The model category axioms

We recall that a map $f: A \to B$ is said to a retract of a map $g: C \to D$ if there exists a commutative diagram

$$\begin{array}{ccc}
A & \longrightarrow C & \longrightarrow A \\
\downarrow^f & \downarrow^g & \downarrow^f \\
B & \longrightarrow D & \longrightarrow B,
\end{array}$$

where the composition of the horizontal maps are equal to the identity maps of A and B, respectively.

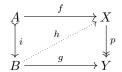
Definition (Quillen) A model category is a category \mathcal{C} together with three classes of maps called the weak equivalences $(\stackrel{\sim}{\longrightarrow})$, the fibrations (\twoheadrightarrow) , and the cofibrations (\rightarrowtail) that satisfy the following axioms:

M1: All small limits and colimits exist in C.

M2: If f and g are two composable maps in C, and if two of the maps f, g, and gf are weak equivalences, then so is the third.

M3: If f and g are maps in C such that f is a retract of g, and if g is a weak equivalence, a fibration, or a cofibration, then so is f.

M4: Given a commutative diagram



where i is a cofibration, p is a fibration, and where, in addition, one of i and p is a weak equivalence, there exists a map

$$h \colon B \to X$$

such that f = hi and g = ph.

M5: Every map f can be factored as a composition

$$f = pi = qj$$

where p is a fibration and i is both a cofibration and a weak equivalence, and where q is both a fibration and a weak equivalence and j a cofibration.

We will discuss the axiom M1 in more detail later. It implies, in particular, that a model category $\mathcal C$ has an initial object \emptyset and a terminal object *. An object X of $\mathcal C$ is called *cofibrant* if the unique map $\emptyset \to X$ is a cofibration and *fibrant* if the unique map $X \to *$ is a fibration.