Shanghai '17

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THE ROYAL SWEDISH ACADEMY OF SCIENCES

$X/k$ smooth $\quad HH(X/k) = \cdots$

(A) $E^2_{i,j} = H^{-i}(X, \Omega^j_{X/k}) \Rightarrow HH_{i+j}(X/k)$

(B) $E_1^{i,j} = H^j(X, \Omega^i_{X/k}) \Rightarrow H^{i+j}_{dR}(X/k)$

$HH(X/k) \circlearrowleft \quad \mathbb{T} = \mathbb{R}/\mathbb{Z}$

~~$HH_*(X/k) \circlearrowleft d$ Connes' op.~~

Connes

$HP_j(X/k) := \hat{H}^{-j}(\mathbb{T}, HH(X/k))$

$\mathbb{N} \rightsquigarrow$ algebra

Waldhausen

$\mathbb{S} \rightsquigarrow$ higher algebra

$$\begin{array}{c} \mathrm{Mod}_{\mathbb{N}} \\ \mathbb{N} \otimes_{\mathbb{S}} - \uparrow \downarrow \text{fgt} \\ \mathrm{Mod}_{\mathbb{S}} \end{array}$$

$=$ Eilenberg-MacLane construction

$\mathrm{Mod}_{\mathbb{S}}$ = Segal's $\Gamma$-sets

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Topological Hochschild homology

$$THH(X) = HH(X/\mathbb{S}).$$

Two miracles happen:

(1) Denominators disappear.

(2) Get Frobenius

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$$\begin{tikzcd} HH_*(\mathbb{F}_p/\mathbb{Z}) \arrow[r, phantom, "="] & \Gamma_{\mathbb{F}_p}\{\alpha\} \\ HH_*(\mathbb{F}_p/\mathbb{S}) \arrow[u] \arrow[r, phantom, "="] & S_{\mathbb{F}_p}\{\alpha\} \arrow[u] \end{tikzcd}$$

Bökstedt periodicity

(ask Guozhen Wang)

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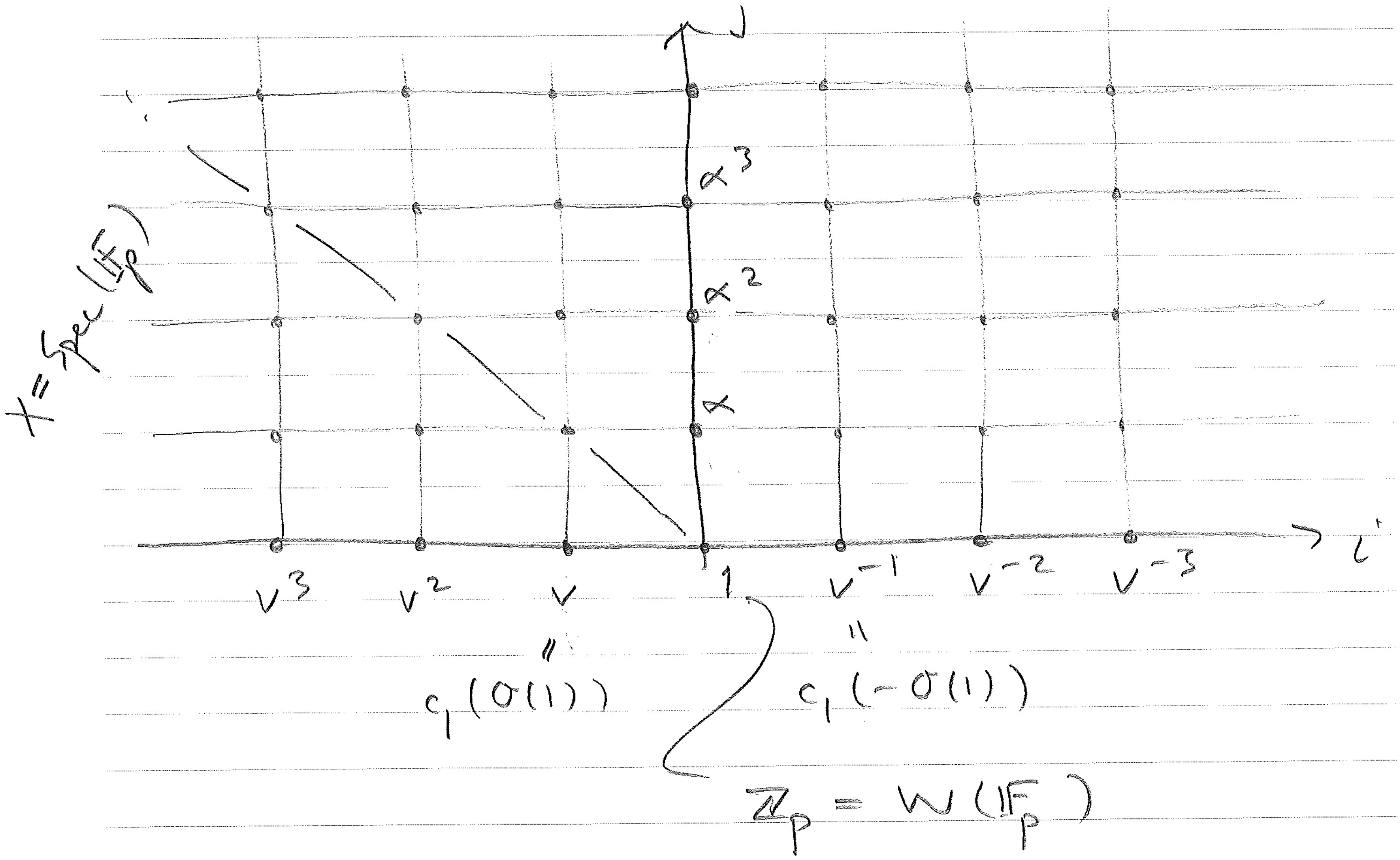
$$TP_j(X) = \hat{H}^{-j}(\mathbb{T}, THH(X))$$

