Oxford '15

A : $\mathbb{Z}$-algebra

Hochschild homology :

$$HH(A) = \left| \begin{array}{c} \downarrow \cdots \downarrow \\ A \otimes_{\mathbb{Z}} A \otimes_{\mathbb{Z}} A \\ \downarrow \uparrow \downarrow \uparrow \downarrow \\ A \otimes_{\mathbb{Z}} A \\ \downarrow \uparrow \downarrow \\ A \end{array} \right| \qquad \otimes \rightsquigarrow \otimes^L$$

Connes : $\mathbb{T}$-spectrum , $\mathbb{T} = \mathbb{R}/\mathbb{Z}$

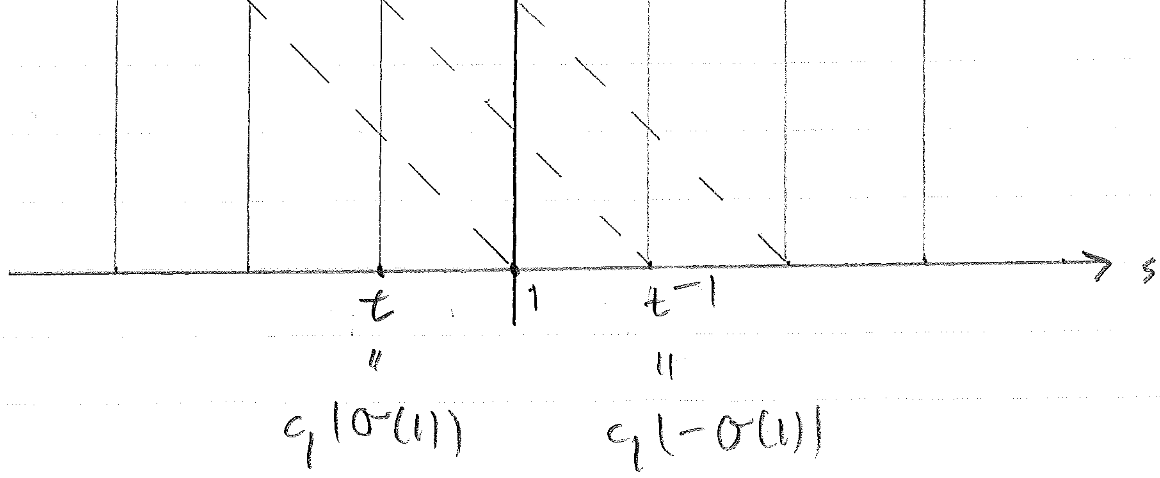
Periodic cyclic homology :

$$HP_m(A) = \hat{H}^{-m}(\mathbb{T}, HH(A))$$

Why periodic ? Spectral sequence

$$E^2_{s,t} = H^{-s}(\mathbb{P}^{\infty}_{-\infty}(\mathbb{C}), HH_t(A))$$

$$\Rightarrow HP_{s+t}(A)$$



Two easy facts:

1) $HP_*(A)$ is an $HP_*(\mathbb{Z})$-module

2) $HH_*(\mathbb{Z}) = \mathbb{Z}$.

By 2), $HP_*(\mathbb{Z}) = S_{\mathbb{Z}}\{t^{\pm 1}\}$, so by 1), $HP_*(A)$ is 2-periodic with

periodicity operator

$= \text{mult. by } t^{-1} \in HP_2(\mathbb{Z})$

A: $\mathbb{S}$-algebra
$$\begin{tikzcd} \mathbb{Z} \\ \mathbb{S} \arrow[u, "\text{Hurewicz}"'] \end{tikzcd}$$

Topological Hochschild homology

$$THH(A) = \left|\begin{array}{c} \downarrow \cdots \downarrow \\ A \otimes_{\mathbb{S}} A \otimes_{\mathbb{S}} A \\ \downarrow\uparrow\downarrow\uparrow\downarrow \\ A \otimes_{\mathbb{S}} A \\ \downarrow\uparrow\downarrow \\ A \end{array}\right|$$

Bökstedt
Breen

Connes: $\mathbb{T}$-spectrum, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$

Define

$$TP_m(A) := \hat{\mathbb{H}}^{-m}(\mathbb{T}, THH(A)).$$

Periodic? Spectral sequence

$$E^2_{s,t} = H^{-s}(\mathbb{P}^{\infty}_{-\infty}(\mathbb{C}), THH_t(A))$$

$$\Rightarrow TP_{s+t}(A).$$

Again,

1) $TP_*(A)$ is a $TP_*(\mathbb{S})$-module.

2) $THH(\mathbb{S}) = \mathbb{S}$.

But $THH_*(\mathbb{S}) = \pi_*(\mathbb{S}) \neq \mathbb{Z}$;
$c_1(\mathcal{O}(1))$ does not survive sp. seq.;
and $TP_*(\mathbb{S})$ is not periodic.

