1 Algebraic K-theory and the p-adic L-function

We let p be an odd prime number and define F_p to be the p-completion of the homotopy fiber of the cyclotomic trace map

$$\mathrm{tr}\colon K(\mathbb{Z}) \to \mathrm{TC}(\mathbb{Z};p).$$

Then we have a long-exact sequence of p-adic homotopy groups

$$\cdots \to \pi_q(F_p) \to K_q(\mathbb{Z};\mathbb{Z}_p) \xrightarrow{\mathrm{tr}_q} \mathrm{TC}_q(\mathbb{Z};p,\mathbb{Z}_p) \to \pi_{q-1}(F_p) \to \cdots$$

We will argue that the following (1)–(3) hold:

1. The value $L_p(\mathbb{Q},\omega^{-2k},1+2k)$ is non-zero if and only if $\pi_{4k+1}(F_p)$ is zero.
2. If $L_p(\mathbb{Q},\omega^{-2k},1+2k) \neq 0$, then $\pi_{4k}(F_p)$ and $\pi_{4k-1}(F_p)$ are finite and
$$|L_p(\mathbb{Q},\omega^{-2k},1+2k)|_p = \#\pi_{4k-1}(F_p)/\#\pi_{4k}(F_p).$$
3. There is an exact sequence
$$0 \to \mathbb{Z}_p \to \pi_{4k-2}(F_p) \to K_{4k-2}(\mathbb{Z},\mathbb{Z}_p) \to 0.$$

Here and below, we write $|a|_p = p^{-v_p(a)}$ for the p-adic absolute value of a.

We briefly recall the p-adic L-function. The Dirichlet series associated with the Dirichlet character χ is defined by

$$L(\mathbb{Q},\chi,s) = \sum_{n=1}^{\infty} \chi(n)n^{-s} = \prod_p (1-\chi(p)p^{-s})^{-1}.$$

The series converges for $\mathrm{Re}(s) > 1$ and admits a unique extension to the meromorphic function $L(\mathbb{Q},\chi,s)$ called the Dirichlet L-function. The values of this function at negative integers are rational numbers. We remark that $L(\mathbb{Q},1,s) = \zeta(s)$ is the Riemann zeta function. Let ω be the Teichmüller character. The Kubota-Leopoldt p-adic L-function associated with χ is the unique continuous function

$$L_p(\mathbb{Q},\chi,-)\colon \mathbb{Z}_p \smallsetminus \{1\} \to \mathbb{Q}_p$$

with the property that

$$L_p(\mathbb{Q},\chi,1-i) = (1-(\chi\omega^{-i})(p)p^{i-1})L(\mathbb{Q},\chi\omega^{-i},1-i)$$

for all integers $i \geq 1$. In this sense, the p-adic L-function interpolates the values of the Dirichlet L-function with the Euler factor corresponding to the prime p removed. In particular, $L_p(\mathbb{Q},\omega^i,1-i) = (1-p^{i-1})\zeta(1-i)$ for $i \geqslant 1$.

We wish to understand the kernel and cokernel of the cyclotomic trace map

$$\mathrm{tr}_q\colon K_q(\mathbb{Z},\mathbb{Z}_p) \to \mathrm{TC}_q(\mathbb{Z};p,\mathbb{Z}_p).$$

Let $f\colon \mathbb{Z} \to \mathbb{Z}_p$ be the canonical ring homomorphism. Then, in the diagram

$$\begin{array}{ccc} K_q(\mathbb{Z}, \mathbb{Z}_p) & \xrightarrow{f_q} & K_q(\mathbb{Z}_p, \mathbb{Z}_p) \\ \downarrow{\scriptstyle \mathrm{tr}_q} & & \downarrow{\scriptstyle \mathrm{tr}_q} \\ \mathrm{TC}_q(\mathbb{Z}; p, \mathbb{Z}_p) & \xrightarrow{f_q} & \mathrm{TC}_q(\mathbb{Z}_p; p, \mathbb{Z}_p), \end{array}$$

the right-hand vertical map tr_q and lower horizontal map f_q are isomorphisms. Hence, the kernel and cokernel of the left-hand vertical map tr_q are isomorphic to the kernel and cokernel, respectively, of the upper horizontal map f_q. The latter, in turn, are isomorphic to the kernel and cokernel of the map

$$K_q(\mathbb{Z}[\tfrac{1}{p}], \mathbb{Z}_p) \xrightarrow{f_q} K_q(\mathbb{Q}_p, \mathbb{Z}_p).$$