"higher Hochschild homology"

$$E^2_{i,j} = H^{-i}(\mathbb{P}^{\infty}_{-\infty}(\mathbb{C}), THH_j(X))$$
$$\Rightarrow TP_{i+j}(X)$$




So for $X/\mathbb{F}_p$, $TP_*(X)$ is 2-periodic, but not in general!

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Thm Let $X/\mathbb{F}_p$ be smooth and proper; let $\iota : \mathbb{Z}_p \hookrightarrow \mathbb{C}$ be an embedding. Then

$$\zeta(X,s) = \frac{\det_\infty\left(\frac{1}{2\pi}(s\cdot\mathrm{id} - \Theta) \mid TP_{od}(X) \otimes_{\mathbb{Z}_p,\iota} \mathbb{C}\right)}{\det_\infty\left(\frac{1}{2\pi}(s\cdot\mathrm{id} - \Theta) \mid TP_{ev}(X) \otimes_{\mathbb{Z}_p,\iota} \mathbb{C}\right)},$$

where

$$\text{"}\ \Theta = \frac{d}{dt} \mathrm{Fr}_t^* \Big|_{t=1} \text{"}$$

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Cyclotomic structure map
(Bökstedt - Hsiang - Madsen)

$$\begin{array}{ccc}
THH(X) & \xrightarrow{\varphi_p} & \hat{H}^\cdot(C_p, THH(X)) \\
\circlearrowleft & & \circlearrowleft \\
\mathbb{T} & \xrightarrow{\sim} & \mathbb{T}/C_p \\
\Psi & & \Psi \\
z & \longmapsto & z^{1/p}\, C_p
\end{array}$$

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$$
\begin{tikzcd}
H^{\cdot}(\mathbb{T}, THH(X)) \arrow[r, "\varphi_p"] \arrow[d, "\sim"', "\leq -d"] & H^{\cdot}(\mathbb{T}/C_p, \hat{H}(C_p, THH(X))) \\
\hat{H}^{\cdot}(\mathbb{T}, THH(X)) \arrow[r, dashed, "\varphi_p"] & \hat{H}^{\cdot}(\mathbb{T}, THH(X)) \arrow[u, "\sim"']
\end{tikzcd}
$$

Calculate : $\varphi_p(v) = p \cdot v$.