Fix prime number p, consider

$$\begin{array}{ccccccc} \mathbb{Q} & \subset & \mathbb{Q}_p & \subset & \overline{\mathbb{Q}}_p & \subset & \mathbb{C}_p \\ \cup & & \cup & & \cup & & \cup \\ \mathbb{Z} & \subset & \mathbb{Z}_p & \subset & \mathcal{O}_{\overline{\mathbb{Q}}_p} & \subset & \mathcal{O}_{\mathbb{C}_p} \end{array}$$

Suslin:

$$S_{K_0(\mathbb{C}_p,\mathbb{Z}_p)}(T_p K_1(\mathbb{C}_p))$$

$$\xrightarrow{\sim} K_*(\mathbb{C}_p, \mathbb{Z}_p)$$

symm.
algebra

$K_0(\mathbb{C}_p, \mathbb{Z}_p) = \mathbb{Z}_p \cdot 1$ $\qquad \beta_\varepsilon \in K_2(\mathbb{C}_p, \mathbb{Z}_p)$

Bott elem.

$T_p K_1(\mathbb{C}_p) = \mathbb{Z}_p \cdot \varepsilon$

$\overset{\shortparallel}{\mathbb{Z}_p(1)} \qquad \varepsilon = (\zeta_p, \zeta_{p^2}, \zeta_{p^3}, \dots)$

H. - Madsen:

compare HH

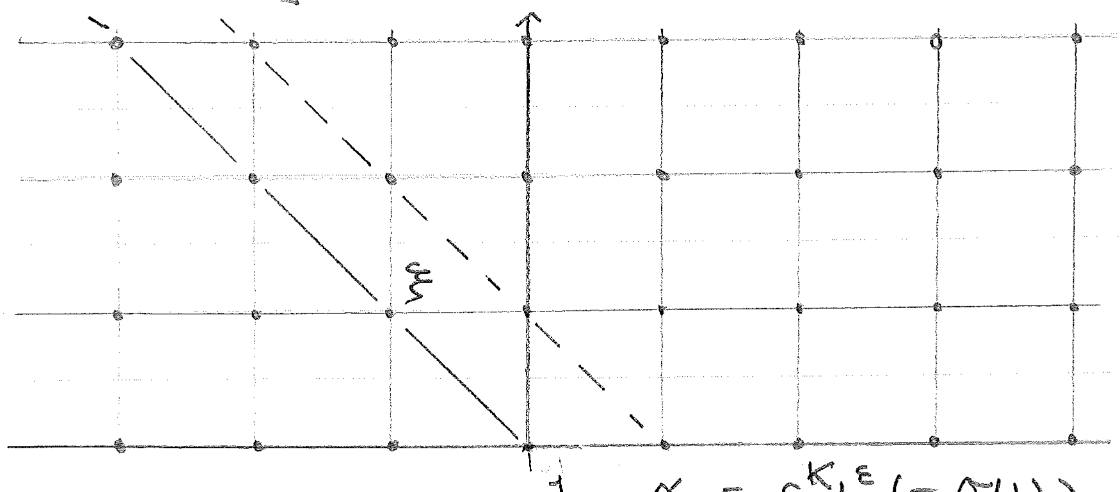
$$S_{THH_0(\mathcal{O}_{\mathbb{C}_p}, \mathbb{Z}_p)}(T_p THH_1(\mathcal{O}_{\mathbb{C}_p}))$$

$$\xrightarrow{\sim} THH_*(\mathcal{O}_{\mathbb{C}_p}, \mathbb{Z}_p)$$

$$THH_0(\mathcal{O}_{\mathbb{C}_p}, \mathbb{Z}_p) = \mathcal{O}_{\mathbb{C}_p} \cdot 1$$

$$T_p THH_1(\mathcal{O}_{\mathbb{C}_p}) = \underbrace{\mathcal{O}_{\mathbb{C}_p} \cdot (\zeta_p - 1)^{-1} d\log \varepsilon}_{\alpha_{\varepsilon,1}}$$

So sp. seq. conc. in even total deg.



$1 \qquad \alpha_\varepsilon = c_1^{K,\varepsilon}(-\mathcal{O}(1))$

$A_{\inf} \qquad A_{\inf}(1) := A_{\inf} \cdot \alpha_\varepsilon$

<u>Question</u> What is won by descending from $\mathbb{Z}$ to $\mathbb{S}$?

<u>Answer</u> An inverse Frobenius operator

(Bökstedt – Hsiang – Madsen)

Cyclotomic trace

$$K(A) \longrightarrow TP(A)$$

$\beta_\varepsilon \mapsto \mu \cdot \lambda_\varepsilon$

lands in eigenspace corresponding to the eigenvalue $\lambda = 1$ of the inverse Frobenius

---

Consider smooth and proper morphism

$$X \xrightarrow{f} \mathrm{Spec}(\mathbb{F}_q).$$

Hasse-Weil zeta function

$$\zeta(X,s) = \prod_{x \in |X|} (1 - \# k(x)^{-s})^{-1}$$

for $\mathrm{Re}(s) > d$. Let $W = W(\mathbb{F}_q)$ and choose $W \hookrightarrow \mathbb{C}$.

