\tau) \arrow[r, two heads] & A(d_3d_4\tau),
\end{tikzcd}$$

the lower left-hand square is a pull-back. Therefore, the lower left-hand vertical map is an admissible epimorphism, and the composition of a kernel of this map and the middle left-hand horizontal map is a kernel of the lower middle vertical map. Now, since $d_2d_3\tau = d_2d_2\tau$ and $d_2d_4\tau = d_3d_2\tau$, the upper middle vertical map is such a kernel. This shows that the left-hand column is an exact sequence in $\mathscr{C}$. We have proved that $h(A)$ satisfies (ii), and hence, that $h(A) \in \operatorname{ob} \tilde{S}^{2,1}\mathscr{C}[n]$. It is clear that the functor $h\colon S^{2,1}\mathscr{C}[n] \to \tilde{S}^{2,1}\mathscr{C}[n]$ is pointed; we proceed to show that it is exact.

Let $A \to B \to C$ be an exact sequence in $S^{2,1}\mathscr{C}[n]$. We wish to show that for every functor $\sigma\colon [3] \to [n]$, the sequence $h(A)(\sigma) \to h(B)(\sigma) \to h(C)(\sigma)$ in $\mathscr{C}$ is exact. By the definition of this sequence, we have the following commutative diagram in which the columns and the top two rows all are exact sequences in $\mathscr{C}$.

$$\begin{tikzcd}
A(d_0\sigma) \arrow[r, tail] \arrow[d, tail] & B(d_0\sigma) \arrow[r, two heads] \arrow[d, tail] & C(d_0\sigma) \arrow[d, tail] \\
A(d_1\sigma) \arrow[r, tail] \arrow[d, two heads] & B(d_1\sigma) \arrow[r, two heads] \arrow[d, two heads] & C(d_1\sigma) \arrow[d, two heads] \\
h(A)(\sigma) \arrow[r, "i"] & h(B)(\sigma) \arrow[r, "p"] & h(C)(\sigma)
\end{tikzcd}$$

We use the redundant axiom c) of [17, §2] as follows to show that the bottow row is exact. The lower right-hand vertical map and the middle right-hand horizontal map are both admissible epimorphisms. Therefore, also their composition is an admissible epimorphism. In this situation, to show that the bottom row is exact, it suffices by said axiom to show that p has a kernel. But a diagram chase based on the diagram

$$\begin{tikzcd}
h(A)(\sigma) \arrow[r, "i"] \arrow[d, tail] & h(B)(\sigma) \arrow[r, "p"] \arrow[d, tail] & h(C)(\sigma) \arrow[d, tail] \\
A(d_2\sigma) \arrow[r, tail] \arrow[d, two heads] & B(d_2\sigma) \arrow[r, two heads] \arrow[d, two heads] & C(d_2\sigma) \arrow[d, two heads] \\
A(d_3\sigma) \arrow[r, tail] & B(d_3\sigma) \arrow[r, two heads] & C(d_3\sigma),
\end{tikzcd}$$

where the columns and the bottom two rows are exact, readily shows that i is a kernel of p. Finally, if $A \to B$ is in $wS^{2,1}\mathscr{C}[n]$, then $h(A) \to h(B)$ is in $w\tilde{S}^{2,1}\mathscr{C}[n]$. Hence, the functor h is exact as stated.

We next define the pointed natural isomorphisms η and ε. If $A \in \operatorname{ob}\tilde{S}^{2,1}\mathscr{C}[n]$, then $h(s_1^*(A))(\sigma)$ is defined to be a choice of cokernel of the admissible monomorphism $a\colon A(s_1d_0\sigma) \rightarrowtail A(s_1d_1\sigma)$. But $s_1d_0\sigma = d_0s_2\sigma$, $s_1d_1\sigma = d_1s_2\sigma$, and $d_2s_2\sigma = \sigma$, so also $A(\sigma)$ is a choice of cokernel of a. It follows that the unique isomorphism of cokernels from $A(\sigma)$ to $h(s_1^*(A))(\sigma)$ form a natural isomorphism $\eta\colon \mathrm{id} \Rightarrow h \circ s_1^*$. Similarly, if $A \in \operatorname{ob}S^{2,1}\mathscr{C}[n]$, then $s_1^*(h(A))(\theta)$ is a choice of cokernel of the admissible monomorphism $a\colon A(d_0s_1\theta) \rightarrowtail A(d_1s_1\theta)$. But $d_0s_1\theta = s_0d_0\theta$ and $d_1s_1\theta = \theta$, so also $A(\theta)$ is a choice of cokernel of f. Hence, the unique isomorphism of cokernels from $s_1^*(h(A))(\theta)$ to $A(\theta)$ form a natural isomorphism $\varepsilon\colon s_1^* \circ h \Rightarrow \mathrm{id}$. Again, by the uniqueness of the isomorphisms of different choices of cokernels, we conclude that the two composite natural transformations

$$s_1^* \xRightarrow{\eta \circ s_1^*} s_1^* \circ h \circ s_1^* \xRightarrow{\varepsilon \circ s_1^*} s_1^* \qquad\qquad h \xRightarrow{h \circ \eta} h \circ s_1^* \circ h \xRightarrow{\varepsilon \circ h} h$$

are equal to the identity natural transformations. This shows that $(s_1^*, h, \eta, \varepsilon)$ forms a pointed exact adjoint equivalence from $\tilde{S}^{2,1}\mathscr{C}[n]$ to $S^{2,1}\mathscr{C}[n]$.

Since s_1^* is duality preserving, we have $s_1^* = (\tilde{D}[n], D[n])(s_1^*)^{\mathrm{op}}$. Moreover, by the uniqueness of cokernels up to unique isomorphism, there is a natural isomorphism $g\colon h \Rightarrow (D[n], \tilde{D}[n])h^{\mathrm{op}}$. It necessarily satisfies $g = (D[n], \tilde{D}[n])g^{\mathrm{op}}$, and it is pointed by our definition of h. Finally, the two diagrams in Definition 1.27 commute, since each morphism in the two diagrams is the unique isomorphism between two choices of cokernels of the same morphism. □

Corollary 2.8. *Let $(\mathscr{C}, w\mathscr{C}, D, 0)$ be a pointed exact category with weak equivalences and strict duality and n a non-negative integer. The pointed real map*

$$|N(w\tilde{S}^{2,1}\mathscr{C}[-], \tilde{D}[-], \tilde{0}[-])[-]|_R \xrightarrow{s_1^*} |N(wS^{2,1}\mathscr{C}[-], D[-], 0[-])[-]|_R.$$

induced by $s_1^*\colon [3] \to [2]$ is a weak equivalence of pointed real spaces.

Proof. It follows from Lemma 2.7 and Corollary 1.34 that for every $n \geqslant 0$, the two composites of the pointed real maps

$$|N(w\tilde{S}^{2,1}\mathscr{C}[n], \tilde{D}[n])[-]|_R \underset{H_*(h,g)}{\overset{H_*(s_1^*)}{\rightleftarrows}} |N(wS^{2,1}\mathscr{C}[n], D[n])[-]|_R$$

are pointed real homotopic to the respective identity maps. Moreover, as n varies, the maps $H_*(s_1^*)$ (but not the maps $H_*(h,g)$) form a map of real simplicial pointed real spaces. Therefore, by the real realization lemma, the induced map of realizations

$$|N(w\tilde{S}^{2,1}\mathscr{C}[-], \tilde{D}[-])[-]|_R \xrightarrow{s_1^*} |N(wS^{2,1}\mathscr{C}[-], D[-])[-]|_R$$

is a weak equivalence of pointed real spaces as stated. □

We end this section by introducing a category $\bar{S}^{2,1}\mathscr{C}[n]$ which is equivalent to the category $S^{2,1}\mathscr{C}[n]$ but is more manageable. The categories $\bar{S}^{2,1}\mathscr{C}[n]$, however, do not form a simplicial category as n varies.

Definition 2.9. Let $\mathscr{C}$ be an exact category. The commutative diagram

$$\begin{tikzcd}
A_{12} \arrow[r, "g_1"] \arrow[d, "h_2"] & A_{13} \arrow[d, "h_3"] \\
A_{22} \arrow[r, "g_2"] & A_{23}
\end{tikzcd}$$

in $\mathscr{C}$ is an admissible square if it can be completed to a commutative diagram

$$\begin{tikzcd}
A_{11} \arrow[r, tail, "f_1"] \arrow[d, tail, "h_1"] & A_{12} \arrow[r, two heads, "g_1"] \arrow[d, tail, "h_2"] & A_{13} \arrow[d, tail, "h_3"] \\
A_{21} \arrow[r, tail, "f_2"] \arrow[d, two heads, "k_1"] & A_{22} \arrow[r, two heads, "g_2"] \arrow[d, two heads, "k_2"] & A_{23} \arrow[d, tail, "k_3"] \\
A_{31} \arrow[r, two heads, "f_3"] & A_{32} \arrow[r, two heads, "g_3"] & A_{33}
\end{tikzcd}$$

in which the rows and columns all are exact. In this case, the latter diagram is said to be a completion of the admissible square.

Remark 2.10. The completion of an admissible square is unique up to canonical isomorphism. Moreover, the square diagram in Definition 2.9 is admissible if and only if the following conditions (i)–(iii) are satisfied.

(i) The morphisms g_1 and g_2 are admissible epimorphisms.
(ii) The morphisms h_2 and h_3 are admissible monomorphisms.
(iii) If $f_1\colon A_{11} \to A_{12}$ and $f_2\colon A_{21} \to A_{22}$ are kernels of g_1 and g_2, respectively, then $h_1\colon A_{11} \to A_{21}$ induced by h_2 and h_3 is an admissible monomorphism.

Here, the condition (iii) is equivalent to the following condition (iv).

(iv) If $k_2\colon A_{22} \to A_{32}$ and $k_3\colon A_{23} \to A_{33}$ are cokernels of h_2 and h_3, respectively, then $g_3\colon A_{32} \to A_{33}$ induced by g_1 and g_2 is an admissible epimorphism.

Here, the equivalence of the conditions (iii) and (iv) uses [17, §2 c)].

Definition 2.11. Let n be a non-negative integer, and let $(\mathscr{C}, w\mathscr{C}, D, 0)$ be a pointed exact category with weak equivalences and strict duality. The degree n restricted real Waldhausen construction of $(\mathscr{C}, w\mathscr{C}, D, 0)$ is the pointed exact category with weak equivalences and strict duality

$$(\bar{S}^{2,1}\mathscr{C}[n], w\bar{S}^{2,1}\mathscr{C}[n], \bar{D}[n], \bar{0}[n]),$$

where

$$(\bar{S}^{2,1}\mathscr{C}[n], D[n]) \subset \mathrm{Cat\,Dual}(\mathrm{Cat\,Dual}(([1], D), ([n], D)), (\mathscr{C}, D))$$

is the full subcategory with strict duality of all functors

$$A\colon\ \mathrm{Cat}([1],[n]) \longrightarrow \mathscr{C}$$

such that

(i) for every functor $\mu\colon\ [1] \to [n]$ with $\mu(0) = 0$ or $\mu(1) = n$,
$$A(\mu) = 0(1),$$
(ii) for every functor $\sigma\colon\ [3] \to [n]$, the diagram
$$\begin{tikzcd} A(d_0d_2\sigma) \arrow[r] \arrow[d] & A(d_1d_2\sigma) \arrow[d] \\ A(d_0d_3\sigma) \arrow[r] & A(d_1d_3\sigma) \end{tikzcd}$$
is an admissible square in $\mathscr{C}$;

where the sequence $A \to \bar{B} \to \bar{C}$ in $\bar{S}^{2,1}\mathscr{C}[n]$ is exact if, for all functors $\mu\colon\ [1] \to [n]$, the sequence $A(\mu) \to \bar{B}(\mu) \to \bar{C}(\mu)$ in $\mathscr{C}$ is exact; where the morphism $A \to \bar{B}$ is in $w\bar{S}^{2,1}\mathscr{C}[n]$ if, for all functors $\mu\colon\ [1] \to [n]$, the morphism $A(\mu) \to \bar{B}(\mu)$ is in $w\mathscr{C}$; and where the basepoint $\bar{0}[n]$ is the constant diagram $\bar{0}[n](1)(\mu) = 0(1)$.

The restricted real Waldhausen construction does not define a real simplicial pointed exact category with weak equivalences and strict duality. For instance,

$$\mathrm{Cat}(\mathrm{Cat}([1],[n]),\mathscr{C}) \xrightarrow{d_n} \mathrm{Cat}(\mathrm{Cat}([1],[n-1]),\mathscr{C})$$

does not map $\bar{S}^{2,1}\mathscr{C}[n]$ to $\bar{S}^{2,1}\mathscr{C}[n-1]$.

Lemma 2.12. *Let $(\mathscr{C}, w\mathscr{C}, D, 0)$ be a pointed exact category with weak equivalences and strict duality. For every integer $n \geqslant 0$, there exists a pointed exact adjunction*

$$(j^*, e, \eta, \varepsilon, 1, g)$$

from $(\tilde{S}^{2,1}\mathscr{C}[n], w\tilde{S}^{2,1}\mathscr{C}[n], \tilde{D}[n], \tilde{0}[n])$ to $(\bar{S}^{2,1}\mathscr{C}[n], w\bar{S}^{2,1}\mathscr{C}[n], \bar{D}[n], \bar{0}[n])$. Moreover, the natural transformations η, ε, and g all are isomorphisms.

Proof. For all non-negative integers m and n, we define

$$\mathrm{Cat}([m],[n]) \xrightarrow{j} \mathrm{Cat}([m+2],[n])$$

to be the functor given by

$$j(\theta)(i) = \begin{cases} 0 & \text{if } i = 0 \\ \theta(i-1) & \text{if } 1 \leqslant i \leqslant m+1 \\ n & \text{if } i = m+2. \end{cases}$$

We note that $j(d_i\theta) = d_{i+1}j(\theta)$ for every functor $\theta\colon [m-1] \to [n]$ and for every integer $0 \leqslant i \leqslant m-1$. Now, we claim that for $m = 1$, the functor j induces a pointed exact duality preserving functor

$$\tilde{S}^{2,1}\mathscr{C}[n] \xrightarrow{j^*} \bar{S}^{2,1}\mathscr{C}[n].$$

To see this, we let $A \in \mathrm{ob}\,\tilde{S}^{2,1}\mathscr{C}[n]$ and show that $j^*(A) = A \circ p$ satisfies (i)–(ii) of Definition 2.11. To prove (i), let $\mu\colon [1] \to [n]$ be a functor. If $\mu(0) = 0$, then we have $j(\mu) = s_0 d_0 j(\mu)$, which implies that $A(j(\mu)) = 0(1)$. Similarly, if $\mu(1) = n$, then $j(\mu) = s_2 d_3 j(\mu)$, so $A(j(\mu)) = 0(1)$. This shows that $j^*(A)$ satisfies (i). To prove (ii), let $\sigma\colon [3] \to [n]$ be a functor. In this situation, the diagram

$$\begin{array}{ccc} A(j(d_0d_2\sigma)) & \longrightarrow & A(j(d_1d_2\sigma)) \\ \downarrow & & \downarrow \\ A(j(d_0d_3\sigma)) & \longrightarrow & A(j(d_1d_3\sigma)) \end{array}$$

can be completed to the diagram

$$\begin{tikzcd}
A(d_0d_3j(\sigma)) \arrow[r, tail] \arrow[d, tail] & A(d_1d_3j(\sigma)) \arrow[r, two heads] \arrow[d, tail] & A(d_2d_3j(\sigma)) \arrow[d, tail] \\
A(d_0d_4j(\sigma)) \arrow[r, tail] \arrow[d, two heads] & A(d_1d_4j(\sigma)) \arrow[r, two heads] \arrow[d, two heads] & A(d_2d_4j(\sigma)) \arrow[d, two heads] \\
A(d_0d_5j(\sigma)) \arrow[r, tail] & A(d_1d_5j(\sigma)) \arrow[r, two heads] & A(d_2d_5j(\sigma))
\end{tikzcd}$$

which shows that $j^*(A)$ satisfies (ii). This shows that the functor j^* is well-defined; it is clear that it is pointed, duality preserving, and exact. For later use, we note that the lower left-hand term is equal to $A(\sigma)$.

We next define the functor $e\colon \bar{S}^{2,1}\mathscr{C}[n] \to \tilde{S}^{2,1}\mathscr{C}[n]$. Let $B\colon \operatorname{Cat}([1],[n]) \to \mathscr{C}$ be an object of $\bar{S}^{2,1}\mathscr{C}[n]$. To define the value of the functor $e(B)\colon \operatorname{Cat}([3],[n]) \to \mathscr{C}$ on the object $\sigma\colon [3] \to [n]$ to be the lower left-hand term in a choice of completion of the following admissible square.

$$\begin{tikzcd}
B(d_0d_2\sigma) \arrow[r, two heads] \arrow[d, tail] & B(d_1d_2\sigma) \arrow[d, tail] \\
B(d_0d_3\sigma) \arrow[r, two heads] & B(d_1d_3\sigma)
\end{tikzcd}$$

If both horizontal morphisms or both vertical morphisms in the admissible square are identity morphisms, then we require that $e(B)(\sigma) = 0(1)$; and if $\sigma = j(\mu)$ with $\mu\colon [1] \to [n]$, then we required that $e(B)(j(\mu)) = B(\mu)$. But in all other cases, the choice of completion of the admissible square is unrestricted. To define the functor $e(B)$ on morphisms, we note that the morphism $\sigma_1 \Rightarrow \sigma_2$ induces a morphism of the admissible squares used to define $e(B)(\sigma_1)$ and $e(B)(\sigma_2)$. This morphism, in turn, extends uniquely to a morphism of the completions of the admissible squares in question, and we define $e(B)(\sigma_1 \Rightarrow \sigma_2)$ to be the morphism $e(B)(\sigma_1) \to e(B)(\sigma_2)$ of lower left-hand terms in the completed diagrams. We claim that $e(B)$ satisfies (i)–(ii) of Definition 2.5, and hence, is an object of $\tilde{S}^{2,1}\mathscr{C}[n]$. To verify (i), we let $\theta\colon [2] \to [n]$ be a functor and consider the following admissible squares.

$$\begin{tikzcd}
B(d_0d_2s_0\theta) \arrow[r] \arrow[d] & B(d_1d_2s_0\theta) \arrow[d] \\
B(d_0d_3s_0\theta) \arrow[r] & B(d_1d_3s_0\theta)
\end{tikzcd}
\qquad
\begin{tikzcd}
B(d_0d_2s_2\theta) \arrow[r] \arrow[d] & B(d_1d_2s_2\theta) \arrow[d] \\
B(d_0d_3s_2\theta) \arrow[r] & B(d_1d_3s_2\theta)
\end{tikzcd}$$

The horizontal morphisms in the left-hand square and the vertical morphisms in the right-hand square are all identity morphisms, and therefore, both $e(B)(s_0\theta)$ and $e(B)(s_2\theta)$ are equal to $0(1)$. This shows that $e(B)$ satisfies (i). To verify (ii), we let

$\tau\colon [4] \to [n]$ be a functor and consider the following sequences.

$$e(B)(d_0\tau) \longrightarrow e(B)(d_1\tau) \longrightarrow e(B)(d_2\tau)$$
$$e(B)(d_2\tau) \longrightarrow e(B)(d_3\tau) \longrightarrow e(B)(d_4\tau)$$

We will show that the top sequence is exact; the proof for the bottom sequence is similar. To this end, we consider the following diagram.

$$\begin{tikzcd}
B(d_0d_2d_0\tau) \arrow[rr, equal] \arrow[dr, two heads] \arrow[dd, tail] & & B(d_0d_2d_1\tau) \arrow[rr, two heads] \arrow[dr, two heads] \arrow[dd, tail] & & B(d_0d_2d_2\tau) \arrow[dr, two heads] \arrow[dd, tail] & \\
& B(d_1d_2d_0\tau) \arrow[rr, two heads] \arrow[dd, tail] & & B(d_1d_2d_1\tau) \arrow[rr, equal] \arrow[dd, tail] & & B(d_1d_2d_2\tau) \arrow[dd, tail] \\
B(d_0d_3d_0\tau) \arrow[rr, equal] \arrow[dr, two heads] & & B(d_0d_3d_1\tau) \arrow[rr, two heads] \arrow[dr, two heads] & & B(d_0d_3d_2\tau) \arrow[dr, two heads] & \\
& B(d_1d_3d_0\tau) \arrow[rr, two heads] & & B(d_1d_3d_1\tau) \arrow[rr, equal] & & B(d_1d_3d_2\tau)
\end{tikzcd}$$

The left-hand vertical square, the middle vertical square, and the right-hand vertical squares are all admissible, and the objects $e(B)(d_0\tau)$, $e(B)(d_1\tau)$, and $e(B)(d_2\tau)$ are the lower left-hand terms in their respective completions. The front left-hand vertical square and the back right-hand vertical square also are admissible. It follows that, by choosing kernels of the admissible epimorphisms from the back rectangular diagram to the front rectangular diagram, we obtain a diagram of the form

$$\begin{tikzcd}
A_{11} \arrow[r, tail] \arrow[d, tail] & B_{11} \arrow[r, two heads] \arrow[d, tail] & C_{11} \arrow[d, tail] \\
A_{21} \arrow[r, tail] & B_{21} \arrow[r, two heads] & C_{21}.
\end{tikzcd}$$

Finally, the sequence in question is the induced sequence of a choice of cokernels of the vertical admissible monomorphisms in this diagram. Hence, the sequence is exact. This shows that $e(B)$ satisfies (ii), proving the claim. We leave it to the reader to verify that the functor $e\colon \bar{S}^{2,1}\mathscr{C}[n] \to \tilde{S}^{2,1}\mathscr{C}[n]$ is exact. It is pointed by definition.

We define η to be the natural isomorphism whose value at $A \in \operatorname{ob}\tilde{S}^{2,1}\mathscr{C}[n]$ is the natural isomorphism $\eta_A\colon A \Rightarrow e(j^*(A))$ whose value at $\sigma\colon [3] \to [n]$, in turn, is the canonical isomorphism from $A(d_2d_5 j(\sigma)) = A(\sigma)$ to $e(j^*(A))(\sigma)$. We define ε to be the identity natural isomorphism. It is clear that both η and ε are pointed. Finally, we define $g\colon e \Rightarrow (\bar{D}[n], \tilde{D}[n])e^{\mathrm{op}}$ to be the natural isomorphism whose value at B is the natural isomorphism $g_B\colon e(B) \Rightarrow ((\bar{D}[n], \tilde{D}[n])e^{\mathrm{op}})(B)$ whose value at σ, in turn, is the canonical isomorphism of lower left-hand terms in two choices of completion of the admissible square used to define $e(B)(\sigma)$. It follows from the definitions that g is pointed, and the uniqueness of kernels and cokernels, up to canonical isomorphism, implies that the two diagrams in Definition 1.27 commute. $\square$

Example 2.13. We spell out the adjoint equivalences of categories

$$S^{2,1}\mathscr{C}[n] \underset{h}{\overset{s_1^*}{\leftrightarrows}} \tilde{S}^{2,1}\mathscr{C}[n] \underset{e}{\overset{j^*}{\rightleftarrows}} \bar{S}^{2,1}\mathscr{C}[n]$$

in the case $n = 3$. An object A of $\tilde{S}^{2,1}\mathscr{C}[n]$ is given by a diagram

$$A(d^0s^1) \overset{a}{\rightarrowtail} A(d^1s^1) \overset{b}{\twoheadrightarrow} A(\mathrm{id}_{[3]}) \overset{c}{\rightarrowtail} A(d^2s^1) \overset{d}{\twoheadrightarrow} A(d^3s^1)$$

such that the sequence (a,b) and (c,d) both are exact. The functor s_1^* takes A to the object $s_1^*(A)$ of $S^{2,1}\mathscr{C}[3]$ given by the 4-term exact sequence

$$A(d^0s^1) \overset{a}{\rightarrowtail} A(d^1s^1) \xrightarrow{c \circ b} A(d^2s^1) \overset{d}{\twoheadrightarrow} A(d^3s^1),$$

and the functor j^* takes A to the object $j^*(A)$ of $\bar{S}^{2,1}\mathscr{C}[3]$ given by the diagram

$$A(d^1s^1) \overset{b}{\twoheadrightarrow} A(\mathrm{id}_{[3]}) \overset{c}{\rightarrowtail} A(d^2s^1).$$

The forgetful functors s_1^* and j^* are equivalences of categories, since A is determined, up to isomorphism, by the either of $s_1^*(A)$ and $j^*(A)$.

3 The real algebraic K-theory spectrum

By analogy with Atiyah's K-theory with reality [2], we associate to the pointed exact category with weak equivalences and strict duality $(\mathscr{C}, w\mathscr{C}, D, 0)$ a real symmetric spectrum $KR(\mathscr{C}, w\mathscr{C}, D, 0)$ that we call the real algebraic K-theory spectrum.

The real Waldhausen construction may be iterated. By applying it r times to the pointed exact category with weak equivalences and strict duality $(\mathscr{C}, w\mathscr{C}, D, 0)$, we obtain an r-real simplicial pointed exact category with weak equivalences and strict duality whose value at the object

$$[n] = [n_1, \ldots, n_r] = [n_1] \times \cdots \times [n_r]$$

of the r-fold product real category $\Delta R \times \cdots \times \Delta R$ is the pointed exact category with weak equivalences and strict duality

$$(S^{2,1}(\ldots S^{2,1}\mathscr{C}[n_r]\ldots)[n_1], wS^{2,1}(\ldots S^{2,1}\mathscr{C}[n_r]\ldots)[n_1], D[n_r]\ldots[n_1], 0[n_r]\ldots[n_1]).$$

Here, and below, we suppress the arrangement of parentheses in the r-fold product; for example, we can choose the arrangement where every pair of parentheses begin on the left. To define the real symmetric spectrum $KR(\mathscr{C}, w\mathscr{C}, D, 0)$, we wish to be able to permutate the r real simplicial directions. With this purpose in mind, we introduce the following construction, which is naturally isomorphic to the r-fold iterated Waldhausen construction.

Definition 3.1. For $r \geqslant 1$, the r-fold real Waldhausen construction of the pointed exact category with weak equivalences and strict duality $(\mathscr{C}, w\mathscr{C}, D, 0)$ is the r-real simplicial pointed exact category with weak equivalences and strict duality

$$(S^{2r,r}\mathscr{C}[-], wS^{2r,r}\mathscr{C}[-], D[-], 0[-])$$

where

$$(S^{2r,r}\mathscr{C}[n], D[n]) \subset \mathrm{Cat\,Dual}(\mathrm{Cat\,Dual}(([2], D), ([n], D)), (\mathscr{C}, D))$$

is the full subcategory with strict duality of all functors

$$A\colon\ \mathrm{Cat}([2], [n]) \to \mathscr{C}$$

such that

(i) for every $1 \leqslant i \leqslant r$, for every $\theta_j\colon\ [2] \to [n_j]$ with $j = 1, \ldots, i-1, i+1, \ldots, r$, and for every $\mu\colon\ [1] \to [n_i]$,

$$A(\theta_1, \ldots, \theta_{i-1}, s_0\mu, \theta_{i+1}, \ldots, \theta_r) = A(\theta_1, \ldots, \theta_{i-1}, s_1\mu, \theta_{i+1}, \ldots, \theta_r) = 0(1),$$

(ii) for every $1 \leqslant i \leqslant r$, for every $\theta_j\colon\ [2] \to [n_j]$ with $j = 1, \ldots, i-1, i+1, \ldots, r$, and for every $\sigma\colon\ [3] \to [n_i]$, the sequence

$$A(\theta_1, \ldots, \theta_{i-1}, d_0\sigma, \theta_{i+1}, \ldots, \theta_r) \longrightarrow A(\theta_1, \ldots, \theta_{i-1}, d_1\sigma, \theta_{i+1}, \ldots, \theta_r)$$
$$\longrightarrow A(\theta_1, \ldots, \theta_{i-1}, d_2\sigma, \theta_{i+1}, \ldots, \theta_r) \longrightarrow A(\theta_1, \ldots, \theta_{i-1}, d_3\sigma, \theta_{i+1}, \ldots, \theta_r)$$

is 4-term exaxt;

where the sequence $A \to B \to C$ in $S^{2,1}\mathscr{C}[n]$ is exact if, for all $\theta\colon [2] \to [n]$, the sequence $A(\theta) \to B(\theta) \to C(\theta)$ in $\mathscr{C}$ is exact; where the morphism $A \to B$ is in $wS^{2,1}\mathscr{C}[n]$ if, for all $\theta\colon [2] \to [n]$, the morphism $A(\theta) \to B(\theta)$ is in $w\mathscr{C}$, and where the basepoint $0[n]$ is the constant diagram $0[n](1)(\theta) = 0(1)$. For $r = 0$, the r-fold real Waldhausen construction of $(\mathscr{C}, w\mathscr{C}, D, 0)$ is equal to $(\mathscr{C}, w\mathscr{C}, D, 0)$.

We have the canonical natural isomorphisms

$$\begin{aligned}
&\mathrm{Cat}(\mathrm{Cat}([2],[n_1]\times[n_2]\times\cdots\times[n_r]),\mathscr{C})\\
\longleftarrow\ &\mathrm{Cat}(\mathrm{Cat}([2],[n_1])\times\mathrm{Cat}([2],[n_2])\times\cdots\times\mathrm{Cat}([2],[n_r]),\mathscr{C})\\
\longrightarrow\ &\mathrm{Cat}(\mathrm{Cat}([2],[n_1]),\mathrm{Cat}(\mathrm{Cat}([2],[n_2]),\ldots,\mathrm{Cat}(\mathrm{Cat}([2],[n_r]),\mathscr{C})\ldots)),
\end{aligned}$$

where the first isomorphism is induced by the canonical isomorphism

$$\mathrm{Cat}([2],[n_1]\times\cdots\times[n_r]) \longrightarrow \mathrm{Cat}([2],[n_1])\times\cdots\times\mathrm{Cat}([2],[n_r]),$$

and where the second isomorphism is determined by the closed symmetric monoidal structure on the category of categories. Comparing Definitions 2.3 and 3.1, we see that the composite natural isomorphism restricts to a natural isomorphism

$$S^{2r,r}\mathscr{C}[n_1,n_2,\ldots,n_r] \xrightarrow{u_r} S^{2,1}(S^{2,1}(\ldots S^{2,1}\mathscr{C}[n_r]\ldots)[n_2])[n_1]$$

through pointed duality preserving exact functors. As n varies, these functors constitute a natural isomorphism from the r-real simplicial pointed exact category with weak equivalences and strict duality defined by the r-fold Waldhausen construction of $(\mathscr{C}, w\mathscr{C}, D, 0)$ to the r-real simplicial pointed exact category with weak equivalences and strict duality defined by the r-fold iterate of the real Waldhausen construction applied to $(\mathscr{C}, w\mathscr{C}, D, 0)$.

Let $\sigma \in \Sigma_r = \mathrm{Aut}(\{1,\ldots,r\})$ be a permutation. It gives rise to the real functor

$$\Delta R\times\cdots\times\Delta R \xrightarrow{r_\sigma} \Delta R\times\cdots\times\Delta R$$

that takes the object $[n_1]\times\cdots\times[n_r]$ to the object $[n_{\sigma(1)}]\times\cdots\times[n_{\sigma(r)}]$ and that acts similarly on morphism real sets. The symmetric monoidal structure on the category of categories gives rise to the canonical isomorphism of categories

$$[n_1]\times\cdots\times[n_r] \longrightarrow r_\sigma([n_1]\times\cdots\times[n_r]),$$

and this isomorphism, in turn, induces a canonical isomorphism of categories

$$\mathrm{Cat}(\mathrm{Cat}([2],r_\sigma([n_1]\times\cdots\times[n_r])),\mathscr{C}) \xrightarrow{l_\sigma} \mathrm{Cat}(\mathrm{Cat}([2],[n_1]\times\cdots\times[n_r]),\mathscr{C}).$$

It follows immediately from Definition 3.1 that the isomorphism l_σ restricts to a natural isomorphism of categories

$$S^{2r,r}\mathscr{C}[n_{\sigma(1)},\ldots,n_{\sigma(r)}] \xrightarrow{l_\sigma} S^{2r,r}\mathscr{C}[n_1,\ldots,n_r]$$

through a pointed duality preserving exact functor. As n varies, these isomorphisms, in turn, form a natural isomorphism of r-real simplicial categories

$$S^{2r,r}\mathscr{C}[-]\circ r_\sigma^{\mathrm{op}} \xrightarrow{l_\sigma} S^{2r,r}\mathscr{C}[-]$$

through pointed duality preserving exact functors. In addition, from the coherence theorem for symmetric monoidal categories [11, Chapter XI, Theorem 1], we conclude that the following diagram commutes.

$$\begin{tikzcd}
S^{2r,r}\mathscr{C}[-]\circ r_\tau^{\mathrm{op}}\circ r_\sigma^{\mathrm{op}} \arrow[r, equal] \arrow[d, "l_\tau\circ r_\sigma^{\mathrm{op}}"] & S^{2r,r}\mathscr{C}[-]\circ r_{\sigma\tau}^{\mathrm{op}} \arrow[d, "l_{\sigma\tau}"] \\
S^{2r,r}\mathscr{C}[-]\circ r_\sigma^{\mathrm{op}} \arrow[r, "l_\sigma"] & S^{2r,r}\mathscr{C}[-]
\end{tikzcd}$$

Finally, we let $\Delta\colon \Delta R \to \Delta R\times\cdots\times\Delta R$ be the diagonal real functor and consider the diagonal real simplicial category $S^{2r,r}\mathscr{C}[-]\circ\Delta^{\mathrm{op}}$.

Lemma 3.2. *Let $(\mathscr{C},w\mathscr{C},D,0)$ be a pointed exact category with weak equivalences and strict duality and let r be a positive integer. The symmetric group Σ_r acts from the left on the diagonal real simplicial category $S^{2r,r}\mathscr{C}[-]\circ\Delta^{\mathrm{op}}$ with $\sigma\in\Sigma_r$ acting through the pointed duality preserving exact functor l_σ.*

Proof. Since $r_\sigma\circ\Delta=\Delta$, this follows from the commutativity of the diagram that precedes the lemma. □

Definition 3.3. Let r be a positive integer and let $X[-]$ be an r-real simplicial set. The geometric realization $|X[-]|_R$ is defined to be the real space $|X[-]\circ\Delta^{\mathrm{op}}|_R$ given by the geometric realization of the diagonal real simplicial set.

We define the rth space in the real algebraic K-theory spectrum to be the pointed real space given by the geometric realization

$$KR(\mathscr{C},w\mathscr{C},T,0)=|N(wS^{2r,r}\mathscr{C}[-],D[-],0[-])[-]|_R$$

of the $(r+1)$-real simplicial set defined by the real nerve of the r-real simplicial pointed category with strict duality $(wS^{2r,r}\mathscr{C}[-],D[-],0[-])$. By Lemma 3.2, there is a left Σ_r-action on $KR(\mathscr{C},w\mathscr{C},D,0)_r$ with $\sigma\in\Sigma_r$ acting through the pointed real map induced by the pointed duality preserving exact functor l_σ.

To define the structure maps in the real algebraic K-theory spectrum, we need a different model for the geometric realization of an r-simplicial set which we first discuss. Spelling out Definition 3.3, the geometric realization $|X[-]|_R$ of the r-real simplicial set $X[-]$ is the following coequalizer in the category of real spaces.

$$\coprod X[n,\dots,n]\times\Delta R([m],[n])\times\Delta R[m] \underset{g}{\overset{f}{\rightrightarrows}} \coprod X[p,\dots,p]\times\Delta R[p]\longrightarrow |X[-]|_R$$

The middle and left-hand coproducts range over the sets of objects and ordered pairs of objects in ΔR, respectively; the real maps f and g were defined in the discussion that precedes Definition 1.18. We now define $|X[-]|'_R$ to be the real space given by the following coequalizer in the category of real spaces.

$$\begin{aligned} &\coprod X[n_1,\dots,n_r] \times \Delta R([m_1],[n_1]) \times \cdots \times \Delta R([m_r],[n_r]) \times \Delta R[m_1] \times \cdots \times \Delta R[m_r] \\ &\underset{g'}{\overset{f'}{\rightrightarrows}} \coprod X[p_1,\dots,p_r] \times \Delta R[p_1] \times \cdots \times \Delta R[p_r] \longrightarrow |X[-]|'_R \end{aligned}$$

The middle and left-hand coproducts range over the sets of objects and ordered pairs of objects in the r-fold product real category $\Delta R \times \cdots \times \Delta R$, respectively, and the real maps f' and g' are defined by

$$\begin{aligned} f' \circ \mathrm{in}_{([m_1,\dots,m_r],[n_1,\dots,n_r])} &= \mathrm{in}_{[m_1,\dots,m_r]} \circ (f_{m_1,\dots,m_r,n_1,\dots,n_r} \times \mathrm{id}) \\ g \circ \mathrm{in}_{([m_1,\dots,m_r],[n_1,\dots,n_r])} &= \mathrm{in}_{[n_1,\dots,n_r]} \circ (\mathrm{id} \times g_{m_1,\dots,m_r,n_1,\dots,n_r}) \end{aligned}$$

where $f_{m_1,\dots,m_r,n_1,\dots,n_r}$ is the composite real map

$$\begin{aligned} &X[n_1,\dots,n_r] \times \Delta R([m_1],[n_1]) \times \cdots \times \Delta R([m_r],[n_r]) \\ &\xrightarrow{\mathrm{id} \times X} X[n_1,\dots,n_r] \times \mathrm{Real\,Set}(X[n_1,\dots,n_r], X[m_1,\dots,m_r]) \\ &\xrightarrow{\varepsilon \circ t} X[m_1,\dots,m_r], \end{aligned}$$

and where $g_{m_1,\dots,m_r,n_1,\dots,n_r}$ is the composition

$$\begin{aligned} &\Delta R([m_1],[n_1]) \times \cdots \times \Delta R([m_r],[n_r]) \times \Delta R[m_1] \times \cdots \times \Delta R[m_r] \\ &\longrightarrow \Delta R([m_1],[n_1]) \times \Delta R[m_1] \times \cdots \times \Delta R([m_r],[n_r]) \times \Delta R[m_r] \\ &\longrightarrow \Delta R[n_1] \times \cdots \times \Delta R[n_r] \end{aligned}$$

of the canonical isomorphism and the map $g_{m_1,n_1} \times \cdots \times g_{m_r,n_r}$. The following result may be proved using the method of [4]; see also [19].

Proposition 3.4. *Let $X[-]$ be an r-real simplicial set. The natural real map*

$$d_r \colon |X[-]|_R \to |X[-]|'_R$$

induced by the diagonal real maps

$$X[p,\dots,p] \times \Delta R[p] \xrightarrow{\mathrm{id} \times \Delta} X[p,\dots,p] \times \Delta R[p] \times \cdots \times \Delta R[p]$$

is a homeomorphism.

We now define the structure maps

$$KR(\mathscr{C}, w\mathscr{C}, D, 0)_r \wedge S^{2s,s} \xrightarrow{\sigma_{r,s}} KR(\mathscr{C}, w\mathscr{C}, D, 0)_{r+s}$$

in the real algebraic K-theory spectrum. We recall that $S^{2s,s}$ is defined to be the s-fold smash product $S^{2,1} \wedge \cdots \wedge S^{2,1}$ of the one-point compactification $S^{2,1}$ of $\mathbb{C}$ with ∞ as the basepoint. We let $\bar{S}^{2s,s}$ be the s-fold smash product $\bar{S}^{2,1} \wedge \cdots \wedge \bar{S}^{2,1}$ of the pointed real space $\bar{S}^{2,1} = \Delta R[2]/\partial\Delta R[2]$ and define the pointed real homeomorphism

$$\bar{S}^{2s,s} \xrightarrow{h_s} S^{2s,s}$$

to be the s-fold smash product of the pointed real homeomorphism h_1 defined by

$$h_1(t_0 \cdot 0 + t_1 \cdot 1 + t_2 \cdot 2) = t_0^{-1} e^{-2\pi i/3} + t_1^{-1} + t_2^{-1} e^{2\pi i/3}.$$

Now the map $\sigma_{r,s}$ is defined to be the composition

$$\begin{aligned}
&|N(wS^{2r,r}\mathscr{C}[-],D[-],0[-])[-]|_R \wedge S^{2,1} \wedge \cdots \wedge S^{2,1} \\
&\xleftarrow{\mathrm{id}\wedge h_s} |N(wS^{2r,r}\mathscr{C}[-],D[-],0[-])[-]|_R \wedge \bar{S}^{2,1} \wedge \cdots \wedge \bar{S}^{2,1} \\
&\xrightarrow{d_r \wedge \mathrm{id}} |N(wS^{2r,r}\mathscr{C}[-],D[-],0[-])[-]|'_R \wedge \bar{S}^{2,1} \wedge \cdots \wedge \bar{S}^{2,1} \\
&\xrightarrow{j_{r,s}} |N(wS^{2(r+s),r+s}\mathscr{C}[-],D[-],0[-])[-]|'_R \\
&\xleftarrow{d_{r+s}} |N(wS^{2(r+s),r+s}\mathscr{C}[-],D[-],0[-])[-]|_R
\end{aligned}$$

where the real map $j_{r,s}$, which is the only map that is not an isomorphism, is defined as follows. Let $A_0 \to \cdots \to A_p$ be an element of $N(wS^{2r,r}\mathscr{C}[m],D[m],0[m])[p]$, let $u_i \in \Delta R[m_i]$ let $v_j \in \Delta R[2]$, and let $w \in \Delta R[p]$. Then

$$\begin{aligned}
&j_{r,s}(\text{class of } (A_0 \to \cdots \to A_p, u_1, \dots, u_r, w, v_1, \dots, v_s)) \\
&\quad = \text{class of } (\tilde{A}_0 \to \cdots \to \tilde{A}_p, u_1, \dots, u_r, v_1, \dots, v_s, w),
\end{aligned}$$

where, if $A_i \colon \mathrm{Cat}([2],[m]) \to \mathscr{C}$, then $\tilde{A}_i \colon \mathrm{Cat}([2],[m] \times [2,\dots,2]) \to \mathscr{C}$ is defined by

$$\tilde{A}_i(\theta,\theta') = \begin{cases} A_i(\theta) & \text{if } \theta' = \Delta \\ 0(1) & \text{otherwise} \end{cases}$$

with $\theta \colon [2] \to [m]$ and $\theta' \colon [2] \to [2,\dots,2]$. It is clear from the definition that $\sigma_{r,s}$ is $\Sigma_r \times \Sigma_s$-equivariant when $\Sigma_r \times \Sigma_s$ acts from the left on the target through the group homomorphism $+ \colon \Sigma_r \times \Sigma_s \to \Sigma_{r+s}$.

