1. Sheaves

Let X be a topological space. We define the category $O(X)$ of opens in X to have objects the open subsets $U \subset X$ and to have the set of morphisms from $U \subset X$ to $V \subset X$ either consists of the canonical inclusion map $i\colon U \to V$ or be empty according as $U \subset V$ or not. Recall that a *presheaf* on X is defined to be a functor

$$O(X)^{\mathrm{op}} \xrightarrow{F} \mathsf{Set}$$

from the opposite category of $O(X)$ to the category of sets. The presheaf F is defined to be a *sheaf* if for every object U in $O(X)$ and every family of morphisms $(U_i \to U)_{i\in I}$ in $O(X)$ with the property that $\bigcup_{i\in I} U_i = U$, the diagram

$$F(U) \xrightarrow{\iota} \prod_{i\in I} F(U_i) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \prod_{(j,k)\in I\times I} F(U_j \cap U_k)$$

is an equalizer in Set. Here ι is the unique map such that for all $i \in I$,

$$\mathrm{pr}_i \circ \iota = F(U_i \to U);$$

and α and β are the unique maps such that for all $(j,k) \in I \times I$,

$$\mathrm{pr}_{(j,k)} \circ \alpha = F(U_j \cap U_k \to U_j) \circ \mathrm{pr}_j,$$
$$\mathrm{pr}_{(j,k)} \circ \beta = F(U_j \cap U_k \to U_k) \circ \mathrm{pr}_k,$$

respectively. We notice that, in the category $O(X)$, the diagram

$$\begin{array}{ccc} U_j \cap U_k & \longrightarrow & U_j \\ \downarrow & & \downarrow \\ U_k & \longrightarrow & U \end{array}$$

is a cartesian square. Therefore, we may write $U_j \times_U U_k$ instead of $U_j \cap U_k$ above. Grothendieck realized that the implicit assumption in the definition of a topology that the morphisms $U \to V$ be inclusions is unnecessary and counterproductive. This realization led to the notion of a Grothendieck topology.

Definition 1.1. Let C be a category which admits fiber products. A *pretopology* on C is a function K that to each object X of C assigns a set

$$K(X) = \{(f_i\colon X_i \to X)_{i\in I}\}$$

of families of morphisms in C, subject to the following axioms.

(PT1) For every object X of C, every family $(f_i\colon X_i \to X)_{i\in I}$ in $K(X)$, and every morphism $g\colon Y \to X$ in C, the family $(f_i'\colon X_i \times_X Y \to Y)_{i\in I}$ is in $K(Y)$.

(PT2) For every object X of C, every family $(f_i\colon X_i \to X)_{i\in I}$ in $K(X)$, and every collection of families $(g_{j_i}\colon X_{j_i} \to X_i)_{j\in J_i}$ in $K(X_i)$, the family

$$(f_i \circ g_{j_i}\colon X_{j_i} \to X)_{(i,j)\in\coprod_{i\in I} J_i}$$

is in $K(X)$.

(PT3) For every object X of C, the family $(\mathrm{id}_X\colon X \to X)_{\emptyset\in\{\emptyset\}}$ is in $K(X)$.

The elements of $K(X)$ are said to be the covering families of X.

With this definition in hand, the definition of sheaves can be repeated mutatis mutandis. A presheaf on C is defined to be a functor

$$\mathsf{C}^{\mathrm{op}} \xrightarrow{F} \mathsf{Set},$$

and this functor is defined to be a sheaf if for every object X of C and every covering family $(f_i \colon X_i \to X)_{i \in I}$ of X, the diagram of sets

$$F(X) \xrightarrow{\iota} \prod_{i \in I} F(X_i) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \prod_{(j,k) \in I \times I} F(X_j \times_X X_k)$$

is an equalizer. Here again ι is the unique map such that for all $i \in I$,

$$\mathrm{pr}_i \circ \iota = F(X_i \xrightarrow{f_i} X);$$

and α and β are the unique maps such that for all $(j,k) \in I \times I$,

$$\mathrm{pr}_{(j,k)} \circ \alpha = F(X_j \times_X X_k \xrightarrow{f'_k} X_j) \circ \mathrm{pr}_j,$$

$$\mathrm{pr}_{(j,k)} \circ \beta = F(X_j \times_X X_k \xrightarrow{f'_j} X_k) \circ \mathrm{pr}_k,$$

respectively.

The definition of a pretopology is not quite satisfying. It requires fiber products to exist. And the indexing sets of the covering families is an annoying baggage to carry around. It is better to think of a pretopology as a basis for a topology, which we proceed to define.

Let C be a category. A *sieve* on an object X of C is a full subcategory S of the slice category C/X with the property that if the morphism $f \colon Y \to X$ is an object of S and if $g \colon Z \to Y$ is any morphism in C, then the composite morphism $f \circ g \colon Z \to X$ also is an object of S. The pullback $h^*(S)$ of a sieve S on X along the morphism $h \colon X' \to X$ of C is the sieve $h^*(S)$ on X' with object set

$$\mathrm{ob}(h^*(S)) = \{ g \colon Y' \to X' \mid f \circ h \colon Y' \to X \text{ is an object of } S \}.$$

The slice category C/X is itself a sieve on X and is maximal among sieves on X.