Cohomological interpretation by crystalline coh. (Berthelot):

$$\zeta(X,s) = \frac{\det(\mathrm{Id} - q^{-s}\mathrm{Fr}_q^* \mid H^{\mathrm{od}}_{\mathrm{crys}}(X/W)\otimes_W \mathbb{C})}{\det(\mathrm{Id} - q^{-s}\mathrm{Fr}_q^* \mid H^{\mathrm{ev}}_{\mathrm{crys}}(X/W)\otimes_W \mathbb{C})}$$

De Rham - Witt cx.

$$H^m_{\mathrm{crys}}(X/W) \xrightarrow{\sim} H^m(X, \cancel{W}\Omega^{\cdot}_X)$$

(Bloch - Deligne - Illusie)

Conjugate spectral sequence

$$(*) \qquad E_2^{s,t} = \lim_{n,F} H^s(X, W_n\Omega^t_X) \Rightarrow H^{s+t}_{\mathrm{crys}}(X/W).$$

The Hodge sp. seq. for $TP_*(X)$ takes the form

$$(**) \qquad E^2_{s,t} = \bigoplus_{u\in\mathbb{Z}} \lim_{n,F} H^{-s}(X, W_n\Omega_X^{t-2u}) \Rightarrow TP_{s+t}(X)$$

(Periodicity + TR ~ DRW)

From (*) – (**), obtain

$$\zeta(X,s) = \frac{\det(\mathrm{id} - q^{-s}\mathrm{Fr}_q^* \mid TP_1(X)\otimes_W \mathbb{C})}{\det(\mathrm{id} - q^{-s}\mathrm{Fr}_q^* \mid TP_0(X)\otimes_W \mathbb{C})}$$

<u>Rmk</u> Up to filtration, Fr_q^* on $TP_m(X)$ is determined by inverse Frob. + Hodge filtr.

Deninger: For $\lambda \in \mathbb{C}^*$ and $\delta \in \mathbb{R}_{>0}$,

$$\prod_{q^\alpha = \lambda} \delta(s-\alpha) = 1 - \lambda q^{-s}$$

using

$$-\pi < \arg(\delta(s-\alpha)) \leq \pi.$$

So define $\mathbb{C}$-linear derivation

$$TP_*(X)\otimes_W \mathbb{C} \xrightarrow{\ \theta\ } TP_*(X)\otimes_W \mathbb{C}$$

by writing

$$\mathrm{Fr}_q^* = \underbrace{\mathrm{Fr}_{q,s}^*}_{\text{semisimple}} \circ \underbrace{\mathrm{Fr}_{q,u}^*}_{\text{unipotent}}$$

and defining

$$\Theta = \log_q(\mathrm{Fr}_q^*)$$

$$= \underbrace{\log_q(\mathrm{Fr}_{q,s}^*)}_{\substack{\text{using } \underline{\text{all}} \\ \text{branches} \\ \text{of } \log}} + \underbrace{\log_q(\mathrm{Fr}_{q,u}^*)}_{\substack{\text{using series} \\ \frac{1}{\log q}\log(-)}}$$

More precisely, define Θ on TP_0, TP_1 by using some branch of log and extend to TP_* by

$$\Theta(\alpha) = \frac{2\pi i}{\log q}\cdot\alpha$$

where $\alpha = \alpha_\varepsilon \in TP_2(\mathbb{F}_q)$.

Thm If $f: X \to \mathrm{Spec}(\mathbb{F}_q)$ is smooth and proper, then

$$\zeta(X,s) = \frac{\det_\infty(s\cdot\mathrm{Id} - \Theta \mid TP_{od}(X)\otimes_W \mathbb{C})}{\det_\infty(s\cdot\mathrm{Id} - \Theta \mid TP_{ev}(X)\otimes_W \mathbb{C})}$$

for all $s \in \mathbb{C}$. //

<u>Rmks</u> 1) For a scheme X smooth and proper over an archimedean field, Connes–Consani have given a similar coh. interpretation of $\zeta(X,s)$ in terms of a Deligne coh. version of $HP_*(X)$ by using that for $\gamma \in \mathbb{C} \setminus (-\infty, 0]$ and $s \in \mathbb{C}$,

$$\prod_{v \in \mathbb{Z}_{\geq 0}} \gamma(s+v) = \gamma^{\frac{1}{2}-s} (2\pi)^{\frac{1}{2}} \Gamma(s)^{-1}.$$

2) For schemes X smooth and proper over a perfectoid base, Bhatt–Morrow–Scholze have constructed a motivic cohomology theory corresponding to $TP_*(X)$.