Indeed, this follows immediately from the localization sequence in K-theory. To understand this map, we consider the induced map of the spectral sequences from motivic cohomology to algebraic K-theory,

$$\begin{aligned} E^2_{s,t} &= H^{t-s}(\operatorname{Spec} \mathbb{Z}[\tfrac{1}{p}], \mathbb{Z}_p(t)) \Rightarrow K_{s+t}(\mathbb{Z}[\tfrac{1}{p}], \mathbb{Z}_p) \\ E^2_{s,t} &= H^{t-s}(\operatorname{Spec} \mathbb{Q}_p, \mathbb{Z}_p(t)) \Rightarrow K_{s+t}(\mathbb{Q}_p, \mathbb{Z}_p). \end{aligned}$$

The motivic cohomology groups in the E^2-terms are isomorphic to the corresponding étale cohomlogy groups, if $s \leqslant t$, and are zero, otherwise. Indeed, this is the statement of the affirmed Beilinson-Lichtenbaum conjectures. We will not distinguish notationally between motivic cohomology groups and the associated étale cohomology groups.

Let $i \neq 0, 1$ be an integer which may be positive or negative. Then the étale cohomology groups $H^q(\operatorname{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(i))$ and $H^q(\operatorname{Spec} \mathbb{Q}_p, \mathbb{Z}_p(i))$ are finitely generated $\mathbb{Z}_p$-modules which are non-zero for $q = 1, 2$ only [4, §1, Satz 5; §2, Satz 2; §3, Satz 4; §4, Lemmas 2 and 5]. Hence, for $i > 1$ and $j = 1, 2$, we have a commutative diagram

$$\begin{array}{ccc} K_{2i-j}(\mathbb{Z}[\tfrac{1}{p}], \mathbb{Z}_p) & \xrightarrow{f_{2i-j}} & K_{2i-j}(\mathbb{Q}_p, \mathbb{Z}_p) \\ \uparrow & & \uparrow \\ H^j(\operatorname{Spec} \mathbb{Z}[\tfrac{1}{p}], \mathbb{Z}_p(i)) & \xrightarrow{f_{2i-j}} & H^j(\operatorname{Spec} \mathbb{Q}_p, \mathbb{Z}_p(i)) \end{array}$$

where the vertical maps are isomorphisms. The lower horizontal map f_{2i-j} appears in the Tate-Poitou duality sequence which takes the following form [4, §2, Satz 5].

$$
\begin{aligned}
0 &\to H^2(\operatorname{Spec}\mathbb{Z}[\tfrac{1}{p}],\mathbb{Q}_p/\mathbb{Z}_p(1-i))^* \\
&\to H^1(\operatorname{Spec}\mathbb{Z}[\tfrac{1}{p}],\mathbb{Z}_p(i)) \xrightarrow{f_{2i-1}} H^1(\operatorname{Spec}\mathbb{Q}_p,\mathbb{Z}_p(i)) \\
&\to H^1(\operatorname{Spec}\mathbb{Z}[\tfrac{1}{p}],\mathbb{Q}_p/\mathbb{Z}_p(1-i))^* \\
&\to H^2(\operatorname{Spec}\mathbb{Z}[\tfrac{1}{p}],\mathbb{Z}_p(i)) \xrightarrow{f_{2i-2}} H^2(\operatorname{Spec}\mathbb{Q}_p,\mathbb{Z}_p(i)) \\
&\to H^0(\operatorname{Spec}\mathbb{Z}[\tfrac{1}{p}],\mathbb{Q}_p/\mathbb{Z}_p(1-i))^* \to 0
\end{aligned}
$$

Here the asterisk indicates the Pontryagin dual. Hence, the kernel and cokernel of the cyclotomic trace map is governed by the groups $H^q(\operatorname{Spec}\mathbb{Z}[\tfrac{1}{p}],\mathbb{Q}_p/\mathbb{Z}_p(1-i))$ with $i > 1$. The following conjecture was made by Lichtenbaum [2, Conjecture 9.1] for $i < 0$ and by Schneider [4, p. 192] in general.