Definition 3.5. The real algebraic K-spectrum of the pointed exact category with weak equivalences and strict duality $(\mathscr{C}, w\mathscr{C}, D, 0)$ is the real symmetric spectrum $KR(\mathscr{C}, w\mathscr{C}, D, 0)$ whose rth space is the pointed real space with left Σ_r-action

$$KR(\mathscr{C}, w\mathscr{C}, D, 0)_r = |N(wS^{2r,r}\mathscr{C}[-],D[-],0[-])[-]|_R$$

and whose structure maps are the $\Sigma_r \times \Sigma_s$-equivariant pointed real maps

$$KR(\mathscr{C}, w\mathscr{C}, D, 0)_r \wedge S^{2s,s} \xrightarrow{\sigma_{r,s}} KR(\mathscr{C}, w\mathscr{C}, D, 0)_{r+s}$$

defined above.

Finally, we prove the following result on the functoriality of the real algebraic K-theory spectrum. We view the topological standard simplex $\Delta[1]$ as a real space with trivial $G_{\mathbb{R}}$-action.

Proposition 3.6. *Let $(\mathscr{C}_1, w\mathscr{C}_1, D_1, 0_1)$ and $(\mathscr{C}_2, w\mathscr{C}_2, D_2, 0_2)$ be two pointed exact categories with weak equivalence and strict duality.*

(1) *A pair (F, f) of a pointed exact functor $F\colon (\mathscr{C}_1, w\mathscr{C}_1, 0_1) \to (\mathscr{C}_2, w\mathscr{C}_2, 0_2)$ and a pointed natural transformation through weak equivalences $f\colon F \Rightarrow (D_1, D_2)F^{\mathrm{op}}$ such that $f = (D_1, D_2)f^{\mathrm{op}}$ induces a map of real symmetric spectra*

$$KR(\mathscr{C}_1, w\mathscr{C}_1, D_1, 0_1) \xrightarrow{(F,f)_*} KR(\mathscr{C}_2, w\mathscr{C}_2, D_2, 0_2).$$

(2) *Let (F_1, f_1) and (F_0, f_0) be two pairs as in* (1). *A pointed natural transformation through weak equivalences $g\colon F_1 \Rightarrow F_0$ such that the diagram*

$$\begin{tikzcd} F_1 \arrow[rr, Rightarrow, "g"] \arrow[d, Rightarrow, "f_1"] & & F_0 \arrow[d, Rightarrow, "f_2"] \\ (D_1,D_2)F_1^{\mathrm{op}} & & (D_1,D_2)F_0^{\mathrm{op}} \arrow[ll, Rightarrow, "(D_1,D_2)g^{\mathrm{op}}"'] \end{tikzcd}$$

commutes induces a map of real symmetric spectra

$$\Delta[1]_+ \wedge KR(\mathscr{C}_1, w\mathscr{C}_1, D_1, 0_1) \xrightarrow{g_*} KR(\mathscr{C}_2, w\mathscr{C}_2, D_2, 0_2)$$

such that $g_* \circ (d^0 \wedge \mathrm{id}) = (F_0, f_0)_*$ and $g_* \circ (d^1 \wedge \mathrm{id}) = (F_1, f_1)_*$.

Proof. We use Addendum 1.29 as explained in Remark 1.31. The pair (F, f) induces the pair $(wS^{2r,r}F[n], wS^{2r,r}f[n])$ which is an object of the category

$$\mathrm{Sym}\,\mathrm{Cat}\,\mathrm{Dual}_*((wS^{2r,r}\mathscr{C}_1[n], D_1[n], 0_1[n]), (wS^{2r,r}\mathscr{C}_2[n], D_2[n], 0_2[n])).$$

The images $H_*(wS^{2r,r}F[n], wS^{2r,r}f[n])$ by the Top_*-functor H_* defined in Remark 1.31 give rise to a Σ_r-equivariant pointed real map that we write

$$KR(\mathscr{C}_1, w\mathscr{C}_1, D_1, 0_1)_r \xrightarrow{(F,f)_{*r}} KR(\mathscr{C}_2, w\mathscr{C}_2, D_2, 0_2)_r.$$

It is clear that the maps $(F, f)_{*r}$ are compatible with the real symmetric spectrum structure maps, and hence, form a map $(F, f)_*$ of real symmetric spectra.

Similarly, the natural transformation $g\colon F_1 \Rightarrow F_0$ induces a natural transformation $wS^{2r,r}g[n]\colon wS^{2r,r}F_1[n] \Rightarrow wS^{2r,r}F_0[n]$ which is a morphism of the above category of symmetric spaces. Hence, from Remark 1.31 we obtain a Σ_r-equivariant real map

$$\Delta[1]_+ \wedge KR(\mathscr{C}_1, w\mathscr{C}_1, D_1, 0_1)_r \xrightarrow{g_{*r}} KR(\mathscr{C}_2, w\mathscr{C}_2, D_2, 0_2)_r$$

such that $g_{*r} \circ (d^0 \wedge \mathrm{id}) = (F_0, f_0)_{*r}$ and $g_{*r} \circ (d^1 \wedge \mathrm{id}) = (F_1, f_1)_{*r}$. Finally, it is clear that the maps g_{*r} form a map g_* of real symmetric spectra. □

Remark 3.7. If $F\colon (\mathscr{C}_1, w\mathscr{C}_1, D_1, 0_1) \to (\mathscr{C}_2, w\mathscr{C}_2, D_2, 0_2)$ is a pointed exact duality preserving functor between pointed exact categories with weak equivalences and strict duality, then the pair $(F, 1_F)$ satisfies the hypothesis of Proposition 3.6 (1). In this case, we abbreviate $(F, 1_F)_*$ as F_*.

4 The real Γ-category construction

In this section, we introduce a variant of Segal's Γ-category that we call the real Γ-category construction. In essence, it associates to the pointed exact category with weak equivalences and strict duality $(\mathscr{C}, w\mathscr{C}, D, 0)$ the real Γ-pointed exact category with weak equivalences and strict duality $(\mathscr{C}(-), w\mathscr{C}(-), D(-), 0(-)$ that takes the pointed finite real set (X,x) to the pointed exact category with weak equivalences and strict duality given by the pointed $\mathscr{C}$-valued sheafs on (X,x).

We recall the closed symmetric monoidal category $\mathrm{Real\,Set}_*$ of (κ-small) pointed real sets. The full subcategory $\mathrm{Fin\,Real\,Set}_*$ of all (κ-small) finite pointed real sets inherits a closed symmetric monoidal structure. We define a pointed real category to be a $\mathrm{Real\,Set}_*$-category, a real pointed functor to be a $\mathrm{Real\,Set}_*$-functor, and a real pointed natural transformation to be a $\mathrm{Real\,Set}_*$-natural transformation. We will abuse notation and write $\mathrm{Fin\,Real\,Set}_*$ for the underlying pointed real category of the closed symmetric monoidal category $\mathrm{Fin\,Real\,Set}_*$.

Definition 4.1. A real Γ-object in the pointed real category $\mathscr{C}$ is a pointed real functor $A\colon \mathrm{Fin\,Real\,Set}_* \to \mathscr{C}$. A morphism $f\colon A \to A'$ between the real Γ-objects is a pointed real natural transformation.

Remark 4.2. We define a real Γ-space to be a real Γ-object in the pointed real category $\mathrm{Real\,Top}_*$ of (κ-small) pointed real spaces. This is essentially the same as a Γ_G-space in the sense of Shimakawa [23, §1] for the group $G = G_{\mathbb{R}} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. We also define a real Γ-category to be a a real Γ-object in the real category $\mathrm{Cat\,Dual}_*$ of (κ-small) pointed categories with strict duality. This, however, is different from the Γ_G-categories of Shimakawa [23, §2].

To define the real Γ-category construction, we define a pointed real functor

$$\mathrm{Fin\,Real\,Set}_*^{\mathrm{op}} \xrightarrow{P} \mathrm{Cat\,Dual}_*$$

by analogy with the power set functor. Let (X,x) be a (κ-small) finite pointed real set. The category $P(X,x)$ has objects all pointed subsets $x \in U \subset X$, and the set $P(X,x)(U,V)$ of morphisms from the object U to object V consists of all subsets $x \in F \subset U \cap V$. We stress that $x \in U, V \subset X$ and $x \in F \subset U \cap V$ are not required to be real subsets. The composition of the morphisms $F\colon U \to V$ and $G\colon V \to W$ is defined to be the morphism $G \circ F = G \cap F\colon U \to W$ and the identity morphism of $x \in X \subset X$ is defined to be $x \in X \subset X$. In particular, the morphism $F\colon U \to V$ is equal to the composition of the morphisms $U \cap V\colon U \to U \cap V$ and $U \cap V\colon U \cap V \to V$ which may be thought of as the map that collapses the complement of $U \cap V \subset U$ to the basepoint and the canonical inclusion of $U \cap V$ into V, respectively. The duality functor

$$P(X,x)^{\mathrm{op}} \xrightarrow{D} P(X,x)$$

takes the object $x \in U \subset X$ to the object $x \in DU = \{tu \mid u \in U\} \subset X$ and takes the morphism $F\colon U \to V$ to the morphism $DF = \{tf \mid f \in F\}\colon DV \to DU$. The basepoint

$\bar{x}\colon (\mathbf{1},\mathrm{id}) \to (P(X,x),D)$ is the duality preserving functor defined by $\bar{x}(1) = \{x\} \subset X$. This defines the pointed real functor P on objects. We define

$$\begin{aligned}&\mathrm{Fin\,Real\,Set}_*((X_1,x_1),(X_2,x_2))\\ &\xrightarrow{P} \mathrm{ob}_R\,\mathrm{Cat\,Dual}_*((P(X_2,x_2),D_2,\bar{x}_2),(P(X_1,x_1),D_1,\bar{x}_1))\end{aligned}$$

to be the pointed real map given by the following variant of the inverse image functor. If $f\colon (X_1,x_1) \to (X_2,x_2)$ is a pointed (but not necessarily real) map, then

$$(P(X_2,x_2),D_2,\bar{x}_2) \xrightarrow{f^*} (P(X_1,x_1),D_1,\bar{x}_1)$$

is the pointed (but not necessarily duality preserving) functor defined on objects and morphisms, respectively, by

$$\begin{aligned}f^*(V) &= f^{-1}(V \smallsetminus \{x_2\}) \cup \{x_1\}\\ f^*(F) &= f^{-1}(F \smallsetminus \{x_2\}) \cup \{x_1\}).\end{aligned}$$

This defines the pointed real functor P on morphism pointed real sets.

We next define a Grothendieck topology on the category $P(X,x)$ and begin by recalling the relevant definitions from [1, Exposé I-II]. A sieve S on the object U is a full subcategory of the overcategory $P(X,x)/U$ such that for every $F\colon V \to U$ in $\mathrm{ob}\,S$ and for every morphism $G\colon W \to V$ in $P(X,x)$, the composite $F \circ G\colon W \to U$ is in $\mathrm{ob}\,S$. The pullback of the sieve S on U along the morphism $F\colon V \to U$ of $P(X,x)$ is the unique sieve F^*S on V such that

$$\mathrm{ob}\,F^*S = \{G\colon W \to V \mid F \circ G\colon W \to U \text{ is in } \mathrm{ob}\,S\}.$$

A Grothendieck topology on $P(X,x)$ is a function that to every object U associates a subset $J(U)$ of the set of sieves on U such that the following axioms (1)–(3) hold. The elements of $J(U)$ are called the covering sieves on U.

(1) If $F\colon V \to U$ is a morphism and if S is a covering sieve on U, then the pullback sieve F^*S is a covering sieve on V.
(2) If U is an object, if S and T are sieves on U, if S is a covering sieve, and if for every morphism $F\colon V \to U$ in $\mathrm{ob}\,S$, the pullback sieve F^*T is a covering sieve on V, then T is a covering sieve on U.
(3) If U is an object of $P(X,x)$, then $P(X,x)/U$ is a covering sieve on U.

Finally, a $\mathscr{C}$-valued sheaf on $P(X,x)$ for the topology J is a functor

$$A\colon P(X,x)^{\mathrm{op}} \to \mathscr{C}$$

such that for every object U and for every $S \in J(U)$, the family of morphisms

$$A(F)\colon A(U) \to A(V),$$

indexed by $F\colon V \to U$ in $\mathrm{ob}\,S$, constitute a limit of the S^{op}-diagram in $\mathscr{C}$ that takes the value $A(V)$ at the object $F\colon V \to U$ of S and the value $A(G)\colon A(V) \to A(W)$ at the morphism $G\colon W \to V$ in S from $F \circ G\colon W \to U$ to $F\colon V \to U$.

Lemma 4.3. *Let (X,x) be a finite pointed set. There is a Grothendieck topology J on the category $P(X,x)$ where the sieve S on the object U is a covering sieve if and only if for every $u \in U$, the morphism $\{x,u\} \colon \{x,u\} \to U$ is in* $\operatorname{ob} S$.

Proof. We first verify axiom (1). Let $F \colon V \to U$ be a morphism and let S be a covering sieve on U. Given $v \in V$, the composite morphism $F \circ \{x,v\} \colon \{x,v\} \to U$ is equal to either $\{x,v\} \colon \{x,v\} \to U$ or $\{x\} \colon \{x,v\} \to U$. The former is in $\operatorname{ob} S$ by definition and the latter is equal to the composition of $\{x\} \colon \{x,v\} \to \{x\}$ and $\{x\} \colon \{x\} \to U$ and hence also is in $\operatorname{ob} S$. This proves (1). We next prove (2). Let S and T be two sieves on U such that S is covering sieve and such that for every $F \colon V \to U$ in $\operatorname{ob} S$, F^*T is a covering sieve on V. We let $u \in U$ and consider the morphism $F = \{x,u\} \colon \{x,u\} \to U$. By assumption, the pullback sieve $\{x,u\}^*T$ is a covering sieve on $\{x,u\}$. Hence, by the definition of covering sieves, $\{x,u\} \colon \{x,u\} \to \{x,u\}$ is in $\operatorname{ob}(\{x,u\}^*T)$. It follows that $\{x,u\} \colon \{x,u\} \to U$ is in $\operatorname{ob} T$, which proves (2). Finally, it is clear that $P(X,x)/U$ is a covering sieve, so that also (3) holds. □

In the following, we will always endow the category $P(X,x)$ with the Grothendieck topology J defined in Lemma 4.3.

Proposition 4.4. *Let (X,x) be a finite pointed set, let $(\mathscr{C},0)$ be a pointed additive category, and let $A \colon (P(X,x),\bar{x})^{\mathrm{op}} \to (\mathscr{C},0)$ be a pointed functor. In this situation, the following properties* (a)–(c) *are equivalent.*

(a) *The functor A is a $\mathscr{C}$-valued sheaf on $P(X,x)$ for the topology J.*
(b) *For every object U of $P(X,x)$, the family of morphisms*

$$A(\{x,u\}) \colon A(U) \to A(\{x,u\}),$$

indexed by $u \in U$, constitute a product of the family of objects $A(\{x,u\})$, indexed by $u \in U$.
(c) *For every object U of $P(X,x)$, the family of morphisms*

$$\{A(\{x,u\}) \colon A(\{x,u\}) \to A(U) \mid u \in U\},$$

indexed by $u \in U$, constitute a coproduct of the family of objects $A(\{x,u\})$, indexed by $u \in U$.

Proof. Let U be an object of $P(X,x)$. If T is a subcategory of $P(X,x)/U$, then we write A_T for the T^{op}-diagram in $\mathscr{C}$ defined by the composition

$$T^{\mathrm{op}} \longrightarrow (P(X,x)/U)^{\mathrm{op}} \longrightarrow P(X,x)^{\mathrm{op}} \xrightarrow{A} \mathscr{C}$$

of the canonical inclusion, the canonical projection, and the given functor A. Now, let S be a covering sieve on U. We define S' to be the full subcategory S that consists of all objects in S of the form $V \colon V \to U$. The subcategory S' is final in S, since there is a unique morphism in S from the general object $F \colon V \to U$ to the object $F \colon F \to U$ which is in the subcategory S'. It follows that the morphisms $A(F) \colon A(U) \to A(V)$, indexed by $F \colon V \to U$ in $\operatorname{ob} S$, constitute a limit of the S^{op}-diagram A_S if and only if the morphisms $A(V) \colon A(U) \to A(V)$, indexed by $V \colon V \to U$ in $\operatorname{ob} S'$, constitute a limit of the S'^{op}-diagram $A_{S'}$. Suppose that property (a) holds. The sieve S on U generated by the set of morphisms $\{\{x,u\}\colon \{x,u\} \to U \mid u \in U\}$ is a covering sieve. This set, in turn, is the set of object in the final subcategory S' of S, and the only non-identity morphisms in S' are the unique morphisms from $\{x\}\colon \{x\} \to U$ to $\{x,u\}\colon \{x,u\} \to U$. Since $A(\{x\})$ is equal to the null object $0(1)$, we conclude that property (b) holds.

Conversely, suppose that property (b) holds. We let S be a covering sieve on the object U and consider a family of morphisms $f_V\colon B \to A(V)$, indexed by $V\colon V \to U$ in $\mathrm{ob}\, S'$, such that for every morphism $W\colon (V \to U) \to (W \to U)$ in S', $f_W = A(W)$. By property (b), this family of morphisms is uniquely determined by the subfamily, indexed by $u \in U$, that consists of the morphisms $f_{\{x,u\}}\colon B \to A(\{x,u\})$. This, in turn, determines a unique morphism $f\colon B \to A(U)$ with the property for all $u \in U$, $f_{\{x,u\}} = A(\{x,u\}) \circ f$. Now, for $V\colon V \to U$ in $\mathrm{ob}\, S'$ and $v \in V$, we have

$$A(\{x,v\}) \circ f_V = f_{\{x,v\}} = A(\{x,v\}) \circ f\colon B \to A(\{x,v\}),$$

which shows that $f_V = A(V) \circ f$. Hence, property (a) holds.

Finally, for every $U \in \mathrm{ob}\, P(X,x)$ and every $u,v \in U$, the composite morphism

$$A(\{x,u\}) \xrightarrow{A(\{x,u\})} A(U) \xrightarrow{A(\{x,v\})} A(\{x,v\})$$

is equal to the identity morphism, if $u = v$, and is the zero morphism, otherwise. In this situation, it follows from [11, Theorem VIII.2.2] that properties (b) and (c) both are equivalent to the property that the sum, indexed by $u \in U$, of the morphisms

$$A(U) \xrightarrow{A(\{x,u\})} A(\{x,u\}) \xrightarrow{A(\{x,u\})} A(U)$$

is equal to the identity morphism. This completes the proof. $\square$

Corollary 4.5. *Let (X,x) be a finite pointed real set, and let $(\mathscr{C},D,0)$ be a pointed additive category with strict duality. In this situation, the duality functor*

$$\mathrm{Cat}_*((P(X,x),\bar{x})^{\mathrm{op}},(\mathscr{C},0))^{\mathrm{op}} \xrightarrow{D} \mathrm{Cat}_*((P(X,x),\bar{x})^{\mathrm{op}},(\mathscr{C},0))$$

preserves sheaves.

Proof. Let the pointed functor $A\colon (P(X,x),\bar{x})^{\mathrm{op}} \to (\mathscr{C},0)$ be a sheaf and let U be an object of $P(X,x)$. In this situation, it follows from Proposition 4.4 that the family of morphisms $A(\{x,Du\})\colon A(\{x,Du\}) \to A(DU)$ indexed by $u \in U$ is a coproduct of the family of objects $A(\{x,Du\})$ indexed by $u \in U$. Hence, the family of morphisms

$$\begin{array}{ccc} D(A(DU)) & \xrightarrow{D(A(\{x,Du\}))} & D(A(\{x,Du\})) \\ \| & & \| \\ (DA)(U) & \xrightarrow{(DA)(\{x,u\})} & (DA)(\{x,u\}) \end{array}$$

indexed by $u \in U$ constitute a product of the family of objects $(DA)(\{x,u\})$ indexed by $u \in U$. By Proposition 4.4, the pointed functor $DA\colon (P(X,x),\bar{x})^{\mathrm{op}} \to (\mathscr{C},0)$ is a sheaf as stated. $\square$

Let $f\colon (X_1,x_1) \to (X_2,x_2)$ is a map of finite pointed sets, and let $(\mathscr{C},0)$ be a pointed additive category. In this case, the pointed functor

$$f^*\colon (P(X_2,x_2),\bar{x}_2) \to (P(X_1,x_1),\bar{x}_1)$$

induces a pointed functor

$$f_p\colon \operatorname{Cat}_*((P(X_1,x_1)^{\mathrm{op}},\bar{x}_1),(\mathscr{C},0)) \longrightarrow \operatorname{Cat}_*((P(X_2,x_2),\bar{x}_2)^{\mathrm{op}},(\mathscr{C},0))$$

called the direct image functor.

Lemma 4.6. *Let $f\colon (X_1,x_1) \to (X_2,x_2)$ be a map of finite pointed sets and let $(\mathscr{C},0)$ be a pointed additive category. The direct image functor*

$$f_p\colon \operatorname{Cat}_*((P(X_1,x_1)^{\mathrm{op}},\bar{x}_1),(\mathscr{C},0)) \longrightarrow \operatorname{Cat}_*((P(X_2,x_2),\bar{x}_2)^{\mathrm{op}},(\mathscr{C},0))$$

preserves sheaves.

Proof. Let the pointed functor $A\colon (P(X_1,x_1),\bar{x}_1)^{\mathrm{op}} \to (\mathscr{C},0)$ be a sheaf. It suffices, by Proposition 4.4, to show that for all $x_2 \in U_2 \subset X_2$, the family of morphisms

$$\begin{tikzcd}
f_p(A)(U_2) \arrow[r, "f_p(A)(\{x_2,u_2\})"] \arrow[d, equal] & f_p(A)(\{x_2,u_2\}) \arrow[d, equal] \\
A(f^*(U_2)) \arrow[r, "A(f^*(\{x_2,u_2\}))"] & A(f^*(\{x_2,u_2\}))
\end{tikzcd}$$

indexed by $u_2 \in U_2$ constitute a product of the family of objects

$$f_p(A)(\{x_2,u_2\}) = A(f^*(\{x_2,u_2\}))$$

indexed by $u_2 \in U_2$. But this readily follows from the family of morphisms

$$f^*(\{x_2,u_2\})\colon f^*(\{x_2,u_2\}) \to f^*(U_2),$$

indexed by $u_2 \in U_2$, being a coproduct of the family of pointed sets $f^*(\{x_2,u_2\})$, indexed by $u_2 \in U_2$, and from the pointed functor $A\colon (P(X_1,x_1),\bar{x}_1)^{\mathrm{op}} \to (\mathscr{C},0)$ being a sheaf. $\square$

Definition 4.7. The real Γ-construction of the pointed exact category with weak equivalences and strict duality $(\mathscr{C},w\mathscr{C},D,0)$ is the real Γ-pointed exact category with weak equivalences and strict duality

$$(\mathscr{C}(-),w\mathscr{C}(-),D(-),0(-))$$

where

$$(\mathscr{C}(X,x),D(X,x),0(X,x)) \subset \operatorname{Cat}\operatorname{Dual}_*((P(X,x),D,\bar{x})^{\mathrm{op}},(\mathscr{C},D,0))$$

is the full sub-pointed category with strict duality whose objects are the sheaves; where the sequence $A \to B \to C$ in $\mathscr{C}(X,x)$ is exact if, for all $x \in U \subset X$, the sequence $A(U) \to B(U) \to C(U)$ in $\mathscr{C}$ is exact; where the morphism $A \to B$ is in

$w\mathscr{C}(X,x)$ if, for every $x \in U \subset X$, the morphism $A(U) \to B(U)$ is in $w\mathscr{C}$; and where for $f\colon (X_1,x_1) \to (X_2,x_2)$ a pointed map,

$$f_*\colon \mathscr{C}(X_1,x_1) \to \mathscr{C}(X_2,x_2)$$

is the direct image functor.

Remark 4.8. In Definition 4.7, we ask the reader to verify that the pointed map

$$\begin{aligned}&\mathrm{Fin\,Real\,Set}_*((X_1,x_1),(X_2,x_2)) \longrightarrow \\ &\mathrm{Cat\,Dual}_*((\mathscr{C}(X_1,x_1),D(X_1,x_1),0(X_1,x_1)),(\mathscr{C}(X_2,x_2),D(X_2,x_2),0(X_2,x_2)))\end{aligned}$$

that takes a pointed map to the associated direct image functor is a real map.

Let (X,x) be a fixed finite pointed real set and let $(\mathscr{C},w\mathscr{C},D,0)$ be a pointed exact category with weak equivalences and strict duality. We view (X,x) as a discrete pointed category with strict duality, where the duality functor $D\colon X^{\mathrm{op}} \to X$ is defined by $Du = tu$, and write

$$\mathrm{Cat\,Dual}_*((X,x),(\mathscr{C},w\mathscr{C},D,0))$$

for the exact category with weak equivalences and strict duality defined as follows. The underlying pointed category with strict duality is $\mathrm{Cat\,Dual}_*((X,D,x),(\mathscr{C},D,0))$; the sequence $A \to B \to C$ is exact if $A(u) \to B(u) \to C(u)$ is exact in $\mathscr{C}$, for all $u \in X$; and the map $A \to B$ is a weak equivalence if $A(u) \to B(u)$ is in $w\mathscr{C}$, for all $u \in X$. Now, the pointed duality preserving functor

$$(X,D,x) \xrightarrow{\;i\;} (P(X,x),D,\bar{x})$$

defined by $i(u) = \{x,u\}$ induces a pointed exact duality preserving functor

$$(\mathscr{C}(X,x),w\mathscr{C}(X,x),D(X,x),0(X,x)) \xrightarrow{\;i^*\;} \mathrm{Cat\,Dual}_*((X,x),(\mathscr{C},w\mathscr{C},D,0)).$$

We note that the domain and target of the real functor i^* are a covariant functor and a contravariant functor, respectively, of (X,x). In particular, the functor i^* is not a natural transformation.

Lemma 4.9. *Let $(\mathscr{C},w\mathscr{C},D,0)$ be a pointed exact category with weak equivalences and strict duality. For every finite pointed real set (X,x), there exists a pointed exact adjunction $(i_!,i^*,1,\varepsilon,f,1)$ from $(\mathscr{C}_1,w\mathscr{C}_1,D_1,0_1)$ to $(\mathscr{C}_2,w\mathscr{C}_2,D_2,0_2)$, where*

$$\begin{aligned}(\mathscr{C}_1,w\mathscr{C}_1,D_1,0_1) &= \mathrm{Cat\,Dual}_*((X,x),(\mathscr{C},w\mathscr{C},D,0))\\ (\mathscr{C}_2,w\mathscr{C}_2,D_2,0_2) &= (\mathscr{C}(X,x),w\mathscr{C}(X,x),D(X,x),0(X,x)).\end{aligned}$$

Moreover, the natural transformations ε and f both are isomorphisms.

Proof. We define the functor $i_!$ as follows. If $B\colon (X,x) \to (\mathscr{C},0)$ is an object of $\mathscr{C}_1$, then we define $i_!(B)\colon (P(X,x),\bar{x}) \to (\mathscr{C},0)$ to be the pointed functor that takes the object U to a choice of sum in $\mathscr{C}$ of the family of objects $B(u)$, $u \in U$, and that takes the morphism $F\colon U \to V$ to the morphism $i_!(B)(F)\colon i_!(B)(U) \to i_!(B)(V)$ whose uth component is the canonical injection $B(u) \to i_!(B)(V)$, if $u \in F \subset U \cap V$, and the zero morphism, otherwise. We require the sum $i_!(B)(U)$ to be equal to $B(u)$, if $U = \{x,u\}$, and to be equal to $0(1)$, if $B(u) = 0(1)$ for all $u \in U$, But in all other cases, the sum may be chosen arbitrarily. It follows from Proposition 4.4 that the pointed functor $i_!(B)$ is a sheaf. If $f\colon B_1 \Rightarrow B_2$ is a morphism in $\mathscr{C}_1$, then $i_!(f)\colon i_!(B_1) \to i_!(B_2)$ is the unique morphism for which $i_!(f)_U$ is the sum of the set of morphisms $\{f_u \mid u \in U\}$. This defines the functor $i_!$; it is pointed and exact. The unit $\eta\colon \mathrm{id}_{\mathscr{C}_1} \Rightarrow i^* \circ i_!$ is the identity natural transformation, and the counit $\varepsilon\colon i_! \circ i^* \Rightarrow \mathrm{id}_{\mathscr{C}_2}$ is the pointed natural isomorphism for which $\varepsilon_A\colon (i_!i^*A)(U) \to A(U)$ is the unique isomorphism in $\mathscr{C}$ whose uth component is the morphism $A(\{x,u\})\colon A(\{x,u\}) \to A(U)$. The two composite pointed natural transformations

$$i_! \xRightarrow{i_! \circ \eta} i_! \circ i^* \circ i_! \xRightarrow{\varepsilon \circ i_!} i_! \qquad\qquad i^* \xRightarrow{\eta \circ i^*} i^* \circ i_! \circ i^* \xRightarrow{i^* \circ \varepsilon} i^*$$

are the respective identity natural transformations. In the case of the former, this follows form the uniqueness of the isomorphism between two choices of sums of a given set of objects. This shows that $(i_!, i^*, 1, \varepsilon)$ is a pointed exact adjunction.

Finally, we define $f\colon i_! \Rightarrow (D_2,D_1)(i_!)^{\mathrm{op}}$ to be the following natural isomorphism. If $x \in U \subset X$ is an object of $P(X,x)$, then $((D_1,D_2)i_!)(B)(U)$ is a choice of product of the family of objects $B(u)$, $u \in U$, with the product equal to $0(1)$ in case $B(u) = 0(1)$ for all $u \in U$. Now we let $f_U\colon (i_!(B))(U) \to ((D_1,D_2)i_!)(B)(U)$ be the canonical isomorphism from the sum to the product of the same finite family of objects in an additive category. We have $f = (D_2,D_1)f^{\mathrm{op}}$, by the uniqueness of the canonical isomorphism from the sum to the product of a finite set of object in an additive category. By the same reason, the two diagrams in Definition 1.27 commute. This completes the proof. □

Remark 4.10. In the proof of Lemma 4.9, the morphism ε_A is an isomorphism in $\mathscr{C}$, and hence, is in $w\mathscr{C}$, because $A\colon P(X,x)^{\mathrm{op}} \to \mathscr{C}$ is a sheaf. The conclusion of the lemma would not hold with the category $\mathscr{C}(X,x)$ of sheaves replaced by the larger category $\mathrm{Cat}_*(P(X,x)^{\mathrm{op}},(\mathscr{C},0))$ of presheaves, unless $w\mathscr{C} = \mathscr{C}$.

We have the real Γ-space $|N(w\mathscr{C}(-),D(-),0(-))[-]|_R$. The following result shows that it is special.

Corollary 4.11. *Let $(\mathscr{C},w\mathscr{C},D,0)$ be a pointed exact category with weak equivalences and strict duality. For every finite pointed real set (X,x), the pointed real map*

$$|N(w\mathscr{C}(X,x),D(X,x),0(X,x))[-]|_R \xrightarrow{i^*} |N\,\mathrm{Cat}\,\mathrm{Dual}_*((X,x),(w\mathscr{C},D,0))[-]|_R$$

is a weak equivalence of pointed real spaces.

Proof. It follows from Lemma 4.9 and Corollary 1.34 that the map of the statement is a pointed real homotopy equivalence with pointed real homotopy inverse the pointed real map $H_*(i_!,f)$. □

5 The real direct sum K-theory spectrum

In this section, we associate to the pointed exact category with weak equivalences and strict duality $(\mathscr{C}, w\mathscr{C}, D, 0)$ a real symmetric spectrum $KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)$ that we call the real direct sum K-theory spectrum. Being defined by a group-completion process, it lends itself more easily to calculation by homological means.

The real Γ-category construction associates to the pointed exact category with weak equivalences and strict duality $(\mathscr{C}, w\mathscr{C}, D, 0)$ the real Γ-pointed exact category with weak equivalences and strict duality $(\mathscr{C}(-), w\mathscr{C}(-), D(-), 0(-))$. It is a pointed real functor from the pointed real category of (κ-small) pointed finite real sets $\mathrm{Fin\,Real\,Set}_*$ to the pointed real category of (κ-small) pointed exact categories with weak equivalences and strict duality $\mathrm{w\,Ex\,Cat\,Dual}_*$. We define

$$S^{2,1}[-] = \Delta R[2][-]/\partial \Delta R[2][-];$$

it is a real simplicial finite pointed real set. For every positive integer r, we define

$$S^{2r,r}[-] = S^{2,1}[-] \wedge \cdots \wedge S^{2,1}[-]$$

to be the r-real simplicial finite pointed real set given by the r-fold smash product.

Definition 5.1. For $r \geqslant 1$, the r-real Segal construction of the pointed exact category with weak equivalences and strict duality $(\mathscr{C}, w\mathscr{C}, D, 0)$ is the r-real simplicial pointed exact category with weak equivalences and strict duality

$$(S^{2r,r}_{\oplus}\mathscr{C}[-], wS^{2r,r}_{\oplus}\mathscr{C}[-], D[-], 0[-]) = (\mathscr{C}(-), w\mathscr{C}(-), D(-), 0(-)) \circ S^{2r,r}[-]$$

given by the composition of the r-real simplicial finite pointed real set $S^{2r,r}[-]$ and the real Γ-category construction of $(\mathscr{C}, w\mathscr{C}, D, 0)$. The 0-real Segal construction of $(\mathscr{C}, w\mathscr{C}, D, 0)$ is equal to $(\mathscr{C}, w\mathscr{C}, D, 0)$.

We compare the r-real Segal construction and the r-fold iterate of the 1-real Segal construction. To this end, we consider the composition

$$\begin{aligned}
&\mathrm{Cat}_*(P(S^{2,1}[n_1] \wedge \cdots \wedge S^{2,1}[n_r])^{\mathrm{op}}, (\mathscr{C}, 0))\\
&\xrightarrow{i_r^*} \mathrm{Cat}_*(P(S^{2,1}[n_1])^{\mathrm{op}} \wedge P(S^{2,1}[n_2])^{\mathrm{op}} \wedge \cdots \wedge P(S^{2,1}[n_r])^{\mathrm{op}}, (\mathscr{C}, 0))\\
&\xrightarrow{k_r} \mathrm{Cat}_*(P(S^{2,1}[n_1])^{\mathrm{op}}, \mathrm{Cat}_*(P(S^{2,1}[n_2])^{\mathrm{op}}, \dots, \mathrm{Cat}_*(P(S^{2,1}[n_r])^{\mathrm{op}}, (\mathscr{C}, 0)) \dots)),
\end{aligned}$$

of the pointed functor i_r^* induced by the pointed functor

$$P(S^{2,1}[n_1]) \wedge \cdots \wedge P(S^{2,1}[n_r]) \xrightarrow{i_r} P(S^{2,1}[n_1] \wedge \cdots \wedge S^{2,1}[n_r])$$

that maps the object $U_1 \wedge \cdots \wedge U_r$ to the object $U_1 \wedge \cdots \wedge U_r$ and the pointed functor k_r defined by the canonical isomorphism of pointed categories given by the closed symmetric monoidal structure on the category of (κ-small) pointed categories. The composite pointed functor $k_r \circ i_r^*$ restricts to a pointed natural transformation

$$S^{2r,r}_{\oplus}\mathscr{C}[n_1, n_2, \dots, n_r] \xrightarrow{i_r^{\#}} S^{2,1}_{\oplus}(S^{2,1}_{\oplus}(\dots S^{2,1}_{\oplus}\mathscr{C}[n_r] \dots)[n_2])[n_1]$$

through pointed duality preserving exact functors. As n varies, the functors $i_r^\#$ form an r-real natural transformation from the r-real simplicial pointed exact category with weak equivalences and strict duality defined by the r-real Segal construction to the r-real simplicial pointed exact category with weak equivalences and strict duality defined by the r-fold iterate of the 1-real Segal construction. The functor i_r is not essentially surjective, since not every pointed subset of $S^{2,1}[n_1]\wedge\cdots\wedge S^{2,1}[n_r]$ is of the form $U_1\wedge\cdots\wedge U_r$, and the functor $i_r^\#$ is not an isomorphism of categories. However, we have the following result.

Lemma 5.2. *For every positive integer r, the pointed real map*

$$|N(wS_\oplus^{2r,r}[-],D[-],0[-])[-]|_R \xrightarrow{i_r^\#} |N(wS_\oplus^{2,1}\dots S_\oplus^{2,1}\mathscr{C}[-],D[-],0[-])[-]|_R$$

is a weak equivalence of pointed real spaces.

Proof. By Proposition 3.4 and the real realization lemma, it suffices to show that for fixed $[n]=[n_1,\dots,n_r]$, the pointed real map

$$|N(wS_\oplus^{2r,r}[n],D[n],0[n])[-]|_R \xrightarrow{i_r^\#} |N(wS_\oplus^{2,1}\dots S_\oplus^{2,1}\mathscr{C}[n],D[n],0[n])[-]|_R$$

is a weak equivalence of pointed real spaces. To this end, we consider the following commutative diagram of pointed categories with strict duality and pointed duality preserving functors in which the vertical functors are induced from the pointed real functor $i\colon (X,x)\to(P(X,x),D,\bar{x})$ defined by $i(u)=\{x,u\}$.

$$\begin{tikzcd}
wS_\oplus^{2r,r}\mathscr{C}[n_1,\dots,n_r] \arrow[r, "i_r^\#"] \arrow[d] & wS_\oplus^{2,1}(\dots S_\oplus^{2,1}\mathscr{C}[n_r]\dots)[n_1] \arrow[d] \\
\mathrm{Cat}_*(S^{2r,r}[n_1,\dots,n_r],(w\mathscr{C},0)) \arrow[r, "k_r"] & \mathrm{Cat}_*(S^{2,1}[n_1],\dots,\mathrm{Cat}_*(S^{2,1}[n_r],(w\mathscr{C},0)),\dots)
\end{tikzcd}$$

Taking real nerves and geometric realization, we obtain a commutative diagram of pointed real spaces. In this diagram, the lower horizontal map is a homeomorphism, since the functor k_r is an isomorphism of categories, and Corollary 4.11 shows that the vertical maps are weak equivalences of pointed real spaces. Hence, also the top horizontal map is a weak equivalence of pointed real spaces as desired. $\square$

If $\sigma\in\Sigma_r$ then the symmetric monoidal structure on the category of finite pointed real sets gives rise to a canonical isomorphism of finite pointed real sets

$$S^{2,1}[n_{\sigma(1)}]\wedge\cdots\wedge S^{2,1}[n_{\sigma(r)}] \xrightarrow{l'_\sigma} S^{2,1}[n_1]\wedge\cdots\wedge S^{2,1}[n_r]$$

and, as n varies, this is a canonical isomorphism of r-real pointed finite real sets

$$S^{2r,r}[-]\circ r_\sigma^{\mathrm{op}} \xrightarrow{l'_\sigma} S^{2r,r}[-],$$

where r_σ is the real functor defined earlier. It induces a canonical natural isomorphism of r-simplicial categories through pointed duality preserving exact functors

$$S^{2r,r}_{\oplus}\mathscr{C}[-] \circ r_\sigma^{\mathrm{op}} \xrightarrow{l_\sigma} S^{2r,r}_{\oplus}\mathscr{C}[-]$$

defined by $l_\sigma = \mathscr{C}(-) \circ l'_\sigma$, and if both $\sigma, \tau \in \Sigma_r$ then the coherence theorem for symmetric monoidal categories shows that the following diagram commutes.

$$\begin{tikzcd}
S^{2r,r}_{\oplus}\mathscr{C}[-] \circ r_\tau^{\mathrm{op}} \circ r_\sigma^{\mathrm{op}} \arrow[r, equal] \arrow[d, "l_\tau \circ r_\sigma^{\mathrm{op}}"] & S^{2r,r}_{\oplus}\mathscr{C}[-] \circ r_{\sigma\tau}^{\mathrm{op}} \arrow[d, "l_{\sigma\tau}"] \\
S^{2r,r}_{\oplus}\mathscr{C}[-] \circ r_\sigma^{\mathrm{op}} \arrow[r, "l_\sigma"] & S^{2r,r}_{\oplus}\mathscr{C}[-]
\end{tikzcd}$$

The following result is an immediate consequence.

Lemma 5.3. *Let $(\mathscr{C}, w\mathscr{C}, D, 0)$ be a pointed exact category with weak equivalences and strict duality and let r be a positive integer. The symmetric group Σ_r acts from the left on the diagonal real simplicial category $S^{2r,r}_{\oplus}\mathscr{C}[-] \circ \Delta^{\mathrm{op}}$ with $\sigma \in \Sigma_r$ acting through the pointed duality preserving exact functor l_σ.*

We define the rth space in the real direct sum K-theory spectrum to be the pointed real space given by the geometric realization

$$KR^{\oplus}(\mathscr{C}, D)_r = |N(wS^{2r,r}_{\oplus}[-], D[-], 0[-])[-]|_R$$

of the $(r+1)$-real simplicial pointed set given by the real nerve of the r-real simplicial pointed category with strict duality $(wS^{2r,r}_{\oplus}\mathscr{C}[-], D[-], 0[-])$. By Lemma 5.3, there is a left Σ_r-action on $KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_r$ with $\sigma \in \Sigma_r$ acting through the pointed real map induced by the pointed duality preserving exact functor l_σ.

