Numbers – New and Old

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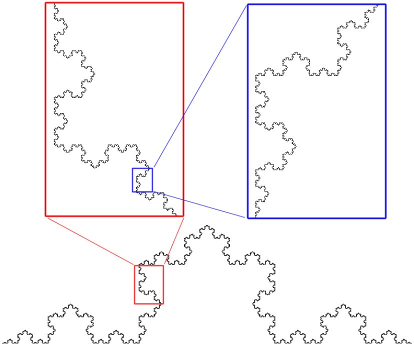


Figure: Scale invariance

Axioms of ZFC set theory

Extensionality: $\forall x \forall y (\forall z (z \in x \Leftrightarrow z \in y) \Rightarrow x = y)$
Empty set: $\exists x \nexists y (y \in x)$
Pairs: $\forall x \forall y \exists z \forall w (w \in z \Leftrightarrow (w = x \vee w = y))$
Power set: $\forall x \exists y \forall z (z \in y \Leftrightarrow \forall w (w \in z \Rightarrow w \in x))$
Union: $\forall x \exists y \forall z (z \in y \Leftrightarrow \exists w (w \in x \wedge z \in w))$
Infinity: $\exists x (\varnothing \in x \wedge \forall y (y \in x \Rightarrow \bigcup \{y, \{y\}\} \in x))$
Separation schema: For every formula φ with v not free,
$\forall u_1 \ldots \forall u_k (\forall w \exists v \forall r (r \in v \Leftrightarrow (r \in w \wedge \varphi(r, u_1, \ldots, u_k))))$
Replacement schema: For every formula φ with v not free,
$\forall u_1 \ldots \forall u_k (\forall x \exists ! y (\varphi(x, y, u_1, \ldots, u_k)$
$\quad \Rightarrow \forall w \exists v \forall r (r \in v \Leftrightarrow \exists s (s \in w \wedge \varphi(s, r, u_1, \ldots, u_k)))))$
Foundation: $\forall x (x \neq \varnothing \Rightarrow \exists y (y \in x \wedge \forall z (z \in x \Rightarrow z \notin y)))$
Choice: $\forall x (\varnothing \notin x \wedge \forall y \in x \forall z \in x (y \neq z \Rightarrow y \cap z = \varnothing))$
$\quad \Rightarrow \exists z \forall y \in x (\exists v \in y \cap z \forall w \in y \cap z (w = v))$

Natural numbers

Call a set x is *inductive* if $\varnothing \in x$ and if $y \in x$ implies $y \cup \{y\} \in x$.

Definition (von Neumann)

The set of natural numbers is the set

$$\mathbb{N} = \{n \in x \mid \forall y (y \text{ is inductive} \Rightarrow n \in y)\},$$

where x is any choice of inductive set.

So the natural numbers are

$$\begin{aligned}
0 &= \varnothing \\
1 &= 0 \cup \{0\} = \{\varnothing\} \\
2 &= 1 \cup \{1\} = \{\varnothing, \{\varnothing\}\} \\
3 &= 2 \cup \{2\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} \\
&\vdots
\end{aligned}$$

Counting

The following result was likely known empirically 20,000 years ago.

Theorem

Let m and n be natural numbers. If there exists a bijection $f : m \to n$, then necessarily $m = n$.

It shows that the following notion of counting is well-defined.

Definition

A set x is finite of cardinality n if there exists a bijection $f : n \to x$ from a natural number n.

For example, the set $\{\mathbb{N}\}$ is finite of cardinality 1, while its unique element set $\mathbb{N}$ itself is an infinite set.

Counting prime numbers

Theorem (Euclid)

There are infinitely many prime numbers.

So the prime number counting function

$$\pi\colon [0,\infty) \to \mathbb{N}$$

defined by

$$\pi(x) = \text{card}(\{p \in \mathbb{N} \mid p \leqslant x \text{ and } p \text{ is a prime number}\})$$

tends to infinity as x tends to infinity.