Conjecture. *The group $H^2(\operatorname{Spec}\mathbb{Z}[\tfrac{1}{p}],\mathbb{Q}_p/\mathbb{Z}_p(1-i))$ is zero for every integer $i \neq 0$.*

The conjecture was proved for $i < 0$ by Soulé [5, Théorème 5] but it remains open for $i > 0$. The conjecture is equivalent to the statement that for every integer $i \neq 0$, the group $H^2(\operatorname{Spec}\mathbb{Z}[\tfrac{1}{p}],\mathbb{Z}_p(1-i))$ is finite. Moreover, by the discussion above, the conjecture for $i > 1$ is equivalent to the statement that the map

$$f_{2i-1}\colon K_{2i-1}(\mathbb{Z},\mathbb{Z}_p) \to K_{2i-1}(\mathbb{Z}_p,\mathbb{Z}_p)$$

is injective.

We next recall the precise relationship between the values of the p-adic L-function and the étale cohomology groups implied by the affirmed Main Conjecture of Iwazawa theory [3,6]. It was proved by Bayer and Neukirch [1, Theorem 6.1] that if $i \neq 1$ is an odd integer and if $L_p(\mathbb{Q},\omega^{1-i},i) \neq 0$, then the groups

$$H^q(\operatorname{Spec}\mathbb{Z}[\tfrac{1}{p}],\mathbb{Z}_p(1-i)$$

are finite for all $q \geqslant 0$, zero for $q \neq 1,2$, and

$$
\begin{aligned}
|L_p(\mathbb{Q},\omega^{1-i},i)|_p &= \frac{\#H^1(\operatorname{Spec}\mathbb{Z}[\tfrac{1}{p}],\mathbb{Z}_p(1-i))}{\#H^2(\operatorname{Spec}\mathbb{Z}[\tfrac{1}{p}],\mathbb{Z}_p(1-i))} \\
&= \frac{\#H^0(\operatorname{Spec}\mathbb{Z}[\tfrac{1}{p}],\mathbb{Q}_p/\mathbb{Z}_p(1-i))}{\#H^1(\operatorname{Spec}\mathbb{Z}[\tfrac{1}{p}],\mathbb{Q}_p/\mathbb{Z}_p(1-i))}.
\end{aligned}
$$

We remark that in loc. cit., the equality above is stated for the so-called Iwazawa zeta function $\zeta_I(\mathbb{Q},\omega^{1-i},i)$ which is introduced in op. cit., §5 and which depends on a choice of topological generator $q \in \mathbb{Z}_p^*$. However, by the affirmed Main Conjecture of Iwazawa theory, it follows that

$$L_p(\mathbb{Q},\omega^{1-i},i) = \zeta_I(\mathbb{Q},\omega^{1-i},i)\cdot u(q^{1-i}-1)$$

where $u(T) \in \mathbb{Z}_p[[T]]^*$. We also remark that for $i < 0$ odd, the p-adic L-function interpolates the values of the Riemann zeta function at negative integers. Hence, in this case, Lichtenbaum's conjecture

$$|\zeta(i)|_p = \frac{\# H^0(\operatorname{Spec}\mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))}{\# H^1(\operatorname{Spec}\mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))}$$

ensues; see [1, Theorem 6.2].