We define the structure maps

$$\sigma^{\oplus}_{r,s} \colon KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_r \wedge S^{2s,s} \to KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_{r+s}$$

in the real direct sum K-theory spectrum to be the composition

$$\begin{aligned}
&|N(wS^{2r,r}_{\oplus}\mathscr{C}[-], D[-], 0[-])[-]|_R \wedge S^{2,1} \wedge \cdots \wedge S^{2,1} \\
&\xleftarrow{\mathrm{id} \wedge h_s} |N(wS^{2r,r}_{\oplus}\mathscr{C}[-], D[-], 0[-])[-]|_R \wedge \bar{S}^{2,1} \wedge \cdots \wedge \bar{S}^{2,1} \\
&\xrightarrow{d_r \wedge \mathrm{id}} |N(wS^{2r,r}_{\oplus}\mathscr{C}[-], D[-], 0[-])[-]|'_R \wedge \bar{S}^{2,1} \wedge \cdots \wedge \bar{S}^{2,1} \\
&\xrightarrow{j^{\oplus}_{r,s}} |N(wS^{2(r+s),r+s}_{\oplus}\mathscr{C}[-], D[-], 0[-])[-]|'_R \\
&\xleftarrow{d_{r+s}} |N(wS^{2(r+s),r+s}_{\oplus}\mathscr{C}[-], D[-], 0[-])[-]|_R
\end{aligned}$$

where the real map $j^{\oplus}_{r,s}$, which is the only map that is not an isomorphism, is defined as follows. Let $A_0 \to \cdots \to A_p$ be an element of $N(wS^{2r,r}_{\oplus}\mathscr{C}[m], D[m], 0[m])[p]$, let $u_i \in \Delta R[m_i]$ let $v_j \in \Delta R[2]$, and let $w \in \Delta R[p]$. Then

$$\begin{aligned}
&j^{\oplus}_{r,s}(\text{class of } (A_0 \to \cdots \to A_p, u_1, \ldots, u_r, w, v_1, \ldots, v_s)) \\
&\quad = \text{class of } (\tilde{A}_0 \to \cdots \to \tilde{A}_p, u_1, \ldots, u_r, v_1, \ldots, v_s, w),
\end{aligned}$$

where, if $A_i\colon P(S^{2r,r}[m])^{\mathrm{op}} \to \mathscr{C}$, then $\tilde{A}_i\colon P(S^{2r,r}[m] \wedge S^{2s,s}[2,\dots,2])^{\mathrm{op}} \to \mathscr{C}$ is the unique pointed functor such that

$$\tilde{A}_i(U \wedge \mathrm{id}_{[2]} \wedge \cdots \wedge \mathrm{id}_{[2]}) = A_i(U).$$

It is clear from the definition that $\sigma_{r,s}^{\oplus}$ is $\Sigma_r \times \Sigma_s$-equivariant when $\Sigma_r \times \Sigma_s$ acts from the left on the target through the group homomorphism $+\colon \Sigma_r \times \Sigma_s \to \Sigma_{r+s}$.

Definition 5.4. The real direct sum K-spectrum of the pointed exact category with weak equivalences and strict duality $(\mathscr{C}, w\mathscr{C}, D, 0)$ is the real symmetric spectrum $KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)$ whose rth space is the pointed real space with left Σ_r-action

$$KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_r = |N(wS_{\oplus}^{2r,r}\mathscr{C}[-], D[-], 0[-])[-]|_R$$

and whose structure maps are the $\Sigma_r \times \Sigma_s$-equivariant pointed real maps

$$KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_r \wedge S^{2s,s} \xrightarrow{\sigma_{r,s}^{\oplus}} KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_{r+s}$$

defined above.

Remark 5.5. The functoriality of the real direct sum K-theory spectrum is completely analogous to that of the real algebraic K-theory spectrum. Indeed, the statement and proof of Proposition 3.6 translates verbatim to the current situation upon substituting $S_{\oplus}^{2r,r}$ and $KR^{\oplus}$ for $S^{2r,r}$ and KR.

The following result, proved by Shimakawa [23, Theorem B], shows that the real direct sum K-theory spectrum is a positively fibrant real symmetric spectrum.

Theorem 5.6. *Let $(\mathscr{C}, w\mathscr{C}, D, 0)$ be a pointed exact category with weak equivalences and strict duality. For all positive integers r and s, the adjoint structure map*

$$KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_r \xrightarrow{\tilde{\sigma}_{r,s}^{\oplus}} \Omega^{2s,s} KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_{r+s}$$

is a weak equivalence of pointed real spaces.

For $r = 0$ and $s > 0$, the adjoint structure map typically is not a weak equivalence of real spaces. However, the real homotopy types of the domain and target are related through group-completion as we now explain. We first define a map

$$KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_r \times KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_r \xrightarrow{\mu_r} KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_r$$

in the homotopy category of pointed real spaces as follows. If we set

$$KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_r^{(2)} = |N((w\mathscr{C}(-), D(-), 0(-)) \circ (S^{2r,r}[-] \vee S^{2r,r}[-]))[-]|_R$$

then Corollary 4.11 shows that the pointed real map

$$KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_r^{(2)} \xrightarrow{q_r} KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_r \times KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_r$$

induced by the pointed real maps $p_1, p_2 \colon S^{2r,r}[-] \vee S^{2r,r}[-] \to S^{2r,r}[-]$ that collaps the second and first summands, respectively, is a weak equivalence of pointed real spaces. The fold map $\nabla \colon S^{2r,r}[-] \vee S^{2r,r}[-] \to S^{2r,r}[-]$ also induces a map

$$KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_r^{(2)} \xrightarrow{a_r} KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_r$$

of pointed real spaces, and we define $\mu_r = \gamma(a_r) \circ \gamma(q_r)^{-1}$, where γ is the functor that localizes with respect to the weak equivalences. The map μ_r is the composition law in a monoid structure on $KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_r$ in the homotopy category of pointed real spaces with respect to the cartesian monoidal structure. Moreover, the basepoint is a two-sided identity element. Indeed, if we let

$$KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_r \vee KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0) \xrightarrow{i_r} KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_r^{(2)}$$

be induced by the canonical inclusions $\mathrm{in}_1, \mathrm{in}_2 \colon S^{2r,r}[-] \to S^{2r,r}[-] \vee S^{2r,r}[-]$, then $q_r \circ i_r$ is the canonical map from the coproduct to the product and $a_r \circ i_r$ is the fold map. Finally, the adjoint structure map $\tilde{\sigma}_{r,s}^{\oplus}$ is a map of monoids with the monoid structures on the domain and target given by μ_r and $\Omega^{2s,s}\mu_{r+s}$, respectively.

We recall that for $s > 0$, the monoid structure on the target of $\tilde{\sigma}_{r,s}^{\oplus}$ is an abelian group structure. Indeed, considered as an object of the homotopy category of real spaces, the sphere $S^{2,1}$ has the structure of a cogroup object with respect to the cocartesian monoidal structure in which the comultiplication is represented by the pointed real map $\psi_1 \colon S^{2,1} \to S^{2,1} \vee S^{2,1}$ defined by

$$\psi_1(x+iy) = \begin{cases} \mathrm{in}_1(x - x^{-1} + iy) & \text{if } x < 0 \\ \mathrm{in}_2(x - x^{-1} + iy) & \text{if } x > 0 \\ \infty & \text{if } x = 0. \end{cases}$$

More generally, the sphere $S^{2s,s}$, considered as an object of the homotopy category of real spaces, has a cogroup structure with the comultiplication represented by the pointed real map $\psi_s \colon S^{2s,s} \to S^{2s,s} \vee S^{2s,s}$ defined by the composite

$$S^{2s,s} = S^{2,1} \wedge S^{2(s-1),s-1} \xrightarrow{\psi_1 \wedge \mathrm{id}} (S^{2,1} \vee S^{2,1}) \wedge S^{2(s-1),s-1} \longrightarrow S^{2s,s} \vee S^{2s,s}$$

where the right-hand map is the canonical isomorphism. Now, the two composition laws on the target of $\tilde{\sigma}_{r,s}^{\oplus}$ induced by ψ_s and μ_{r+s} are mutually distributive and both have the basepoint as two-sided identity element. This implies that the composition laws are equal and commutative; see [24, Theorem 1.6.8].

Let k be a commutative ring, and let M be a monoid object in the homotopy category of pointed real spaces with respect to the cartesian monoidal structure. For every subgroup $H \subset G_{\mathbb{R}}$, the homology $H_*(M^H, k)$ is a graded k-algebra with respect to the Pontryagin product, and the set of components $\pi_0(M^H)$ is a monoid which we view as a multiplicative subset of the Pontryagin algebra. If M is a group object, then $\pi_0(M^H)$ is a group. The following result is the real group-completion theorem.

Theorem 5.7. *Let k be a commutative ring, and let $(\mathscr{C}, w\mathscr{C}, D, 0)$ be a pointed exact category with weak equivalences and strict duality. For every positive integer s and for every subgroup $H \subset G_{\mathbb{R}}$, the map induced by the adjoint structure map*

$$\begin{aligned}&H_*((KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_0)^H, k)[\pi_0((KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_0)^H)^{-1}]\\ &\xrightarrow{\tilde{\sigma}^{\oplus}_{0,s*}} H_*((\Omega^{2s,s} KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0)_s)^H, k)\end{aligned}$$

is an isomorphism of graded k-algebras.

The rest of this section is devoted to the proof of Theorem 5.7.

Remark 5.8. The map μ_r in the homotopy category of pointed real spaces defines a commutative monoid structure on the object $KR_r = KR(\mathscr{C}, w\mathscr{C}, D, 0)_r$. Indeed, the following diagram in the category of pointed real spaces, commutes, since $p_1 \circ t = p_2$ and $\nabla \circ t = \nabla$ as maps from $S^{2r,r}[-] \vee S^{2r,r}[-]$ to $S^{2r,r}[-]$.

$$\begin{tikzcd} KR_r \times KR_r \arrow[d, "t"] & KR_r^{(2)} \arrow[l, "q_r"'] \arrow[r, "a_r"] \arrow[d, "t_*"] & KR_r \arrow[d, equal] \\ KR_r \times KR_r & KR_r^{(2)} \arrow[l, "q_r"'] \arrow[r, "a_r"] & KR_r \end{tikzcd}$$

It follows that the Pontryagin ring $H_*((KR_r)^H)$ is an anti-symmetric graded ring with the multiplicative subset $\pi_0((KR_r)^H)$ contained in the center. Hence, the localization $H_*((KR_r)^H)[(\pi_0((KR_r)^H))^{-1}]$ admits calculation by right fractions.

Definition 5.9. A strict sum on the pointed exact category with weak equivalence and strict duality $(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})$ is a pointed exact duality preserving functor

$$(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0}) \times (\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0}) \xrightarrow{\oplus} (\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})$$

such that the following (i)–(iii) hold.

(i) For all $\tilde{A} \in \operatorname{ob}\tilde{\mathscr{C}}$, $\tilde{0}(1) \oplus \tilde{A} = \tilde{A} = \tilde{A} \oplus \tilde{0}(1)$.
(ii) For all $\tilde{A}, \tilde{B}, \tilde{C} \in \tilde{\mathscr{C}}$, $(\tilde{A} \oplus \tilde{B}) \oplus \tilde{C} = \tilde{A} \oplus (\tilde{B} \oplus \tilde{C})$.
(iii) For all $\tilde{A}, \tilde{B} \in \operatorname{ob}\tilde{\mathscr{C}}$, the morphisms

$$\tilde{A} = \tilde{A} \oplus \tilde{0}(1) \xrightarrow{\mathrm{id}\oplus 0} \tilde{A} \oplus \tilde{B} \xleftarrow{0\oplus\mathrm{id}} \tilde{0}(1) \oplus \tilde{B} = \tilde{B}$$

form a sum in $\tilde{\mathscr{C}}$ of $\tilde{A}$ and $\tilde{B}$.

We first show that a structure of strict sum on $(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})$ makes it possible to lift the monoid structure on $KR(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})_r$ from the homotopy category of pointed real spaces to the category of pointed real spaces.

Lemma 5.10. *A strict sum on the pointed exact category with weak equivalences and strict duality $(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})$ gives rise to a map of pointed real spaces*

$$KR^{\oplus}(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})_r \times KR^{\oplus}(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})_r \xrightarrow{m_r} KR^{\oplus}(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})_r$$

which is the composition law of a monoid structure on $KR^{\oplus}(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})_r$ in the category of pointed real spaces with respect to the cartesian monoidal structure and which satisfies that $\gamma(m_r) = \mu_r$. Here r is a non-negative integer.

Proof. We define the pointed real map m_r to be the composition of the inverse of the canonical pointed real homeomorphism from the realization of the product of two $(r+1)$-real simplicial pointed real sets to the product of their realizations and the pointed real map induced by the strict sum functor $\oplus$. It follows from Proposition 4.4 and Definition 5.9 (iii) that m_r is well-defined and from loc. cit. (i)–(ii) that it is the composition law of a monoid structure on $KR^{\oplus}(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})_r$ in the cartesian category of pointed real spaces.

To prove that $\gamma(m_r) = \mu_r$, we consider the pointed real map

$$KR^{\oplus}(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})_r^{(2)} \times KR^{\oplus}(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})_r^{(2)} \xrightarrow{m_r^{(2)}} KR^{\oplus}(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})_r^{(2)}$$

defined by substituting $S^{2r,r}[-] \vee S^{2r,r}[-]$ for $S^{2r,r}[-]$ in the definition of m_r. For every map of r-real simplicial pointed real sets $f\colon S^{2r,r}[-] \vee S^{2r,r}[-] \to S^{2r,r}$, the diagram

$$\begin{array}{ccc}
KR^{\oplus}(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})_r^{(2)} \times KR^{\oplus}(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})_r^{(2)} & \xrightarrow{m_r^{(2)}} & KR^{\oplus}(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})_r^{(2)} \\
\downarrow{\scriptstyle f_* \times f_*} & & \downarrow{\scriptstyle f_*} \\
KR^{\oplus}(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})_r \times KR^{\oplus}(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})_r & \xrightarrow{m_r} & KR^{\oplus}(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})_r
\end{array}$$

commutes, and we define

$$KR^{\oplus}(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})_r \times KR^{\oplus}(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})_r \xrightarrow{s_r} KR^{\oplus}(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})_r^{(2)}$$

to be the pointed real map $s_r = m_r^{(2)} \circ (\mathrm{in}_{1*} \times \mathrm{in}_{2*})$. From the calculation

$$\begin{aligned}
p_{1*} \circ s_r &= p_{1*} \circ m_r^{(2)} \circ (\mathrm{in}_{1*} \times \mathrm{in}_{2*}) = m_r \circ ((p_{1*} \circ \mathrm{in}_{1*}) \times (p_{1*} \circ \mathrm{in}_{2*})) \\
&= m_r \circ (\mathrm{id} \times \tilde{0}) = \mathrm{pr}_1 \\
p_{2*} \circ s_r &= p_{2*} \circ m_r^{(2)} \circ (\mathrm{in}_{1*} \times \mathrm{in}_{2*}) = m_r \circ ((p_{2*} \circ \mathrm{in}_{1*}) \times (p_{2*} \circ \mathrm{in}_{2*})) \\
&= m_r \circ (\tilde{0} \times \mathrm{id}) = \mathrm{pr}_2,
\end{aligned}$$

where $\tilde{0}$ is the constant map, we find that

$$q_r \circ s_r = (p_{1*}, p_{2*}) \circ s_r = (\mathrm{pr}_1, \mathrm{pr}_2) = \mathrm{id}$$

which shows that s_r is a section of q_r. In particular, we have $\gamma(s_r) = \gamma(q_r)^{-1}$, and hence, the calculation

$$a_r \circ s_r = \nabla_* \circ m_r^{(2)} \circ (\mathrm{in}_{1_*} \times \mathrm{in}_{2*}) = m_r \circ ((\nabla_* \circ \mathrm{in}_{1*}) \times (\nabla_* \circ \mathrm{in}_{2*})) = m_r$$

shows that

$$\gamma(m_r) = \gamma(a_r \circ s_r) = \gamma(a_r) \circ \gamma(s_r) = \gamma(a_r) \circ \gamma(q_r)^{-1} = \mu_r$$

as desired. This completes the proof. $\square$

Following [13, Proposition 4.2], we next prove the that, up to equivalence in the appropriate sense, every pointed exact category with weak equivalences and strict duality admits a strict sum.

Proposition 5.11. *Let $(\mathscr{C}, w\mathscr{C}, D, 0)$ be a pointed exact category with weak equivalences and strict duality. There exists a pointed exact category with weak equivalences and strict duality $(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})$ equipped with a strict sum*

$$(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0}) \times (\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0}) \xrightarrow{\oplus} (\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})$$

and a pointed exact adjunction $(F, G, \eta, 1, f, 1)$ from $(\tilde{\mathscr{C}}, w\tilde{\mathscr{C}}, \tilde{D}, \tilde{0})$ to $(\mathscr{C}, w\mathscr{C}, D, 0)$ such that the natural transformations η and f both are natural isomorphisms.

Proof. We first define the category $\tilde{\mathscr{C}}$ and the functor $F: \tilde{\mathscr{C}} \to \mathscr{C}$. We let $\operatorname{ob}\tilde{\mathscr{C}}$ be the set of all tuples $\tilde{A} = (A_1, \ldots, A_n)$ of objects in $\mathscr{C}$ and let $F\colon \operatorname{ob}\tilde{\mathscr{C}} \to \operatorname{ob}\mathscr{C}$ be the map that takes the 0-tuple $(\ )$ to the nullobject $0(1)$, takes the 1-tuple (A) to the object A, and takes the n-tuple $(A_1, \ldots, A_n)$ with $n \geqslant 2$ to an arbitrary choice of sum in $\mathscr{C}$ of the objects $A_1, \ldots, A_n$. We define the morphism sets in $\tilde{\mathscr{C}}$ by

$$\tilde{\mathscr{C}}(\tilde{A}, \tilde{B}) = \{\tilde{B}\} \times \mathscr{C}(F(\tilde{A}), F(\tilde{B}) \times \{\tilde{A}\},$$

where the two singleton sets are included in order to ensure that the morphism sets corresponding to distinct pairs of objects are disjoint, and define

$$\tilde{\mathscr{C}}(\tilde{A}, \tilde{B}) \xrightarrow{F} \mathscr{C}(F(\tilde{A}), F(\tilde{B}))$$

to be the projection onto the middle factor. The identity and composition in $\tilde{\mathscr{C}}$ are defined in the unique way that renders F a functor. We note that with these defintions $\tilde{\mathscr{C}}$ is an additive category and F an additive functor.

We next define $G\colon \mathscr{C} \to \tilde{\mathscr{C}}$ to be the unique functor with $F \circ G = \mathrm{id}_{\mathscr{C}}$ that on object sets is given by the map $G\colon \operatorname{ob}\mathscr{C} \to \operatorname{ob}\tilde{\mathscr{C}}$ defined by

$$G(A) = \begin{cases} (\ ) & \text{if } A = 0(1) \\ (A) & \text{if } A \neq 0(1); \end{cases}$$

it is an additive functor. The functor $G \circ F$ takes $(\ )$ to $(\ )$ and takes $(A_1, \ldots, A_n)$ with $n \geqslant 1$ to $(F((A_1, \ldots, A_n)))$. We note that $(A_1, \ldots, A_n)$ and $(F((A_1, \ldots, A_n)))$ both are sums in $\tilde{\mathscr{C}}$ of the objects $(A_1), \ldots, (A_n)$. The unit $\eta\colon \mathrm{id}_{\tilde{\mathscr{C}}} \Rightarrow G \circ F$ is defined to be the unique natural isomorphism between two choices of sum of the same objects, and the counit $\varepsilon\colon F \circ G \Rightarrow \mathrm{id}_{\mathscr{C}}$ is defined to be the identity natural isomorphism.

We define $\tilde{A} \to \tilde{B} \to \tilde{C}$ to be an exact sequence in $\tilde{\mathscr{C}}$ if $F(\tilde{A}) \to F(\tilde{B}) \to F(\tilde{C})$ is an exact sequence in $\mathscr{C}$ and define $\tilde{A} \to \tilde{B}$ to be in $w\tilde{\mathscr{C}}$ if $F(\tilde{A}) \to F(\tilde{B})$ is in $w\mathscr{C}$. With these definitions, the functors F and G are exact.

We proceed to define the duality functor $\tilde{D}\colon \tilde{\mathscr{C}}^{\mathrm{op}} \to \tilde{\mathscr{C}}$ together with the natural isomorphism $f\colon F \Rightarrow (\tilde{D}, D)F^{\mathrm{op}}$. We define

$$\operatorname{ob}\tilde{\mathscr{C}}^{\mathrm{op}} \xrightarrow{\tilde{D}} \operatorname{ob}\tilde{\mathscr{C}}$$

to be the map that to $(A_1,\dots,A_n)^{\mathrm{op}}$ associates $(D(A_1^{\mathrm{op}}),\dots,D(A_n^{\mathrm{op}}))$ and define

$$F(\tilde{A}) \xrightarrow{f_{\tilde{A}}} D(F^{\mathrm{op}}(\tilde{D}^{\mathrm{op}}(\tilde{A})))$$

with $\tilde{A} = (A_1,\dots,A_n)$ to be the unique isomorphism between two choices of sum in $\mathscr{C}$ of the objects $A_1,\dots,A_n$. Since the morphisms $f_{\tilde{A}}$ are isomorphisms and since the functor F is fully faithful, there is a unique way to define a collection of maps

$$\tilde{\mathscr{C}}^{\mathrm{op}}(\tilde{A}^{\mathrm{op}},\tilde{B}^{\mathrm{op}}) \xrightarrow{\tilde{D}} \tilde{\mathscr{C}}(\tilde{D}(\tilde{A}^{\mathrm{op}}),\tilde{D}(\tilde{B}^{\mathrm{op}}))$$

such that $\tilde{D}\colon \tilde{\mathscr{C}}^{\mathrm{op}} \to \tilde{\mathscr{C}}$ is a functor and the collection of isomorphisms $f_{\tilde{A}}$ is a natural isomorphism $f\colon F \Rightarrow (\tilde{D},D)F^{\mathrm{op}}$. We have $f = (\tilde{D},D)f^{\mathrm{op}}$ by the uniqueness of the isomorphism between two choices of sum of the same family of objects. We define the basepoint $\tilde{0}\colon (\mathbf{1},\mathbf{1},\mathrm{id}) \to (\tilde{\mathscr{C}},w\tilde{\mathscr{C}},\tilde{D})$ by $\tilde{0}(1) = (\ )$. It is an exact duality preserving functor. Finally, in the case at hand, all morphisms in the two diagrams in Definition 1.27 are the unique isomorphisms between possibly different choices of sum of the same family of objects. Hence, the diagrams necessarily commute.

Finally, the strict sum functor $\oplus$ is defined, on objects, by

$$(A_1,\dots,A_m)\oplus(B_1,\dots,B_n) = (A_1,\dots,A_m,B_1,\dots,B_n).$$

The object $\tilde{A}\oplus\tilde{B}$ is a preferred choice of sum in $\tilde{\mathscr{C}}$ of the objects $\tilde{A}$ and $\tilde{B}$ and we define the functor $\oplus$ on morphism sets to be the categorical sum of morphisms. It is clear that $\oplus$ is a strict sum in the sense of Definition 5.9. □

Remark 5.12. If M is a (κ-small) category such that $\mathrm{ob}(M) = \{\emptyset\}$, then the triple of the set $M(\emptyset,\emptyset)$, the composition map $\circ\colon M(\emptyset,\emptyset)\times M(\emptyset,\emptyset) \to M(\emptyset)$, and the identity element $\mathrm{id}_\emptyset \in M(\emptyset,\emptyset)$ is a (κ-small) monoid, and every (κ-small) monoid is of this form. We describe the nerve $N(M)[-]$ explicitly as follows. The set $N(M)[m]$ consists of all functors $a\colon [m] \to M$. Moreover, the map $\theta^*\colon N(M)[m] \to N(M)[l]$ induced by the functor $\theta\colon [l] \to [m]$ takes the functor $a\colon [m] \to M$ given by the diagram

$$\emptyset \xleftarrow{a_1} \emptyset \xleftarrow{a_2} \ \dots\ \xleftarrow{a_m} \emptyset$$

to the functor $b = a\circ\theta\colon [l] \to M$ given by the diagram

$$\emptyset \xleftarrow{b_1} \emptyset \xleftarrow{b_2} \ \dots\ \xleftarrow{b_l} \emptyset$$

whose components $b_1,\dots,b_l$ are given as follows. We let $I_1,\dots,I_l$ be the pairwise disjoint intervals in the target of the object map $\theta\colon \mathrm{ob}[l] \to \mathrm{ob}[m]$ defined by

$$I_i = \{j \in \mathrm{ob}[m] \mid \theta(i-1) < j \leqslant \theta(i)\}.$$

The intervals need not be non-empty and they need not partition $\mathrm{ob}[m]$. Now, if the interval $I_i = \{j_1,\dots,j_r\}$ with $j_1 < \dots < j_r$ is non-empty, then $b_i = a_{j_1}\circ\dots\circ a_{j_r}$, and if I_i is the empty interval, then $b_i = \mathrm{id}_\emptyset$.

We recall that the intervals $I_1,\dots,I_l$ above have the following interpretation in terms of the simplicial circle $S^1[-] = \Delta[1][-]/\partial\Delta[1][-]$. The complement of the basepoint in $S^1[m]$ consists of the m elements $\bar{\sigma}_{1,m},\dots,\bar{\sigma}_{m,m}$, where $\bar{\sigma}_{j,m}=\{\sigma_{j,m}\}$ is the class of the functor $\sigma_{j,m}\colon [m]\to[1]$ defined by

$$\sigma_{j,m}(s)=\begin{cases}0 & \text{if } s<j\\ 1 & \text{if } s\geqslant j.\end{cases}$$

Now, the pointed map $\theta^*\colon S^1[m]\to S^1[l]$ is characterized as follows: For all $1\leqslant i\leqslant l$,

$$(\theta^*)^{-1}(\bar{\sigma}_{i,l})=\{\bar{\sigma}_{j,m}\mid j\in I_i\}.$$

A strict sum on the pointed exact category with weak equivalences and strict duality $(\tilde{\mathscr{C}},w\tilde{\mathscr{C}},\tilde{D},\tilde{0})$ gives rise to the map of pointed real simplicial sets

$$N(w\tilde{\mathscr{C}},\tilde{D},\tilde{0})[-]\times N(w\tilde{\mathscr{C}},\tilde{D},\tilde{0})[-]\xrightarrow{\oplus[-]} N(w\tilde{\mathscr{C}},\tilde{D},\tilde{0})[-]$$

that to the pair $(\tilde{A},\tilde{B})$ of functors from $[n]$ to $w\tilde{\mathscr{C}}$ assigns the composite functor

$$[n]\xrightarrow{\Delta}[n]\times[n]\xrightarrow{\tilde{A}\times\tilde{B}} w\tilde{\mathscr{C}}\times w\tilde{\mathscr{C}}\xrightarrow{\oplus} w\tilde{\mathscr{C}}.$$

It is the composition law of a monoid structure with respect to the cartesian monoidal structure on the object $N(w\tilde{\mathscr{C}},\tilde{D},\tilde{0})[-]$ of the category of (κ-small) real simplicial pointed real sets. The identity element for the monoid structure is the basepoint $\tilde{0}[-]$. The nerve of this monoid as defined in Remark 5.12 gives rise to the simplicial real simplicial pointed real set

$$N(N(w\tilde{\mathscr{C}},\tilde{D},\tilde{0})[-],\oplus[-],\tilde{0}[-])[-].$$

We define the geometric realization of the simplicial real simplicial set $X[-,-]$ to be the real space, pointed if $X[-,-]$ is, given by

$$|X[-,-]|_R=|[m]\mapsto|[n]\mapsto X[m,n]|_R|.$$

Lemma 5.13. *Let $(\tilde{\mathscr{C}},w\tilde{\mathscr{C}},\tilde{D},\tilde{0})$ be a pointed exact category with weak equivalences and strict duality equipped with the strict sum*

$$(\tilde{\mathscr{C}},w\tilde{\mathscr{C}},\tilde{D},\tilde{0})\times(\tilde{\mathscr{C}},w\tilde{\mathscr{C}},\tilde{D},\tilde{0})\xrightarrow{\oplus}(\tilde{\mathscr{C}},w\tilde{\mathscr{C}},\tilde{D},\tilde{0}).$$

There is a canonical weak equivalence of pointed real spaces

$$\begin{aligned}&|N(N(w\tilde{\mathscr{C}},\tilde{D},\tilde{0})[-],\oplus[-],\tilde{0}[-])[-]|_R\\ &\xrightarrow{i_!'}|N(w\tilde{\mathscr{C}}(S^1[-]),\tilde{D}(S^1[-]),\tilde{0}(S^1[-]))[-]|_R.\end{aligned}$$

Proof. We define the map $i'_!$ as the composition of two pointed real maps

$$\begin{aligned}
&|N(N(w\tilde{\mathscr{C}},\tilde{D},\tilde{0})[-],\oplus[-],\tilde{0}[-])[-]|_R \\
&\xrightarrow{u} |N\,\mathrm{Cat\,Dual}_*(S^1[-],(w\tilde{\mathscr{C}},\tilde{D},\tilde{0}))[-]|_R \\
&\xrightarrow{i_!} |N(w\tilde{\mathscr{C}}(S^1[-]),\tilde{D}(S^1[-]),\tilde{0}(S^1[-]))[-]|_R,
\end{aligned}$$

each of which is the realization of a map of simplicial real simplicial pointed real sets, and begin by defining the middle simplicial real simplicial pointed set. More generally, let (X,x) and (Y,y) be two finite pointed sets equipped with total orderings of the complements of the respective basepoints, say, $X \smallsetminus \{x\} = \{x_1,\dots,x_m\}$ and $Y \smallsetminus \{y\} = \{y_1,\dots,y_l\}$, and let $f\colon (X,x) \to (Y,y)$ be a pointed map that is pointed order preserving in the sense that its restriction to $f^{-1}(Y \smallsetminus \{y\})$ is order preserving. In this situation, we define the pointed functor

$$\mathrm{Cat\,Dual}_*((X,x),(w\tilde{\mathscr{C}},\tilde{D},\tilde{0})) \xrightarrow{f_*} \mathrm{Cat\,Dual}_*((Y,y),(w\tilde{\mathscr{C}},\tilde{D},\tilde{0}))$$

as follows. If $f^{-1}(y_i) = \{x_{j_1},\dots,x_{j_r}\}$ with $j_1 < \dots < j_r$ is non-empty, then

$$f_*(\tilde{B})(y_i) = \tilde{B}(x_{j_1}) \oplus \dots \oplus \tilde{B}(x_{j_r})$$

with the strict sum on the right-hand side taken in the indicated order; and if $f^{-1}(y_i)$ is empty, then $f_*(\tilde{B})(y_i) = \tilde{0}(1)$. The functor f_* is duality preserving, since (X,x) and (Y,y) have trivial real structures. Moreover, if also $g\colon (Y,y) \to (Z,z)$ is a pointed order preserving pointed map, then we have $(g \circ f)_* = g_* \circ f_*$.

In the case at hand, we give the complement of the basepoint in $S^1[m]$ the total ordering such that $\bar{\sigma}_{1,m} < \dots < \bar{\sigma}_{m,m}$; see Remark 5.12 for the definition of $\bar{\sigma}_{j,m}$. One verifies that for every functor $\theta\colon [l] \to [m]$, the pointed map $\theta^*\colon S^1[m] \to S^1[l]$ is pointed order preserving; whence, we obtain the simplicial pointed category with strict duality $\mathrm{Cat\,Dual}_*(S^1[-],(w\tilde{\mathscr{C}},\tilde{D},\tilde{0}))$, and the simplicial real simplicial pointed set $N\,\mathrm{Cat\,Dual}_*(S^1[-],(w\tilde{\mathscr{C}},\tilde{D},\tilde{0}))[-]$ is the degreewise real nerve.

We next define the pointed real map u and prove that it is an isomorphism. In fact, it readily follows from Remark 5.12 that there is an isomorphism of simplicial real simplicial pointed sets

$$N(N(w\tilde{\mathscr{C}},\tilde{D},\tilde{0})[-],\oplus[-],\tilde{0}[-])[-] \xrightarrow{u[-,-]} N\,\mathrm{Cat\,Dual}_*(S^1[-],(w\tilde{\mathscr{C}},\tilde{D},\tilde{0}))[-]$$

that, with $\tilde{A}_1,\dots,\tilde{A}_m \in N(w\tilde{\mathscr{C}},\tilde{D},\tilde{0})[n]$, is defined by

$$u[m,n](\ \emptyset \xleftarrow{\tilde{A}_1} \ \dots \ \xleftarrow{\tilde{A}_m} \emptyset\ )(\bar{\sigma}_{j,m}) = \tilde{A}_j.$$

Accordingly, the induced map of the geometric realizations, $u = |u[-,-]|_R$, is an isomorphism of pointed real spaces as desired.

It remains to define the map $i_!$ and prove that it is a weak equivalence of real pointed spaces. To this end, we recall the pointed adjunction $(i_!,i^*,\eta,\varepsilon)$ given by

Lemma 4.9. Let again (X,x) be a finite pointed set equipped a total ordering of the complement of the basepoint, say, $X \smallsetminus \{x\} = \{x_1,\dots,x_m\}$. We define

$$\operatorname{Cat}\operatorname{Dual}_*((X,x),(w\tilde{\mathscr{C}},\tilde{D},\tilde{0})) \xrightarrow{i_!} (w\tilde{\mathscr{C}}(X,x),\tilde{D}(X,x),\tilde{0}(X,x))$$

as in the proof of Lemma 4.9, making the following particular choice of the sum that defines $i_!(\tilde{B})(U)$, where U is an object of $P(X,x)$. If $U \smallsetminus \{x\} = \{x_{j_1},\dots,x_{j_r}\}$ with $j_1 < \dots < j_r$ non-empty, then we define

$$i_!(\tilde{B})(U) = \tilde{B}(x_{j_1}) \oplus \tilde{B}(x_{j_2}) \oplus \dots \oplus \tilde{B}(x_{j_r}),$$

where the strict sum is formed in the indicated order; and if $U = \{x\}$, then we define $i_!(\tilde{B})(\{x\}) = \tilde{0}(1)$ as before. With this definition, the functor $i_!$ is duality preserving. Moreover, if also (Y,y) is a finite pointed set equipped with a total ordering of the complement of the basepoint, and if $f\colon (X,x) \to (Y,y)$ is a pointed order preserving pointed map, then the following diagram commutes.

$$\begin{array}{ccc}
\operatorname{Cat}\operatorname{Dual}_*((X,x),(w\tilde{\mathscr{C}},\tilde{D},\tilde{0})) & \xrightarrow{i_!} & (w\tilde{\mathscr{C}}(X,x),\tilde{D}(X,x),\tilde{0}(X,x)) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
\operatorname{Cat}\operatorname{Dual}_*((Y,y),(w\tilde{\mathscr{C}},\tilde{D},\tilde{0})) & \xrightarrow{i_!} & (w\tilde{\mathscr{C}}(Y,y),\tilde{D}(Y,y),\tilde{0}(Y,y))
\end{array}$$

It follows that, in the case at hand, the functor $i_!$ gives rise to a map of simplicial real simplicial pointed sets

$$N\operatorname{Cat}\operatorname{Dual}_*(S^1[-],(w\tilde{\mathscr{C}},\tilde{D},\tilde{0}))[-] \xrightarrow{i_!} N(w\tilde{\mathscr{C}}(S^1[-]),\tilde{D}(S^1[-]),\tilde{0}(S^1[-]))[-],$$

and Corollary 4.11 shows that the induced map of geometric realizations is a weak equivalence of pointed real spaces. This completes the proof. $\square$

Proof of Theorem 5.7. By Theorem 5.6, we may assume that $s = 1$. We construct a commutative diagram of pointed real spaces

$$\begin{array}{ccc}
KR^{\oplus}(\mathscr{C},w\mathscr{C},D,0)_0 & \xrightarrow{\tilde{\sigma}^{\oplus}_{0,1}} & \Omega^{2,1}KR^{\oplus}(\mathscr{C},w\mathscr{C},D,0)_1 \\
\downarrow{\scriptstyle \tilde{\sigma}^{\oplus}_{(0,0),(1,0)}} & & \downarrow{\scriptstyle \tilde{\sigma}_{(0,1),(1,0)}} \\
\Omega^{1,0}KR^{\oplus}(\mathscr{C},w\mathscr{C},D,0)_{1,0} & \xrightarrow{\tilde{\sigma}_{(1,0),(0,1)}} & \Omega^{3,1}KR^{\oplus}(\mathscr{C},w\mathscr{C},D,0)_{1,1}
\end{array}$$

with $KR^{\oplus}(\mathscr{C},w\mathscr{C},D,0)_{1,r}$ defined analogously to $KR^{\oplus}(\mathscr{C},w\mathscr{C},D,0)_r$, substituting the simplicial r-real simplicial finite pointed set $S^{2r+1,r}[-] = S^1[-] \wedge S^{2r,r}[-]$ for the r-simplicial finite pointed set $S^{2r,r}[-]$. The right-hand vertical map and the lower horizontal map are both weak equivalences of pointed real spaces by the theorem of Shimakawa [23, Theorem B]. Hence, it suffice to show that the map of graded k-algebras

$$\begin{aligned}
&H_*((KR^{\oplus}(\mathscr{C},w\mathscr{C},D,0)_0)^H,k)[\pi_0((KR^{\oplus}(\mathscr{C},w\mathscr{C},D,0)_0)^H)^{-1}]\\
&\longrightarrow H_*((\Omega^{1,0}KR^{\oplus}(\mathscr{C},w\mathscr{C},D,0)_{(1,0)})^H;k)\\
&=\!=\!= H_*(\Omega^1((KR^{\oplus}(\mathscr{C},w\mathscr{C},D,0)_{(1,0)})^H);k)
\end{aligned}$$

induced adjoint structure map $\tilde{\sigma}_{(0,0),(1,0)}$ is an isomorphism. This, in turn, follows from Quillen [6, Appendix] as we now explain. We consider the following commutative diagram.

$$\begin{tikzcd}
KR^{\oplus}(\mathscr{C},w\mathscr{C},D,0)_0 \arrow[r, "\tilde{\sigma}_{(0,0),(1,0)}"] & \Omega^{1,0}KR^{\oplus}(\mathscr{C},w\mathscr{C},D,0)_{(1,0)} \\
KR^{\oplus}(\tilde{\mathscr{C}},w\tilde{\mathscr{C}},\tilde{D},\tilde{0})_0 \arrow[u, "(F,f)_*"] \arrow[r, "\tilde{\sigma}_{(0,0),(1,0)}"] \arrow[d, equal] & \Omega^{1,0}KR^{\oplus}(\tilde{\mathscr{C}},w\tilde{\mathscr{C}},\tilde{D},\tilde{0})_{(1,0)} \arrow[u, "\Omega^{1,0}(F,f)_*"'] \\
KR^{\oplus}(\tilde{\mathscr{C}},w\tilde{\mathscr{C}},\tilde{D},\tilde{0})_0 \arrow[r, "\tilde{\sigma}"] & \Omega^{1,0}BKR^{\oplus}(\tilde{\mathscr{C}},w\tilde{\mathscr{C}},\tilde{D},\tilde{0})_0 \arrow[u, "\Omega^{1,0}i'_!"']
\end{tikzcd}$$

Here $(\tilde{\mathscr{C}},w\tilde{\mathscr{C}},\tilde{D},\tilde{0})$ and (F,f) are as in Proposition 5.11; hence, Remark 5.5 shows that the top vertical maps are weak equivalences of pointed real spaces. The lower right-hand term is the loop space of the pointed real space

$$BKR^{\oplus}(\tilde{\mathscr{C}},w\tilde{\mathscr{C}},\tilde{D},\tilde{0})_0 = |N(N(w\tilde{\mathscr{C}},\tilde{D},\tilde{0})[-],\oplus[-],\tilde{0}[-])[-]|,$$

and Lemma 5.13 shows that the lower right-hand vertical map is a weak equivalence of pointed real spaces. Therefore, it suffices to show that the map of graded k-algebras

$$\begin{aligned}
&H_*((KR^{\oplus}(\mathscr{C},w\mathscr{C},D,0)_0)^H,k)[\pi_0((KR^{\oplus}(\mathscr{C},w\mathscr{C},D,0)_0)^H)^{-1}]\\
&\longrightarrow H_*((\Omega^{1,0}BKR^{\oplus}(\mathscr{C},w\mathscr{C},D,0)_0)^H,k)\\
&=\!=\!= H_*(\Omega^1(BKR^{\oplus}(\mathscr{C},w\mathscr{C},D,0)_0)^H,k)
\end{aligned}$$

induced by the lower horizontal map $\tilde{\sigma}$ is an isomorphism. Now, if $H=\{1\}$, then this follows from the group-completion theorem in [6, Appendix] applied to the simplicial monoid $(N(w\tilde{\mathscr{C}})[-],\oplus[-],\tilde{0}[-])$; and if $H=G_{\mathbb{R}}$, then it follows from loc. cit. applied to the simplicial monoid $(N\,\mathrm{Sym}(w\tilde{\mathscr{C}},\tilde{D})[-],\mathrm{Sym}\oplus[-],\mathrm{Sym}\,\tilde{0}[-])$. This completes the proof. □

Remark 5.14. Let $(\mathscr{C},w\mathscr{C},D,0)$ be a pointed exact category with weak equivalences and strict duality, and let $H\subset G_{\mathbb{R}}$ be a subgroup. The canonical map

$$\pi_0((KR^{\oplus}(\mathscr{C},w\mathscr{C},D,0)_0)^G_{\mathbb{R}}) \longrightarrow KR^{\oplus}_{0,0}(\mathscr{C},w\mathscr{C},D,0)$$

is a group-completion of the domain considered as a commutative monoid with the map $\pi_0((\mu_0)^G_{\mathbb{R}})$ as the composition law and with the basepoint as the identity element.

In addition, Addendum 1.29 identifies the domain with the pointed set of equivalence classes of symmetric spaces in $(w\mathscr{C}, D)$ under the equivalence relation generated by the relation that identifies the symmetric spaces (c_1, f_1) and (c_2, f_2) if there is a map of symmetric spaces in $(w\mathscr{C}, D)$ from (c_1, f_1) to (c_2, f_2).