Using periodicity, get

$$TP_*(X)[1/p] \xrightarrow{\varphi_p} TP_*(X)[1/p]$$

The Frobenius $\varphi_p$ and the geometric Frobenius $Fr_p^*$ are related by

$$Fr_p^* = p^j \cdot \varphi_p$$

where

$$j = \text{weight}$$

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Weight filtrations:

$$E^2_{i,j} = H^{j-i}(X, \mathbb{Z}(j)) \Rightarrow K_{i+j}(X)$$

$$\circlearrowleft$$

$$\psi^k = k^j \cdot \mathrm{id}$$

Atiyah-Hirzebruch
Voevodsky

$$E^2_{i,j} = H^{j-i}(X, W(j)) \Rightarrow TP_{i+j}(X)$$

Scholze

What is $W(j)$? Two cases known:

If $X/\mathbb{F}_p$ is smooth, then

$$W(j) \simeq W\Omega^{\cdot}_X$$

is the deRham-Witt cx. (This uses Cartier isom.)

If $X/\mathcal{O}_{\mathbb{C}_p}$ is smooth, then

$$W(j) \simeq A\Omega^{\cdot}_X$$

is the Bhatt-Morrow-Scholze cx.

$\rightsquigarrow$ Explain this

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Fontaine's ring of p-adic periods

$$A_{\inf} = \lim_{n, F} W_n(\mathcal{O}_{\mathbb{C}_p})$$

appears naturally as

$$A_{\inf} = TP_0(\mathcal{O}_{\mathbb{C}_p}, \mathbb{Z}_p).$$

Moreover,

$$TP_*(\mathcal{O}_{\mathbb{C}_p}, \mathbb{Z}_p) = S_{\mathcal{O}_{\mathbb{C}_p}}\{\alpha_\varepsilon^{\pm 1}\}$$

where the generator $\alpha_\varepsilon$ is the divided Bott element:

[illegible]

$$\begin{array}{ccc} K_*(\mathcal{O}_{\mathbb{C}_p}, \mathbb{Z}_p) & \longrightarrow & TP_*(\mathcal{O}_{\mathbb{C}_p}, \mathbb{Z}_p) \\ \| & & \| \\ S_{\mathbb{Z}_p}\{\beta_\varepsilon\} & \longrightarrow & S_{A_{\inf}}\{\alpha_\varepsilon^{\pm 1}\} \\ \cup & & \cup \\ \beta_\varepsilon & \longmapsto & \mu_\varepsilon \cdot \alpha_\varepsilon \end{array}$$

Depends on $\varepsilon \in \mathbb{Z}_p(1)$ generator.

Suppose $X/\mathbb{Z}$ smooth; choose local coord.

$$X \xrightarrow{\square} \mathrm{Spec}(\mathbb{Z}[t_1, \dots, t_d]).$$

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Define $q$-deformation of DR cx.

$$q\text{-}\Omega^{\bullet}_{X[[q-1]]/\mathbb{Z}[[q-1]], \square}$$

$$\|$$

$$\Omega^{\bullet}_{X/\mathbb{Z}} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[[q-1]] \circlearrowleft \nabla_q \quad \text{new diff.}$$

If $X = \mathrm{Spec}(\mathbb{Z}[t])$, then

$$\mathbb{Z}[[q-1]][t] \xrightarrow{\nabla_q} \mathbb{Z}[[q-1]][t] \cdot dt$$

$$f(t) \longmapsto \frac{f(qt) - f(t)}{(q-1)\,t} \cdot dt$$

Then for $X \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{C}_p} / \mathcal{O}_{\mathbb{C}_p}$,

$$W(j) \simeq q\text{-}\Omega^{\bullet}_{X[[q-1]]/\mathbb{Z}[[q-1]]} \underset{\substack{\mathbb{Z}[[q-1]] \to A_{\inf} \\ q-1 \mapsto \mu_\varepsilon}}{\otimes} A_{\inf}$$

In particular, the homotopy type of RHS does not depend on choice of local coord. $\square$ !

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