The model category axioms

We recall that a map $f\colon A \to B$ is said to a *retract* of a map $g\colon C \to D$ if there exists a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & D & \longrightarrow & B, \end{array}$$

where the composition of the horizontal maps are equal to the identity maps of A and B, respectively.

Definition (Quillen) A *model category* is a category $\mathcal{C}$ together with three classes of maps called the *weak equivalences* ($\xrightarrow{\sim}$), the *fibrations* ($\twoheadrightarrow$), and the *cofibrations* ($\rightarrowtail$) that satisfy the following axioms:

M1: All small limits and colimits exist in $\mathcal{C}$.

M2: If f and g are two composable maps in $\mathcal{C}$, and if two of the maps f, g, and gf are weak equivalences, then so is the third.

M3: If f and g are maps in $\mathcal{C}$ such that f is a retract of g, and if g is a weak equivalence, a fibration, or a cofibration, then so is f.

M4: Given a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & X \\ {\scriptstyle i}\big\downarrow & \overset{h}{\nearrow} & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow[\ g\ ]{} & Y \end{array}$$

where i is a cofibration, p is a fibration, and where, in addition, one of i and p is a weak equivalence, there exists a map

$$h\colon B \to X$$

such that $f = hi$ and $g = ph$.

M5: Every map f can be factored as a composition

$$f = pi = qj$$

where p is a fibration and i is both a cofibration and a weak equivalence, and where q is both a fibration and a weak equivalence and j a cofibration.

We will discuss the axiom M1 in more detail later. It implies, in particular, that a model category $\mathcal{C}$ has an initial object $\emptyset$ and a terminal object $*$. An object X of $\mathcal{C}$ is called *cofibrant* if the unique map $\emptyset \to X$ is a cofibration and *fibrant* if the unique map $X \to *$ is a fibration.