6 The forgetful map

In this section, we define for every pointed exact category with weak equivalences and strict duality $(\mathscr{C}, w\mathscr{C}, D, 0)$ a real simplicial pointed exact duality preserving functor

$$(S^{2,1}_{\oplus}\mathscr{C}[-], wS^{2,1}_{\oplus}\mathscr{C}[-], D[-], 0[-]) \xrightarrow{\phi^*} (S^{2,1}\mathscr{C}[-], wS^{2,1}\mathscr{C}[-], D[-], 0[-]).$$

from the real Segal construction to the real Waldhausen construction. The functor ϕ^* is given by forgetting a large part of the information contained in the sheaves that constitute the objects in the categories $S^{2,1}_{\oplus}\mathscr{C}[n]$, so we call it the forgetful functor.

We define the functor $\phi\colon \operatorname{Cat}([2],[n]) \to P(S^{2,1}[n])^{\mathrm{op}}$, on objects, by

$$\phi([2] \xrightarrow{\theta} [n]) = \{[n] \xrightarrow{\rho} [2] \mid \rho \circ \theta = \mathrm{id}_{[2]}\} \cup \{\infty[n]\}$$

and, on morphisms, by

$$\phi\Big(\begin{tikzcd}[2] \arrow[r, bend left, "\theta_1"{name=U}] \arrow[r, bend right, "\theta_2"'{name=D}] & {[n]} \arrow[Rightarrow, from=D, to=U]\end{tikzcd}\Big) = \phi([2] \xrightarrow{\theta_1} [n]) \cap \phi([2] \xrightarrow{\theta_2} [n]) \qquad \begin{tikzcd} \phi([2] \xrightarrow{\theta_1} [n]) \arrow[d] \\ \phi([2] \xrightarrow[\theta_2]{} [n]). \end{tikzcd}$$

Here $\infty[n] = \partial \Delta R[2][n]$ is the basepoint of $S^{2,1}[n]$. We note that the complement of the basepoint in $\phi(\theta\colon [2] \to [n])$ has cardinality $(\theta(2) - \theta(1))(\theta(1) - \theta(0))$.

Lemma 6.1. *Let $(\mathscr{C}, w\mathscr{C}, D, 0)$ be a pointed exact category with weak equivalences and strict duality. The functor ϕ induces a real simplicial pointed duality preserving exact functor*

$$(S^{2,1}_{\oplus}\mathscr{C}[-], wS^{2,1}_{\oplus}\mathscr{C}[-], D[-], 0[-]) \xrightarrow{\phi^*} (S^{2,1}\mathscr{C}[-], wS^{2,1}\mathscr{C}[-], D[-], 0[-]).$$

Proof. The functor ϕ induces a functor

$$\operatorname{Cat}_*(P(S^{2,1}[n])^{\mathrm{op}}, (\mathscr{C}, 0)) \xrightarrow{\phi^*} \operatorname{Cat}(\operatorname{Cat}([2],[n]), \mathscr{C}),$$

and we first prove that this functor maps the full subcategory $S^{2,1}_{\oplus}\mathscr{C}[n]$ of the domain to the full subcategory $S^{2,1}\mathscr{C}[n]$ of the target. So we let $B\colon P(S^{2,1}[n])^{\mathrm{op}} \to \mathscr{C}$ be an object of $S^{2,1}_{\oplus}\mathscr{C}[n]$ and show that $A = B \circ \phi\colon \operatorname{Cat}([2],[n]) \to \mathscr{C}$ satisfies (i)–(ii) of Definition 2.3. First, for every functor $\mu\colon [1] \to [n]$, we have directly from the definition of ϕ that $\phi(s_0\mu) = \phi(s_1\mu) = \{*\}$. Since the functor B is pointed, we find that $A(s_0\mu) = A(s_1\mu) = 0(1)$, which shows that A satisfies (i). Next, we wish to show that for every functor $\tau\colon [3] \to [n]$, the sequence

$$B(\phi(d_0\tau)) \longrightarrow B(\phi(d_1\tau)) \longrightarrow B(\phi(d_2\tau)) \longrightarrow B(\phi(d_3\tau))$$

is 4-term exact. Since B is a sheaf, it will suffice to show that $\phi(d_0\tau) \subset \phi(d_1\tau)$, $\phi(d_2\tau) \supset \phi(d_3\tau)$, and $\phi(d_1\tau) \smallsetminus \phi(d_0\tau) = \phi(d_2\tau) \smallsetminus \phi(d_3\tau)$. We have

$$\begin{aligned}
\phi(d_0\tau) &= \{[n] \xrightarrow{\rho} [2] \mid \rho([0,\tau(1)]) = 0, \rho(\tau(2)) = 1, \rho([\tau(3),n]) = 2\}\\
\phi(d_1\tau) &= \{[n] \xrightarrow{\rho} [2] \mid \rho([0,\tau(0)]) = 0, \rho(\tau(2)) = 1, \rho([\tau(3),n]) = 2\}\\
\phi(d_2\tau) &= \{[n] \xrightarrow{\rho} [2] \mid \rho([0,\tau(0)]) = 0, \rho(\tau(1)) = 1, \rho([\tau(3),n]) = 2\}\\
\phi(d_3\tau) &= \{[n] \xrightarrow{\rho} [2] \mid \rho([0,\tau(0)]) = 0, \rho(\tau(1)) = 1, \rho([\tau(2),n]) = 2\}
\end{aligned}$$

which shows, first, that $\phi(d_0\tau) \subset \phi(d_1\tau)$ and $\phi(d_2\tau) \supset \phi(d_3\tau)$, and, second, that the two complements agree and equal

$$\{[n] \xrightarrow{\rho} [2] \mid \rho([0,\tau(0)]) = 0, \rho([\tau(1),\tau(2)]) = 1, \rho([\tau(3),n]) = 2\}.$$

This shows that A satisfies (ii). Finally, it is clear from the definitions that the resulting functor $\phi^*\colon S_{\oplus}^{2,1}\mathscr{C}[n] \to S^{2,1}\mathscr{C}[n]$ is pointed, exact, and duality preserving. $\square$

Example 6.2. The forgetful functor $\phi^*\colon S_{\oplus}^{2,1}\mathscr{C}[n] \to S^{2,1}\mathscr{C}[n]$ is an isomorphism of categories, for $n \leqslant 2$. We illustrate the case $n = 3$ in detail. The pointed set $S^{2,1}[3]$ has elements $\bar{s}^0$, $\bar{s}^1$, $\bar{s}^2$, and $\infty[3]$, which we abbreviate s^0, s^1, s^2, and ∞. Now, the object B of the domain category $S_{\oplus}^{2,1}\mathscr{C}[3]$ is given by a diagram of the form

$$\begin{tikzcd}
 & & B(\{s^2,\infty\}) \arrow[dl, shift left] \arrow[dl, leftarrow, shift right] \arrow[dd, shift left] \arrow[dd, leftarrow, shift right] \arrow[dr, shift left] \arrow[dr, leftarrow, shift right] & & \\
 & B(\{s^0,s^2,\infty\}) \arrow[dr, shift left] \arrow[dr, leftarrow, shift right] \arrow[ddl, shift left] \arrow[ddl, leftarrow, shift right] & & B(\{s^1,s^2,\infty\}) \arrow[dl, shift left] \arrow[dl, leftarrow, shift right] \arrow[ddr, shift left] \arrow[ddr, leftarrow, shift right] & \\
 & & B(\{s^0,s^1,s^2,\infty\}) \arrow[dll, shift left] \arrow[dll, leftarrow, shift right] \arrow[d, shift left] \arrow[d, leftarrow, shift right] \arrow[drr, shift left] \arrow[drr, leftarrow, shift right] & & \\
B(\{s^0,\infty\}) \arrow[rr, shift right] \arrow[rr, leftarrow, shift left] & & B(\{s^0,s^1,\infty\}) \arrow[rr, shift left] \arrow[rr, leftarrow, shift right] & & B(\{s^1,\infty\})
\end{tikzcd}$$

which is subject to the sheaf condition that every "linear" subdiagram is a biproduct diagram in $\mathscr{C}$. The forgetful functor ϕ^* takes B to the object $A = B \circ \phi$ of $S^{2,1}\mathscr{C}[3]$ given by the subdiagram

$$\begin{tikzcd}
A(d^0) \arrow[r] \arrow[d, equal] & A(d^1) \arrow[r] \arrow[d, equal] & A(d^2) \arrow[r] \arrow[d, equal] & A(d^3) \arrow[d, equal] \\
B(\{s^0,\infty\}) \arrow[r] & B(\{s^0,s^1,\infty\}) \arrow[r] & B(\{s^1,s^2,\infty\}) \arrow[r] & B(\{s^2,\infty\})
\end{tikzcd}$$

We remark that this subdiagram does not contain the object $B(\{s^1,\infty\})$.

We next factor the functor ϕ^* through the extended real Waldhausen construction defined in Definition 2.5. To this end, we factor the functor ϕ as the composition

$$\mathrm{Cat}([2],[n]) \xrightarrow{s_1} \mathrm{Cat}([3],[n]) \xrightarrow{\tilde{\phi}} P(S^{2,1}[n])^{\mathrm{op}},$$

where the functor $\tilde{\phi}$ is defined, on objects, by

$$\tilde{\phi}([3] \xrightarrow{\sigma} [n]) = \{[n] \xrightarrow{\rho} [2] \mid \rho \circ \sigma = s^1\} \cup \{\infty[n]\}$$

and, on morphisms, by

$$\tilde{\phi}\Big(\begin{tikzcd}{[3]} \arrow[r, bend left=50, "\sigma_1"] \arrow[r, bend right=50, "\sigma_2"'] \arrow[r, phantom, "\Uparrow"] & {[n]}\end{tikzcd}\Big) = \tilde{\phi}([3] \xrightarrow{\sigma_1} [n]) \cap \tilde{\phi}([3] \xrightarrow{\sigma_2} [n]) \begin{tikzcd} \tilde{\phi}([3] \xrightarrow{\sigma_1} [n]) \arrow[d] \\ \tilde{\phi}([3] \xrightarrow[\sigma_2]{} [n]). \end{tikzcd}$$

Lemma 6.3. *Let $(\mathscr{C}, w\mathscr{C}, D, 0)$ be a pointed exact category with weak equivalences and strict duality. The functor $\tilde{\phi}$ induces a real simplicial pointed duality preserving exact functor*

$$(S^{2,1}_{\oplus}\mathscr{C}[-], wS^{2,1}_{\oplus}\mathscr{C}[-], D[-], 0[-]) \xrightarrow{\tilde{\phi}^*} (\tilde{S}^{2,1}\mathscr{C}[-], w\tilde{S}^{2,1}\mathscr{C}[-], \tilde{D}[-], \tilde{0}[-]).$$

Proof. We show that for every object $B\colon P(S^{2,1}[n])^{\mathrm{op}} \to \mathscr{C}$ of $S^{2,1}_{\oplus}\mathscr{C}[n]$, the composite functor $\tilde{A} = B \circ \tilde{\phi}\colon \mathrm{Cat}([3],[n]) \to \mathscr{C}$ satisfies (i)–(ii) of Definition 2.5. First, for every functor $\theta\colon [2] \to [n]$, we have $\tilde{\phi}(s_0\theta) = \tilde{\phi}(s_2\theta) = \{\infty[n]\}$, since $s^1\colon [3] \to [2]$ does not factor through $s^0\colon [3] \to [2]$ or $s^2\colon [3] \to [2]$. This shows that $\tilde{A}$ satisfies (i). Next, for every functor $\tau\colon [4] \to [n]$, we have

$$\tilde{\phi}(d_0\tau) = \{[n] \xrightarrow{\rho} [2] \mid \rho([0,\tau(1)]) = 0, \rho([\tau(2),\tau(3)]) = 1, \rho([\tau(4),n]) = 2\}$$

$$\tilde{\phi}(d_1\tau) = \{[n] \xrightarrow{\rho} [2] \mid \rho([0,\tau(0)]) = 0, \rho([\tau(2),\tau(3)]) = 1, \rho([\tau(4),n]) = 2\}$$

$$\tilde{\phi}(d_2\tau) = \{[n] \xrightarrow{\rho} [2] \mid \rho([0,\tau(0)]) = 0, \rho([\tau(1),\tau(3)]) = 1, \rho([\tau(4),n]) = 2\}$$

$$\tilde{\phi}(d_3\tau) = \{[n] \xrightarrow{\rho} [2] \mid \rho([0,\tau(0)]) = 0, \rho([\tau(1),\tau(2)]) = 1, \rho([\tau(4),n]) = 2\}$$

$$\tilde{\phi}(d_4\tau) = \{[n] \xrightarrow{\rho} [2] \mid \rho([0,\tau(0)]) = 0, \rho([\tau(1),\tau(2)]) = 1, \rho([\tau(3),n]) = 2\},$$

and hence, $\tilde{\phi}(d_1\tau) = \tilde{\phi}(d_0\tau) \vee \tilde{\phi}(d_2\tau)$ and $\tilde{\phi}(d_3\tau) = \tilde{\phi}(d_2\tau) \vee \tilde{\phi}(d_4\tau)$. Since B is a sheaf, we conclude that $\tilde{A}$ satisfies (ii). ☐

Remark 6.4. Let $(\mathscr{C}, w\mathscr{C}, D, 0)$ be a pointed exact category with weak equivalences and strict duality and let n be a fixed non-negative integer. The pointed exact duality preserving functor

$$S^{2,1}_{\oplus}\mathscr{C}[n] \xrightarrow{i^*} \operatorname{Cat}\operatorname{Dual}_*(S^{2,1}[n], (\mathscr{C}, 0)),$$

from Lemma 4.9 factors as the composition

$$S^{2,1}_{\oplus}\mathscr{C}[n] \xrightarrow{\tilde{\phi}^*} \tilde{S}^{2,1}\mathscr{C}[n] \xrightarrow{v} \operatorname{Cat}\operatorname{Dual}_*(S^{2,1}[n], (\mathscr{C}, 0)),$$

where the functor v is defined as follows. Given an element $\bar{\rho}$ of $S^{2,1}[n]$ different from the basepoint, there is unique functor $\sigma = \sigma(\bar{\rho})\colon [3] \to [n]$ such that $\rho \circ \sigma = s^1$ and such that $\sigma(1) - \sigma(0) = 1 = \sigma(3) - \sigma(2)$. We now define the functor v on objects and morphisms, respectively, by $v(A)(\bar{\rho}) = A(\sigma)$ and $v(f)_{\bar{\rho}} = f_{\sigma}$. The definition of $\sigma(\bar{\rho})$ shows that $\tilde{\phi}(\sigma(\bar{\rho})) = \{\bar{\rho}, \infty[n]\} = i(\bar{\rho})$, so i^* factors as $v \circ \tilde{\phi}^*$. We also note that the functor v is pointed, exact, and duality preserving.

We end this section by introducing a companion $\bar{S}^{2,1}_{\oplus}\mathscr{C}[n]$ of the restricted real Waldhausen construction.

Definition 6.5. Let n be a non-negative integer. Then $(C[n], D)$ is the category with strict duality defined as follows. The set of objects is the set of all pairs of integers (a, b) with $0 \leqslant a \leqslant b \leqslant n$, and the set of morphisms is generated by

$$(a,b) \xrightarrow{p} (a-1,b) \qquad (\text{with } 0 < a \leqslant b \leqslant n)$$

$$(a-1,b) \xrightarrow{s} (a,b) \qquad (\text{with } 0 < a \leqslant b \leqslant n)$$

$$(a,b+1) \xrightarrow{i} (a,b) \qquad (\text{with } 0 \leqslant a \leqslant b < n)$$

$$(a,b) \xrightarrow{r} (a,b+1) \qquad (\text{with } 0 \leqslant a \leqslant b < n)$$

subject to the relations that

$$i \circ p = p \circ i\colon\ (a,b+1) \longrightarrow (a-1,b) \qquad (\text{for all } 0 < a \leqslant b < n)$$

$$r \circ s = s \circ r\colon\ (a-1,b) \longrightarrow (a,b+1) \qquad (\text{for all } 0 < a \leqslant b < n)$$

$$p \circ s = \mathrm{id}\colon\ (a-1,b) \longrightarrow (a-1,b) \qquad (\text{for all } 0 < a \leqslant b \leqslant n)$$

$$r \circ i = \mathrm{id}\colon\ (a,b+1) \longrightarrow (a,b+1) \qquad (\text{for all } 0 \leqslant a \leqslant b < n).$$

The functor $D\colon C[n]^{\mathrm{op}} \to C[n]$ is defined, on objects, by

$$D((a,b)^{\mathrm{op}}) = (n-b, n-a),$$

where $0 \leqslant a \leqslant b \leqslant n$, and, on morphisms, by

$$\begin{tikzcd} D((a,b)^{\mathrm{op}}) \arrow[d, shift left, "D(s^{\mathrm{op}})=r"] \\ D((a-1,b)^{\mathrm{op}}) \arrow[u, shift left, "D(p^{\mathrm{op}})=i"] \end{tikzcd} \qquad \begin{tikzcd} D((a,b+1)^{\mathrm{op}}) \arrow[d, shift left, "D(r^{\mathrm{op}})=s"] \\ D((a,b)^{\mathrm{op}}), \arrow[u, shift left, "D(i^{\mathrm{op}})=p"] \end{tikzcd}$$

where $0 < a \leqslant b \leqslant n$ and $0 \leqslant a \leqslant b < n$, respectively.

Definition 6.6. Let n be a non-negative integer, and let $(\mathscr{C}, w\mathscr{C}, D, 0)$ be a pointed exact category with weak equivalences and strict duality. The degree n restricted real Segal construction of $(\mathscr{C}, w\mathscr{C}, D, 0)$ is the pointed exact category with weak equivalences and strict duality

$$(\bar{S}^{2,1}_{\oplus}\mathscr{C}[n], w\bar{S}^{2,1}_{\oplus}\mathscr{C}[n], \bar{D}[n], \bar{0}[n]),$$

where

$$(\bar{S}^{2,1}_{\oplus}\mathscr{C}[n], D[n]) \subset \operatorname{Cat}\operatorname{Dual}((C[n], D)), (\mathscr{C}, D))$$

is the full subcategory with strict duality of all functors

$$A \colon C[n] \longrightarrow \mathscr{C}$$

such that

(i) for all integers $0 \leqslant a \leqslant b \leqslant n$ with $a = 0$ or $b = n$,

$$A(a,b) = 0(1),$$

(ii) for all integers $0 < a < b \leqslant n$, the diagram

$$\begin{tikzcd} A(a,b) \arrow[r, "A(p)"] \arrow[d, "A(i)"] & A(a-1,b) \arrow[d, "A(i)"] \\ A(a,b-1) \arrow[r, "A(p)"] & A(a-1,b-1) \end{tikzcd}$$

is an admissible square in $\mathscr{C}$;

where the sequence $A \to \bar{B} \to \bar{C}$ in $\bar{S}^{2,1}\mathscr{C}[n]$ is exact if, for all integers $0 \leqslant a \leqslant b \leqslant n$, the sequence $A(a,b) \to \bar{B}(a,b) \to \bar{C}(a,b)$ in $\mathscr{C}$ is exact; where the morphism $A \to \bar{B}$ is in $w\bar{S}^{2,1}\mathscr{C}[n]$ if, for all integers $0 \leqslant a \leqslant b \leqslant n$, the morphism $A(a,b) \to \bar{B}(a,b)$ is in $w\mathscr{C}$; and where the basepoint $\bar{0}[n]$ is the constant diagram $\bar{0}[n](1)(a,b) = 0(1)$.

We recall from [21, Section 6.2] that an exact category $\mathscr{C}$ is semi-idempotent complete if every morphism that admits a section is an admissible epimorphism. For such $\mathscr{C}$, the axiom (ii) in Definition 6.6 is redundant.

Lemma 6.7. *Let $(\mathscr{C}, w\mathscr{C}, D, 0)$ be a pointed exact category with weak equivalences and strict duality. For every integer $n \geqslant 0$, there exists a pointed exact adjunction*

$$(j^*_{\oplus}, e_{\oplus}, \eta, \varepsilon, 1, g)$$

from $(S^{2,1}_{\oplus}\mathscr{C}[n], wS^{2,1}_{\oplus}\mathscr{C}[n], D[n], 0[n])$ to $(\bar{S}^{2,1}_{\oplus}\mathscr{C}[n], w\bar{S}^{2,1}_{\oplus}\mathscr{C}[n], \bar{D}[n], \bar{0}[n])$. Moreover, the natural transformations η, ε, and g all are isomorphisms.

Proof. We define the functor $j_\oplus \colon C[n] \to P(S^{2,1}[n])^{\mathrm{op}}$ as follows. On objects,

$$j_\oplus(a,b) = \{[n] \xrightarrow{\rho} [2] \mid \rho(0) = 0, \rho([a,b]) = 1, \rho(n) = 2\} \cup \{\infty[n]\}$$

and, on morphisms, by

$$\begin{tikzcd} j_\oplus(a,b) \arrow[d, shift left, "j_\oplus(s)=j_\oplus(a-1,b)"] \\ j_\oplus(a-1,b) \arrow[u, shift left, "j_\oplus(p)=j_\oplus(a-1,b)"] \end{tikzcd} \qquad \begin{tikzcd} j_\oplus(a,b) \arrow[d, shift left, "j_\oplus(r)=j_\oplus(a,b)"] \\ j_\oplus(a,b-1). \arrow[u, shift left, "j_\oplus(i)=j_\oplus(a,b)"] \end{tikzcd}$$

The functor $j_\oplus$ is well-defined, since for all $0 < a < b \leqslant n$,

$$j_\oplus(a-1,b) = j_\oplus(a-1,b-1) \cap j_\oplus(a,b).$$

Moreover, as in the proof of Lemma 2.12, one readily verifies that it induces a pointed exact duality preserving functor $j_\oplus^* \colon S_\oplus^{2,1}\mathscr{C}[n] \to \bar{S}_\oplus^{2,1}\mathscr{C}[n]$.

We next define the functor $e_\oplus \colon \bar{S}_\oplus^{2,1}\mathscr{C}[n] \to S_\oplus^{2,1}\mathscr{C}[n]$. So let $B \colon C[n] \to \mathscr{C}$ be an object of $\bar{S}_\oplus^{2,1}\mathscr{C}[n]$. We define the functor $e_\oplus(B) \colon P(S^{2,1}[n]) \to \mathscr{C}$ as follows. First, if $\bar{\rho} \in S^{2,1}[n]$ is different from the basepoint, then we define $e_\oplus(B)(\{\bar{\rho}, \infty[n]\})$ to the lower left-hand term in a choice of completion of the following admissible square in which $a = \min \rho^{-1}(1)$ and $b = \max \rho^{-1}(1)$.

$$\begin{tikzcd} B(a,b+1) \arrow[r, two heads, "B(p)"] \arrow[d, tail, "B(i)"] & B(a-1,b+1) \arrow[d, tail, "B(i)"] \\ B(a,b) \arrow[r, two heads, "B(p)"] & B(a-1,b) \end{tikzcd}$$

The completion can be chosen arbitrarily except that for $B = \bar{0}[n]$, we required it to be the unique diagram in which every object is equal to $0(1)$. Second, if U is an object of $P(S^{2,1}[n])$ such that $U \subset S^{2,1}[n]$ contains two or more elements different from the basepoint, then we define $e_\oplus(B)(U)$ to be a choice of sum in $\mathscr{C}$ of the family of objects $e_\oplus(B)(\{\bar{\rho}, \infty[n]\})$ indexed by $\bar{\rho} \in U$. If $B = \bar{0}[n]$, then we require the sum to be equal to $0(1)$, but otherwise, it can be chosen arbitrarily. This defines the functor $e_\oplus(B)$ on objects. Thirdly, if $F \colon U \to V$ is a morphism in $P(S^{2,1}[n])$, then we define the morphism $e_\oplus(B)(F) \colon e_\oplus(B)(V) \to e_\oplus(B)(U)$ to be the composition of the canonical projection $e_\oplus(B)(V) \twoheadrightarrow e_\oplus(B)(F)$ followed by the canonical inclusion $e_\oplus(B)(F) \rightarrowtail e_\oplus(B)(U)$. To prove compatibility with composition, we let $F \colon U \to V$ and $G \colon V \to W$ be composable morphisms. Then the diagram

$$\begin{tikzcd} e_\oplus(B)(W) \arrow[r, two heads] \arrow[dr, two heads] & e_\oplus(B)(G) \arrow[r, tail] \arrow[d, two heads] & e_\oplus(B)(V) \arrow[d, two heads] \\ & e_\oplus(B)(F \cap G) \arrow[r, tail] \arrow[dr, tail] & e_\oplus(B)(F) \arrow[d, tail] \\ & & e_\oplus(B)(U) \end{tikzcd}$$

of canonical projections and canonical inclusions commutes. This shows that $e_\oplus(B)$ is indeed a functor. Finally, it follows from Proposition 4.4 that $e_\oplus(B)$ is an object of $S^{2,1}_\oplus\mathscr{C}[n]$. This defines the functor $e_\oplus$ on objects; it is defined on morphisms by using the uniqueness, up to canonical isomorphism, of kernels, cokernels, and sums. It is clear that $e_\oplus$ is pointed; we leave it to the reader to verify that it is exact.

We define η to be the natural isomorphism whose value at $A \in \operatorname{ob} S^{2,1}_\oplus\mathscr{C}[n]$ is the natural isomorphism $\eta_A \colon A \Rightarrow e_\oplus(A \circ j_\oplus)$ defined as follows. We first define η_A at the objects U of $P(S^{2,1}[n])$ of the form $U = \{\bar\rho, \infty[n]\}$ with $\bar\rho \in S^{2,1}[n]$ different from the basepoint. To this end, we let $a = \min \rho^{-1}(1)$ and $b = \max \rho^{-1}(1)$ and write

$$\begin{aligned} j_\oplus(a,b+1) &= j_\oplus(a-1,b+1) \vee V \\ j_\oplus(a-1,b) &= j_\oplus(a-1,b+1) \vee W. \end{aligned}$$

In this situation, we have the commutative diagram

$$\begin{tikzcd} A(V) \arrow[r, "V_*"] \arrow[d, "V_*"] & A(j_\oplus(a,b+1)) \arrow[r, "p_*"] \arrow[d, "i_*"] & A(j_\oplus(a-1,b+1)) \arrow[d, "i_*"] \\ A(U \vee V) \arrow[r, "(U\vee V)_*"] \arrow[d, "U_*"] & A(j_\oplus(a,b)) \arrow[r, "p_*"] \arrow[d, "(U\vee W)_*"] & A(j_\oplus(a-1,b)) \arrow[d, "W_*"] \\ A(U) \arrow[r, "U_*"] & A(U \vee W) \arrow[r, "W_*"] & A(W) \end{tikzcd}$$

with all rows and columns exact. This diagram is a completion of the upper right-hand square. Therefore, the objects $A(U)$ and $e_\oplus(j^*_\oplus(A))(U)$ are canonically isomorphic, and we define $\eta_A \colon A(U) \to e_\oplus(j^*_\oplus(A))(U)$ to be the canonical isomorphism. Next, we define η_A at an object U of $P(S^{2,1}[n])$ such that $U \subset S^{2,1}[n]$ contains two or more elements different from the basepoint. The objects $A(U)$ and $e_\oplus(j^*_\oplus(A))(U)$ are sums in $\mathscr{C}$ of the families of objects $A(\{\bar\rho, \infty[n]\})$ and $e_\oplus(j^*_\oplus(A))(\{\bar\rho, \infty[n]\})$, respectively, indexed by $\bar\phi \in U$, and we define the isomorphism $\eta_A \colon A(U) \to e_\oplus(j^*_\oplus(A))(U)$ to be the sum of the isomorphisms $\eta_A \colon A(\{\bar\rho, \infty[n]\}) \to e_\oplus(j^*_\oplus(A))(\{\bar\rho, \infty[n]\})$ already defined. It is clear that η is pointed.

We define ε to be the natural isomorphism whose value at $B \in \operatorname{ob} \bar S^{2,1}_\oplus\mathscr{C}[n]$ is the natural isomorphism $\varepsilon_B \colon e_\oplus(B) \circ j_\oplus \Rightarrow B$ defined as follows. Let $N(a,b) = a(n-b)$ be the number of elements different from the basepoint in $j_\oplus(a,b)$. We proceed by recursion on $N(a,b) \geqslant 0$ to define the morphism ε_B on the object (a,b) in such a way that in the diagram

$$\begin{tikzcd} e_\oplus(B)(j_\oplus(a,b+1)) \arrow[r, shift right, "i_*"'] \arrow[d, "\varepsilon_B"] & e_\oplus(B)(j_\oplus(a,b)) \arrow[l, shift right, "r_*"'] \arrow[r, shift left, "p_*"] \arrow[d, "\varepsilon_B"] & e_\oplus(B)(j_\oplus(a-1,b)) \arrow[l, shift left, "s_*"] \arrow[d, "\varepsilon_B"] \\ B(a,b+1) \arrow[r, shift right, "i_*"'] & B(a,b) \arrow[l, shift right, "r_*"'] \arrow[r, shift left, "p_*"] & B(a-1,b), \arrow[l, shift left, "s_*"] \end{tikzcd}$$

the two left-hand squares and the two right-hand squares commute. If $N(a,b) = 0$, then $e_\oplus(B)(j_\oplus(a,b)) = B(a,b) = 0(1)$, and we define ε_B to be the unique map. So we fix the object (a,b) and assume that ε_B has been defined at all objects (c,d) with $N(c,d) < N(a,b)$ in such a way that the required squares commute. To define the ε_B at the object (a,b), we consider the following diagram.

$$\begin{tikzcd}
e_\oplus(B)(V) \arrow[r, shift left, "V_*"] \arrow[d, shift left, "V_*"] & e_\oplus(B)(j_\oplus(a,b+1)) \arrow[l, shift left, "V_*"] \arrow[r, shift left, "p_*"] \arrow[d, shift left, "i_*"] & e_\oplus(B)(j_\oplus(a-1,b+1)) \arrow[l, shift left, "s_*"] \arrow[d, shift left, "i_*"] \\
e_\oplus(B)(U\vee V) \arrow[u, shift left, "V_*"] \arrow[r, shift left, "(U\vee V)_*"] \arrow[d, shift left, "U_*"] & e_\oplus(B)(j_\oplus(a,b)) \arrow[u, shift left, "r_*"] \arrow[l, shift left, "(U\vee V)_*"] \arrow[r, shift left, "p_*"] \arrow[d, shift left, "(U\vee W)_*"] & e_\oplus(B)(j_\oplus(a-1,b)) \arrow[u, shift left, "r_*"] \arrow[l, shift left, "s_*"] \arrow[d, shift left, "W_*"] \\
e_\oplus(B)(U) \arrow[u, shift left, "U_*"] \arrow[r, shift left, "U_*"] & e_\oplus(B)(U\vee W) \arrow[u, shift left, "(U\vee W)_*"] \arrow[l, shift left, "U_*"] \arrow[r, shift left, "W_*"] & e_\oplus(B)(W) \arrow[u, shift left, "W_*"] \arrow[l, shift left, "W_*"]
\end{tikzcd}$$

Here, V and W were defined earlier, and $U = \{\bar\rho, \infty[n]\}$ with $\rho\colon [n] \to [2]$ given by $\rho([0,a-1]) = 0$, $\rho([a,b]) = 1$, and $\rho([b+1,n]) = 2$. We compare this diagram to the following diagram, where the subdiagram of rightward pointing and downward pointing morphisms is the chosen completion of the admissible square consisting of the maps p_* and i_*, and where the remaining morphisms are the splittings induced from the given sections s_* of p_* and retractions r_* of i_*.

$$\begin{tikzcd}
B_{11} \arrow[r, shift left] \arrow[d, shift left] & B(a,b+1) \arrow[l, shift left] \arrow[r, shift left, "p_*"] \arrow[d, shift left, "i_*"] & B(a-1,b+1)) \arrow[l, shift left, "s_*"] \arrow[d, shift left, "i_*"] \\
B_{12} \arrow[u, shift left] \arrow[r, shift left] \arrow[d, shift left] & B(a,b) \arrow[u, shift left, "r_*"] \arrow[l, shift left] \arrow[r, shift left, "p_*"] \arrow[d, shift left] & B(a-1,b) \arrow[u, shift left, "r_*"] \arrow[l, shift left, "s_*"] \arrow[d, shift left] \\
e_\oplus(B)(U) \arrow[u, shift left] \arrow[r, shift left] & B_{32} \arrow[u, shift left] \arrow[l, shift left] \arrow[r, shift left] & B_{33} \arrow[u, shift left] \arrow[l, shift left]
\end{tikzcd}$$

There is a unique morphism of diagrams from the first diagram to the second diagram that, on upper middle terms, upper right-hand terms, and middle right-hand terms are given by the morphisms ε_B already defined, and that, on lower left-hand terms, is given by the identity morphism. Now, we take ε_B on the object (a,b) to be the morphism of central terms. This completes the recursive definition of the natural isomorphism $\varepsilon_B\colon e_\oplus(B) \circ j_\oplus \Rightarrow B$. It is clear that ε is pointed.

Finally, we define $g\colon e \Rightarrow (\bar D[n], D[n])e^{\mathrm{op}}$ to be the pointed natural isomorphism whose value at B is the natural isomorphism $g_B\colon e_\oplus(B) \Rightarrow ((\bar D[n], D[n])e^{\mathrm{op}})(B)$ whose value at (a,b), in turn, is the canonical isomorphism of sums of lower left-hand terms in different choices of completion of the same admissible squares. It follows from the uniqueness of kernels, cokernels, and sums, up to canonical isomorphism, implies that the two diagrams in Definition 1.27 commute. $\square$

We define $\bar\phi\colon \mathrm{Cat}([1],[n]) \to C[n]$ to be the unique functor that is given, on objects, by $\bar\phi(\mu) = (\mu(0), \mu(1))$ and, on morphisms, by

$$\bar\phi(\mu_1 \Rightarrow \mu_2) = p \circ \cdots \circ p \circ i \circ \cdots \circ i\colon \bar\phi(\mu_1) \to \bar\phi(\mu_2),$$

where there are $\mu_1(0)-\mu_2(0)$ copies of p and $\mu_2(1)-\mu_1(1)$ copies of i. The functor $\bar{\phi}$ is an isomorphism of the category $\mathrm{Cat}([1],[n])$ onto the subcategory of $C[n]$ generated by the morphisms p and i.

Lemma 6.8. *Let $(\mathscr{C}, w\mathscr{C}, D, 0)$ be a pointed exact category with weak equivalences and strict duality and let n be a non-negative integer. The functor $\bar{\phi}$ induces a pointed duality preserving exact functor*

$$(\bar{S}^{2,1}_{\oplus}\mathscr{C}[n], w\bar{S}^{2,1}_{\oplus}\mathscr{C}[n], \bar{D}[n], \bar{0}[n]) \xrightarrow{\bar{\phi}^*} (\bar{S}^{2,1}\mathscr{C}[n], w\bar{S}^{2,1}\mathscr{C}[n], \bar{D}[n], \bar{0}[n]).$$

Proof. The functor $\bar{\phi}$ induces a functor

$$\mathrm{Cat}(C[n],\mathscr{C}) \xrightarrow{\bar{\phi}^*} \mathrm{Cat}(\mathrm{Cat}([1],[n]),\mathscr{C})$$

and comparing Definitions 2.11 and 6.6, we see that it restricts to the stated pointed exact duality preserving functor. □

Remark 6.9. Let $(\mathscr{C}, w\mathscr{C}, D, 0)$ be a pointed exact category with weak equivalences and strict duality and let n be a non-negative integer. The following diagram of pointed exact duality preserving functors commute.

$$\begin{tikzcd}
S^{2,1}_{\oplus}\mathscr{C}[n] \arrow[r, equal] \arrow[d, "\phi^*"] & S^{2,1}_{\oplus}\mathscr{C}[n] \arrow[r, "j^*_{\oplus}"] \arrow[d, "\tilde{\phi}^*"] & \bar{S}^{2,1}_{\oplus}\mathscr{C}[n] \arrow[d, "\bar{\phi}^*"] \\
S^{2,1}\mathscr{C}[n] & \tilde{S}^{2,1}\mathscr{C}[n] \arrow[l, "s_1^*"'] \arrow[r, "j^*"] & \bar{S}^{2,1}\mathscr{C}[n]
\end{tikzcd}$$

Here, we recall, the horizontal functors all are equivalences of categories. By contrast, the vertical functors are not equivalences of categories except in trivial cases.

Finally, we define for every pointed exact category with weak equivalences and strict duality $(\mathscr{C}, w\mathscr{C}, D, 0)$ a map of real symmetric spectra

$$KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0) \xrightarrow{\phi^*} KR(\mathscr{C}, w\mathscr{C}, D, 0)$$

from the real direct sum K-theory spectrum to the real algebraic K-theory spectrum.

Let r be a positive integer, and let $[n]=[n_1]\times\cdots\times[n_r]$ be an object of the r-fold product real category $\Delta R\times\cdots\times\Delta R$. We define the functor

$$\mathrm{Cat}([2],[n]) \xrightarrow{\phi_r} P(S^{2r,r}[n])$$

to be the composition

$$\begin{aligned}
&\mathrm{Cat}([2],[n]) \longrightarrow \mathrm{Cat}([2],[n_1])\times\cdots\times\mathrm{Cat}([2],[n_r]) \\
&\longrightarrow P(S^{2,1}[n_1])\times\cdots\times P(S^{2,1}[n_r]) \longrightarrow P(S^{2r,r}[n])
\end{aligned}$$

of the canonical isomorphism $(\mathrm{pr}_{1*},\dots,\mathrm{pr}_{r*})$, the product functor $\phi\times\cdots\times\phi$, and the functor that to the object $(U_1,\dots,U_r)$ and the morphism $(F_1,\dots,F_r)$ associate the object $U_1\wedge\cdots\wedge U_r$ and the morphism $F_1\wedge\cdots\wedge F_r$, respectively.

Lemma 6.10. *Let* $(\mathscr{C}, w\mathscr{C}, D, 0)$ *be a pointed exact category with weak equivalences and strict duality. For every positive integer r, the functor* ϕ_r *induces an r-real simplicial pointed duality preserving exact functor*

$$(S_{\oplus}^{2r,r}\mathscr{C}[-], wS_{\oplus}^{2r,r}\mathscr{C}[-], D[-], 0[-]) \xrightarrow{\phi_r^*} (S^{2r,r}\mathscr{C}[-], wS^{2r,r}\mathscr{C}[-], D[-], 0[-]).$$

Proof. The proof is similar to the proof of Lemma 6.1 □

Definition 6.11. Let $(\mathscr{C}, w\mathscr{C}, D, 0)$ be a pointed exact category with weak equivalences and strict duality. The forgetful map is the map of real symmetric spectra

$$KR^{\oplus}(\mathscr{C}, w\mathscr{C}, D, 0) \xrightarrow{\phi^*} KR(\mathscr{C}, w\mathscr{C}, D, 0)$$

that in level $r \geqslant 1$ is given by the pointed real map

$$|N(wS_{\oplus}^{2r,r}\mathscr{C}[-], D[-], 0[-])[-]|_R \xrightarrow{\phi_r^*} |N(wS^{2r,r}\mathscr{C}[-], D[-], 0[-])[-]|_R$$

induced by the r-real simplicial pointed exact duality preserving functor ϕ_r^* and that in level $r = 0$ is the identity map.

7 The split-exact case

In this section, we explictly describe the category $S^{2,1}\mathscr{C}[n]$ in the case, where the exact category $\mathscr{C}$ is split-exact. We begin by proving some basic results about split-exact categories.

Given any exact category $\mathscr{C} = (\mathscr{C}, \mathscr{E})$, we define the exact category of exact sequences in $\mathscr{C}$ to be the exact category $E(\mathscr{C})$ whose set of objects is the given set $\mathscr{E}$ of exact sequences in $\mathscr{C}$ and whose set of morphisms from the object $A_1 \to A_2 \to A_3$ to the object $B_1 \to B_2 \to B_3$ consists of all triples (f_1, f_2, f_3) of morphisms $f_i \colon A_i \to B_i$ that make the left-hand diagram below commute. The sequence in $E(\mathscr{C})$ indicated by the right-hand diagram below is defined to be exact if the three horizontal sequences in the diagram are exact sequences in $\mathscr{C}$.

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d] & B_1 \arrow[d] \\
A_2 \arrow[r, "f_2"] \arrow[d] & B_2 \arrow[d] \\
A_3 \arrow[r, "f_3"] & B_3
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d] & B_1 \arrow[r, "g_1"] \arrow[d] & C_1 \arrow[d] \\
A_2 \arrow[r, "f_2"] \arrow[d] & B_2 \arrow[r, "g_2"] \arrow[d] & C_2 \arrow[d] \\
A_3 \arrow[r, "f_3"] & B_3 \arrow[r, "g_3"] & C_3
\end{tikzcd}$$

Now, for split-exact $\mathscr{C}$, we have the following result.

Lemma 7.1. *If the exact category $\mathscr{C}$ is split-exact, then the exact category $E(\mathscr{C})$ of exact sequences in $\mathscr{C}$ is again split-exact.*

Proof. We must show that the exact sequence in $E(\mathscr{C})$ given by the left-hand diagram below can be completed to a biproduct diagram in $E(\mathscr{C})$. To this end, we first choose biproduct diagrams in $\mathscr{C}$ completing the three columns and the top and bottom row as indicated by the right-hand diagram below.