Graph of $\pi(x)$ for $x \leqslant 25$

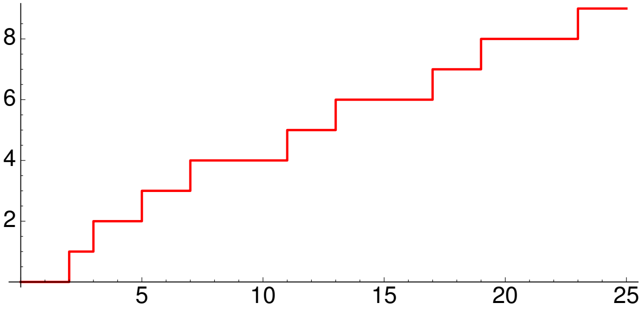


Figure: Source: Mazur-Stein

Graph of $\pi(x)$ for $x \leqslant 100$

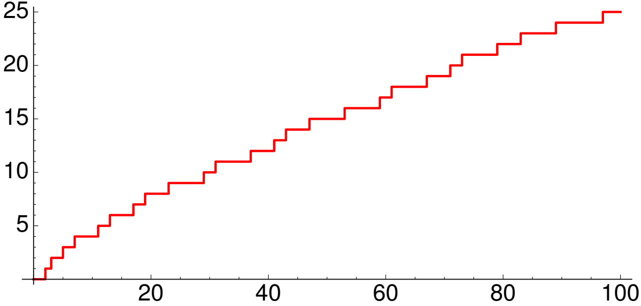


Figure: Source: Mazur-Stein

Graph of $\pi(x)$ for $x \leqslant 1000$

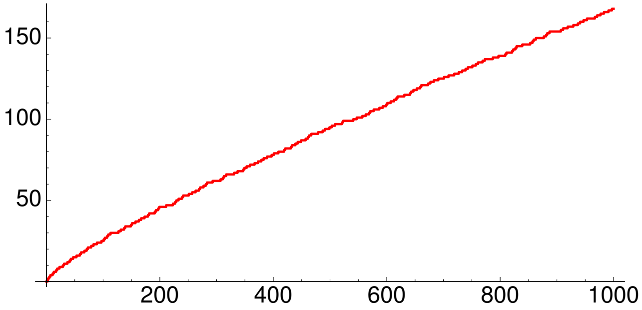


Figure: Source: Mazur-Stein

Graph of $\pi(x)$ for $x \leqslant 10000$

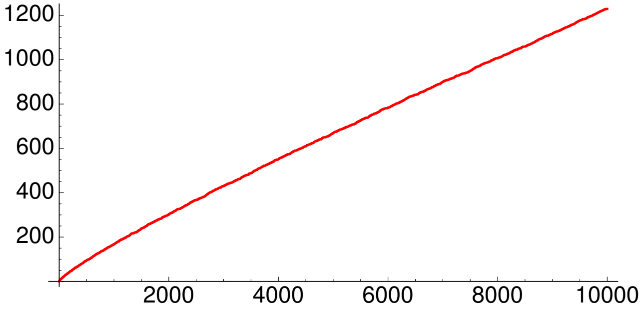


Figure: Source: Mazur-Stein

Gauss' guess

Unter	gezählte Primzahlen	Integral $\int \frac{dn}{\log n}$	Differ	Ihre Formel	Abweich.
500 000	41 556	41606,4	+50,4	41596,9	+40,9
1000 000	78 501	78627,5	+126,5	78672,7	+171,7
1500 000	114 112	114263,1	+151,1	114374,0	+264,0
2000 000	148883	149054,8	+171,8	149233,0	+350,0
2500 000	183016	183245,0	+229,0	183495,1	+479,1
3000 000	216745	216970,6	+225,6	217308,5	+563,6

Dass Legendre sich auch mit diesem Gegenstande beschäftigt hat, war mir nicht bekannt; auf Veranlassung Ihres Briefes habe ich in seiner Théorie des Nombres nachgesehen, und in der zweiten Ausgabe einige darauf bezügliche Seiten gefunden, die ich früher übersehen (oder seitdem vergessen) haben muß. Legendre gebraucht die Formel

$$\frac{n}{\log n - A}$$

Figure: Letter from Gauss to Encke (1849)

Riemann's conjecture

Gauss' guess was that $\pi(x)$ approximately is equal to

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}.$$

Riemann, his student, made the following more precise conjecture.