Now let $i > 1$ be odd and assume that $L_p(\mathbb{Q}, \omega^{1-i}, i)$ is non-zero. Then we conclude from the theorem of Bayer-Neukirch and from the Tate-Poitou sequence that the group $H^2(\operatorname{Spec}\mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))$ is zero, that

$$f_{2i-1} \colon K_{2i-1}(\mathbb{Z}, \mathbb{Z}_p) \to K_{2i-1}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is injective, and that the cokernel is related to the p-adic L-function as follows.

$$\begin{aligned} |L_p(\mathbb{Q}, \omega^{1-i}, i)|_p &= \frac{\# H^0(\operatorname{Spec}\mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))}{\# H^1(\operatorname{Spec}\mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))} \\ &= \frac{\# H^2(\operatorname{Spec}\mathbb{Q}_p, \mathbb{Z}_p(i))}{\#\operatorname{coker}(f_{2i-1}) \cdot \# H^2(\operatorname{Spec}\mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(i))} \\ &= \frac{\# K_{2i-2}(\mathbb{Z}_p, \mathbb{Z}_p)}{\#\operatorname{coker}(f_{2i-1}) \cdot \# K_{2i-2}(\mathbb{Z}, \mathbb{Z}_p)}. \end{aligned}$$

The group $K_{2i-2}(\mathbb{Z}, \mathbb{Z}_p)$ is zero for all odd integers $i > 1$ if and only if the Kummer-Vandiver conjecture holds for the prime number p. The group $K_{2i-2}(\mathbb{Z}_p, \mathbb{Z}_p)$ is finite cyclic of order $p^{v_p(i-1)+1}$ if $p-1$ divides $i-1$, and is zero otherwise. Therefore, if the Kummer-Vandiver conjecture holds for the prime p, and if $i > 1$ is odd, we find

$$\#\operatorname{coker}(f_{2i-1}) = \begin{cases} p^{v_p(i-1)+1} |L_p(\mathbb{Q}, \omega^{1-i}, i)|_p^{-1} & \text{if } p-1 \text{ divides } i-1 \\ |L_p(\mathbb{Q}, \omega^{1-i}, i)|_p^{-1} & \text{otherwise,} \end{cases}$$

or equivalently,

$$\operatorname{length}(\operatorname{coker}(f_{2i-1})) = \begin{cases} v_p(L_p(\mathbb{Q}, \omega^{1-i}, i)) + v_p(i-1) + 1 & \text{if } p-1 \text{ divides } i-1 \\ v_p(L_p(\mathbb{Q}, \omega^{1-i}, i)) & \text{otherwise.} \end{cases}$$

Next, we let $i > 1$ be even. Then it follows from [4, §4, Satz 6] that

$$\operatorname{rk}_{\mathbb{Z}_p} H^1(\operatorname{Spec}\mathbb{Z}[\tfrac{1}{p}], \mathbb{Z}_p(i)) = \operatorname{rk}_{\mathbb{Z}_p} H^2(\operatorname{Spec}\mathbb{Z}[\tfrac{1}{p}], \mathbb{Z}_p(i)).$$

But the right-hand group is finite by Soulé's theorem [5, Théorème 5], and hence, also the left-hand group is finite. This, we claim, implies that

$$H^2(\operatorname{Spec}\mathbb{Z}[\tfrac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i)) = 0.$$

Indeed, by the Tate-Poitou sequence, the Pontryagin dual of this group is a subgroup of the finite group $H^1(\operatorname{Spec}\mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(i))$, and hence, is finite. But it also a quotient of the group $H^2(\operatorname{Spec}\mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p(i))$, and hence, a divisible group. Therefore, the group is zero as claimed. In addition, it follows that the boundary map

$$H^j(\operatorname{Spec}\mathbb{Z}[\tfrac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(i)) \to H^{j+1}(\operatorname{Spec}\mathbb{Z}[\tfrac{1}{p}], \mathbb{Z}_p(i))$$

is an isomorphism for all $j \geqslant 0$. For every integer $i \neq 0$, we have

$$\begin{aligned} \#H^0(\operatorname{Spec}\mathbb{Z}[\tfrac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(i)) &= \max\{p^v \mid [\mathbb{Q}(\mu_{p^v})\colon \mathbb{Q}] \text{ divides } i\} \\ \#H^0(\operatorname{Spec}\mathbb{Q}_p, \mathbb{Q}_p/\mathbb{Z}_p(i)) &= \max\{p^v \mid [\mathbb{Q}_p(\mu_{p^v})\colon \mathbb{Q}_p] \text{ divides } i\} \end{aligned}$$

and both orders are equal to $p^{v_p(i)+1}$, if $p-1$ divides i, and equal to 1, otherwise.

References

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