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & B_1 \arrow[r, "g_1"] \arrow[d, "h_2"] & C_1 \arrow[d, "h_3"] \\
A_2 \arrow[r, "f_2"] \arrow[d, "k_1"] & B_2 \arrow[r, "g_2"] \arrow[d, "k_2"] & C_2 \arrow[d, "k_3"] \\
A_3 \arrow[r, "f_3"] & B_3 \arrow[r, "g_3"] & C_3
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A_1 \arrow[r, shift left, "f_1"] \arrow[d, shift left, "h_1"] & B_1 \arrow[l, shift left, "u_1"] \arrow[r, shift left, "g_1"] \arrow[d, shift left, "h_2"] & C_1 \arrow[l, shift left, "s_1"] \arrow[d, shift left, "h_3"] \\
A_2 \arrow[u, shift left, "r_1"] \arrow[r, "f_2"] \arrow[d, shift left, "k_1"] & B_2 \arrow[u, shift left, "r'_2"] \arrow[r, "g_2"] \arrow[d, shift left, "k_2"] & C_2 \arrow[u, shift left, "r_3"] \arrow[d, shift left, "k_3"] \\
A_3 \arrow[u, shift left, "v_1"] \arrow[r, shift left, "f_3"] & B_3 \arrow[u, shift left, "v'_2"] \arrow[l, shift left, "u_3"] \arrow[r, shift left, "g_3"] & C_3 \arrow[u, shift left, "v_3"] \arrow[l, shift left, "s_3"]
\end{tikzcd}$$

We do not claim any compatibility between these maps other than all columns and the top and bottom row being biproduct diagrams in $\mathscr{C}$. However, using these biproduct diagrams, we can express the morphism $g_2 \colon B_2 \to C_2$ as

$$\begin{aligned}
g_2 &= (h_3 r_3 + v_3 k_3) g_2 (h_2 r'_2 + v'_2 k_2) \\
&= h_3 r_3 g_2 h_2 r'_2 + h_3 r_3 g_2 v'_2 k_2 + v_3 k_3 g_2 h_2 r'_2 + v_3 k_3 g_2 v'_2 k_2 \\
&= h_3 r_3 h_3 g_1 r'_2 + h_3 r_3 g_2 v'_2 k_2 + v_3 g_3 k_2 h_2 r'_2 + v_3 g_3 k_2 v'_2 k_2 \\
&= h_3 g_1 r'_2 + h_3 r_3 g_2 v'_2 k_2 + v_3 g_3 k_2.
\end{aligned}$$

We now define $s_2 \colon C_2 \to B_2$ by

$$s_2 = h_2 s_1 r_3 - h_2 s_1 r_3 g_2 v_2' s_3 k_3 + v_2' s_3 k_3$$

and calculate

$$\begin{aligned}
g_2 s_2 &= (h_3 g_1 r_2' + h_3 r_3 g_2 v_2' k_2 + v_3 g_3 k_2)(h_2 s_1 r_3 - h_2 s_1 r_3 g_2 v_2' s_3 k_3 + v_2' s_3 k_3) \\
&= (h_3 g_1 r_2' + h_3 r_3 g_2 v_2' k_2 + v_3 g_3 k_2)(h_2 s_1 r_3) \\
&\quad - (h_3 g_1 r_2' + h_3 r_3 g_2 v_2' k_2 + v_3 g_3 k_2) h_2 s_1 r_3 g_2 v_2' s_3 k_3 \\
&\quad + (h_3 g_1 r_2' + h_3 r_3 g_2 v_2' k_2 + v_3 g_3 k_2) v_2' s_3 k_3 \\
&= h_3 r_3 - h_3 r_3 g_2 v_2' s_3 k_3 + h_3 r_3 g_2 v_2' s_3 k_3 + v_3 k_3 \\
&= h_3 r_3 + v_3 k_3 = \mathrm{id}_{C_2} .
\end{aligned}$$

This shows that s_2 is a section of g_2. Moreover, the calculation

$$\begin{aligned}
s_2 h_3 &= (h_2 s_1 r_3 - h_2 s_1 r_3 g_2 v_2' s_3 k_3 + v_2' s_3 k_3) h_3 = h_2 s_1 \\
k_2 s_2 &= k_2 (h_2 s_1 r_3 - h_2 s_1 r_3 g_2 v_2' s_3 k_3 + v_2' s_3 k_3) = s_3 k_3
\end{aligned}$$

shows that the triple (s_1, s_2, s_3) is a morphism of exact sequences. This shows that the morphism of exact sequences (s_1, s_2, s_3) is a section of the morphism of exact sequences (g_1, g_2, g_3). Finally, if we define $u_2 \colon B_2 \to A_2$ to be the unique morphism that makes the middle row in the diagram below a biproduct diagram in $\mathscr{C}$, then said diagram is the desired biproduct diagram in $E(\mathscr{C})$ completing the exact sequence in $E(\mathscr{C})$ given by the left-hand diagram at the beginning of the proof.

$$\begin{tikzcd}
A_1 \arrow[r, shift left, "f_1"] \arrow[d, "h_1"] & B_1 \arrow[l, shift left, "u_1"] \arrow[r, shift left, "g_1"] \arrow[d, "h_2"] & C_1 \arrow[l, shift left, "s_1"] \arrow[d, "h_3"] \\
A_2 \arrow[r, shift left, "f_2"] \arrow[d, "k_1"] & B_2 \arrow[l, shift left, "u_2"] \arrow[r, shift left, "g_2"] \arrow[d, "k_2"] & C_2 \arrow[l, shift left, "s_2"] \arrow[d, "k_3"] \\
A_3 \arrow[r, shift left, "f_3"] & B_3 \arrow[l, shift left, "u_3"] \arrow[r, shift left, "g_3"] & C_3 \arrow[l, shift left, "s_3"]
\end{tikzcd}$$

This completes the proof. □

Addendum 7.2. *Let $\mathscr{C}$ be a split-exact category and let*

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & B_1 \arrow[r, "g_1"] \arrow[d, "h_2"] & C_1 \arrow[d, "h_3"] \\
A_2 \arrow[r, "f_2"] \arrow[d, "k_1"] & B_2 \arrow[r, "g_2"] \arrow[d, "k_2"] & C_2 \arrow[d, "k_3"] \\
A_3 \arrow[r, "f_3"] & B_3 \arrow[r, "g_3"] & C_3
\end{tikzcd}$$

be a commutative diagram in $\mathscr{C}$ in which all rows and columns are exact. Then there exists a commutative diagram in $\mathscr{C}$ of the form

$$\begin{tikzcd}
A_1 & B_1 \arrow[l, "u_1"'] & C_1 \arrow[l, "s_1"'] \\
A_2 \arrow[u, "r_1"'] & B_2 \arrow[l, "u_2"'] \arrow[u, "r_2"'] & C_2 \arrow[l, "s_2"'] \arrow[u, "r_3"'] \\
A_3 \arrow[u, "v_1"'] & B_3 \arrow[l, "u_3"'] \arrow[u, "v_2"'] & C_3 \arrow[l, "s_3"'] \arrow[u, "v_3"']
\end{tikzcd}$$

such that, in the combined diagram,

$$\begin{tikzcd}
A_1 \arrow[r, shift left, "f_1"] \arrow[d, shift left, "h_1"] & B_1 \arrow[l, shift left, "u_1"] \arrow[r, shift left, "g_1"] \arrow[d, shift left, "h_2"] & C_1 \arrow[l, shift left, "s_1"] \arrow[d, shift left, "h_3"] \\
A_2 \arrow[u, shift left, "r_1"] \arrow[r, shift left, "f_2"] \arrow[d, shift left, "k_1"] & B_2 \arrow[u, shift left, "r_2"] \arrow[l, shift left, "u_2"] \arrow[r, shift left, "g_2"] \arrow[d, shift left, "k_2"] & C_2 \arrow[u, shift left, "r_3"] \arrow[l, shift left, "s_2"] \arrow[d, shift left, "k_3"] \\
A_3 \arrow[u, shift left, "v_1"] \arrow[r, shift left, "f_3"] & B_3 \arrow[u, shift left, "v_2"] \arrow[l, shift left, "u_3"] \arrow[r, shift left, "g_3"] & C_3 \arrow[u, shift left, "v_3"] \arrow[l, shift left, "s_3"]
\end{tikzcd}$$

all rows and columns are bi-product diagrams in $\mathscr{C}$. Moreover, the sections s_1 and s_3 and the retractions r_1 and r_3 may be predescribed freely.

Proof. First, we use the predescribed sections s_1 and s_3 and retractions r_1 and r_3 to complete the given diagram to the diagram on the left-hand side below in which the top and bottom rows and the left-hand and right-hand columns are biproduct diagrams. Second, we follow the proof of Lemma 7.1 to further complete this diagram to the diagram on the right-hand side below, where also the middle row is a biproduct diagram, and where the map of exact sequence (s_1, s_2, s_3) is a section of the map of exact sequences (g_1, g_2, g_3).

$$\begin{tikzcd}
A_1 \arrow[r, shift left, "f_1"] \arrow[d, shift left, "h_1"] & B_1 \arrow[l, shift left, "u_1"] \arrow[r, shift left, "g_1"] \arrow[d, "h_2"] & C_1 \arrow[l, shift left, "s_1"] \arrow[d, shift left, "h_3"] \\
A_2 \arrow[u, shift left, "r_1"] \arrow[r, "f_2"] \arrow[d, shift left, "k_1"] & B_2 \arrow[r, "g_2"] \arrow[d, "k_2"] & C_2 \arrow[u, shift left, "r_3"] \arrow[d, shift left, "k_3"] \\
A_3 \arrow[u, shift left, "v_1"] \arrow[r, shift left, "f_3"] & B_3 \arrow[l, shift left, "u_3"] \arrow[r, shift left, "g_3"] & C_3 \arrow[u, shift left, "v_3"] \arrow[l, shift left, "s_3"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[r, shift left, "f_1"] \arrow[d, shift left, "h_1"] & B_1 \arrow[l, shift left, "u_1"] \arrow[r, shift left, "g_1"] \arrow[d, "h_2"] & C_1 \arrow[l, shift left, "s_1"] \arrow[d, shift left, "h_3"] \\
A_2 \arrow[u, shift left, "r_1"] \arrow[r, shift left, "f_2"] \arrow[d, shift left, "k_1"] & B_2 \arrow[l, shift left, "u_2"] \arrow[r, shift left, "g_2"] \arrow[d, "k_2"] & C_2 \arrow[u, shift left, "r_3"] \arrow[l, shift left, "s_2"] \arrow[d, shift left, "k_3"] \\
A_3 \arrow[u, shift left, "v_1"] \arrow[r, shift left, "f_3"] & B_3 \arrow[l, shift left, "u_3"] \arrow[r, shift left, "g_3"] & C_3 \arrow[u, shift left, "v_3"] \arrow[l, shift left, "s_3"]
\end{tikzcd}$$

We now define $r_2\colon B_2 \to B_1$ and $v_2\colon B_3 \to B_2$ by

$$r_2 = f_1 r_1 u_2 + s_1 r_3 g_2$$
$$v_2 = f_2 v_1 u_3 + s_2 v_3 g_3$$

and calculate

$$\begin{aligned}
r_2h_2 &= (f_1r_1u_2+s_1r_3g_2)h_2 = f_1r_1h_1u_1+s_1r_3h_3g_1\\
&= f_1u_1+s_1g_1 = \mathrm{id}_{B_1}\\
k_2v_2 &= k_2(f_2v_1u_3+s_2v_3g_3) = f_3k_1v_1u_3+s_3k_3v_3g_3\\
&= f_3u_3+s_3g_3 = \mathrm{id}_{B_3}\\
h_2r_2+v_2k_2 &= h_2(f_1r_1u_2+s_1r_3g_2)+(f_2v_1u_3+s_2v_3g_3)k_2\\
&= f_2h_1r_1u_2+s_2h_3r_3g_2+f_2v_1k_1u_2+s_2v_3k_3g_2\\
&= f_2(h_1r_1+v_1k_1)u_2+s_2(h_3r_3+v_3k_3)g_2\\
&= f_2u_2+s_2g_2 = \mathrm{id}_{B_2}\,.
\end{aligned}$$

This shows that the middle column in the bottom diagram in the statement is a biproduct diagram in $\mathscr{C}$. Finally, we calculate

$$\begin{aligned}
u_1r_2 &= u_1(f_1r_1u_2+s_1r_3g_2) = r_1u_2\\
r_2s_2 &= (f_1r_1u_2+s_1r_3g_2)s_2 = s_1r_3\\
u_2v_2 &= u_2(f_2v_1u_3+s_2v_3g_3) = v_1u_3\\
v_2s_3 &= (f_2v_1u_3+s_2v_3g_3)s_3 = s_2v_3
\end{aligned}$$

which shows that the middle diagram in the statement commutes. □

The basis for the description of the category $S^{2,1}\mathscr{C}[n]$ for split-exact $\mathscr{C}$ is the following result. Here, for $\mathscr{D}$ a category, we write $\pi_0(i\mathscr{D})$ for the set of isomorphism classes of objects.

Proposition 7.3. *Let $(\mathscr{C},0)$ be a pointed split-exact category. For every non-negative integer n, the map induced by the forgetful functor*

$$\pi_0(iS^{2,1}_{\oplus}\mathscr{C}[n]) \xrightarrow{\pi_0(\phi^*)} \pi_0(iS^{2,1}\mathscr{C}[n]).$$

is a bijection.

Proof. We fix the non-negative integer n. It suffices by Remark 6.9 to show that the map $\pi_0(\tilde{\phi}^*)$ is injective and that the map $\pi_0(\bar{\phi}^*)$ is surjective.

First, we recall from Remark 6.4 that the composite functor

$$S^{2,1}_{\oplus}\mathscr{C}[n] \xrightarrow{\tilde{\phi}^*} \tilde{S}^{2,1}\mathscr{C}[n] \xrightarrow{\nu} \mathrm{Cat}_*(S^{2,1}[n],(\mathscr{C},0))$$

is equal to the functor i^* defined in the proof of Proposition 4.11. Moreover, the latter functor was proved in said proof to be an equivalence of categories. Hence, we find that $\pi_0(\tilde{\phi}^*)$ is injective as desired.

Second, to prove that the map $\pi_0(\bar{\phi}^*)$ is surjective, we will prove the stronger statement that given an object $B\colon \mathrm{Cat}([1],[n]) \to \mathscr{C}$ of $\bar{S}^{2,1}\mathscr{C}[n]$, there exists an object $A\colon C[n] \to \mathscr{C}$ of $\bar{S}^{2,1}_{\oplus}\mathscr{C}[n]$ such that $B = A\circ\bar{\phi}$. This forces us to define

$$A(a,b) = B(\mu(a,b)) \qquad (\text{for all } 0 \leqslant a \leqslant b \leqslant n)$$

with $\mu = \mu(a,b)$ is determined by $\bar{\phi}(\mu) = (a,b)$, and to define

$$A(a,b) \xrightarrow{p_*} A(a-1,b) \qquad \text{(with } 0 < a \leqslant b \leqslant n\text{)}$$

$$A(a,b+1) \xrightarrow{i_*} A(a,b) \qquad \text{(with } 0 \leqslant a \leqslant b < n\text{)}$$

to be the images by the functor B of the unique morphisms $\mu(a,b) \Rightarrow \mu(a-1,b)$ and $\mu(a,b+1) \Rightarrow \mu(a,b)$, respectively. It remains to define the morphisms

$$A(a-1,b) \xrightarrow{s_*} A(a,b) \qquad \text{(with } 0 < a \leqslant b \leqslant n\text{)}$$

$$A(a,b) \xrightarrow{r_*} A(a,b+1) \qquad \text{(with } 0 \leqslant a \leqslant b < n\text{)}$$

such that s_* is a section of p_*, such that r_* is a retraction of i_*, and such that for all $0 < a \leqslant b < n$, the diagram following diagram commutes.

$$\begin{tikzcd}
A(a,b+1) & A(a-1,b+1) \arrow[l, "s_*"'] \\
A(a,b) \arrow[u, "r_*"] & A(a-1,b) \arrow[l, "s_*"'] \arrow[u, "r_*"]
\end{tikzcd}$$

To define the morphisms s_* and r_*, we proceed by induction on $N(a,b) = a(n-b)$, beginning from the trivial case $N(a,b) = 0$. So we fix (a,b) and assume that the morphisms $s_*\colon A(c-1,d) \to A(c,d)$ and $r_*\colon A(c,d) \to A(c,d+1)$ with the required properties have been defined for all (c,d) with $N(c,d) < N(a,b)$. In this situation, we apply Addendum 7.2 to a completion of the admissible square

$$\begin{tikzcd}
A(a,b+1) \arrow[r, "p_*"] \arrow[d, "i_*"] & A(a-1,b+1) \arrow[d, "i_*"] \\
A(a,b) \arrow[r, "p_*"] & A(a-1,b)
\end{tikzcd}$$

with the predescribed section s_* of the top horizontal map p_* and retraction r_* of the right-hand vertical map i_* provided by the inductive hypothesis. We conclude that there exists a section s_* of the bottom horizontal map p_* and a retraction r_* of the left-hand vertical map i_* that make the top square diagram commute. This completes the proof. $\square$

Corollary 7.4. *Let $(\mathscr{C},0)$ be a pointed exact category and let n be a non-negative integer. If $\mathscr{C}$ is split-exact, then so are the exact categories $S^{2,1}\mathscr{C}[n]$ and $S^{2,1}_{\oplus}\mathscr{C}[n]$.*

Proof. We first consider the exact category $S^{2,1}_{\oplus}\mathscr{C}[n]$. From the proof of Proposition 4.11, we have the exact adjoint equivalence of categories $(i_!, i^*, \eta, \varepsilon)$ from the exact category $S^{2,1}_{\oplus}\mathscr{C}[n]$ to the exact category $\mathrm{Cat}_*(S^{2,1}[n], (\mathscr{C},0))$. The latter exact category is clearly split-exact, and therefore, so is $S^{2,1}_{\oplus}\mathscr{C}[n]$.

We proceed to prove that $S^{2,1}\mathscr{C}[n]$ is split-exact. We recall from Lemma 7.1 that the exact category $E(\mathscr{C})$ of exact sequence in $\mathscr{C}$ again is split-exact. Therefore, it follows from Proposition 7.3 that the map

$$\pi_0(iS^{2,1}_{\oplus}E(\mathscr{C})[n]) \xrightarrow{\pi_0(\phi^*)} \pi_0(iS^{2,1}E(\mathscr{C})[n])$$

is a bijection. Moreover, since the exact categories $S^{2,1}_{\oplus}E(\mathscr{C})[n]$ and $E(S^{2,1}_{\oplus}\mathscr{C}[n])$ and the exact categories $S^{2,1}E(\mathscr{C})[n]$ and $E(S^{2,1}\mathscr{C}[n])$ are canonically isomorphic, we conclude that the map

$$\pi_0(iE(S^{2,1}_{\oplus}\mathscr{C}[n])) \xrightarrow{\pi_0(E(\phi^*))} \pi_0(iE(S^{2,1}\mathscr{C}[n])$$

is a bijection. This shows that every exact sequence in $S^{2,1}\mathscr{C}[n]$ is isomorphic to an exact sequence that is the image by ϕ^* of an exact sequence in $S^{2,1}_{\oplus}\mathscr{C}[n]$. Now, since every exact sequence in $S^{2,1}_{\oplus}\mathscr{C}[n]$ is split-exact, we conclude that every exact sequence in $S^{2,1}\mathscr{C}[n]$ is isomorphic to a split-exact sequence. But an exact sequence that is isomorphic to a split-exact sequence is itself split-exact. $\square$

Let $A,B \in \operatorname{ob} S^{2,1}_{\oplus}\mathscr{C}[n]$. We proceed to describe the set of morphisms

$$\phi^*(A) \xrightarrow{f} \phi^*(B)$$

from $\phi^*(A)$ to $\phi^*(B)$ in the category $S^{2,1}\mathscr{C}[n]$. The morphism f is determined by the family of morphisms $f_\theta\colon \phi^*(A)(\theta) \to \phi^*(B)(\theta)$ indexed by the set of functors $\theta\colon [2] \to [n]$. In addition, since A and B are sheaves, the morphism f_θ, in turn, is determined by the family of morphisms

$$A(\{\rho',\infty\}) \xrightarrow{f_{\theta,\rho,\rho'}} B(\{\rho,\infty\})$$

indexed by the set of pairs (ρ,ρ') of elements of $\phi(\theta)$ different from the basepoint with $f_{\theta,\rho,\rho'}$ defined as the composite morphism

$$A(\{\rho',\infty\}) \xrightarrow{A(\{\rho',\infty\})} \phi^*(A)(\theta) \xrightarrow{\quad f_\theta \quad} \phi^*(B)(\theta) \xrightarrow{B(\{\rho,\infty\})} B(\{\rho,\infty\}).$$

We call the morphism $f_{\theta,\rho,\rho'}$ the (ρ,ρ')th component of the morphism f_θ.

Lemma 7.5. *Let n be a non-negative integer and let $(\mathscr{C},0)$ be a pointed exact category. Let $A,B \in \operatorname{ob} S^{2,1}_{\oplus}\mathscr{C}[n]$ and let*

$$\phi^*(A)(\theta) \xrightarrow{f_\theta} \phi^*(B)(\theta)$$

be a family of morphisms in $\mathscr{C}$ indexed $\theta \in \operatorname{ob}\mathrm{Cat}([2],[n])$. The following (i)–(ii) *are equivalent.*

(i) *The family of morphisms* $\{f_\theta\}$ *constitute a natural transformation*

$$\phi^*(A) \xrightarrow{f} \phi^*(B).$$

(ii) *The family of component morphisms*

$$A(\{\rho',\infty\}) \xrightarrow{f_{\theta,\rho,\rho'}} B(\{\rho,\infty\}),$$

indexed by $\theta \in \operatorname{ob}\mathrm{Cat}([2],[n])$ *and* $\rho,\rho' \in \phi(\theta)$ *different from the basepoint, satisfies the following* (a)–(c)*, for every morphism* $\theta' \Rightarrow \theta$ *in* $\mathrm{Cat}([2],[n])$.

(a) *If both* $\rho,\rho' \in \phi(\theta)$ *and* $\rho,\rho' \in \phi(\theta')$*, then* $f_{\theta,\rho,\rho'} = f_{\theta',\rho,\rho'}$.

(b) *If* $\rho,\rho' \in \phi(\theta)$ *and* $\rho \in \phi(\theta')$ *but* $\rho' \notin \phi(\theta')$*, then* $f_{\theta,\rho,\rho'} = 0$.

(c) *If* $\rho,\rho' \in \phi(\theta')$ *and* $\rho \notin \phi(\theta)$ *but* $\rho' \in \phi(\theta)$*, then* $f_{\theta',\rho,\rho'} = 0$.

Proof. The family of morphisms $f_\theta\colon \phi^*(A)(\theta) \to \phi^*(B)(\theta)$ with $\theta \in \operatorname{ob}\mathrm{Cat}([2],[n])$ form a natural transformation $f\colon \phi^*(A) \to \phi^*(B)$ if and only if the diagram

$$\begin{tikzcd}
A(\phi(\theta')) \arrow[r, "f_{\theta'}"] & B(\phi(\theta')) \\
A(\phi(\theta)) \arrow[r, "f_\theta"] \arrow[u, "A(\phi(\theta)\cap\phi(\theta'))"'] & B(\phi(\theta)) \arrow[u, "B(\phi(\theta)\cap\phi(\theta'))"']
\end{tikzcd}$$

commutes, for every morphism $\theta' \Rightarrow \theta$ in $\mathrm{Cat}([2],[n])$. But since A and B are sheaves, this diagram commutes precisely if the conditions (a)–(c) are satisfied. □

Lemma 7.6. *Let n be a non-negative integer and let $(\mathscr{C},0)$ be a pointed exact category. Let $A,B \in \operatorname{ob}S^{2,1}_{\oplus}\mathscr{C}[n]$ and let $f\colon \phi^*(A) \to \phi^*(B)$ is a morphism in $S^{2,1}\mathscr{C}[n]$. If $\theta,\theta' \in \operatorname{ob}\mathrm{Cat}([2],[n])$ and if $\rho,\rho' \in \phi(\theta)\cap\phi(\theta')$ are different from the basepoint, then the two component morphisms*

$$A(\{\rho',\infty\}) \xrightarrow{f_{\theta,\rho,\rho'}} B(\{\rho,\infty\}), \qquad A(\{\rho',\infty\}) \xrightarrow{f_{\theta',\rho,\rho'}} B(\{\rho,\infty\}),$$

are equal.

Proof. If, in addition, there exists a morphism $\theta' \Rightarrow \theta$ in $\mathrm{Cat}([2],[n])$, then it follows from Lemma 7.5 that $f_{\theta,\rho,\rho'} = f_{\theta',\rho,\rho'}$. In general, we assume that $\theta'(1) \leqslant \theta(1)$ and consider the following morphisms in $\mathrm{Cat}([2],[n])$, where we represent the functor $\varphi\colon [2] \to [n]$ by the vertical array of the numbers $\varphi(0)$, $\varphi(1)$, and $\varphi(2)$.

$$\begin{array}{ccccccccccc}
\theta(0) & & 0 & & 0 & & 0 & & 0 & & \theta'(0) \\
\theta(1) & \Longleftarrow & \theta(1) & \Longrightarrow & \theta(1) & \Longleftarrow & \theta'(1) & \Longleftarrow & \theta'(1) & \Longrightarrow & \theta'(1) \\
\theta(2) & & \theta(2) & & n & & n & & \theta'(2) & & \theta'(2)
\end{array}$$

This shows that $f_{\theta,\rho,\rho'} = f_{\theta',\rho,\rho'}$ as stated. □

Lemma 7.7. *Let n be a non-negative integer, and let $\rho,\rho'\colon [n] \to [2]$ be two functors that are surjective on objects. The following* (i)–(ii) *are equivalent.*

(i) *There exists a morphism* $\rho' \Rightarrow \rho$ *in* $\mathrm{Cat}([n],[2])$.
(ii) *For every morphism* $\theta' \Rightarrow \theta$ *in* $\mathrm{Cat}([2],[n])$*, if both* $\rho \in \phi(\theta)$ *and* $\rho' \in \phi(\theta')$*, then also* $\rho \in \phi(\theta')$ *and* $\rho' \in \phi(\theta)$*.*

Proof. There exists a morphism $\rho' \Rightarrow \rho$ in $\mathrm{Cat}([n],[2])$ if and only if $\rho'(s) \leqslant \rho(s)$, for all $s \in \mathrm{ob}([n])$; and there exists a morphism $\theta' \Rightarrow \theta$ in $\mathrm{Cat}([2],[n])$ if and only if $\theta'(s) \leqslant \theta(s)$, for all $s \in \mathrm{ob}([2])$. We first assume that (i) holds and prove (ii). So we let $\theta' \Rightarrow \theta$ be a morphism in $\mathrm{Cat}([2],[n])$. For all $s \in \mathrm{ob}[2]$, we have

$$s = \rho'(\theta'(s)) \leqslant \rho'(\theta(s)) \leqslant \rho(\theta(s)) = s$$
$$s = \rho'(\theta'(s)) \leqslant \rho(\theta'(s)) \leqslant \rho(\theta(s)) = s$$

which shows that $\rho' \in \phi(\theta)$ and $\rho \in \phi(\theta')$ as desired. This proves (ii).

Next, we show that if (i) fails, then also (ii) fails. If a morphism $\rho' \Rightarrow \rho$ does not exist, then either $\min \rho'^{-1}(1) < \min \rho^{-1}(1)$ or $\max \rho'^{-1}(1) < \max \rho^{-1}(1)$ or both. If the first inequality holds, then we consider $\theta' \Rightarrow \theta$ with $\theta, \theta' \colon [2] \to [n]$ defined by

$$\theta(s) = \begin{cases} 0 & \text{if } s = 0 \\ \min \rho^{-1}(1) & \text{if } s = 1 \\ n & \text{if } s = 2 \end{cases} \qquad \theta'(s) = \begin{cases} 0 & \text{if } s = 0 \\ \min \rho'^{-1}(1) & \text{if } s = 1 \\ n & \text{if } s = 2 \end{cases}$$

We now have $\rho \in \phi(\theta)$, $\rho' \in \phi(\theta')$, but $\rho \notin \phi(\theta')$ since $\rho(\theta'(1)) = 0$, so (ii) fails. Similarly, if the second of the two inequalities above holds, then we consider the morphism $\theta' \Rightarrow \theta$ with $\theta, \theta' \colon [2] \to [n]$ defined by

$$\theta(s) = \begin{cases} 0 & \text{if } s = 0 \\ \max \rho^{-1}(1) & \text{if } s = 1 \\ n & \text{if } s = 2 \end{cases} \qquad \theta'(s) = \begin{cases} 0 & \text{if } s = 0 \\ \max \rho'^{-1}(1) & \text{if } s = 1 \\ n & \text{if } s = 2 \end{cases}$$

We now have $\rho \in \phi(\theta)$, $\rho' \in \phi(\theta')$, but $\rho' \notin \phi(\theta)$ since $\rho'(\theta(1)) = 2$, so again (ii) fails. This proves that (ii) implies (i). $\square$

Definition 7.8. Let n be a non-negative integer. A morphism $\rho' \Rightarrow \rho$ in $\mathrm{Cat}([n],[2])$ is admissible if there exists $\theta \colon [2] \to [n]$ such that $\rho, \rho' \in \phi(\theta)$.

Remark 7.9. Let $\rho, \rho' \colon [n] \to [2]$ be a pair of surjective functors. There exists a morphism $\rho' \Rightarrow \rho$ in $\mathrm{Cat}([n],[2])$ if and only if both $\min \rho^{-1}(1) \leqslant \min \rho'^{-1}(1)$ and $\max \rho^{-1}(1) \leqslant \max \rho'^{-1}(1)$. The morphism is admissible if and only if, in addition, $\min \rho'^{-1}(1) \leqslant \max \rho^{-1}(1)$. Hence, there exists an admissible morphism $\rho' \Rightarrow \rho$ in $\mathrm{Cat}([n],[2])$ if and only if the following three inequalities hold:

$$\min \rho^{-1}(1) \leqslant \min \rho'^{-1}(1) \leqslant \max \rho^{-1}(1) \leqslant \max \rho'^{-1}(1).$$

We conclude that the admissible morphisms are saturated in the sense that if the composite morphism $\rho'' \Rightarrow \rho' \Rightarrow \rho$ is admissible, then both $\rho'' \Rightarrow \rho'$ and $\rho' \Rightarrow \rho$ are admissible. Indeed, we have

$$\min \rho'^{-1}(1) \leqslant \min \rho''^{-1}(1) \leqslant \max \rho^{-1}(1) \leqslant \max \rho'^{-1}(1),$$

where the left-hand and right-hand inequalities hold because there are morphisms $\rho'' \Rightarrow \rho'$ and $\rho' \Rightarrow \rho$, respectively, and where the middle inequality holds by the assumption that $\rho'' \Rightarrow \rho$ is admissible. In general, however, the composition $\rho'' \Rightarrow \rho$ of two admissible morphisms $\rho'' \Rightarrow \rho'$ and $\rho' \Rightarrow \rho$ need not be admissible.

Definition 7.10. Let n be a non-negative integer, let $(\mathscr{C}, 0)$ be a pointed exact category, and let $A, B \in \operatorname{ob} S^{2,1}_{\oplus}\mathscr{C}[n]$. The matrix of a morphism

$$\phi^*(A) \xrightarrow{f} \phi^*(B)$$

in $S^{2,1}\mathscr{C}[n]$ is the family of morphisms $(f_{\rho \Leftarrow \rho'})$ indexed by the set of the admissible morphisms $\rho' \Rightarrow \rho$ in $\operatorname{Cat}([n],[2])$ in which $f_{\rho \Leftarrow \rho'}$ is the (ρ, ρ')th component

$$A(\{\rho', \infty\}) \xrightarrow{A(\{\rho',\infty\})} \phi^*(A)(\theta) \xrightarrow{f_\theta} \phi^*(B)(\theta) \xrightarrow{B(\{\rho,\infty\})} B(\{\rho, \infty\})$$

of the map $f_\theta \colon \phi^*(A)(\theta) \to \phi^*(B)(\theta)$, for any $\theta \in \operatorname{ob}\operatorname{Cat}([2],[n])$ with $\rho, \rho' \in \phi(\theta)$.

It follows form Lemma 7.6 that the matrix of f is well-defined.

Remark 7.11. The matrix $(\mathrm{id}_{\rho \Leftarrow \rho'})$ of the identity morphism is given by

$$\mathrm{id}_{\rho \Leftarrow \rho'} = \begin{cases} \mathrm{id} & \text{if } \rho = \rho' \\ 0 & \text{if } \rho \neq \rho'. \end{cases}$$

The matrix of the composition $g \circ f \colon \phi^*(A) \to \phi^*(C)$ of $f \colon \phi^*(A) \to \phi^*(B)$ and $g \colon \phi^*(B) \to \phi^*(C)$ is given by the matrix multiplication formula

$$(g \circ f)_{\rho \Leftarrow \rho''} = \sum_{\rho \Leftarrow \rho' \Leftarrow \rho''} g_{\rho \Leftarrow \rho'} \circ f_{\rho' \Leftarrow \rho''}.$$

Here the sum on the right-hand side is indexed by all factorizations $\rho'' \Rightarrow \rho' \Rightarrow \rho$ of the admissible morphism $\rho'' \Rightarrow \rho$ in $\operatorname{Cat}([n],[2])$.

Proposition 7.12. *Let n be a non-negative integer, let $(\mathscr{C}, 0)$ be a pointed exact category, and let $A, B \in \operatorname{ob} S^{2,1}_{\oplus}\mathscr{C}[n]$. The map*

$$S^{2,1}\mathscr{C}[n](\phi^*(A), \phi^*(B)) \longrightarrow \prod \mathscr{C}(A(\{\rho', \infty\}), B(\{\rho, \infty\}))$$

that to the morphism f associates its matrix $(f_{\rho \Leftarrow \rho'})$ is an isomorphism of abelian groups. Here, the product on the right-hand side is indexed by the set of admissible morphisms $\rho' \Rightarrow \rho$ in $\operatorname{Cat}([n],[2])$.

Proof. If $\rho' \Rightarrow \rho$ is an admissible morphism, and if $\rho, \rho' \in \phi(\theta)$, then Lemmas 7.5 and 7.7 show that every morphism $g \colon A(\{\rho', \infty\}) \to B(\{\rho, \infty\})$ in $\mathscr{C}$ appears as the component $g = f_{\rho \Leftarrow \rho'} = f_{\theta, \rho, \rho'}$ of a morphism $f \colon \phi^*(A) \to \phi^*(B)$ in $S^{2,1}\mathscr{C}[n]$. This shows that the map in the statement is surjective. Similarly, if $f \colon \phi^*(A) \to \phi^*(B)$ is a morphism in $S^{2,1}\mathscr{C}[n]$, if $\theta \in \operatorname{ob}\operatorname{Cat}([2],[n])$, and if $\rho, \rho' \in \phi(\theta)$, then Lemmas 7.5 and 7.7 show that $f_{\theta,\rho,\rho'} = f_{\rho \Leftarrow \rho'}$, if there exists a (necessarily admissible) morphism $\rho' \Rightarrow \rho$ in $\operatorname{Cat}([n],[2])$, and that $f_{\theta,\rho,\rho'} = 0$, otherwise. This shows that the map in the statement is also injective. $\square$

Remark 7.13. We recall from the proof of Proposition 7.3 that the forgetful functor ϕ^* is fully faithful, and hence, the map

$$S^{2,1}_{\oplus}\mathscr{C}[n](A,B) \xrightarrow{\phi^*} S^{2,1}\mathscr{C}[n](\phi^*(A),\phi^*(B))$$

is injective. The image is precisely the morphisms f whose matrix $(f_{\rho \Leftarrow \rho'})$ is diagonal in the sense that $f_{\rho \Leftarrow \rho'}$ is zero for $\rho \neq \rho'$.

We next consider the map of the sets of isomorphism classes of symmetric objects

$$\pi_0(\mathrm{Sym}(iS^{2,1}_{\oplus}\mathscr{C}[n],D[n])) \xrightarrow{\pi_0(\mathrm{Sym}(\phi^*))} \pi_0(\mathrm{Sym}(iS^{2,1}\mathscr{C}[n],D[n]))$$

induced by the forgetful functor. The map is injective, by an argument similar to the proof of Proposition 7.3, but it is generally not surjective. However, orthogonal sum of symmetric spaces gives rise to abelian monoid structures on the domain and target, and we proceed to show that the induced map of the abelian groups obtained by group-completion is a bijection. In preparation, we recall from [20] some classical theory concerning symmetric spaces.

Definition 7.14. Let $(\mathscr{C},\mathscr{E},D,\eta)$ be an exact category with duality, and let (A,φ) be a non-degenerate symmetric object in the category with duality $(\mathscr{C},D,\eta)$.

(1) An admissible monomorphism $i\colon L \to A$ such that the composite morphism

$$L \xrightarrow{i} A \xrightarrow{\varphi} D(A) \xrightarrow{D(i)} D(L)$$

is zero is said to admit an orthogonal complement with respect to (A,φ), and a kernel $i^{\perp}\colon L^{\perp} \to A$ of the admissible epimorphism $D(i)\circ\varphi\colon A \to D(L)$ is said to be an orthogonal complement of $i\colon L \to A$ with respect to (A,φ).

(2) An admissible monomorphism $i\colon L \to A$ is said to be a sub-Lagrangian of (A,φ) if it admits an orthogonal complement $i^{\perp}\colon L^{\perp} \to A$ and if the unique morphism $j\colon L \to L^{\perp}$ such that $i = i^{\perp}\circ j$ is an admissible monomorphism.

(3) A sub-Lagrangian $i\colon L \to A$ of (A,φ) is a Lagrangian of (A,φ) if the admissible monomorphism $j\colon L \to L^{\perp}$ is an isomorphism.

(4) A non-degenerate symmetric object (A,φ) in $(\mathscr{C},D,\eta)$ is metabolic if it admits a Lagrangian $i\colon L \to A$.

Remark 7.15. (i) The assumption that $\varphi\colon A \to D(A)$ be an isomorphism is used to conclude in (2) that $D(i)\circ\varphi\colon A \to D(L)$ is an admissible epimorphism, and hence, admits a kernel. This kernel, $i^{\perp}\colon L^{\perp} \to A$, is unique, up to unique isomorphism.
(ii) An admissible monomorphism $i\colon L \to A$ is a Lagrangian for the non-degenerate symmetric object (A,φ) if and only if the sequence

$$L \xrightarrow{i} A \xrightarrow{D(i)\circ\varphi} D(L)$$

is exact.

Example 7.16. Let $(\mathscr{C},\mathscr{E},D,\eta)$ be an exact category with duality, let L be an object of $\mathscr{C}$, and let $\gamma\colon D(L)\to D(D(L))$ be a symmetric form on $D(L)$, not necessarily non-degenerate. We define the associated split metabolic object $H(L,\gamma)$ to be the non-degenerate symmetric space (A,φ) in $(\mathscr{C},D,\eta)$, where A is a choice of sum of L and $D(L)$ in $\mathscr{C}$, and where the morphism $\varphi\colon A\to D(A)$ is given by

$$\begin{pmatrix} D(\mathrm{in}_1)\circ\varphi\circ\mathrm{in}_1 & D(\mathrm{in}_1)\circ\varphi\circ\mathrm{in}_2 \\ D(\mathrm{in}_2)\circ\varphi\circ\mathrm{in}_1 & D(\mathrm{in}_2)\circ\varphi\circ\mathrm{in}_2 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id}_{D(L)} \\ \eta_L & \gamma \end{pmatrix}.$$

The symmetric object $H(L,\gamma)$ is metabolic with $\mathrm{in}_1\colon L\to A$ as a Lagrangian. We write $H(L)$ instead of $H(L,0)$ and call it the hyperbolic object associated with L.

Lemma 7.17. *Let $(\mathscr{C},\mathscr{E},D,\eta)$ be an exact category with duality and let (A,φ) be a symmetric object in $(i\mathscr{C},D,\eta)$. Suppose that $i\colon L\to A$ is a sub-Lagrangian of (A,φ) and let $p\colon L^\perp\to L^\perp/L$ be a cokernel of $j\colon L\to L^\perp$. In this situation, there is a unique non-degenerate symmetric from $\bar{\varphi}\colon L^\perp/L\to D(L^\perp/L)$ making the diagram*

$$\begin{tikzcd} L^\perp/L \arrow[d, "\bar{\varphi}"] & L^\perp \arrow[l, "p"'] \arrow[r, "i^\perp"] \arrow[d, "D(i^\perp)\circ\varphi\circ i^\perp"] & A \arrow[d, "\varphi"] \\ D(L^\perp/L) \arrow[r, "D(p)"] & D(L^\perp) & D(A) \arrow[l, "D(i^\perp)"'] \end{tikzcd}$$

commute. Moreover, the orthogonal sum $(A,\varphi)\perp(L^\perp/L,-\bar{\varphi})$ is metabolic with the morphism $(i^\perp,p)\colon L^\perp\to A\oplus(L^\perp/L)$ as a Lagrangian.

Proof. See [20, Lemma 2.6]. □

Example 7.18. Taking $L=0$ in Lemma 7.17, we find that $(A,\varphi)\perp(A,-\varphi)$ is metabolic with the diagonal morphism $\Delta\colon A\to A\oplus A$ as a Lagrangian.

Lemma 7.19. *Let $(\mathscr{C},\mathscr{E},D,\eta)$ be a split-exact category with duality and let (A,φ) be a metabolic symmetric object in $(\mathscr{C},D,\eta)$. The symmetric object (A,φ) together with choices of a Lagrangian $i\colon L\to A$ of (A,φ) and a retraction $r\colon A\to L$ of i determine a metabolic symmetric object (A',φ') in $(\mathscr{C},D,\eta)$ and an isomorphism*

$$H(L)\perp(A',\varphi')\xrightarrow{\;g\;}(A,\varphi)\perp(A',\varphi')$$

of symmetric objects in $(\mathscr{C},D,\eta)$.

Proof. The morphism $s=\varphi^{-1}\circ D(r)$ is a section of $D(i)\circ\varphi$, but the composite morphism $r\circ s$ need not be zero. It satisfies $D(r\circ s)=\eta_L\circ r\circ s$, however, so the common morphism $\gamma\colon D(L)\to D(D(L))$ is a symmetric form on $D(L)$. Moreover, the morphism $f=i+s\colon H(L,\gamma)\to(A,\varphi)$ is an isomorphism of symmetric objects in $(\mathscr{C},D,\eta)$. So we may assume that (A,φ) is equal to the split metabolic object $H(L,\gamma)$ defined in Example 7.16. Now,

$$H(L,0)\perp H(L,-\gamma)\xrightarrow{\;g\;}H(L,\gamma)\perp H(L,-\gamma)$$

defined by

$$\big(D(\mathrm{in}_i)\circ g\circ \mathrm{in}_j\big) = \begin{pmatrix} \mathrm{id}_L & 0 & -\mathrm{id}_L & \eta_L^{-1}\circ\gamma \\ 0 & \mathrm{id}_{D(L)} & 0 & 0 \\ 0 & 0 & \mathrm{id}_L & 0 \\ 0 & \mathrm{id}_{D(L)} & 0 & \mathrm{id}_{D(L)} \end{pmatrix}$$

is the desired isomorphism of symmetric objects in $(\mathscr{C},D,\eta)$. $\square$

If $(\mathscr{C},\mathscr{E},D,\eta)$ is an exact category with duality, then orthogonal sum gives rise to a symmetric monoidal structure on the groupoid $\mathrm{Sym}(i\mathscr{C},D,\eta)$ of non-degenerate symmetric objects in $(\mathscr{C},D,\eta)$. This, by turn, gives rise to an abelian monoid structure on the set $\pi_0(\mathrm{Sym}(i\mathscr{C},D,\eta))$ of isomorphism classes of objects, and we let

$$\pi_0(\mathrm{Sym}(i\mathscr{C},D,\eta)) \xrightarrow{\iota} \pi_0(\mathrm{Sym}(i\mathscr{C},D,\eta))^{\mathrm{gp}}$$

be the group-completion. We write $[(A,\varphi)]$ for the image by ι of the isomorphism class of the object (A,φ) of $\mathrm{Sym}(i\mathscr{C},D,\eta)$.