Conjecture (Riemann hypothesis)

For every real number $\epsilon > 0$, $\pi(x) = \mathrm{Li}(x) + O(x^{\frac{1}{2}+\epsilon})$.

Figure: Carl Friedrich Gauss (left) and Bernhard Riemann (right)

The Riemann zeta function

Riemann was the first to consider Euler's product

$$\sum_{n=1}^{\infty} n^{-s} = \prod_{p} (1 - p^{-s})^{-1}$$

as a function of a complex variable s, and he showed that there is a meromorphic function $\zeta(s)$, necessarily unique, such that

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1}$$

for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. He introduced the Euler factor

$$\zeta_{\infty}(s) = 2^{-\frac{1}{2}} \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})$$

at $p = \infty$ and the completed zeta function

$$\hat{\zeta}(s) = \zeta(s) \cdot \zeta_{\infty}(s) = \zeta(\text{``}\mathrm{Spec}(\mathbb{Z}) \cup \{\infty\}\text{''}, s).$$

The Riemann zeta function

It satisfies the following functional equation.

Theorem (Riemann)

For all $s \in \mathbb{C}$,

$$\hat{\zeta}(1-s) = \hat{\zeta}(s).$$

So for all $s \in \mathbb{C}$ with $\bar{s} = 1 - s$, or equivalently, $\mathrm{Re}(s) = \frac{1}{2}$,

$$\overline{\hat{\zeta}(s)} = \hat{\zeta}(\bar{s}) = \hat{\zeta}(1-s) = \hat{\zeta}(s).$$

The Riemann hypothesis takes the following equivalent form.

Conjecture (Riemann hypothesis)

All zeroes of $\hat{\zeta}(s)$ are located on the line $\mathrm{Re}(s) = \frac{1}{2}$.

More numbers

$$\mathbb{C} \;—\; \mathbb{R} \;—\; \mathbb{Q} \;—\; \mathbb{Q}_p \;—\; \mathbb{C}_p \;—\; B_{dR}$$

$$\mathbb{Q} \;—\; \mathbb{Z} \;—\; \mathbb{N}$$

"Zeta functions are able to travel to the p-adic world, and to understand their relation to arithmetic, it is important to study their lifestyles in the p-adic world"
— Kazuya Kato (Plenary lecture at 2006 ICM)

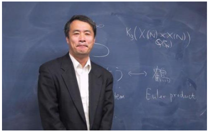


Figure: Kazuya Kato

The Weil conjectures

Let X be a scheme smooth and proper over a finite field of relative dimension d. The Hasse-Weil zeta function of X is defined to be the unique meromorphic function $\zeta(X, s)$ such that

$$\zeta(X, s) = \prod_{x \in |X|} (1 - \mathrm{card}(k(x))^{-s})^{-1}$$

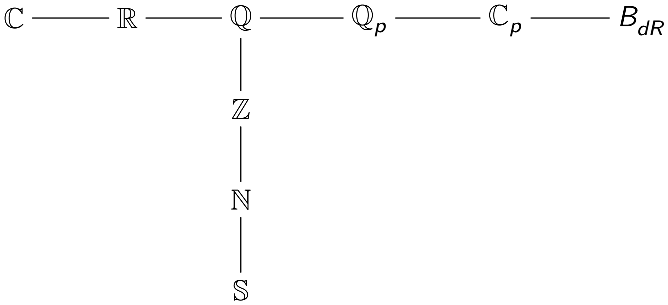
for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > d$. The following analog of the Riemann hypothesis was proved by Pierre Deligne in 1974, building on the twenty-year long effort of the Grothendieck school.

Theorem (Deligne)

All zeroes (resp. poles) of $\zeta(X, s)$ are located on the lines $\mathrm{Re}(s) = \frac{k}{2}$ with $0 \leqslant k \leqslant 2d$ odd (resp. even).

More basic numbers

There is a notion of numbers more basic than that of natural numbers. It emerges by encoding all possible ways of counting finite sets, as opposed to only remembering their cardinalities.



We are only beginning to understand their arithmetic.