Corollary 7.20. *Let $(\mathscr{C},\mathscr{E},D,\eta)$ be a split-exact category with duality, let (A,φ) be a non-degenerate symmetric object in $(\mathscr{C},D,\eta)$, and let $i\colon L\to A$ be a sub-Lagrangian of (A,φ). In this situation, the following identities hold in $\pi_0(\mathrm{Sym}(i\mathscr{C},D,\eta))^{\mathrm{gp}}$.*

(i) $[(A,\varphi)]+[(A,-\varphi)] = [H(A)]$
(ii) $[(A,\varphi)] = [H(L)]+[(L^\perp/L,\bar\varphi)]$

Proof. The left-hand side of (i) is equal to $[(A,\varphi)\perp(A,-\varphi)]$, by definition, and $(A,\varphi)\perp(A,-\varphi)$ is metabolic with Lagragian $\Delta\colon A\to A\oplus A$, by Example 7.18. The identity (i) now follows from Lemma 7.19. To prove (ii), we similarly conclude from Lemmas 7.17 and 7.19 that

$$[(A,\varphi)]+[(L^\perp/L,-\bar\varphi)] = [H(L^\perp)].$$

Hence, adding $[(L^\perp/L,\bar\varphi)]$ on both sides and applying (i), we obtain that

$$[(A,\varphi)]+[H(L^\perp/L)] = H(L^\perp)+[(L^\perp/L,\bar\varphi)].$$

Finally, since $\mathscr{C}$ is split-exact, $L^\perp$ is a sum of L and $L^\perp/L$, and therefore, $H(L^\perp)$ is an orthogonal sum of $H(L)$ and $H(L^\perp/L)$. Hence, subtracting $H(L^\perp/L)$ on both sides, the identity (ii) follows. $\square$

We return to the problem at hand. Let $\rho\colon[n]\to[2]$ be a surjective functor. There always exists a morphism $D\rho\Rightarrow\rho$ or $\rho\Rightarrow D\rho$. If both exists, then $\rho=D\rho$. If the former but not the latter exists, then we say that ρ is positive; and if the latter but not the former exists, then we say that ρ is negative.

Theorem 7.21. *Let $(\mathscr{C},\mathscr{E},D,\eta)$ be a split-exact category with duality. For every non-negative integer n, the map induced by the forgetful functor*

$$\pi_0(\mathrm{Sym}(iS_\oplus^{2,1}(\mathscr{C},\mathscr{E},D,\eta)[n]))^{\mathrm{gp}} \xrightarrow{\phi^*} \pi_0(\mathrm{Sym}(iS^{2,1}(\mathscr{C},\mathscr{E},D,\eta)[n]))^{\mathrm{gp}}$$

is an isomorphism.

Proof. Only surjectivity is at issue. We assume that $(\mathscr{C},D)$ is a strict category with duality and consider the diagram of additive categories with strict duality

$$\begin{array}{ccccc}
S_{\oplus}^{2,1}(\mathscr{C},D)[n] & & \xrightarrow{\phi^*} & & S^{2,1}(\mathscr{C},D)[n] \\
\big\downarrow{\scriptstyle \phi'} & & & & \big\uparrow{\scriptstyle \phi''} \\
\mathrm{Fil}^m S^{2,1}(\mathscr{C},D)[n] & \xrightarrow{i_m} & \cdots & \xrightarrow{i_1} & \mathrm{Fil}^0 S^{2,1}(\mathscr{C},D)[n]
\end{array}$$

defined as follows. We choose a linear order $\rho_m \leqslant \cdots \leqslant \rho_1$ on the set of positive surjective functors $\rho\colon [n] \to [2]$ in such a way that $\rho' \leqslant \rho$ whenever there exists a morphism $\rho' \Rightarrow \rho$. This is possible, since every partial order on a set can be extended to a linear order. The category $\mathrm{Fil}^u S^{2,1}\mathscr{C}[n]$ has the same object set as $S_{\oplus}^{2,1}\mathscr{C}[n]$, and if $A,B \in \mathrm{ob}(\mathrm{Fil}^u S^{2,1}\mathscr{C}[n])$, then

$$\mathrm{Fil}^u S^{2,1}\mathscr{C}[n](A,B) \subset S^{2,1}\mathscr{C}[n](\phi^*(A),\phi^*(B))$$

is the subset of all morphisms $f\colon \phi^*(A) \to \phi^*(B)$ with the property that the matrix entry $f_{\rho\Leftarrow\rho'}$ is zero whenever ρ is positive and $\rho_u \leqslant \rho$ or ρ' is negative and $\rho_u \leqslant D\rho'$ or both. For $u = 0$, the matrix entries $f_{\rho\Leftarrow\rho'}$ are unrestricted. The functors i_u are given by the identity maps on object sets and by the canonical inclusions on morphism sets; the functor ϕ' is given by the identity map on object sets and by the map ϕ^* on morphism sets; and the functor ϕ'' is given by the map ϕ^* on object sets and by the identity map on morphism sets. Both ϕ' and ϕ'' are equivalences of exact categories with duality.

By easy induction, it suffices to show that for all $1 \leqslant u \leqslant m$, the map

$$\pi_0(\mathrm{Sym}(\mathrm{Fil}^u S^{2,1}(\mathscr{C},D)[n]))^{\mathrm{gp}} \longrightarrow \pi_0(\mathrm{Sym}(\mathrm{Fil}^{u-1} S^{2,1}(\mathscr{C},D)[n]))^{\mathrm{gp}}$$

induced by i_u is surjective. To this end, we let (A,φ) be a non-degenerate symmetric space in $\mathrm{Fil}^{u-1} S^{2,1}(\mathscr{C},D)[n]$ and proceed to show that the class $[(A,\varphi)]$ is in the image. Let $i\colon L \to A$ be the admissible monomorphism in $S_{\oplus}^{2,1}(\mathscr{C},D)[n]$ given by

$$L(U) = A(U \cap \{\rho_u,\infty\}) \xrightarrow{A(U\cap\{\rho_u,\infty\})} A(U).$$

We claim that the admissible monomorphism $\phi^*(i)\colon L \to A$ is a sub-Lagrangian of the symmetric space (A,φ) in $\mathrm{Fil}^{u-1} S^{2,1}(\mathscr{C},D)[n]$. To prove the claim, we first note that, since ρ_u is positive, there is no morphism $\rho_u \Rightarrow D\rho_u$. In particular, there is no admissible morphism $\rho_u \Rightarrow D\rho_u$, and therefore, the composite

$$L \xrightarrow{\phi^*(i)} A \xrightarrow{\varphi} D(A) \xrightarrow{D(\phi^*(i))} D(L)$$

is the zero morphism. It follows that $\phi^*(i)$ admits an orthogonal complement with respect to (A,φ) in $\mathrm{Fil}^{u-1} S^{2,1}(\mathscr{C},D)[n]$. We next let $i^{\perp}\colon L^{\perp} \to A$ be the admissible monomorphism in $S_{\oplus}^{2,1}(\mathscr{C},D)[n]$ defined by

$$L^{\perp}(U) = A(U \smallsetminus \{D\rho_u\}) \xrightarrow{A(U\smallsetminus\{D\rho_u\})} A(U).$$

The admissible monomorphism $\phi^*(i^\perp)\colon L^\perp \to A$ is a kernel of $D(\phi^*(i)) \circ \varphi$. Indeed, if $\rho' \Rightarrow D\rho_u$ is an admissible morphism, then ρ' is negative and $\rho_u \leqslant D\rho'$, and hence, the only non-zero matrix entry of $D(\phi^*(i)) \circ \varphi$ is

$$(D(\phi^*(i)) \circ \varphi)_{D\rho_u \Leftarrow D\rho_u} = \varphi_{D\rho_u \Leftarrow D\rho_u},$$

which is an isomorphism. This shows that $\phi^*(i)^\perp = \phi^*(i^\perp)$ is an orthogonal complement of $\phi^*(i)$ with respect to (A, φ). Finally, we have $\phi^*(i) = \phi^*(i)^\perp \circ \phi^*(j)$, where $j\colon L \to L^\perp$ is the admissible monomorphism in $S^{2,1}_\oplus(\mathscr{C}, D)[n]$ defined by

$$L(U) = A(U \cap \{\rho_u, \infty\}) \xrightarrow{A(U \cap \{\rho_u, \infty\})} A(U \smallsetminus \{D\rho_u\}) = L^\perp(U).$$

This shows that $\phi^*(i)\colon L \to A$ is a sub-Lagrangian of (A, φ) as claimed.

We conclude from Corollary 7.20 (ii) that the equality

$$[(A, \varphi)] = [H(L)] + [(L^\perp/L, \bar{\varphi})]$$

holds in $\pi_0(\mathrm{Sym}(\mathrm{Fil}^{u-1} S^{2,1}(\mathscr{C}, D)[n]))^{\mathrm{gp}}$. We further claim that both summands on the right-hand side are contained in the image of $\pi_0(\mathrm{Sym}(\mathrm{Fil}^{u} S^{2,1}(\mathscr{C}, D)[n]))^{\mathrm{gp}}$. For the first summand, this is clear, and for the second summand, we must show that the matrix entry $(D(\phi^*(i)^\perp) \circ \varphi \circ \phi^*(i)^\perp)_{\rho \Leftarrow \rho'}$ is zero whenever ρ is positive and $\rho_u \leqslant \rho$ or ρ' is negative and $\rho_u \leqslant D\rho'$ or both. But only the cases $\rho = \rho_u$ and $\rho' = D\rho_u$ need proof, and both follow immediately from $L^\perp(\{D\rho_u, \infty\})$ being trivial. This completes the proof. $\square$

8 The comparison theorem

This section is devoted to proof of the following comparison theorem. In outline, the proof is similar to Quillen's proof of [18, Theorem 2], but the details are somewhat more involved.

Theorem 8.1. *Let $(\mathscr{C},D,0)$ be a pointed split-exact category with strict duality, and let $i\mathscr{C} \subset \mathscr{C}$ be the subcategory of isomorphisms. Then the forgetful map*

$$KR^{\oplus}(\mathscr{C},i\mathscr{C},D,0) \xrightarrow{\phi^*} KR(\mathscr{C},i\mathscr{C},D,0)$$

is a level weak equivalence of real symmetric spectra.

We begin the proof with the following reduction.

Lemma 8.2. *The following* (i)–(ii) *are equivalent.*

(i) *For every pointed split-exact category with strict duality $(\mathscr{C},D,0)$, the map*

$$KR^{\oplus}(\mathscr{C},i\mathscr{C},D,0)_1 \xrightarrow{\phi^*} KR(\mathscr{C},i\mathscr{C},D,0)_1$$

is a weak equivalence of pointed real spaces.

(ii) *For every pointed split-exact category with strict duality $(\mathscr{C},D,0)$, the map*

$$KR^{\oplus}(\mathscr{C},i\mathscr{C},D,0) \xrightarrow{\phi^*} KR(\mathscr{C},i\mathscr{C},D,0)$$

is a level weak equivalence of real symmetric spectra.

Proof. The statement (i) is a special case of the statement (ii). So we assume (i) and prove (ii). We wish to show that for every pointed split-exact category with strict duality $(\mathscr{C},D,0)$ and for every non-negative integer r, the map

$$|N(iS^{2r,r}_{\oplus}[-],D[-],0[-])[-]|_R \xrightarrow{\phi_r^*} |N(iS^{2r,r}\mathscr{C}[-],D[-],0[-])[-]|_R$$

is a weak equivalence of pointed real spaces. The statement is trivial for $r=0$ and holds for $r=1$ by assumption. For $r \geqslant 2$, we consider the commutative diagram

$$\begin{array}{ccc}
|N(iS^{2r,r}_{\oplus}[-],D[-],0[-])[-]|_R & \xrightarrow{i_r^{\#}} & |N(iS^{2,1}_{\oplus}\dots S^{2,1}_{\oplus}\mathscr{C}[-],D[-],0[-])[-]|_R \\
\Big\downarrow{\scriptstyle \phi_r^*} & & \Big\downarrow{\scriptstyle \phi^* \dots \phi^*} \\
|N(iS^{2r,r}\mathscr{C}[-],D[-],0[-])[-]|_R & \xrightarrow{u_r} & |N(iS^{2,1}\dots S^{2,1}\mathscr{C}[-],D[-],0[-])[-]|_R,
\end{array}$$

where the right-hand map is given by applying the forgetful functor ϕ^* in each of the r real simplicial directions in the r-fold iterate of the 1-real Segal construction, where the upper horizontal map is the weak equivalence of pointed real spaces from Lemma 5.2, and where the lower horizontal map is the isomorphism of pointed real spaces defined in the discussion following Definition 3.1. It follows that it will suffice to show that the right-hand vertical map is a weak equivalence of pointed real spaces. We write this map as the composite map

$$\begin{aligned}
&|N(iS^{2,1}_{\oplus}S^{2,1}_{\oplus}\dots S^{2,1}_{\oplus}\mathscr{C}[-],D[-],0[-])[-]|_R\\
&\xrightarrow{\phi^*_{(1)}} |N(iS^{2,1}S^{2,1}_{\oplus}\dots S^{2,1}_{\oplus}\mathscr{C}[-],D[-],0[-])[-]|_R\\
&\xrightarrow{\phi^*_{(2)}} |N(iS^{2,1}S^{2,1}\dots S^{2,1}_{\oplus}\mathscr{C}[-],D[-],0[-])[-]|_R\\
&\quad\vdots\\
&\xrightarrow{\phi^*_{(r)}} |N(iS^{2,1}S^{2,1}\dots S^{2,1}\mathscr{C}[-],D[-],0[-])[-]|_R
\end{aligned}$$

with the map $\phi^*_{(i)}$ given by applying the forgetful functor ϕ^* in the ith real simplicial direction. We first show that the map $\phi^*_{(1)}$ is a weak equivalence of pointed real spaces. By Proposition 3.4, we may instead show that the map

$$\begin{aligned}
&|N(iS^{2,1}_{\oplus}S^{2,1}_{\oplus}\dots S^{2,1}_{\oplus}\mathscr{C}[-],D[-],0[-])[-]|'_R\\
&\xrightarrow{\phi^*_{(1)}} |N(iS^{2,1}S^{2,1}_{\oplus}\dots S^{2,1}_{\oplus}\mathscr{C}[-],D[-],0[-])[-]|'_R
\end{aligned}$$

is a weak equivalence of pointed real spaces, and by the realization lemma, this map, in turn, is a weak equivalence of pointed real spaces if and only if the map

$$\begin{aligned}
&|N(iS^{2,1}_{\oplus}(S^{2,1}_{\oplus}(\dots S^{2,1}_{\oplus}\mathscr{C}[n_r]\dots)[n_2])[-])[-]|_R\\
&\xrightarrow{\phi^*} |N(iS^{2,1}(S^{2,1}_{\oplus}(\dots S^{2,1}_{\oplus}\mathscr{C}[n_r]\dots)[n_2])[-])[-]|_R
\end{aligned}$$

is a weak equivalence of pointed real spaces, for all non-negative integers $n_2,\dots,n_r$. Since Corollary 7.4 shows that $S^{2,1}_{\oplus}(\dots S^{2,1}_{\oplus}\mathscr{C}[n_r]\dots)[n_2]$ is split-exact, this follows from the assumption that (i) holds. Hence, the map $\phi^*_{(1)}$ is a weak equivalence of pointed real spaces. Finally, to prove that $\phi^*_{(i)}$ is a weak equivalence of pointed real spaces, we consider the following commutative diagram of real simplicial pointed exact categories with strict duality, where the vertical functors are the isomorphisms of categories induced by the transposition $(1,i)\in\Sigma_r$ as in Lemmas 3.2 and 5.3.

$$\begin{tikzcd}
S^{2,1}\dots S^{2,1}S^{2,1}_{\oplus}S^{2,1}_{\oplus}\dots S^{2,1}_{\oplus}\mathscr{C}[-]\circ\Delta^{\mathrm{op}} \arrow[r, "\phi^*_{(i)}"] \arrow[d, "l_{(1,i)}"] & S^{2,1}\dots S^{2,1}S^{2,1}S^{2,1}_{\oplus}\dots S^{2,1}_{\oplus}\mathscr{C}[-]\circ\Delta^{\mathrm{op}} \arrow[d, "l_{(1,i)}"] \\
S^{2,1}_{\oplus}\dots S^{2,1}S^{2,1}S^{2,1}_{\oplus}\dots S^{2,1}_{\oplus}\mathscr{C}[-]\circ\Delta^{\mathrm{op}} \arrow[r, "\phi^*_{(1)}"] & S^{2,1}\dots S^{2,1}S^{2,1}S^{2,1}_{\oplus}\dots S^{2,1}_{\oplus}\mathscr{C}[-]\circ\Delta^{\mathrm{op}}
\end{tikzcd}$$

This shows that it will suffice to prove that the map

$$\begin{aligned}
&|N(iS^{2,1}_{\oplus}S^{2,1}\dots S^{2,1}S^{2,1}_{\oplus}\dots S^{2,1}_{\oplus}\mathscr{C}[-],D[-],0[-])[-]|_R\\
&\xrightarrow{\phi^*_{(1)}} |N(iS^{2,1}S^{2,1}\dots S^{2,1}S^{2,1}_{\oplus}\dots S^{2,1}_{\oplus}\mathscr{C}[-],D[-],0[-])[-]|_R
\end{aligned}$$

is a weak equivalence of pointed real spaces. But this follows from the argument above, since, by Corollary 7.4, the exact categories

$$S^{2,1}(\dots S^{2,1}(S^{2,1}(S^{2,1}_{\oplus}(\dots S^{2,1}_{\oplus}\mathscr{C}[n_r]\dots)[n_{i+1}])[n_1])[n_{i-1}]\dots)[n_2]$$

are split-exact. This completes the proof. $\square$

Before we proceed, we recall the following general result.

Lemma 8.3. *Let $f\colon X \to Y$ be a map of spaces and suppose that for every prime field k, the induced map $f_*\colon H_*(X,k) \to H_*(Y,k)$ of singular homology groups with k-coefficients is an isomorphism. Then the induced map*

$$f_*\colon H_*(X,\mathbb{Z}) \to H_*(Y,\mathbb{Z})$$

of singular homology groups with $\mathbb{Z}$-coefficients is an isomorphism.

Proof. Let HA be a choice of Eilenberg-Mac Lane spectrum for the abelian group A. The abelian groups $\pi_q(HA \wedge^L X_+)$ and $H_q(X,A)$ are isomorphic and the isomorphism may be chosen to be natural both in X and A. We recall the arithmetic square

$$\begin{tikzcd} H\mathbb{Z}\wedge^L X_+ \arrow[r] \arrow[d] & \prod(H\mathbb{Z}\wedge^L X_+)_p \arrow[d] \\ (H\mathbb{Z}\wedge^L X_+)_{\mathbb{Q}} \arrow[r] & (\prod(H\mathbb{Z}\wedge^L X_+)_p)_{\mathbb{Q}} \end{tikzcd}$$

where $(-)_p$ and $(-)_{\mathbb{Q}}$ indicates p-completion and rationalization, respectively, and where the products range over all prime numbers. It is a homotopy cartesian square by [3, Proposition 2.9]. Therefore, it suffices to prove the following (1)–(2).

(1) For all integers q, the map f induces isomorphisms

$$\pi_q((H\mathbb{Z}\wedge^L X_+)_{\mathbb{Q}}) \xrightarrow{\sim} \pi_q((H\mathbb{Z}\wedge^L Y_+)_{\mathbb{Q}}).$$

(2) For all integers q and for all prime numbers p, the map f induces isomorphisms

$$\pi_q((H\mathbb{Z}\wedge^L X_+)_p) \to \pi_q((H\mathbb{Z}\wedge Y_+)_p).$$

The statement (1) holds because f induces an isomorphism of singular homology groups with $\mathbb{Q}$-coefficients. To prove (2), we recall that p-completion is defined to be Bousfield localization with respect to a Moore spectrum M_p. Therefore, the statement (2) is equivalent to the statement that the map f induces an isomorphism

$$\pi_q(M_p \wedge^L H\mathbb{Z}\wedge^L X_+) \to \pi_q(M_p \wedge^L H\mathbb{Z}\wedge^L Y_+)$$

for all integers q and prime numbers p. But $M_p \wedge^L H\mathbb{Z}$ is an Eilenberg-Mac Lane spectrum for the prime field $\mathbb{F}_p$, so this holds because f induces an isomorphism of singular homology with $\mathbb{F}_p$-coefficients. $\square$

Remark 8.4. For every field extension k'/k, we have a natural isomorphism

$$H_*(X,k)\otimes_k k' \xrightarrow{\sim} H_*(X,k').$$

Therefore, since the functor $-\otimes_k k'$ is fully faithful, the map $f\colon X\to Y$ induces an isomorphism of singular homology groups with k-coefficients if and only if it induces an isomorphism of singular homology groups with k'-coefficients. It follows that, in the hypothesis of Lemma 8.3, we may replace the prime field k by any extension field k'/k. In particular, we may replace k by an algebraic closure of k.

Lemma 8.5. *Let $(\mathscr{C},D,0)$ be a pointed split-exact category with strict duality, and suppose that for every non-negative integer n, for every subgroup $H\subset G_{\mathbb{R}}$, and for every algebraically closed field k, the forgetful map*

$$\begin{aligned}&H_*(|N(iS^{2,1}_{\oplus}\mathscr{C}[n],D[n])[-]|^H_R,k)[\pi_0(|N(iS^{2,1}_{\oplus}\mathscr{C}[n],D[n])[-]|^H_R)^{-1}]\\ &\xrightarrow{\phi^*} H_*(|N(iS^{2,1}\mathscr{C}[n],D[n])[-]|^H_R)[\pi_0(|N(iS^{2,1}\mathscr{C}[n],D[n],k)[-]|^H_R)^{-1}]\end{aligned}$$

is an isomorphism of graded k-algebras. Then the forgetful map

$$KR^{\oplus}(\mathscr{C},i\mathscr{C},D,0)_1 \xrightarrow{\phi^*} KR(\mathscr{C},i\mathscr{C},D,0)_1$$

is a weak equivalence of pointed real spaces.

Proof. We wish to prove that the top horizontal map in the following commutative diagram is a weak equivalence of pointed real spaces.

$$\begin{tikzcd}
|N(iS^{2,1}_{\oplus}\mathscr{C}[-],D[-])[-]|_R \arrow[r, "\phi^*"] \arrow[d, "\tilde{\sigma}^{\oplus}_{0,1}"] & |N(iS^{2,1}\mathscr{C}[-],D[-])[-]|_R \arrow[d, "\tilde{\sigma}^{\oplus}_{0,1}"] \\
\Omega^{2,1}(|N(iS^{2,1}_{\oplus}S^{2,1}_{\oplus}\mathscr{C}[-],D[-])[-]|_R) \arrow[r, "\phi^*"] & \Omega^{2,1}(|N(iS^{2,1}_{\oplus}S^{2,1}\mathscr{C}[-],D[-])[-]|_R)
\end{tikzcd}$$

We claim that the vertical maps are weak equivalences of pointed real spaces. Indeed, this follows from Shimakawa [23, Theorem B] once we show that the two top terms are real connected in the sense that both the underlying pointed spaces and the subspaces of $G_{\mathbb{R}}$-fixed points are connected. We prove that the top left-hand term is real connected; the proof for the top right-hand term is analogous. First,

$$N(iS^{2,1}_{\oplus}\mathscr{C}[0])[0]=\{0[0](1)\}$$

which shows that the underlying pointed space is connected. Second, by Lemma 1.20, we have the homeomorphism

$$|N\mathrm{Sym}(\mathrm{sd}\,iS^{2,1}_{\oplus}\mathscr{C}[-],\mathrm{sd}D[-])[-]| \xrightarrow{d} |N(iS^{2,1}_{\oplus}\mathscr{C}[-],D[-])[-]|^{G_{\mathbb{R}}}_R$$

and since

$$N\mathrm{Sym}(\mathrm{sd}\,iS^{2,1}_{\oplus}\mathscr{C}[0],\mathrm{sd}D[0])[0]=\{(0[1](1),\mathrm{id})\}$$

we conclude that also the subspace of $G_{\mathbb{R}}$-fixed points is connected. This proves the claim.

We proceed to show that the bottom horizontal map in the diagram at the top of the proof is a weak equivalence of pointed real spaces. By Proposition 3.4, we may instead show that the map

$$\begin{aligned}&\Omega^{2,1}(|N(iS^{2,1}_{\oplus}(S^{2,1}_{\oplus}\mathscr{C}[-])[-],D[-][-])[-]|'_R)\\ &\xrightarrow{\phi^*}\Omega^{2,1}(|N(iS^{2,1}_{\oplus}(S^{2,1}\mathscr{C}[-])[-],D[-][-])[-]|'_R)\end{aligned}$$

is a weak equivalence of pointed real spaces. To prove this, it suffices to show that for every non-negative integer n, the map

$$\begin{aligned}&\Omega^{2,1}(|N(iS^{2,1}_{\oplus}(S^{2,1}_{\oplus}\mathscr{C}[n])[-],D[n][-])[-]|_R)\\ &\xrightarrow{\phi^*}\Omega^{2,1}(|N(iS^{2,1}_{\oplus}(S^{2,1}\mathscr{C}[n])[-],D[n][-])[-]|_R)\end{aligned}$$

is a weak equivalence of pointed real spaces; compare [8, Lemma 2.4]. This map, in turn, is a map of group objects in the homotopy category of pointed real spaces with respect to the cartesian monoidal structure. Therefore, it will suffice to show that for every subgroup $H \subset G_{\mathbb{R}}$, the induced map

$$\begin{aligned}&H_*((\Omega^{2,1}|N(iS^{2,1}_{\oplus}(S^{2,1}_{\oplus}\mathscr{C}[n])[-],D[n][-])[-]|_R)^H)\\ &\xrightarrow{\phi^*}H_*((\Omega^{2,1}|N(iS^{2,1}_{\oplus}(S^{2,1}\mathscr{C}[n])[-],D[n][-])[-]|_R)^H)\end{aligned}$$

is an isomorphism; compare [24, Chapter 7, Section 3, Theorem 9]. By Lemma 8.3 and Remark 8.4, it will suffice to show that for every subgroup $H \subset G_{\mathbb{R}}$ and for every algebraically closed field k, the induced map

$$\begin{aligned}&H_*((\Omega^{2,1}|N(iS^{2,1}_{\oplus}(S^{2,1}_{\oplus}\mathscr{C}[n])[-],D[n][-])[-]|_R)^H,k)\\ &\xrightarrow{\phi^*}H_*((\Omega^{2,1}|N(iS^{2,1}_{\oplus}(S^{2,1}\mathscr{C}[n])[-],D[n][-])[-]|_R)^H,k)\end{aligned}$$

is an isomorphism. Finally, the real group-completion theorem, Theorem 5.7, identifies the latter map with the map

$$\begin{aligned}&H_*(|N(iS^{2,1}_{\oplus}\mathscr{C}[n],D[n])[-]|^H_R,k)[\pi_0(|N(iS^{2,1}_{\oplus}\mathscr{C}[n],D[n])[-]|^H_R)^{-1}]\\ &\xrightarrow{\phi^*}H_*(|N(iS^{2,1}\mathscr{C}[n],D[n])[-]|^H_R,k)[\pi_0(|N(iS^{2,1}\mathscr{C}[n],D[n])[-]|^H_R)^{-1}]\end{aligned}$$

which we assumed to be an isomorphism. This completes the proof. □

We follow the strategy of Quillen [18] in proving that the hypothesis of Lemma 8.5 is satisfied. It is helpful to first discuss the strategy more generally.

Let $\mathscr{G}$ be a (κ-small) symmetric monoidal groupoid, and let k be a (κ-small) field. The graded k-vector space $H_*(B\mathscr{G},k)$ has the structure of a bi-antisymmetric graded k-bialgebra, where the product and unit maps are induced by the monoidal product $\oplus\colon \mathscr{G}\times\mathscr{G}\to\mathscr{G}$ and the unit $0\in\operatorname{ob}\mathscr{G}$, respectively. The localization

$$T(\mathscr{G},k)=H_*(B\mathscr{G},k)[\pi_0(B\mathscr{G})^{-1}]$$

has an induced bi-antisymmetric graded k-bialgebra structure. Moreover, the latter k-bialgebra has an antipode, and therefore, is a bi-antisymmetric k-Hopf algebra.

Let G be a κ-small group. We view G as a groupoid with $\mathrm{ob}(G) = \{\emptyset\}$ and say that a functor $E\colon G \to \mathscr{G}$ is a representation of G in $\mathscr{G}$. The set

$$\mathrm{Rep}(G,\mathscr{G}) = \pi_0(B\,\mathrm{Cat}(G,\mathscr{G}))$$

of isomorphism classes of such representations forms a commutative monoid with composition law $[E_1] + [E_2] = [E_1 \oplus E_2]$ and with identity element the class $[0]$ of the trivial representation of G on the object 0. Given a κ-small graded k-vector space V, we write $H^0(BG,V) = \prod H^i(BG,V_i)$ for the product k-vector space, the product indexed by the set of non-negative integers, and define a characteristic class of representations in $\mathscr{G}$ with coefficients in V to be a natural transformation

$$\mathrm{Rep}(-,\mathscr{G}) \xRightarrow{\theta} H^0(B(-),V)$$

of functors from the category of κ-small groups to the category of κ-small sets. The characteristic class θ is said to be additive if it is a natural homomorphism of monoids. The natural k-vector space structure of $H^0(B(-),V)$ induces to a k-vector space structure on the sets $\mathrm{Char\,Cl}(\mathscr{G},V)$ and $\mathrm{Add\,Char\,Cl}(\mathscr{G},V)$ of characteristic classes and additive characteristic classes, respectively, of representations in $\mathscr{G}$ with coefficients in V.

Lemma 8.6. *Let $\mathscr{G}$ be a symmetric monoidal groupoid, let k be a field, and let V be a graded k-vector space. There is a natural k-linear isomorphism*

$$\mathrm{Gr\,Vect}_k(QT(\mathscr{G},k),V) \xrightarrow{h_V} \mathrm{Add\,Char\,Cl}(\mathscr{G},V)$$

from the k-vector space of graded k-linear maps from the graded k-vector space of indecomposables in $T(\mathscr{G},k)$ to V and onto the k-vector space of additive characteristic classes of representations in $\mathscr{G}$ with coefficients in V.

Proof. Let G be a group. Since k is a field, the canonical map

$$H^i(BG,V_i) \xrightarrow{\alpha_i} \mathrm{Hom}_k(H_i(BG,k),V_i)$$

is an isomorphism, and hence, an element $x \in H^0(BG,V)$ is uniquely determined by the map of graded k-vector spaces $\alpha(x)\colon H_*(BG,k) \to V$. In particular, given a functor $E\colon G \to \mathscr{G}$, the induced map $E_*\colon H_*(BG,k) \to H_*(B\mathscr{G},k)$ determines a unique element $u_G(E) \in H^0(BG,H_*(B\mathscr{G},k))$. The natural transformation

$$\mathrm{Rep}(-,\mathscr{G}) \xRightarrow{u} H^0(B(-),H_*(B\mathscr{G},k))$$

defined in this way is a characteristic class of representations in $\mathscr{G}$ with coefficients in $H_*(B\mathscr{G},k)$. It is the universal characteristic class in the sense that the map

$$\mathrm{Gr\,Vect}_k(H_*(B\mathscr{G},k),V) \xrightarrow{\theta_V} \mathrm{Char\,Cl}(\mathscr{G},V)$$

that takes the map of graded k-vector spaces $f\colon H_*(B\mathscr{G},k) \to V$ to the characteristic class $\theta_V(f)$ defined by the composite natural transformation

$$\operatorname{Rep}(-,\mathscr{G}) \xRightarrow{u} H^0(B(-),H_*(B\mathscr{G},k)) \xRightarrow{f_*} H^0(B(-),V)$$

is an isomorphism of k-vector spaces. Moreover, it induces an isomorphism

$$\operatorname{Gr}\operatorname{Der}_k(H_*(B\mathscr{G},k),V) \xrightarrow{\theta_V} \operatorname{Add}\operatorname{Char}\operatorname{Cl}(\mathscr{G},V)$$

from the k-vector space of graded k-linear derivations from $H_*(B\mathscr{G},k)$ to V viewed as a left $H_*(B\mathscr{G},k)$-module via the augmentation and onto the k-vector space of additive characteristic classes of representations in $\mathscr{G}$ with coefficients in V.

Next, there is a natural isomorphism

$$\operatorname{Gr}\operatorname{Vect}_k(QH_*(B\mathscr{G},k),V) \xrightarrow{\delta_V} \operatorname{Gr}\operatorname{Der}_k(H_*(B\mathscr{G},k),V)$$

that to the map of graded k-vector spaces $f\colon QH_*(B\mathscr{G},k) \to V$ associates the graded k-linear derivation $\delta_V(f)\colon H_*(B\mathscr{G},k) \to V$ defined by the composite map

$$H_*(B\mathscr{G},k) \xrightarrow{\mathrm{id}-\eta\varepsilon} IH_*(B\mathscr{G},k) \xrightarrow{\mathrm{pr}} QH_*(B\mathscr{G},k) \xrightarrow{f} V.$$

Finally, the localization induces an isomorphism of graded k-vector spaces

$$QH_*(B\mathscr{G},k) \xrightarrow{\gamma} QT(\mathscr{G},k).$$

Now, the natural isomorphism in the statement is defined to be the composition

$$\begin{aligned}\operatorname{Gr}\operatorname{Vect}_k(QT(\mathscr{G},k),V) &\xleftarrow{\gamma^*} \operatorname{Gr}\operatorname{Vect}_k(QH_*(B\mathscr{G},k),V)\\ &\xrightarrow{\delta_V} \operatorname{Gr}\operatorname{Der}_k(H_*(B\mathscr{G},k),V) \xrightarrow{\theta_V} \operatorname{Add}\operatorname{Char}\operatorname{Cl}(\mathscr{G},V)\end{aligned}$$

of the indicated natural isomorphisms. □

In the following, we write $k[d]$ for the graded k-vector space that is equal to k in degree d and otherwise zero.

Proposition 8.7. *Let k be a field, let $\mathscr{G}_1$ and $\mathscr{G}_2$ be symmetric monoidal groupoids, and let $i\colon \mathscr{G}_1 \to \mathscr{G}_2$ be a strong symmetric monoidal functor. Suppose that the induced map of abelian groups $i_*\colon (\pi_0(B\mathscr{G}_1))^{\mathrm{gp}} \to (\pi_0(B\mathscr{G}_2))^{\mathrm{gp}}$ is surjective and that for every positive integer d, the induced map of k-vector spaces*

$$\operatorname{Add}\operatorname{Char}\operatorname{Cl}(\mathscr{G}_2,k[d]) \xrightarrow{i^*} \operatorname{Add}\operatorname{Char}\operatorname{Cl}(\mathscr{G}_1,k[d])$$

is injective. Then the induced map of k-Hopf algebras

$$T(\mathscr{G}_1,k) \xrightarrow{i_*} T(\mathscr{G}_2,k)$$

is surjective.

Proof. The degree zero part of the graded k-Hopf algebra $T(\mathscr{G}_i,k)$ is canonically isomorphic to the group k-Hopf algebra $k[(\pi_0(B\mathscr{G}_i))^{\mathrm{gp}}]$. Therefore, by assumption, the map $i_*\colon T(\mathscr{G}_1,k) \to T(\mathscr{G}_2,k)$ is surjective in degree zero. Hence, it will suffice to show that the induced map of the graded k-vector spaces of indecomposables

$$QT(\mathscr{G}_1,k) \xrightarrow{i_*} QT(\mathscr{G}_2,k)$$

is surjective; compare [15, Proposition 3.8]. Equivalently, it suffices to show that for every positive integer d, the top horizontal map in the following commutative diagram of k-vector spaces and k-linear maps is injective.

$$\begin{tikzcd}
\operatorname{Gr}\operatorname{Vect}_k(QT(\mathscr{G}_1,k),k[d]) \arrow[d, "h_{k[d]}"] & \operatorname{Gr}\operatorname{Vect}_k(QT(\mathscr{G}_2,k),k[d]) \arrow[l, "i^*"'] \arrow[d, "h_{k[d]}"] \\
\operatorname{Add}\operatorname{Char}\operatorname{Cl}(\mathscr{G}_1,k[d]) & \operatorname{Add}\operatorname{Char}\operatorname{Cl}(\mathscr{G}_2,k[d]) \arrow[l, "i^*"']
\end{tikzcd}$$

Here, the vertical maps are the canonical isomorphisms of Lemma 8.6, and the lower horizontal map is injective by assumption. This completes the proof. □

We apply Proposition 8.7 to the strong symmetric monoidal functor

$$iS^{2,1}_{\oplus}\mathscr{C}[n] \xrightarrow{\phi^*} iS^{2,1}\mathscr{C}[n],$$

where the symmetric monoidal structure of the domain and target categories are given by a choice of sum, and to the strong symmetric monoidal functor

$$\operatorname{Sym}(iS^{2,1}_{\oplus}\mathscr{C}[n],D[n]) \xrightarrow{\phi^*} \operatorname{Sym}(iS^{2,1}\mathscr{C}[n],D[n]),$$

where the symmetric monoidal structure of the domain and target categories are given by a choice of orthogonal sum. We note that both of these strong symmetric monoidal functors admit a strong symmetric monoidal retraction, up to monoidal natural isomorphism.

We will need the following generalization of [18, Lemma of Theorem 2].

Lemma 8.8. *Let k be an algebraically closed field and r a positive integer. Then there exists an order R in a number field of degree r over $\mathbb{Q}$ with the following properties: Let N_1 be a right R-module, let R^* act on N_1 by multiplication, and let the group homology $H_*(N_1,k)$ be endowed with the induced right $k[R^*]$-module structure. Let N_2 be a right module over the subring $S \subset R\otimes R$ of elements fixed by the symmetry isomorphism, let $a \in R^*$ act on N_2 by multiplication by $a\otimes a \in S$, and let $H_*(N_2,k)$ be given the induced right $k[R^*]$-module structure. Then, for all non-negative integers i and j, the right $k[R^*]$-module $H_i(N_1,k)\otimes_k H_j(N_2,k)$ decomposes as a direct sum of eigenspaces belonging to a finite set of characters $\chi\colon R^* \to k^*$. Moreover, if the non-negative integers i and j satisfy $0 < i+2j < r$, then the trivial character does not occur in this decomposition.*

Proof. Suppose first that k is of characteristic zero. We may assume that k is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. We let R be the ring of integers in a totally real field F of degree r over $\mathbb{Q}$. To see that such a field exists, we recall that, by Dirichlet, there exists an odd prime ℓ such that r divides $(\ell-1)/2$. Hence, we can take F to be the subfield of $\mathbb{Q}(\mu_\ell)\cap\mathbb{R}$ fixed by the subgroup of order $(\ell-1)/2r$. Now, for abelian groups N_1 and N_2, we have a natural isomorphism

$$\Lambda_k^i(N_1\otimes_{\mathbb{Z}}k)\otimes_k\Lambda_k^j(N_2\otimes_{\mathbb{Z}}k)\longrightarrow H_i(N_1,k)\otimes_k H_j(N_2,k).$$

Hence, if the abelian groups N_1 and N_2 are endowed with right R^*-actions, then this isomorphism is an isomorphism of right $k[R^*]$-modules. Let N_1 and N_2 be as in the statement. The group $N_1\otimes_{\mathbb{Z}}k$ is a right module over the ring $R\otimes_{\mathbb{Z}}k$, and the right R^*-action on this group is induced by the right multiplication by R on N_1. The ring $R\otimes_{\mathbb{Z}}k$ is semi-simple. Indeed, if $\sigma_1,\dots,\sigma_r$ is an enumeration of the embeddings of F in k, then, by Galois theory, the map $R\otimes_{\mathbb{Z}}k\to\prod_{1\leqslant u\leqslant r}k$ whose uth component takes $x\otimes y$ to $\sigma_u(x)y$ is an isomorphism. We have the corresponding idempotent decomposition of the right $R\otimes_{\mathbb{Z}}k$-module $N_1\otimes_{\mathbb{Z}}k$, and we see that, on the uth factor, the group R^* acts through the character $\sigma_u\colon R^*\to k^*$. Similarly, the group $N_2\otimes_{\mathbb{Z}}k$ is a right module over the ring $S\otimes_{\mathbb{Z}}k$, and the right action by $a\in R^*$ on this group is induced by the right multiplication by $a\otimes a\in S$ on N_2. The ring $S\otimes_{\mathbb{Z}}k$ again is semi-simple, the ring homomorphism $S\otimes_{\mathbb{Z}}k\to\prod_{1\leqslant v\leqslant w\leqslant r}k$ whose (v,w)th factor takes $x_1\otimes x_2\otimes y$ to $\sigma_v(x_1)\sigma_w(x_2)y$ being an isomorphism. Under the corresponding idempotent decomposition of the right $S\otimes_{\mathbb{Z}}k$-module $N_2\otimes_{\mathbb{Z}}k$, the group R^* acts on (v,w)th factor through the character $\sigma_v\sigma_w\colon R^*\to k^*$. Hence, the right $k[R^*]$-module in question, $H_i(N_1,k)\otimes_k H_j(N_2,k)$, decomposes as a direct sum of eigenspaces belonging to the characters $\chi\colon R^*\to k^*$ of the form

$$\chi=\prod_{1\leqslant u\leqslant r}\sigma_u^{n_u}\cdot\prod_{1\leqslant v\leqslant w\leqslant r}(\sigma_v\sigma_w)^{n_{v,w}}$$

with n_u and $n_{v,w}$ non-negative integers that satisfy $\sum_u n_u=i$ and $\sum_{v,w}n_{v,w}=j$. Suppose that the character $\tau=\prod_{1\leqslant u\leqslant r}\sigma_u^{m_u}$, where the exponents m_u are non-negative integers, is the trivial character. Then, for all $a\in R^*$, we have

$$\sum_{1\leqslant u\leqslant r}m_u\log(|\sigma_u(a)|)=0,$$

where $|-|$ is the absolute value in $\mathbb{C}$. By Dirichlet's unit theorem, this happens only if the exponents m_u are all equal. In the case as hand, we see that if $0<i+2j<r$, then the character χ cannot be trivial. This completes the proof in the case where k is of characteristic zero.

We next suppose that k is of characteristic $p>0$ and let $k_r\subset k$ be the subfield of order p^r. The norm map $N\colon k_r^*\to k_1^*$ is surjective, since $N(a)=a^{p^r}=a$ for all $a\in k_1^*$, and hence, its kernel U is a cyclic group of order $(p^r-1)/(p-1)$. We choose a generator $x\in U$. The group of units in the subfield $k_1(x)\subset k_r$ contains U as a subgroup, and comparing orders, we see that $k_1(x)=k_r$. Hence, the minimal polynomial

$$g(X)=X^r+b_1X^{r-1}+\cdots+b_{r-1}X+b_r$$

of x over k_1 has order r, and its constant term is $b_r = (-1)^r N(x) = (-1)^r$. We choose a monic polynomial with integer coefficients

$$f(X) = X^r + a_1 X^{r-1} + \cdots + a_{r-1} X + a_r$$

that reduces to $g(X)$ modulo p and that has constant term is $a_r = (-1)^r$, and define $R = \mathbb{Z}[X]/(f(X))$. The polynomial $f(X)$ is irreducible over $\mathbb{Z}$ because the polymial $g(X)$ is irreducible over k_1, and therefore, the ring R is an order in the number field $F = \mathbb{Q}[X]/(f(X))$ of degree r over $\mathbb{Q}$. Let $\tilde{x} \in R$ be the class of X. The unique ring homomorphism $\lambda : R \to k_r$ that takes $\tilde{x}$ to x induces an isomorphism of R/pR onto k_r. We abuse language and write $\lambda : R^* \to k^*$ for the character defined as the composition of λ and the canonical inclusion of k_r^* into k^*. The image of the character λ contains the subgroup $U \subset k_r^* \subset k^*$. Indeed, $\tilde{x}$ is a unit in R, because the constant term of the polynomial $f(X)$ is a unit in $\mathbb{Z}$, and $\lambda(\tilde{x}) = x$ which generates U.

Now, let N_1 and N_2 be as in the statement. The argument in [18, p. 214] shows that for $s = 1$ and for $s = 2$, there exist isomorphisms of graded $k[R^*]$-algebras

$$\Lambda_k(N_s \otimes_{\mathbb{Z}} k) \otimes_k \Gamma_k({}_pN_s \otimes_{\mathbb{Z}} k) \longrightarrow H_*(N_s, k).$$

Here $N_s \otimes_{\mathbb{Z}} k$ and ${}_pN_s \otimes_{\mathbb{Z}} k$ are located in degrees 1 and 2, respectively, and ${}_pN_s \subset N_s$ is the subgroup of elements annihilated by p. The isomorphisms are canonical if $p \neq 2$ or if $p = 2$ and ${}_2N_s \subset 2N_s$, but otherwise depend on choices. The case $s = 1$ is proved in loc. cit., and the proof in the case $s = 2$ is entirely analogous once one notices that the ring S/pS is semi-simple and that the order of its group of units is not divisible by p. To see this, we note that the ring homomorphism $\lambda : R \to k_r$ induces an isomorphism of S/pS onto the subring of $k_r \otimes k_r$ fixed by the symmetry isomorphism. This subring is readily identified by Galois theory. The result is that, if $r = 2m$ is even, then S/pS is isomorphic to a product of m copies of k_r and one copy of k_{r-1}; and that if $r = 2m-1$ is odd, then S/pS is isomorphic to a product of m copies of k_r.

The groups $N_1 \otimes_{\mathbb{Z}} k$ and ${}_pN_1 \otimes_{\mathbb{Z}} k$ are right modules over the ring $R \otimes_{\mathbb{Z}} k$, and the right actions by R^* on these groups are induced by the right multiplication by R on N_1. The ring $R \otimes_{\mathbb{Z}} k$ is semi-simple. Indeed, if $\varphi : k_r \to k_r$ is the Frobenius, then Galois theory shows that the ring homomorphism $R \otimes_{\mathbb{Z}} k \to \prod_{0 \leqslant u < r} k$ whose uth component takes $x_1 \otimes y$ to $\varphi^u(\lambda(x_1))y$ is an isomorphism. It follows that, in the corresponding idempotent decomposition of the right $R \otimes_{\mathbb{Z}} k$-modules $N_1 \otimes_{\mathbb{Z}} k$ and ${}_pN_1 \otimes_{\mathbb{Z}} k$, the group R^* acts on the uth factors through the character $\lambda^{p^u} : R^* \to k^*$. Similarly, the groups $N_2 \otimes_{\mathbb{Z}} k$ and ${}_pN_2 \otimes_{\mathbb{Z}} k$ are right modules over the ring $S \otimes_{\mathbb{Z}} k$, and the right actions by $a \in R^*$ on these groups are induced by the right multiplications by $a \otimes a \in S$ on N_2. The ring $S \otimes_{\mathbb{Z}} k$ again is semi-simple. Indeed, the ring homomorphism $S \otimes_{\mathbb{Z}} k \to \prod_{0 \leqslant v \leqslant w < r} k$ whose (v,w)th factor takes $x_1 \otimes x_2 \otimes y$ to $\varphi^v(\lambda(x_1))\varphi^w(\lambda(x_2))y$ is an isomorphism. Therefore, in the corresponding idempotent decomposition of the right $S \otimes_{\mathbb{Z}} k$-modules $N_2 \otimes_{\mathbb{Z}} k$ and ${}_pN_2 \otimes_{\mathbb{Z}} k$, the group R^* acts on the (v,w)th factor through the character $\lambda^{p^v + p^w} : R^* \to k^*$. By the above reasoning, we conclude that the right $k[R^*]$-module $H_i(N_1, k) \otimes H_j(N_2, k)$ decomposes as a sum of eigenspaces belonging to the characters $\chi : R^* \to k^*$ of the form $\chi = \lambda^d$ with

$$d = \sum_{0 \leqslant u < r} (m_u + n_u) p^u + \sum_{0 \leqslant v \leqslant w < r} (m_{v,w} + n_{v,w})(p^v + p^w),$$

where m_u, n_u, $m_{v,w}$, and $n_{v,w}$ are non-negative integers such that $\sum_u (m_u + 2n_u) = i$ and $\sum_{v,w}(m_{v,w} + 2n_{v,w}) = j$. Now, if the character $\tau = \lambda^d$ is the trivial character, then d is divisible by $(p^r - 1)/(p-1)$. Indeed, the image of $\lambda\colon R^* \to k^*$ contains the cyclic subgroup U of order $(p^r-1)/(p-1)$. Therefore, we see that if $0 < i + 2j < r$, then the character χ cannot be trivial. $\square$

Remark 8.9. Let r be positive integer, let A be an object of an additive category $\mathscr{C}$, and suppose that the morphisms $i_v\colon A \to A^r$ and $p_v\colon A^r \to A$ indexed by $1 \leqslant v \leqslant r$ is a biproduct diagram of r copies of A. There is a natural ring homomorphism

$$M_r(\mathbb{Z}) \xrightarrow{\eta_*} M_r(\mathrm{End}(A)) \xrightarrow{s_A} \mathrm{End}(A^r)$$

defined by the composition of the ring homomorphism induced by the unique ring homomorphism $\eta\colon \mathbb{Z} \to \mathrm{End}(A)$ and the natural ring isomorphism that to the matrix of endomorphisms (f_{uv}) associates the endomorphism $\sum_{u,v}(i_u \circ f_{uv} \circ p_v)$. This defines a left $M_r(\mathbb{Z})$-module structure on A^r which is natural in the sense that if $f\colon A_1 \to A_2$ is a morphism in $\mathscr{C}$, then its r-fold sum $f^r\colon A_1^r \to A_2^r$ is a $M_r(\mathbb{Z})$-linear morphism. Now, let R be an order in a number field of degree r over $\mathbb{Q}$. Choosing an ordered basis of R over $\mathbb{Z}$, we obtain an embedding of R as a subring of $M_r(\mathbb{Z})$. Hence, the natural left $M_r(\mathbb{Z})$-module structure on A^r gives rise to a natural left R-module structure on A^r by restriction.

We first consider $\phi^*\colon iS^{2,1}_{\oplus}\mathscr{C}[n] \to iS^{2,1}\mathscr{C}[n]$. To this end, we consider the commutative diagram of additive categories

$$\begin{tikzcd}
S^{2,1}_{\oplus}\mathscr{C}[n] \arrow[rrrr, "\phi^*"] \arrow[d, "\phi'"] & & & & S^{2,1}\mathscr{C}[n] \\
\mathrm{Fil}^m S^{2,1}\mathscr{C}[n] \arrow[r, "i_m"] & \cdots \arrow[r, "i_2"] & \mathrm{Fil}^1 S^{2,1}\mathscr{C}[n] \arrow[rr, "i_1"] & & \mathrm{Fil}^0 S^{2,1}\mathscr{C}[n] \arrow[u, "\phi''"']
\end{tikzcd}$$

defined as follows. We choose a linear ordering

$$\rho_1 \geqslant \rho_2 \geqslant \cdots \geqslant \rho_m$$

of the set of surjective functors $\rho\colon [n] \to [2]$ with the property that if there exists a morphism $\rho' \Rightarrow \rho$, then $\rho' \geqslant \rho$. This is possible since every partial order can be extended to a linear order. Now, the categories $\mathrm{Fil}^u S^{2,1}\mathscr{C}[n]$ have the same set of objects as the category $S^{2,1}_{\oplus}\mathscr{C}[n]$, and if $A, B \in \mathrm{ob}\,\mathrm{Fil}^u S^{2,1}\mathscr{C}[n]$, then

$$\mathrm{Fil}^u S^{2,1}\mathscr{C}[n](A,B) \subset S^{2,1}\mathscr{C}(\phi^*(A), \phi^*(B))$$

is defined to be the subset of all morphisms $f\colon \phi^*(A) \to \phi^*(B)$ with the property that the matrix entries $f_{\rho \Leftarrow \rho'}$ with $\rho_u \leqslant \rho'$ and $\rho \neq \rho'$ are zero. Here, for $u = 0$, none of the matrix entries $f_{\rho \Leftarrow \rho'}$ are required to be zero. The functors i_u are given by the identity maps on object sets and by the canonical inclusions on morphism sets. The functor ϕ' is given by the identity map on object sets and by the map ϕ^* on morphism sets. The functor ϕ'' is given by the map ϕ^* on object sets and by the identity map on morphism sets. We further define the functor

$$\mathrm{Fil}^{u-1} S^{2,1}\mathscr{C}[n] \xrightarrow{r_u} \mathrm{Fil}^{u} S^{2,1}\mathscr{C}[n]$$

to be the retraction of the inclusion functor i_u that is given on morphism sets by

$$r_u(f)_{\rho \Leftarrow \rho'} = \begin{cases} 0 & \text{if } \rho \neq \rho' = \rho_u \\ f_{\rho \Leftarrow \rho'} & \text{otherwise.} \end{cases}$$

Finally, we note that the functors ϕ' and ϕ'' both are equivalences of categories.

Proposition 8.10. *Let $(\mathscr{C}, 0)$ be a pointed split-exact category, and let n be a non-negative integer. Let k be an algebraically closed field and let d be a positive integer. The map induced by the forgetful functor*

$$\mathrm{Add\,Char\,Cl}(iS^{2,1}\mathscr{C}[n], k[d]) \xrightarrow{\phi_*} \mathrm{Add\,Char\,Cl}(iS^{2,1}_{\oplus}\mathscr{C}[n], k[d])$$

is injective.

Proof. It will suffice to show that for every integer $1 \leqslant u \leqslant m$, the map

$$\mathrm{Add\,Char\,Cl}(i\,\mathrm{Fil}^{u-1} S^{2,1}\mathscr{C}[n], k[d]) \xrightarrow{i_u^*} \mathrm{Add\,Char\,Cl}(i\,\mathrm{Fil}^{u} S^{2,1}\mathscr{C}[n], k[d])$$

is injective. So we fix an element θ of the kernel of this map and proceed to show that this element is zero. We must show that for every representation

$$G \xrightarrow{E} i\,\mathrm{Fil}^{u-1} S^{2,1}\mathscr{C}[n],$$

the class $\theta_G(E) \in H^d(G, k)$ is zero. Let

$$i\,\mathrm{Fil}^{u-1} S^{2,1}\mathscr{C}[n] \xrightarrow{v} \mathrm{Fil}^{u-1} S^{2,1}\mathscr{C}[n]$$

be the canonical inclusion. We choose an integer $r > d$ that is not divisible by the characteristic of k and let R be as in Lemma 8.8. Let E^r be the r-fold monoidal product of the representation E. Since θ is additive, we have $\theta_G(E^r) = r\theta_G(E)$, and hence, it will suffice to show that $\theta_G(E^r)$ is zero. Let $A^r = (v \circ E^r)(\emptyset)$. It follows from Remark 8.9 that A^r has a natural left R-module structure and that $v \circ E^r$ is a representation of G on A^r through R-linear automorphisms.

By the above reasoning and by the naturality of θ, it will suffice to consider the following situation. Let A be an object of $i\,\mathrm{Fil}^{u-1} S^{2,1}\mathscr{C}[n]$ and let the object $A(\{\rho_u, \infty\})$ of $\mathscr{C}$ be endowed with a left R-module structure; let G be the full group of automorphisms $g \colon A \to A$ with the property that the component map $g_{\rho_u \Leftarrow \rho_u}$ is R-linear; and let E be the canonical representation of G on A. We must show that the class $\theta_G(E) \in H^d(G, k)$ is zero. To this end, we let

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] & G \arrow[r, "r_u"'] & \bar{G} \arrow[l, bend right, "i_u"'] \arrow[r] & 1 \end{tikzcd}$$

be the group extension, where $\bar{G}$ is the group of automorphisms $\bar{g}\colon r_u(A) \to r_u(A)$ such that the component map $\bar{g}_{\rho_u \Leftarrow \rho_u}$ is R-linear. The map

$$H^d(G,k) \xrightarrow{i_u^*} H^d(\bar{G},k)$$

takes $\theta_G(E)$ to $\theta_{\bar{G}}(i_u^*(E))$ which is zero since θ is in the kernel of i_u^*. Hence, it suffices to prove that the map $i_{u*}\colon H_d(\bar{G},k) \to H_d(G,k)$ is an isomorphism. Equivalently, it suffices to prove that the map $r_{u*}\colon H_d(G,k) \to H_d(\bar{G},k)$ is an isomorphism. Up to canonical isomorphism, the latter map is equal to the edge homomorphism of the Hochschild-Serre spectral sequence

$$E^2_{s,t} = H_s(\bar{G}, H_t(N,k)) \Rightarrow H_{s+t}(G,k).$$

Therefore, it will suffice to show that the groups $E^2_{s,t}$ with $0 < t < r$ are zero. To this end, we employ Lemma 8.8 with $N_1 = N$ and with $N_2 = 0$ as follows.

The subgroup $N \subset G$ consists of the automorphisms $h\colon A \to A$ with the property that $h_{\rho \Leftarrow \rho'} = (\mathrm{id}_A)_{\rho \Leftarrow \rho'}$ unless $\rho' = \rho_u$. It follows that the map

$$N \longrightarrow \prod \mathscr{C}(A(\{\rho_u,\infty\}), A(\{\rho,\infty\}))$$

that to g associates the partial matrix $(g_{\rho \Leftarrow \rho_u})$ is a group isomorphism. Here, the product is indexed by the set of admissible morphisms $\rho_u \Rightarrow \rho$ with $\rho \neq \rho_u$ and is an abelian group under matrix addition. The left R-module structure on $A(\{\rho_u,\infty\})$ gives rise to a right R-module structure on $\mathscr{C}(A(\{\rho_u,\infty\}), A(\{\rho,\infty\}))$ and we give N the right R-module structure that makes the isomorphism above R-linear.

We define the group homomorphism

$$(R^*)^{\mathrm{op}} \xrightarrow{\alpha} \mathrm{Inn}(G)$$

by $\alpha(a)(g) = \gamma(a)^{-1} \circ g \circ \gamma(a)$ where $\gamma(a) \in G$ is given by the diagonal matrix

$$\gamma(a)_{\rho \Leftarrow \rho'} = \begin{cases} l_a & \text{if } \rho = \rho' = \rho_u \\ \mathrm{id} & \text{if } \rho = \rho' \neq \rho_u \\ 0 & \text{if } \rho \neq \rho'. \end{cases}$$

Here l_a denotes left multiplication by $a \in R$ on $A(\{\rho_u,\infty\})$. We claim that R^* acts trivially on the subgroup $\bar{G} \subset G$ and that R^* acts on the subgroup $N \subset G$ through the right R-module structure defined above. To see this, we note the following.

(1) If $\bar{g} \in \bar{G}$ and $\bar{g}_{\rho \Leftarrow \rho_u}$ is non-zero, then $\rho = \rho_u$.
(2) If $g \in G$ and $g_{\rho_u \Leftarrow \rho'}$ is non-zero, then $\rho' = \rho_u$.

Here (1) follows immediately from the definition of $\mathrm{Fil}^u S^{2,1}\mathscr{C}[n]$ and (2) follows from the definition of $\mathrm{Fil}^{u-1} S^{2,1}\mathscr{C}[n]$ and from the inequality $\rho_u \leqslant \rho'$, which, in turn, holds by our choice of linear order. Now, for $\bar{g} \in \bar{G}$, we have

$$\begin{aligned} \alpha(a)(\bar{g})_{\rho \Leftarrow \rho'} &= \gamma(a)^{-1}_{\rho \Leftarrow \rho} \circ \bar{g}_{\rho \Leftarrow \rho'} \circ \gamma(a)_{\rho' \Leftarrow \rho'} \\ &= \begin{cases} l_a^{-1} \circ \bar{g}_{\rho_u \Leftarrow \rho_u} \circ l_a & \text{if } \rho = \rho' = \rho_u \\ \bar{g}_{\rho \Leftarrow \rho'} & \text{otherwise} \end{cases} \end{aligned}$$

and since $\bar{g}_{\rho_u \Leftarrow \rho_u}$ is R-linear, we have $\alpha(a)(\bar{g}) = \bar{g}$ as claimed. Similarly, for $g \in N$ and for $\rho_u \Rightarrow \rho$ admissible with $\rho \neq \rho_u$, we have

$$\alpha(a)(g)_{\rho \Leftarrow \rho_u} = \gamma(a)^{-1}_{\rho \Leftarrow \rho} \circ g_{\rho \Leftarrow \rho_u} \circ \gamma(a)_{\rho_u \Leftarrow \rho_u} = g_{\rho \Leftarrow \rho_u} \circ l_a$$

as claimed.

We now fix $0 < t < r$ and show that, in the Hochschild-Serre spectral sequence, the groups $E^2_{s,t} = H_s(\bar{G}, H_t(N,k))$ are zero. To this end, we evaluate the right R^*-action on $E^2_{s,t}$ induces by the right R^*-action on N in two different ways and, by comparing the two results, conclude that $E^2_{s,t}$ is zero. The right action by $\bar{g} \in \bar{G}$ on

$$V = H_t(N,k) = H_t(B(k,k[N],k))$$

is given by the map induced by the chain map $B(\mathrm{id}, c_g, \mathrm{id})$, where $g \in G$ is any lifting of $\bar{g}$, and where $c_g(h) = g^{-1}hg$. Therefore, if we let R^* act from the right on V via the group homomorphism $\bar{\gamma} = r_u \circ \gamma \colon R^* \to \bar{G}$, then by what was said above, this right R^*-action is equal to the right R^*-action on V induced from the right R^*-action on N. It follows that the induced right action by $a \in R^*$ on $E^2_{s,t}$ is equal to the map

$$H_s(B(V,k[\bar{G}],k)) \xrightarrow{h(a)_*} H_s(B(V,k[\bar{G}],k)),$$

where $h(a)$ is the chain map defined, up to canonical isomorphism, as the composite

$$\begin{aligned} B(V,k[\bar{G}],k[\bar{G}]) \otimes_{k[\bar{G}]} k &\xrightarrow{\tilde{r}_{\bar{\gamma}(a)} \otimes \mathrm{id}} (c^*_{\bar{\gamma}(a)} B(V,k[\bar{G}],k[\bar{G}])) \otimes_{k[\bar{G}]} k \\ &\xrightarrow{\varphi} B(V,k[\bar{G}],k[\bar{G}]) \otimes_{k[\bar{G}]} k. \end{aligned}$$

Here $\varphi(x \otimes y) = x \otimes y$ and $\tilde{r}_{\bar{\gamma}(a)}$ is any choice of a chain map

$$\begin{tikzcd} B(V,k[\bar{G}],k[\bar{G}]) \arrow[r, "\tilde{r}_{\bar{\gamma}(a)}"] \arrow[d, "\varepsilon"] & B(V,k[\bar{G}],k[\bar{G}]) \arrow[d, "\varepsilon"] \\ V \arrow[r, "r_{\bar{\gamma}(a)}"] & V \end{tikzcd}$$

that lifts the right multiplication by $\bar{\gamma}(a)$ on V. First, we have the choice

$$\tilde{r}_{\bar{\gamma}(a)} = \tilde{r}^{(1)}_{\bar{\gamma}(a)} = B(\mathrm{id}, \mathrm{id}, r_{\bar{\gamma}(a)})$$

which gives $h(a) = h^{(1)}(a) = B(\mathrm{id},\mathrm{id},\mathrm{id}) = \mathrm{id}$. This shows that the right R^*-action on the k-vector space $E^2_{s,t}$ is trivial. Second, we have the choice

$$\tilde{r}_{\bar{\gamma}(a)} = \tilde{r}^{(2)}_{\bar{\gamma}(a)} = B(r_{\bar{\gamma}(a)}, c_{\bar{\gamma}(a)}, c_{\bar{\gamma}(a)}) = B(r_{\bar{\gamma}(a)}, \mathrm{id}, \mathrm{id})$$

which gives $h(a) = h^{(2)}(a) = B(r_{\bar{\gamma}(a)}, \mathrm{id}, \mathrm{id})$. This, we claim, implies that the right $k[R^*]$-module $E^2_{s,t}$ decomposes as a direct sum of eigenspaces attached to non-trivial characters. Indeed, the category $\mathscr{A}$ of right $k[R^*]$-modules of this kind is abelian, and by Lemma 8.8, it contains V. Therefore, the complex $B(k,k[\bar{G}],V)$ considered as a complex of right $k[R^*]$-modules with $a \in R^*$ acting through the chain map $h^{(2)}(a)$ is a complex in $\mathscr{A}$, and hence, its homology groups are in $\mathscr{A}$. The claim follows. Comparing the two descriptions of the right $k[R^*]$-module $E^2_{s,t}$, we conclude that it is zero. This shows that the map $r_{u*}\colon H_d(G,k) \to H_d(\bar{G},k)$ is an isomorphism, which, in turn, shows that the characteristic class θ is zero as desired. □

Remark 8.11. Let r be a positive integer and let A be an object in an additive category with strict duality $(\mathscr{C},D)$. If the morphisms $i_u\colon A \to A^r$ and $p_u\colon A^r \to A$ indexed by $1 \leqslant u \leqslant r$ form a biproduct diagram of r copies of A, then the morphisms $Dp_u^{\mathrm{op}}\colon DA^{\mathrm{op}} \to D(A^r)^{\mathrm{op}}$ and $Di_u^{\mathrm{op}}\colon D(A^r)^{\mathrm{op}} \to DA^{\mathrm{op}}$ indexed by $1 \leqslant u \leqslant r$ form a biproduct diagram of r copies of DA^{op}. In this situation, the diagram

$$\begin{tikzcd}
M_r(\mathbb{Z})^{\mathrm{op}} \arrow[r, "\eta_*^{\mathrm{op}}"] \arrow[d, "(-)^t"] & M_r(\mathrm{End}(A))^{\mathrm{op}} \arrow[r, "s_A^{\mathrm{op}}"] \arrow[d, "(-)^t"] & \mathrm{End}(A^r)^{\mathrm{op}} \arrow[dd, "D"] \\
M_r(\mathbb{Z}) \arrow[r, "\eta_*"] \arrow[d, equal] & M_r(\mathrm{End}(A)^{\mathrm{op}}) \arrow[d, "M_r(D)"] & \\
M_r(\mathbb{Z}) \arrow[r, "\eta_*"] & M_r(\mathrm{End}(DA^{\mathrm{op}})) \arrow[r, "s_{DA^{\mathrm{op}}}"] & \mathrm{End}(D(A^r)^{\mathrm{op}})
\end{tikzcd}$$

commutes. Indeed, the two left-hand squares commute by naturality, and it follows readily from the definition of the map s_A that the right-hand rectangular diagram commutes. The object $D(A^r)^{\mathrm{op}}$ has both a natural left $M_r(\mathbb{Z})$-module structure and natural right $M_r(\mathbb{Z})$-module structure. The left $M_r(\mathbb{Z})$-module structure is defined by Remark 8.9 using the biproduct morphisms Dp_u^{op} and Di_u^{op}. The right $M_r(\mathbb{Z})$-module structure is induced by the left $M_r(\mathbb{Z})$-module structure on A^r defined by Remark 8.9 using the biproduct morphisms i_u and p_u. Now, by the commutativity of the diagram above, we conclude that left multiplication by the matrix $a \in M_r(\mathbb{Z})$ is equal to right multiplication by its transpose $a^t \in M_r(\mathbb{Z})$. In particular, if $f\colon A_1 \to A_2$ is a morphism in $\mathscr{C}$, then the morphism $D(f^r)^{\mathrm{op}}\colon D(A_2^r)^{\mathrm{op}} \to D(A_1^{\mathrm{op}})$ is linear with respect to both module structures.

We next consider the strong symmetric monoidal functor

$$\mathrm{Sym}(iS^{2,1}_{\oplus}\mathscr{C}[n],D[n]) \xrightarrow{\phi^*} \mathrm{Sym}(iS^{2,1}\mathscr{C}[n],D[n])$$

and the commutative following diagram of additive categories with strict duality

$$\begin{tikzcd}
(S^{2,1}_{\oplus}\mathscr{C}[n],D[n]) \arrow[rrr, "\phi^*"] \arrow[d, "\phi'"] & & & (S^{2,1}\mathscr{C}[n],D[n]) \\
(\mathrm{Fil}^m S^{2,1}\mathscr{C}[n],D[n]) \arrow[r, "i_m"] & \cdots \arrow[rr, "i_1"] & & (\mathrm{Fil}^0 S^{2,1}\mathscr{C}[n],D[n]) \arrow[u, "\phi''"]
\end{tikzcd}$$

defined as follows. We say that a surjective functor $\rho\colon [n] \to [2]$ is positive, if there exists a non-identity morphism $\rho \Rightarrow D\rho^{\mathrm{op}}$, and choose a linear ordering

$$\rho_1 \geqslant \rho_2 \geqslant \cdots \geqslant \rho_m$$

of the set of positive surjective functors $\rho\colon [n] \to [2]$ such that $\rho' \geqslant \rho$ whenever there exists a morphism $\rho' \Rightarrow \rho$. Now, the categories $\operatorname{Fil}^u S^{2,1}\mathscr{C}[n]$ have the same objects as the category $S^{2,1}_{\oplus}\mathscr{C}[n]$, and if $A, B \in \operatorname{ob}\operatorname{Fil}^u S^{2,1}\mathscr{C}[n]$, then

$$\operatorname{Fil}^u S^{2,1}\mathscr{C}[n](A,B) \subset S^{2,1}\mathscr{C}[n](\phi^*(A), \phi^*(B))$$

is the subset of all morphisms $f\colon \phi^*(A) \to \phi^*(B)$ with the property that the matrix entry $f_{\rho \Leftarrow \rho'}$ is zero whenever $\rho \neq \rho'$ and either $\rho \leqslant D\rho_u^{\mathrm{op}}$ or $\rho_u \leqslant \rho'$ or both. For $u = 0$, the matrix entries $f_{\rho \Leftarrow \rho'}$ are unrestricted. The functors i_u are given by the identity maps on object sets and by the canonical inclusions on morphism sets; the functor ϕ' is given by the identity map on object sets and by the map ϕ^* on morphism sets; and the functor ϕ'' is given by the map ϕ^* on object sets and by the identity map on morphism sets. We further define

$$(\operatorname{Fil}^{u-1} S^{2,1}\mathscr{C}[n], D[n]) \xrightarrow{r_u} (\operatorname{Fil}^u S^{2,1}\mathscr{C}[n], D[n])$$

to be the retraction of the inclusion functor i_u that is given on morphism sets by

$$r_u(f)_{\rho \Leftarrow \rho'} = \begin{cases} 0 & \text{if } \rho \neq \rho' \text{ and } \rho = D\rho_u^{\mathrm{op}} \text{ or } \rho' = \rho_u \\ f_{\rho \Leftarrow \rho'} & \text{otherwise.} \end{cases}$$

The functors i_u and r_u are additive and duality preserving, and the functors ϕ' and ϕ'' are adjoint equivalences of additive categories with strict duality.

Proposition 8.12. *Let $(\mathscr{C}, D, 0)$ be a pointed split-exact category with strict duality and let n be a non-negative integer. Let k be an algebraically closed field and let d be a positive integer. Then the map induced by the forgetful functor*

$$\begin{aligned}&\operatorname{Add}\operatorname{Char}\operatorname{Cl}(\operatorname{Sym}(iS^{2,1}\mathscr{C}[n], D[n]), k[d]) \\ &\xrightarrow{\phi_*} \operatorname{Add}\operatorname{Char}\operatorname{Cl}(\operatorname{Sym}(iS^{2,1}_{\oplus}\mathscr{C}[n], D[n]), k[d])\end{aligned}$$

is injective.

Proof. It will suffice to show that for every integer $1 \leqslant u \leqslant m$, the map

$$\begin{aligned}&\operatorname{Add}\operatorname{Char}\operatorname{Cl}(\operatorname{Sym}(i\operatorname{Fil}^{u-1} S^{2,1}\mathscr{C}[n], D[n]), k[d]) \\ &\xrightarrow{i_u^*} \operatorname{Add}\operatorname{Char}\operatorname{Cl}(\operatorname{Sym}(i\operatorname{Fil}^u S^{2,1}\mathscr{C}[n], D[n]), k[d])\end{aligned}$$

is injective. So we fix an element θ of the kernel of this map and proceed to show that this element is zero. We must show that for every representation

$$G \xrightarrow{E} \operatorname{Sym}(i\operatorname{Fil}^{u-1} S^{2,1}\mathscr{C}[n], D[n]),$$

the class $\theta_G(E) \in H^d(G,k)$ is zero. Let

$$\operatorname{Sym}(i\operatorname{Fil}^{u-1} S^{2,1}\mathscr{C}[n], D[n]) \xrightarrow{v} \operatorname{Fil}^{u-1} S^{2,1}\mathscr{C}[n]$$

be the forgetful functor. We choose an integer $r > 2d$ that is not divisible by the characteristic of k and let R be as in Lemma 8.8. Let E^r be the r-fold monoidal product of the representation E. Since θ is additive, we have $\theta_G(E^r) = r\theta_G(E)$, so it will suffice to show that $\theta_G(E^r)$ is zero. Let $E^r(\emptyset) = (A^r, f^r)$. It follows from Remark 8.9 that A^r and $D(A^r)^{\mathrm{op}}$ are left $M_r(\mathbb{Z})$-modules, that $f^r\colon A^r \to D(A^r)^{\mathrm{op}}$ is an $M_r(\mathbb{Z})$-linear isomorphism, and that $\nu \circ E^r$ is a representation of G on A^r through left $M_r(\mathbb{Z})$-module automorphisms. Moreover, Remark 8.11 shows that the right $M_r(\mathbb{Z})$-module structure on A^r induced from the left $M_r(\mathbb{Z})$-module structure on $D(A^r)^{\mathrm{op}}$ is equal to the transpose of the left $M_r(\mathbb{Z})$-module structure on A^r. It follows that $\nu \circ E^r$ also is a representation of G on A^r through right $M_r(\mathbb{Z})$-module automorphisms.

By the above reasoning and by the naturality of θ, it will suffice to consider the following situation. Let (A, f) be an object of $\mathrm{Sym}(i\,\mathrm{Fil}^{u-1} S^{2,1}\mathscr{C}[n], D[n])$ and let the objects $A(\{\rho_u, \infty\})$ and $A(\{D\rho_u^{\mathrm{op}}, \infty\})$ of $\mathscr{C}$ be endowed with a left R-module structure and a right R-module structure, respectively, such that the isomorphism

$$A(\{\rho_u, \infty\}) \xrightarrow{f_{\rho_u \Leftarrow \rho_u}} (DA^{\mathrm{op}})(\{\rho_u, \infty\}) = D(A(\{D\rho_u^{\mathrm{op}}, \infty\})^{\mathrm{op}})$$

is R-linear; let G be the full group of automorphisms $g\colon (A, f) \to (A, f)$ for which the component maps $g_{\rho_u \Leftarrow \rho_u}$ and $g_{D\rho_u^{\mathrm{op}} \Leftarrow D\rho_u^{\mathrm{op}}}$ are R-linear; and let E be the canonical representation of G on (A, f). We must show that $\theta_G(E) \in H^d(G, k)$ is zero. Let

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] & G \arrow[r, "r_u"'] & \bar{G} \arrow[r] \arrow[l, bend right, "i_u"'] & 1 \end{tikzcd}$$

be the group extension, where $\bar{G}$ is the group of automorphisms

$$(r_u(A), r_u(f)) \xrightarrow{\bar{g}} (r_u(A), r_u(f))$$

such that the component maps $\bar{g}_{\rho_u \Leftarrow \rho_u}$ and $\bar{g}_{D\rho_u^{\mathrm{op}} \Leftarrow D\rho_u^{\mathrm{op}}}$ are D-linear. The map

$$H^d(G, k) \xrightarrow{i_u^*} H^d(\bar{G}, k)$$

takes $\theta_G(E)$ to $\theta_{\bar{G}}(i_u^*(E))$ which is zero since θ is in the kernel of i_u^*. Hence, it suffices to prove that the map $i_{u*}\colon H_d(\bar{G}, k) \to H_d(G, k)$ is an isomorphism. Equivalently, it suffices to prove that the map $r_{u*}\colon H_d(G, k) \to H_d(\bar{G}, k)$ is an isomorphism. Up to canonical isomorphism, the latter map is equal to the edge homomorphism of the Hochschild-Serre spectral sequence

$$E^2_{s,t} = H_s(\bar{G}, H_t(N, k)) \Rightarrow H_{s+t}(G, k).$$

Therefore, it will suffice to show that the groups $E^2_{s,t}$ with $0 < 2t < r$ are zero. We will employ Lemma 8.8 prove that this is so, but first we analyze the subgroup $N \subset G$ more carefully.

The subgroup $N \subset G$ is the full group of automorphisms $h\colon (A, f) \to (A, f)$ such that $h_{\rho \Leftarrow \rho'} = (\mathrm{id}_A)_{\rho \Leftarrow \rho'}$ unless either $\rho = D\rho_u^{\mathrm{op}}$ or $\rho' = \rho_u$ or both. Moreover, the equation $f = Dh^{\mathrm{op}} \circ f \circ h$, that every automorphism $f\colon (A, f) \to (A, f)$ satisfies, is equivalent to the following additional restrictions (1)–(2) on the family of component maps $h_{\rho \Leftarrow \rho'}$ of the element $h \in N$.

(1) For every admissible morphism $\rho_u \Rightarrow \rho$ with $\rho_u > \rho > D\rho_u^{\mathrm{op}}$,

$$f_{\rho \Leftarrow \rho} \circ h_{\rho \Leftarrow \rho_u} = -D(h_{D\rho_u^{\mathrm{op}} \Leftarrow D\rho^{\mathrm{op}}})^{\mathrm{op}} \circ f_{\rho_u \Leftarrow \rho_u}.$$

(2) If there exists an admissible morphism $\rho_u \Rightarrow D\rho_u^{\mathrm{op}}$, then

$$\begin{aligned} & f_{D\rho_u^{\mathrm{op}} \Leftarrow D\rho_u^{\mathrm{op}}} \circ h_{D\rho_u^{\mathrm{op}} \Leftarrow \rho_u} + D(f_{D\rho_u^{\mathrm{op}} \Leftarrow D\rho_u^{\mathrm{op}}} \circ h_{D\rho_u^{\mathrm{op}} \Leftarrow \rho_u})^{\mathrm{op}} \\ & = -\sum D(h_{D\rho^{\mathrm{op}} \Leftarrow \rho_u})^{\mathrm{op}} \circ f_{\rho \Leftarrow \rho'} \circ h_{\rho' \Leftarrow \rho_u}, \end{aligned}$$

where the sum is indexed by the set of factorizations $\rho_u \Rightarrow \rho' \Rightarrow \rho \Rightarrow D\rho_u^{\mathrm{op}}$ such that $\rho_u \neq \rho'$ and $\rho \neq D\rho_u^{\mathrm{op}}$.

We note that the morphisms $f_{\rho \Leftarrow \rho}$ and $f_{\rho_u \Leftarrow \rho_u}$ in (1) are automorphisms. Hence, the component maps $h_{\rho \Leftarrow \rho_u}$ and $h_{D\rho_u^{\mathrm{op}} \Leftarrow D\rho^{\mathrm{op}}}$ determine each other. We also note that the sum on the right-hand side of the equation in (2) is a Tate 0-cocycle in the left $G_{\mathbb{R}}$-module $\mathscr{C}(A(\{\rho_u,\infty\}), A(D\rho_u^{\mathrm{op}},\infty\}))$. Here the action of the generator $\sigma \in G_{\mathbb{R}}$ is given by $\sigma x = D(x^{\mathrm{op}})$. This shows that the map

$$N \longrightarrow \prod \mathscr{C}(A(\{\rho_u,\infty\}), A(\{\rho,\infty\}))$$

that to h associates the partial matrix $(h_{\rho \Leftarrow \rho_u})$ defines a bijection of N onto the subset consisting of all tuples $(h_{\rho \Leftarrow \rho_u})$ that satisfy (2). Here, the product is indexed by the set of admissible morphisms $\rho_u \Rightarrow \rho$ with $\rho_u > \rho \geqslant D\rho_u^{\mathrm{op}}$.

We define $N_2 \subset N$ to be the full group of automorphism $h\colon (A,f) \to (A,f)$ with the property that $g_{\rho \Leftarrow \rho'} = (\mathrm{id}_A)_{\rho \Leftarrow \rho'}$ unless both $\rho = D\rho_u^{\mathrm{op}}$ and $\rho' = \rho_u$. If $g \in G$ and $h \in N_2$, then $(g \circ h \circ g^{-1})_{\rho \Leftarrow \rho'} = (\mathrm{id}_A)_{\rho \Leftarrow \rho'}$ unless $\rho = D\rho_u^{\mathrm{op}}$ and $\rho' = \rho_u$ in which case we find that

$$(g \circ h \circ g^{-1})_{D\rho_u^{\mathrm{op}} \Leftarrow \rho_u} = g_{D\rho_u^{\mathrm{op}} \Leftarrow D\rho_u^{\mathrm{op}}} \circ h_{D\rho_u^{\mathrm{op}} \Leftarrow \rho_u} \circ (g_{\rho_u \Leftarrow \rho_u})^{-1}.$$

This shows that $N_2 \subset G$ is normal and that $N_2 \subset N$ is central. We let $N_1 = N/N_2$ be the quotient and consider the central group extension

$$1 \longrightarrow N_2 \longrightarrow N \longrightarrow N_1 \longrightarrow 1.$$

We evaluate the groups N_2 and N_1.

First, we have the injective group homomorphism

$$N_1 \xrightarrow{e_1} \prod \mathscr{C}(A(\{\rho_u,\infty\}), A(\{\rho,\infty\}))$$

that to h associates the partial matrix $(h_{\rho \Leftarrow \rho_u})$. Here, the product is indexed by the set of admissible morphisms $\rho_u \Rightarrow \rho$ with $\rho_u > \rho > D\rho_u^{\mathrm{op}}$ and is an abelian group under matrix addition. The map e_1 is an isomorphism onto the kernel of the map

$$\prod \mathscr{C}(A(\{\rho_u,\infty\}), A(\{\rho,\infty\})) \xrightarrow{\mathrm{inv}} \hat{H}^0(G_{\mathbb{R}}, \mathscr{C}((A(\{\rho_u,\infty\}), D(A(\{\rho_u,\infty\})^{\mathrm{op}}))$$

that to the tuple $(h_{\rho \Leftarrow \rho_u})$ associates the class of the Tate 0-cocycle

$$-\sum D(h_{D\rho^{\mathrm{op}} \Leftarrow \rho_u})^{\mathrm{op}} \circ f_{\rho \Leftarrow \rho'} \circ h_{\rho' \Leftarrow \rho_u} \in \mathscr{C}(A(\{\rho_u,\infty\}), D(A(\{\rho_u,\infty\})^{\mathrm{op}})).$$

Here, the sum is indexed by the set of factorizations $\rho_u \Rightarrow \rho' \Rightarrow \rho \Rightarrow D\rho_u^{\mathrm{op}}$ such that both $\rho_u > \rho'$ and $\rho > D\rho_u^{\mathrm{op}}$. This isomorphism gives the abelian group N_1 the structure of a right R-module.

Second, we have the injective group homomorphism

$$N_2 \xrightarrow{e_2} \mathscr{C}(A(\{\rho_u,\infty\}),D(A(\{\rho_u,\infty\})^{\mathrm{op}}))$$

defined by $e_2(h) = f_{D\rho_u^{\mathrm{op}} \Leftarrow D\rho_u^{\mathrm{op}}} \circ h_{D\rho_u^{\mathrm{op}} \Leftarrow \rho_u}$. It follows from the restriction (2) above that the map e_2 is an isomorphism onto the kernel of the norm map

$$\mathscr{C}(A(\{\rho_u,\infty\}),D(A(\{\rho_u,\infty\})^{\mathrm{op}})) \xrightarrow{1+\sigma} \mathscr{C}(A(\{\rho_u,\infty\}),D(A(\{\rho_u,\infty\})^{\mathrm{op}})).$$

This isomorphism gives the abelian group N_2 the structure of a right module over the subring $S \subset R \otimes R$ fixed by the symmetry isomorphism.

We will apply Lemma 8.8 with N_1 and N_2 defined as above. To this end, we define the group homomorphism

$$(R^*)^{\mathrm{op}} \xrightarrow{\alpha} \mathrm{Inn}(G)$$

by $\alpha(a)(g) = \gamma(a)^{-1} \circ g \circ \gamma(a)$ where $\gamma(a) \in G$ is given by the diagonal matrix

$$\gamma(a)_{\rho \Leftarrow \rho'} = \begin{cases} l_a & \text{if } \rho = \rho' = \rho_u \\ r_a^{-1} & \text{if } \rho = \rho' = D\rho_u^{\mathrm{op}} \\ \mathrm{id} & \text{if } \rho = \rho' \neq \rho_u, D\rho_u^{\mathrm{op}} \\ 0 & \text{otherwise.} \end{cases}$$

Here l_a denotes left multiplication by $a \in R$ on $A(\{\rho_u,\infty\})$, r_a denotes right multiplication by $a \in R$ on $A(\{D\rho_u^{\mathrm{op}},\infty\})$, and the assumption that $f_{\rho_u \Leftarrow \rho_u}$ be R-linear implies that $\gamma(a) \in G$. We claim that $a \in R^*$ acts on trivially on the subgroup $\bar{G} \subset G$, that it acts on $N_1 \subset G/N_2$ by right multiplication by $a \in R$, and that it acts on $N_2 \subset G$ by right multiplication by $a \otimes a \in S$. To see this, we note the following.

(1) If $\bar{g} \in \bar{G}$, then $\bar{g}_{\rho \Leftarrow \rho_u}$ is zero unless $\rho = \rho_u$ and $\bar{g}_{D\rho_u^{\mathrm{op}} \Leftarrow \rho}$ is zero unless $\rho = D\rho_u^{\mathrm{op}}$.

(2) If $g \in G$, then $g_{\rho_u \Leftarrow \rho'}$ is zero unless $\rho' = \rho_u$ and $g_{\rho \Leftarrow D\rho_u^{\mathrm{op}}}$ is zero unless $\rho = D\rho_u^{\mathrm{op}}$.

Here (1) follows immediately from the definition of $\mathrm{Fil}^u S^{2,1}\mathscr{C}[n]$, and (2) follows from the definition of $\mathrm{Fil}^{u-1} S^{2,1}\mathscr{C}[n]$ and from the inequalities $\rho_u \leqslant \rho'$ and $\rho \leqslant D\rho_u^{\mathrm{op}}$, which, in turn, hold by the choice of linear ordering. Now, for $\bar{g} \in \bar{G}$,

$$\begin{aligned} \alpha(a)(\bar{g})_{\rho \Leftarrow \rho'} &= \gamma(a)^{-1}_{\rho \Leftarrow \rho} \circ \bar{g}_{\rho \Leftarrow \rho'} \circ \gamma(a)_{\rho' \Leftarrow \rho'} \\ &= \begin{cases} l_a^{-1} \circ \bar{g}_{\rho_u \Leftarrow \rho_u} \circ l_a & \text{if } \rho = \rho' = \rho_u \\ r_a \circ \bar{g}_{D\rho_u^{\mathrm{op}} \Leftarrow D\rho_u^{\mathrm{op}}} \circ r_a^{-1} & \text{if } \rho = \rho' = D\rho_u^{\mathrm{op}} \\ \bar{g}_{\rho \Leftarrow \rho'} & \text{otherwise,} \end{cases} \end{aligned}$$

and since $\bar{g}_{\rho_u \Leftarrow \rho_u}$ and $\bar{g}_{D\rho_u^{\mathrm{op}} \Leftarrow D\rho_u^{\mathrm{op}}}$ are both R-linear, we have $\alpha(a)(\bar{g}) = \bar{g}$ as claimed. Similarly, given $\bar{h} \in N_1$, we have

$$\bar{\alpha}(a)(\bar{h})_{\rho \Leftarrow \rho_u} = \bar{\gamma}(a)^{-1}_{\rho \Leftarrow \rho} \circ \bar{h}_{\rho \Leftarrow \rho_u} \circ \bar{\gamma}(a)_{\rho_u \Leftarrow \rho_u} = \bar{h}_{\rho \Leftarrow \rho_u} \circ l_a$$

which, as claimed, is right multiplication by $a \in R$. Finally, for $h \in N_2$, we have

$$\alpha(a)(h)_{D\rho_u^{\mathrm{op}} \Leftarrow \rho_u} = \gamma(a)^{-1}_{D\rho_u^{\mathrm{op}} \Leftarrow D\rho_u^{\mathrm{op}}} \circ h_{D\rho_u^{\mathrm{op}} \Leftarrow \rho_u} \circ \gamma(a)_{\rho_u \Leftarrow \rho_u} = r_a \circ h_{D\rho_u^{\mathrm{op}} \Leftarrow \rho_u} \circ l_a$$

which is right multiplication by $a \otimes a \in S$ as claimed.

We now consider the Hochschild-Serre spectral sequence

$$E^2_{i,j} = H_i(N_1, k) \otimes_k H_j(N_2, k) \Rightarrow H_{i+j}(N, k).$$

Here, the E^2-term takes the stated form because $N_2 \subset N$ is central and because k is a field. The right action by $\bar{G}$ on N makes the spectral sequence a spectral sequence of right $k[\bar{G}]$-modules. We also view the spectral sequence as a spectral sequence of right $k[R^*]$-modules via the group homomorphism $\bar{\gamma} = r_u \circ \gamma \colon R^* \to \bar{G}$. The above calculation shows that, as a right $k[R^*]$-module, $E^2_{i,j}$ is the tensor product of $H_i(N_1, k)$ and $H_j(N_2, k)$ with the right $k[R^*]$-module structures induced by the right actions of R^* on N_1 and N_2. Therefore, Lemma 8.8 shows that if $0 < i + 2j < r$, then the right $k[R^*]$-module $E^2_{i,j}$ decomposes as a direct sum of eigenspaces associated with non-trivial characters of R^* over k. It follows that if $0 < 2t < r$, then the right $k[\bar{G}]$-module $V = H_t(N, k)$ admits a finite filtration

$$0 = V_{-1} \subset V_0 \subset \cdots \subset V_t = V$$

such that, viewed as right $k[R^*]$-modules, the filtration quotients $\mathrm{gr}_i V$ decompose as direct sums of eigenspaces belonging to non-trivial characters of R^* over k. Therefore, the argument at the end of the proof of Proposition 8.10 shows that the group homology groups $H_s(\bar{G}, \mathrm{gr}_i V)$ all are zero. It follows, by easy induction, that the group homology groups $H_s((\bar{G}, H_t(N, k))$ are zero whenever $0 < 2t < r$. This implies that the map $r_{u*} \colon H_d(G, k) \to H_d(\bar{G}, k)$ is an isomorphism, which, in turn, implies that the characteristic class θ is zero. This completes the proof. $\square$

Proof of Theorem 8.1. By Lemmas 8.2, 8.3, and 8.5, it will suffice to show that for every algebraically closed field k, the following maps of k-Hopf algebras induced by the forgetful functor are isomorphisms.

(1) $T(iS^{2,1}\mathscr{C}[n], k) \xrightarrow{\phi_*} T(iS^{2,1}_{\oplus}\mathscr{C}[n], k)$

(2) $T(\mathrm{Sym}(iS^{2,1}\mathscr{C}[n], D[n]), k) \xrightarrow{\phi_*} T(\mathrm{Sym}(iS^{2,1}_{\oplus}\mathscr{C}[n], D[n]), k)$

Both maps are split injective, so only the surjectivity is at issue. To prove surjectivity, it will suffice to show the hypotheses of Proposition 8.7 are satisfied. In the case of the map (1), this follows from Propositions 7.3 and 8.10, and in the case of the map (2), it follows from Theorem **??** and Proposition 8.12. This completes the proof. $\square$

9 Agreement with Schlichting's construction

We use real additivity theorems and real group completion theorems to show that (the underlying infinite loop space of the G-fixed spectrum of) real algebraic K-theory agrees with Schlichting's Grothendieck-Witt space.

Theorem 9.1. *If $(\mathscr{C}, D, \eta)$ is an exact category with duality, then the pointed real spaces $\Omega^{1,1}B(iS^{1,1}(\mathscr{C}, D, \eta))$ and $\Omega^{2,1}B(iS^{2,1}(\mathscr{C}, D, \eta))$ are canonically naturally weakly equivalent.*

Proof. We will argue that, in the following diagram of pointed real spaces, the pointed real maps, which we specify below, all are real weak equivalences.

$$\begin{array}{ccc}
\Omega^{1,1}B(iS^{1,1}(\mathscr{C},D,\eta)) & & \Omega^{2,1}B(iS^{2,1}(\mathscr{C},D,\eta)) \\
\downarrow & & \downarrow \\
\Omega^{3,2}B(iS^{2,1}_{\oplus}S^{1,1}(\mathscr{C},D,\eta)) & & \Omega^{3,2}B(iS^{1,1}_{\oplus}S^{2,1}(\mathscr{C},D,\eta)) \\
\downarrow & & \downarrow \\
\Omega^{3,2}B(iS^{1,1}S^{2,1}_{\oplus}(\mathscr{C},D,\eta)) & & \Omega^{3,2}B(iS^{2,1}S^{1,1}_{\oplus}(\mathscr{C},D,\eta)) \\
\downarrow & & \downarrow \\
\Omega^{3,2}B(iS^{1,1}S^{2,1}(\mathscr{C},D,\eta)) & \longrightarrow & \Omega^{3,2}B(iS^{2,1}S^{1,1}(\mathscr{C},D,\eta))
\end{array}$$

We first explain the map in the diagram. The two top vertical maps are the spectrum structure maps in the respective direct sum K-theory spectra; the middle vertical maps are the canonical isomorphisms that interchange the two real simplicial directions; the bottom vertical maps are the forgetful maps from the respective direct sum K-theory constructions to the corresponding K-theory construction; and the bottom horizontal map is the canonical isomorphism that interchages the two real simplicial directions. The top left-hand vertical map is a real weak equivalence by the real group completion theorem, since it is a monoid object in the homotopy category of pointed real spaces with respect to orthogonal sum, and since, by a result of Schlichting [21, Proposition 3], for every subgroup $H \subset G$, the pointed set of components $\pi_0((\Omega^{1,1}B(iS^{1,1}(\mathscr{C},D,\eta)))^H)$ with the induced monoid structure is a group. Similarly, by a theorem of Moi [16, Theorem 5.8], the top right-hand vertical map is a real weak equivalence, since it is a monoid object in the homotopy category of real pointed spaces, and since the pointed set of components $\pi_0(\Omega^{2,1}B(iS^{2,1}(\mathscr{C},D,\eta)))$ with the induced monoid structure is a group. Finally, it follows from Schlichting's real additivity theorem [21, Theorem 4] that the lower left-hand vertical map is a real equivalence, and it follows similarly from the real additivity theorem for real algebraic K-theory that the lower right-hand vertical map is a real weak equivalence. Since all remaining maps in the diagram are isomorphisms, the theorem follows. □

The following surprising result is due to Schlichting [21, Proposition 3]. We present a more direct proof, which we learned from Schlichting.

Proposition 9.2. *If $(\mathscr{C}, D, \eta, 0)$ is a pointed exact category with duality, the abelian monoid structure on $\pi_0((\Omega^{1,1}B(iS^{1,1}(\mathscr{C}, D, \eta)))^G)$ induced from orthogonal sum is an abelian group structure.*

Proof. We consider the cofibration sequence of pointed real spaces

$$S^{0,0} \wedge G_+ \xrightarrow{f} S^{0,0} \xrightarrow{i} S^{1,1} \xrightarrow{h} S^{1,0} \wedge G_+,$$

where f collapses G onto the non-basepoint in $S^{0,0}$, where i is the inclusion of the subspace fixed by the G-action, and where h takes the class of iy to the class of $(y - y^{-1}, 1)$, if $0 < y < \infty$, to the class of $(y - y^{-1}, -1)$, if $-\infty < y < 0$, and to the basepoint, otherwise. It induces a sequence of pointed sets

$$\cdots \longrightarrow \pi_1(X) \xrightarrow{h^*} \pi_0((\Omega^{1,1}(X))^G) \xrightarrow{i^*} \pi_0(X^G) \xrightarrow{f^*} \pi_0(X),$$

which is exact in the sense that, at every term in the sequence, the subset of elements that are mapped to the basepoint by the map leaving the term is equal to the image of the map entering the term; see [24, Theorem 7.1.3]. Moreover, orthogonal sum induces an abelian monoid structure on each term in the sequence and the maps in the sequence are monoid homomorphisms. It follows from [24, Theorem 1.6.8] that said monoid structure $\pi_1(X)$ is equal to the underlying monoid structure underlying the groups structure of the fundamental group. In particular, it is an abelian group structure. We claim that also the abelian monoid structure on $\pi_0(X^G)$ is an abelian group structure. Granting this for the moment, it follows the statement follows. Indeed, let x be an element of $\pi_0((\Omega^{1,1}(X))^G)$. The claim implies that $i^*(x)$ has an inverse y, and since $\pi_0(X)$ is trivial, we can, by the exactness of the sequence, find an element x' of $\pi_0((\Omega^{1,1}(X))^G)$ with $i^*(x') = y$. Now,

$$i^*(x + x') = i^*(x) + i^*(x') = i^*(x) + y = 0,$$

and appealing again to the exactness of the sequence, we conclude that there exists an element z of $\pi_1(X)$ with $h^*(z) = x + x'$. But then $x' + h^*(-z)$ is an inverse of x, since

$$x + x' + h^*(-z) = h^*(z) + h^*(-z) = h^*(z + (-z)) = h^*(0) = 0.$$

It remains to prove the claim. To this end, we let $Y[-,-]$ be the pointed bisimplicial set obtained from $NiS^{1,1}(\mathscr{C}, D, \eta)$ by applying Segal's subdivision in both simplicial directions and taking G-fixed points. The realization of $Y[-,-]$, we recall, is canonically pointed homeomorphic to X^G. We have the following coequalizer diagram of pointed sets

$$\pi_0(|Y[-,1]|) \overset{d_0}{\underset{d_1}{\rightrightarrows}} \pi_0(|Y[-,0]|) \xrightarrow{e} \pi_0(|Y[-,-]|).$$

Again, orthogonal sum gives rise to an abelian monoid structure on each term and the three maps are monoid homomorphisms. The left half of the diagram is canonically identified with the diagram of isomorphism classes of objects obtained from the diagram

$$\mathrm{Sym}(iS^{1,1}(\mathscr{C},D,\eta)[3]) \underset{\mathrm{Sym}(d_0d_3)}{\overset{\mathrm{Sym}(d_1d_2)}{\rightrightarrows}} \mathrm{Sym}(iS^{1,1}(\mathscr{C},D,\eta)[1])$$

of groupoids and functors. So let (c,b) be an object of the right-hand groupoid. We define an object (c',b') of the left-hand groupoid which by $\mathrm{Sym}(d_1d_2)$ is mapped to the orthogonal sum of (c,b) and $(c,-b)$ and which by $\mathrm{Sym}(d_0d_3)$ is mapped to $(0[0],\mathrm{id}_{0[0]})$. The object (c',b'), in turn, is defined to be the image by $\mathrm{Sym}(s_1)$ of the object (c'',b'') of the groupoid $\mathrm{Sym}(iS^{1,1}(\mathscr{C},D,\eta)[2])$, where c'' is the unique diagram with

$$\begin{tikzcd}
c''_{01} \arrow[r] \arrow[d, equal] & c''_{02} \arrow[r] \arrow[d, equal] & c''_{12} \arrow[d, equal] \\
c_{01} \arrow[r, "(\mathrm{id},\mathrm{id})"] & c_{01} \oplus c_{01} \arrow[r, "\mathrm{id}+(-\mathrm{id})"] & c_{01},
\end{tikzcd}$$

and where $b''\colon c'' \to D[2](c'')$ is the unique natural transformation given by

$$\begin{tikzcd}
c''_{01} \arrow[r] \arrow[d, "b_{01}"] & c''_{02} \arrow[r] \arrow[d, "b_{01}\oplus(-b_{01})"] & c''_{12} \arrow[d, "b_{01}"] \\
D(c''_{12}) \arrow[r] & D(c''_{02}) \arrow[r] & D(c''_{01}).
\end{tikzcd}$$

Finally, using the simplicial identities $d_1d_2s_1 = d_1$ and $d_0d_3s_1 = s_0d_0d_2$, we find

$$\begin{aligned}
\mathrm{Sym}(d_1d_2)(c',b') &= \mathrm{Sym}(d_1)(c'',b'') = (c,b)\oplus(c,-b),\\
\mathrm{Sym}(d_0d_3)(c'',b'') &= \mathrm{Sym}(s_0)(0[0],\mathrm{id}_{0[0]}) = (0[1],\mathrm{id}_{0[1]}),
\end{aligned}$$

which completes the proof. □

We consider pairs $(\mathscr{D}^s, T)$, where $\mathscr{D}^s$ is a category enriched in the symmetric monoidal category of symmetric spectra and smash product, and where $T\colon \mathscr{D}^s \to (\mathscr{D}^s)^{\mathrm{op}}$ is an enriched functor such that the composite functor $T \circ T$ is equal to the identity functor of $\mathscr{D}^s$. We abbreviate the symmetric spectrum of maps in $\mathscr{D}^s$ from P to Q by

$$\mathscr{D}^s(P,Q) = \mathrm{Hom}_{\mathscr{D}^s}(P,Q).$$

That T preserves composition means that for all objects P, Q, and R, the following diagram of symmetric spectra commutes.

$$\begin{tikzcd}
\mathscr{D}^s(Q,R) \wedge \mathscr{D}^s(P,Q) \arrow[r, "T \wedge T"] \arrow[dd, "\circ"] & \mathscr{D}^s(R,Q) \wedge \mathscr{D}^s(Q,P) \arrow[d, "\gamma^s"] \\
 & \mathscr{D}^s(Q,P) \wedge \mathscr{D}^s(R,Q) \arrow[d, "\circ"] \\
\mathscr{D}^s(P,R) \arrow[r, "T"] & \mathscr{D}^s(R,P)
\end{tikzcd}$$

Here, the map γ^s is the symmetry isomorphism which is part of the structure of symmetric monoidal category. This, in turn, means that for all objects P, Q, and R, and all non-negative integers i and j, the following diagrams commute.

$$\begin{tikzcd}
\mathscr{D}^s(Q,R)_j \wedge \mathscr{D}^s(P,Q)_i \arrow[r, "T \wedge T"] \arrow[ddd, "\circ"] & \mathscr{D}^s(R,Q)_j \wedge \mathscr{D}^s(Q,P)_i \arrow[d, "\gamma"] \\
 & \mathscr{D}^s(Q,P)_i \wedge \mathscr{D}^s(R,Q)_j \arrow[d, "\circ"] \\
 & \mathscr{D}^s(R,P)_{i+j} \arrow[d, "\chi_{i,j}"] \\
\mathscr{D}^s(P,R)_{j+i} \arrow[r, "T"] & \mathscr{D}^s(R,P)_{j+i}
\end{tikzcd}$$

Here, the map γ is the canonical homeomorphism that permutes the two smash factors, and $\chi_{i,j} \in \Sigma_{i+j}$ is the permutation defined by

$$\chi_{i,j}(s) = \begin{cases} j+s & (1 \leqslant s \leqslant i) \\ s-i & (i+1 \leqslant s \leqslant i+j). \end{cases}$$

We leave it to the reader to spell out the easier statement that T preserves the identity.

We proceed to define a dihedral pointed space $\mathrm{THH}(\mathscr{D}^s, T)[-]$ with underlying cyclic pointed space $\mathrm{THH}(\mathscr{D}^s)[-]$ as defined in [5, Def. 1.3.6].

Let I be the category where the objects are the positive integers and where the morphisms from i to j is the set of all injective maps

$$\alpha\colon \{1,2,\dots,i\} \to \{1,2,\dots,j\}.$$

In particular, there is a unique morphism from 0 to each j. The category I has a strict monoidal structure given by the functor $+\colon I \times I \to I$ defined on objects by addition and morphisms by concatenation. In more detail, if $\alpha\colon i \to i'$ and $\beta\colon j \to j'$ are two morphisms, then $\alpha + \beta\colon i + j \to i' + j'$ is the morphism defined by

$$(\alpha+\beta)(s) = \begin{cases} \alpha(s) & (1 \leqslant s \leqslant i) \\ \beta(s-i)+i' & (i+1 \leqslant s \leqslant i+j). \end{cases}$$

The category I is equivalent to the category of finite sets and injective maps.

We first define a dihedral category $I[-]$ with

$$I[k] = I \times \cdots \times I \qquad (k+1 \text{ factors}).$$

The dihedral structure maps are generated by the functors

$$\begin{aligned} d_u\colon I[k] &\to I[k-1] && (0 \leqslant u \leqslant k) \\ s_u\colon I[k] &\to I[k+1] && (0 \leqslant u \leqslant k) \\ t_k\colon I[k] &\to I[k] \\ w_k\colon I[k] &\to I[k] \end{aligned}$$

defined as follows. The cyclic structure maps d_u, s_u, and t_k are defined on objects by

$$\begin{aligned} d_u(i_0,i_1,\dots,i_k) &= \begin{cases} (i_0,\dots,i_u+i_{u+1},\dots,i_k) & (0 \leqslant u < k) \\ (i_k+i_0,i_1,\dots,i_{k-1}) & (u = k) \end{cases} \\ s_u(i_0,i_1,\dots,i_k) &= (i_0,\dots,i_u,0,i_{u+1},\dots,i_k) \qquad (0 \leqslant u \leqslant k) \\ t_k(i_0,i_1,\dots,i_k) &= (i_k,i_0,i_1,\dots,i_{k-1}) \end{aligned}$$

and similarly on morphisms. To define the functor w_k, we let $\omega = \omega_i\colon i \to i$ be the involution $\omega(s) = i - s$ and, given a morphism $\alpha\colon i \to i'$, define

$$\alpha^{\omega} = \omega_{i'} \circ \alpha \circ \omega_i^{-1}\colon i \to i'.$$

The functor $w_k\colon I[k] \to I[k]$ is now defined on objects and morphisms by the following formulas, respectively.

$$\begin{aligned} w_k(i_0,i_1,\dots,i_k) &= (i_0,i_k,i_{k-1},\dots,i_1) \\ w_k(\alpha_0,\alpha_1,\dots,\alpha_k) &= (\alpha_0^{\omega},\alpha_k^{\omega},\alpha_{k-1}^{\omega},\dots,\alpha_1^{\omega}) \end{aligned}$$

Lemma 10.1. *The functors d_u, s_u, t_k, and w_k satisfy the cyclic relations and the following additional dihedral relations.*

$$\begin{aligned} d_u w_k &= w_{k-1} d_{k-u} & \qquad s_u w_k &= w_{k+1} s_{k-u} \\ w_k t_k &= t_k^{-1} w_k & \qquad w_k w_k &= \mathrm{id} \end{aligned}$$

Proof. Only the relations $d_u w_k = w_{k-1} d_{k-u}$ need proof. We consider the case $u = 0$; the remaining cases are similar. The functors $d_0 w_k$ and $w_{k-1} d_k$ take the morphism

$$(\alpha_0, \alpha_1, \dots, \alpha_k) \colon (i_0, i_1, \dots, i_k) \to (i'_0, i'_1, \dots, i'_k)$$

to the following morphisms, respectively.

$$\begin{gathered}(\alpha_0^\omega + \alpha_k^\omega, \alpha_{k-1}^\omega, \dots, \alpha_1^\omega) \colon (i_0 + i_k, i_{k-1}, \dots, i_1) \to (i'_0 + i'_k, i'_{k-1}, \dots, i'_1) \\ ((\alpha_k + \alpha_0)^\omega, \alpha_{k-1}^\omega, \dots, \alpha_1^\omega) \colon (i_k + i_0, i_{k-1}, \dots, i_1) \to (i'_k + i'_0, i'_{k-1}, \dots, i'_1)\end{gathered}$$

The relation $d_0 w_k = w_{k-1} d_k$ now follows from the identity

$$(\alpha + \beta)^\omega = \beta^\omega + \alpha^\omega$$

which is readily verified from the definitions. □

Remark 10.2. Taking the nerve and geometric realization of the categories $I[k]$, we obtain the dihedral space $[k] \mapsto BI[k]$. One wonders about the equivariant homotopy type of the $O(2)$-space $|[k] \mapsto BI[k]|$ defined by its geometric realization.

Let $\mathscr{T}$ be the category of pointed spaces. We recall the functor

$$G(\mathscr{D}^s)[k] \colon I[k] \to \mathscr{T}$$

that to the object $(i_0, \dots, i_k)$ associates the pointed mapping space

$$F(S^{i_0} \wedge S^{i_1} \wedge \cdots \wedge S^{i_k}, \bigvee \mathscr{D}^s(P_0, P_k)_{i_0} \wedge \mathscr{D}_s(P_1, P_0)_{i_1} \wedge \cdots \wedge \mathscr{D}^s(P_k, P_{k-1})_{i_k}),$$

where the wedge sum ranges over all $k+1$-tuples $(P_0, P_1, \dots, P_k)$ of objects in $\mathscr{D}^s$. To give the definition of the functor $G(\mathscr{D}^s)[k]$ on morphisms, we first let $\iota \colon i_r \to i'_r$ be the standard inclusion $\iota \colon \{1, 2, \dots, i_r\} \to \{1, 2, \dots, i'_r\}$ and write $i'_r = i_r + j_r$. Then the map of pointed spaces $G(\mathscr{D}^s)[k](i_0, \dots, \iota, \dots, i_k)$ takes the map

$$f \colon S^{i_0} \wedge \cdots \wedge S^{i_k} \to \bigvee \mathscr{D}^s(P_0, P_k)_{i_0} \wedge \cdots \wedge \mathscr{D}^s(P_k, P_{k-1})_{i_k}$$

to the composition

$$\begin{aligned} & S^{i_0} \wedge \cdots \wedge S^{i'_r} \wedge \cdots \wedge S^{i_k} \to S^{i_0} \wedge \cdots \wedge S^{i_r} \wedge \cdots \wedge S^{i_k} \wedge S^{j_r} \\ & \to \bigvee \mathscr{D}^s(P_k, P_{k-1})_{i_0} \wedge \cdots \wedge \mathscr{D}^s(P_r, P_{r-1})_{i_r} \wedge \cdots \wedge \mathscr{D}^s(P_k, P_{k-1})_{i_k} \wedge S^{j_r} \\ & \to \bigvee \mathscr{D}^s(P_k, P_{k-1})_{i_0} \wedge \cdots \wedge \mathscr{D}^s(P_r, P_{r-1})_{i_r} \wedge S^{j_r} \cdots \wedge \mathscr{D}^s(P_k, P_{k-1})_{i_k} \\ & \to \bigvee \mathscr{D}^s(P_k, P_{k-1})_{i_0} \wedge \cdots \wedge \mathscr{D}^s(P_r, P_{r-1})_{i_r + j_r} \cdots \wedge \mathscr{D}^s(P_k, P_{k-1})_{i_k} \end{aligned}$$

of the canonical homeomorphism, the map $f \wedge \mathrm{id}_{S^{j_r}}$, the canonical homeomorphism, and the map induced by the structure map in the symmetric spectrum $\mathscr{D}^s(P_r, P_{r-1})$. A general morphism $\alpha \colon i_r \to i'_r$ may be written, non-uniquely, as the composition $\alpha = \sigma \circ \iota \colon i_r \to i'_r$ of the standard inclusion and a bijection

$$\sigma \in \mathrm{Aut}(\{1, 2, \dots, i'_r\}) = \Sigma_{i'_r}.$$

Now, the left actions of the symmetric group $\Sigma_{i'_r}$ on $S^{i'_r}$ and $\mathscr{D}^s(P_r, P_{r-1})_{i'_r}$ induce left actions on the domain and target of the mapping space $G(\mathscr{D}^s)[k](i_0, \ldots, i'_r, \ldots, i_k)$. We define $G(\mathscr{D}^s)[k](i_0, \ldots, \sigma, \ldots, i_k)$ to be the conjugation by σ on the mapping space.

We define the dihedral pointed space $\mathrm{THH}(\mathscr{D}^s, T)[-]$ as follows. The pointed space of k-simplices is defined to be the homotopy colimit

$$\mathrm{THH}(\mathscr{D}^s, T)[k] = \operatorname*{hocolim}_{I[k]} G(\mathscr{D}^s)[k]$$

and the cyclic structure maps are defined as in [7, Sect. 3.2]. The additional dihedral structure map, which depends on the functor $T\colon \mathscr{D}^s \to (\mathscr{D}^s)^{\mathrm{op}}$, is the map

$$w_k\colon \mathrm{THH}(\mathscr{D}^s, T)[k] \to \mathrm{THH}(\mathscr{D}^s, T)[k]$$

defined to be the composition

$$\operatorname*{hocolim}_{I[k]} G(\mathscr{D}^s)[k] \to \operatorname*{hocolim}_{I[k]} G(\mathscr{D}^s)[k] \circ w_k \to \operatorname*{hocolim}_{I[k]} G(\mathscr{D}^s)[k]$$

of the map of homotopy colimits induced by a natural transformation

$$\mathrm{w}_k\colon G(\mathscr{D}^s)[k] \to G(\mathscr{D}^s)[k] \circ w_k$$

defined by the following diagram and the canonical map.

$$\begin{tikzcd}
S^{i_0} \wedge S^{i_1} \wedge \cdots \wedge S^{i_k} \arrow[r, "f"] \arrow[d, "\omega_{i_0} \wedge \omega_{i_1} \wedge \cdots \wedge \omega_{i_k}"] & \bigvee \mathscr{D}^s(P_0, P_k)_{i_0} \wedge \mathscr{D}^s(P_1, P_0)_{i_1} \wedge \cdots \wedge \mathscr{D}^s(P_k, P_{k-1})_{i_k} \arrow[d, "\bigvee \omega_{i_0} \wedge \omega_{i_1} \wedge \cdots \wedge \omega_{i_k}"] \\
S^{i_0} \wedge S^{i_1} \wedge \cdots \wedge S^{i_k} \arrow[d, equal] & \bigvee \mathscr{D}^s(P_0, P_k)_{i_0} \wedge \mathscr{D}^s(P_1, P_0)_{i_1} \wedge \cdots \wedge \mathscr{D}^s(P_k, P_{k-1})_{i_k} \arrow[d, "\bigvee T \wedge T \wedge \cdots \wedge T"] \\
S^{i_0} \wedge S^{i_1} \wedge \cdots \wedge S^{i_k} \arrow[d] & \bigvee \mathscr{D}^s(P_k, P_0)_{i_0} \wedge \mathscr{D}^s(P_0, P_1)_{i_1} \wedge \cdots \wedge \mathscr{D}^s(P_{k-1}, P_k)_{i_k} \arrow[d] \\
S^{i_0} \wedge S^{i_k} \wedge \cdots \wedge S^{i_1} \arrow[r, "\mathrm{w}_k(f)"] & \bigvee \mathscr{D}^s(P_k, P_0)_{i_0} \wedge \mathscr{D}^s(P_{k-1}, P_k)_{i_k} \wedge \cdots \wedge \mathscr{D}^s(P_0, P_1)_{i_1}
\end{tikzcd}$$

Here, the lower left-hand vertical map is the canonical homeomorphism that permutes the smash factors as indicated, and the lower right-hand vertical map is the map that takes the summand $(P_0, P_1, \ldots, P_k)$ to the summand $(P_k, P_{k-1}, \ldots, P_0)$ by the canonical homeomorphism that permutes the smash factors. The top vertical maps are given by the actions of $\omega_i \in \Sigma_i$ on S^i and $\mathscr{D}^s(P, Q)_i$ which are part of the symmetric spectrum structures.

Proposition 10.3. *The pointed maps d_u, s_u, t_k, and w_k satisfy the cyclic relations and the following additional dihedral relations.*

$$d_u w_k = w_{k-1} d_{k-u} \qquad s_u w_k = w_{k+1} s_{k-u}$$
$$w_k t_k = t_k^{-1} w_k \qquad w_k w_k = \mathrm{id}$$

Proof. We verify the relation $d_0 w_k = w_{k-1} d_k$; the reader may verify the remaining relations in a similar manner. We have already proved in Lemma 10.1 that the functors $d_0 w_k$ and $w_{k-1} d_k$ from $I[k]$ to $I[k-1]$ are equal. Hence, it suffices to also show that the natural transformations

$$\mathrm{d}_0\mathrm{w}_k, \mathrm{w}_{k-1}\mathrm{d}_k \colon G(\mathscr{D}^s)[k] \to G(\mathscr{D}^s)[k-1] \circ d_0 w_k = G(\mathscr{D}^s)[k-1] \circ w_{k-1} d_k$$

are equal. The natural transformation $\mathrm{d}_0\mathrm{w}_k$ is defined by the following diagram.

$$\begin{tikzcd}
S^{i_0} \wedge S^{i_1} \wedge \cdots \wedge S^{i_k} \arrow[r, "f"] \arrow[d, "\omega_{i_0} \wedge \omega_{i_1} \wedge \cdots \wedge \omega_{i_k}"] & \bigvee \mathscr{D}^s(P_0, P_k)_{i_0} \wedge \mathscr{D}^s(P_1, P_0)_{i_1} \wedge \cdots \wedge \mathscr{D}^s(P_k, P_{k-1})_{i_k} \arrow[d, "\bigvee \omega_{i_0} \wedge \omega_{i_1} \wedge \cdots \wedge \omega_{i_k}"] \\
S^{i_0} \wedge S^{i_1} \wedge \cdots \wedge S^{i_k} \arrow[d, equal] & \bigvee \mathscr{D}^s(P_0, P_k)_{i_0} \wedge \mathscr{D}^s(P_1, P_0)_{i_1} \wedge \cdots \wedge \mathscr{D}^s(P_k, P_{k-1})_{i_k} \arrow[d, "\bigvee T \wedge T \wedge \cdots \wedge T"] \\
S^{i_0} \wedge S^{i_1} \wedge \cdots \wedge S^{i_k} \arrow[d] & \bigvee \mathscr{D}^s(P_k, P_0)_{i_0} \wedge \mathscr{D}^s(P_0, P_1)_{i_1} \wedge \cdots \wedge \mathscr{D}^s(P_{k-1}, P_k)_{i_k} \arrow[d] \\
S^{i_0} \wedge S^{i_k} \wedge \cdots \wedge S^{i_1} \arrow[r, "\mathrm{w}_k(f)"] \arrow[d] & \bigvee \mathscr{D}^s(P_k, P_0)_{i_0} \wedge \mathscr{D}^s(P_{k-1}, P_k)_{i_k} \wedge \cdots \wedge \mathscr{D}^s(P_0, P_1)_{i_1} \arrow[d, "\bigvee \circ_{i_0, i_k} \wedge \mathrm{id}"] \\
S^{i_0 + i_k} \wedge \cdots \wedge S^{i_1} \arrow[r, "\mathrm{d}_0\mathrm{w}_k(f)"] & \bigvee \mathscr{D}^s(P_{k-1}, P_0)_{i_0 + i_k} \wedge \cdots \wedge \mathscr{D}^s(P_0, P_1)_{i_1}
\end{tikzcd}$$

Here, the lower left-hand vertical map is the canonical homeomorphism, and the lower right-hand vertical map is the map that takes the summand $(P_k, P_{k-1}, \dots, P_0)$ to the summand $(P_{k-1}, \dots, P_0)$ by the map induced from the composition map

$$\circ_{i_0, i_k} \colon \mathscr{D}^s(P_k, P_0)_{i_0} \wedge \mathscr{D}^s(P_{k-1}, P_k)_{i_k} \to \mathscr{D}^s(P_{k-1}, P_0)_{i_0 + i_k}.$$

which is part of the structure of a category enriched in symmetric spectra. We recall that this is a $\Sigma_{i_0} \times \Sigma_{i_k}$-equivariant map, where $\Sigma_{i_0} \times \Sigma_{i_k}$ acts on the target via the group homomorphism

$$+ \colon \Sigma_{i_0} \times \Sigma_{i_k} \to \Sigma_{i_0 + i_k}$$

defined by

$$(\sigma + \tau)(s) = \begin{cases} \sigma(s) & (1 \leqslant s \leqslant i_0) \\ \tau(s - i_0) + i_0 & (i_0 + 1 \leqslant s \leqslant i_0 + i_k). \end{cases}$$

We wish to compare the diagram above to the following diagram that defines the natural transformation $\mathrm{w}_{k-1}\mathrm{d}_k$. In this diagram, the top vertical maps and the lower left-hand vertical map in this diagram are the canonical homeomorphisms that permute the smash factors as indicated, and the lower right-hand vertical map takes the summand $(P_0,\dots,P_{k-1})$ to the summand $(P_{k-1},\dots,P_0)$ by the canonical map that permutes the smash factors as indicated.

$$\begin{tikzcd}
S^{i_0}\wedge S^{i_1}\wedge\cdots\wedge S^{i_k} \arrow[r, "f"] \arrow[d] & \bigvee \mathscr{D}^s(P_0,P_k)_{i_0}\wedge\mathscr{D}^s(P_1,P_0)_{i_1}\wedge\cdots\wedge\mathscr{D}^s(P_k,P_{k-1})_{i_k} \arrow[d] \\
S^{i_k}\wedge S^{i_0}\wedge\cdots\wedge S^{i_{k-1}} \arrow[d] & \bigvee \mathscr{D}^s(P_k,P_{k-1})_{i_k}\wedge\mathscr{D}^s(P_0,P_k)_{i_0}\wedge\cdots\wedge\mathscr{D}^s(P_{k-1},P_{k-2})_{i_{k-1}} \arrow[d, "\bigvee\circ_{i_k,i_0}\wedge\mathrm{id}"] \\
S^{i_k+i_0}\wedge\cdots\wedge S^{i_{k-1}} \arrow[r, "\mathrm{d}_k(f)"] \arrow[d, "\omega_{i_k+i_0}\wedge\cdots\wedge\omega_{i_{k-1}}"] & \bigvee \mathscr{D}^s(P_0,P_{k-1})_{i_k+i_0}\wedge\cdots\wedge\mathscr{D}^s(P_{k-1},P_{k-2})_{i_{k-1}} \arrow[d, "\bigvee\omega_{i_k+i_0}\wedge\cdots\wedge\omega_{i_{k-1}}"] \\
S^{i_k+i_0}\wedge\cdots\wedge S^{i_{k-1}} \arrow[d, equal] & \bigvee \mathscr{D}^s(P_0,P_{k-1})_{i_k+i_0}\wedge\cdots\wedge\mathscr{D}^s(P_{k-1},P_{k-2})_{i_{k-1}} \arrow[d, "\bigvee T\wedge\cdots\wedge T"] \\
S^{i_k+i_0}\wedge\cdots\wedge S^{i_{k-1}} \arrow[d] & \bigvee \mathscr{D}^s(P_0,P_{k-1})_{i_k+i_0}\wedge\cdots\wedge\mathscr{D}^s(P_{k-1},P_{k-2})_{i_{k-1}} \arrow[d] \\
S^{i_k+i_0}\wedge\cdots\wedge S^{i_1} \arrow[r, "\mathrm{w}_{k-1}\mathrm{d}_k(f)"] & \bigvee \mathscr{D}^s(P_0,P_{k-1})_{i_k+i_0}\wedge\cdots\wedge\mathscr{D}^s(P_0,P_1)_{i_1}
\end{tikzcd}$$

The compositions of the left-hand vertical maps in the two diagrams are readily seen to agree. Therefore, it suffices to show that the same holds for the compositions of the right-hand vertical maps in the two diagrams. This, in turn, follows from the commutativity of the outer square in the following diagram.

$$\begin{tikzcd}
\mathscr{D}^s(P_k,P_{k-1})_{i_k}\wedge\mathscr{D}^s(P_0,P_k)_{i_0} \arrow[rr, "\omega_{i_k}\wedge\omega_{i_0}"] \arrow[dd, "\circ_{i_k,i_0}"] & & \mathscr{D}^s(P_k,P_{k-1})_{i_k}\wedge\mathscr{D}^s(P_0,P_k)_{i_0} \arrow[d, "\gamma"] \arrow[ddl, "\circ_{i_k,i_0}"'] \\
& & \mathscr{D}^s(P_0,P_k)_{i_0}\wedge\mathscr{D}^s(P_k,P_{k-1})_{i_k} \arrow[dd, "T\wedge T"] \\
\mathscr{D}^s(P_0,P_{k-1})_{i_k+i_0} \arrow[dd, "\omega_{i_k+i_0}"] & \mathscr{D}^s(P_0,P_{k-1})_{i_k+i_0} \arrow[l, "\omega_{i_k}+\omega_{i_0}"'] \arrow[ddl, "\chi_{i_k,i_0}"] & \\
& & \mathscr{D}^s(P_k,P_0)_{i_0}\wedge\mathscr{D}^s(P_{k-1},P_0)_{i_k} \arrow[d, "\circ_{i_0,i_k}"] \\
\mathscr{D}^s(P_0,P_{k-1})_{i_0+i_k} \arrow[rr] & & \mathscr{D}^s(P_{k-1},P_0)_{i_0+i_k}
\end{tikzcd}$$

Here, the top trapezoidal diagram commutes since $\circ_{i_k,i_0}$ is $\Sigma_{i_k}\times\Sigma_{i_0}$-equivariant; the lower right-hand triangular diagram commutes since $T\colon\mathscr{D}^s\to(\mathscr{D}^s)^{\mathrm{op}}$ is an enriched functor; finally, the lower left-hand triangular diagram commutes since the equality $\chi_{i_k,i_0} = \omega_{i_k+i_0}(\omega_{i_k} + \omega_{i_0})$ holds in $\Sigma_{i_k+i_0}$. □

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