Example 1.2. A family $(f_i \colon X_i \to X)_{i \in I}$ of morphisms determines the sieve S on X whose objects are all morphisms $f \colon Y \to X$ for which there exists an element $i \in I$ and a morphism $g \colon Y \to X_i$ in C such that $f = f_i \circ g$.

Definition 1.3. Let C be a category. A *topology* on C is a function J that to each object X of C assigns a subset $J(X)$ of the set of sieves on X, subject to the following axioms.

- (T1) If $f \colon Y \to X$ is a morphism of C and if S is an element of $J(X)$, then the pullback sieve $f^*(S)$ is an element of $J(Y)$.
- (T2) If X is an object of C, if S and T are sieves on X, if S is an element of $J(X)$, and if for every morphism $f \colon Y \to X$ in S, the pullback sieve $f^*(T)$ is an element of $J(Y)$, then T is a covering sieve on X.
- (T3) If X is an object of C, then the maximal sieve C/X on X is an element of $J(X)$.

The elements of $J(X)$ are called the covering sieves of X. A *site* is a pair (C, J) of a category C and a topology J on C.

Example 1.4. Let C be a category. We obtain a topology J on C called the discrete topology by declaring all sieves to be covering sieves. Similarly, we obtain a topology J' on C called the chaotic topology by declaring only the maximal sieves to be covering sieves.

Lemma 1.5. *Let J be a topology on the category C and let X be an object of C.*

(1) *If $S \in J(X)$ and if T is a sieve on X with the property that $\mathrm{ob}(S) \subset \mathrm{ob}(T)$, then $T \in J(X)$.*
(2) *If $S \in J(X)$ and if, for every $f\colon Y \to X$ in $\mathrm{ob}(S)$, $T_f \in J(Y)$, then the unique sieve T on X with $\mathrm{ob}(T) = \{f \circ g \mid g \in T_f\}$ is in $J(X)$.*

Proof. To prove (1), it suffices by (T2) to show for every $f\colon Y \to X$ in $\mathrm{ob}(S)$, $f^*(T) \in J(Y)$. Since $S \subset T$, also $f^*(S) \subset f^*(T)$. But $\mathrm{id}_Y \in f^*(S)$, so $f^*(S) = \mathsf{C}/Y$ is the maximal sieve. Therefore, also $f^*(T) = \mathsf{C}/Y$ is the maximal sieve, and hence, $f^*(T) \in J(Y)$ as desired.

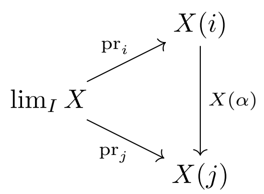
To prove (2), we again show that for every $f\colon Y \to X$ in $\mathrm{ob}(S)$, $f^*(T) \in J(Y)$. We have $\mathrm{ob}(T_f) \subset \mathrm{ob}(f^*(T))$. Indeed, if $g\colon Z \to Y$ is in $\mathrm{ob}(T_f)$, then, by the definition of T, we have $f \circ g\colon Z \to X$, so $g \in f^*(T)$ as desired. Therefore, we conclude from (1) that $f^*(T) \in J(Y)$ as desired. $\square$

We pause to discuss limits and colimits. There are two ways to construct new sets out of olds ones: As solution sets to systems of equations; and by gluing sets together. These two procedures are encoded in the notions of limits and colimits, respectively.

Definition 1.6. Let $X\colon I \to \mathsf{C}$ be a diagram. A *limit* of X is a pair

$$(\lim_I X, (\lim_I X \xrightarrow{\mathrm{pr}_i} X(i))_{i \in \mathrm{ob}(I)})$$

of an object of C and a family of of morphisms in C with the property that for every morphism $\alpha\colon i \to j$ in I, the diagram



commutes and such that if also $(Y, (f_i\colon Y \to X(i))_{i \in \mathrm{ob}(I)})$ is such a pair, then there exists a unique morphism

$$Y \xrightarrow{f} \lim_I X$$

such that for all $i \in \mathrm{ob}(I)$, $f_i = \mathrm{pr}_i \circ f$.

Dually, a *colimit* of a diagram $X\colon I \to \mathsf{C}$ is a pair

$$(\operatorname*{colim}_I X, (X_i \xrightarrow{\mathrm{in}_i} \operatorname{colim}_I X)_{i \in \mathrm{ob}(I)})$$

of an object of C and a family of morphisms in C such that for every morphism $\alpha\colon i \to j$ in I, the diagram

$$\begin{array}{ccc} X(i) & & \\ \Big\downarrow{\scriptstyle X(\alpha)} & \searrow^{\mathrm{in}_i} & \\ & & \operatorname{colim}_I X \\ & \nearrow_{\mathrm{in}_j} & \\ X(j) & & \end{array}$$

commutes and such that if also $(Y, (f_i\colon X(i) \to Y)_{i \in \mathrm{ob}(I)})$ is such a pair, then there exists a unique morphism

$$\operatorname{colim}_I X \xrightarrow{\ f\ } Y$$

such that for all $i \in \mathrm{ob}(I)$, $f_i = f \circ \mathrm{in}_i$.

We will write $(f_i)\colon Y \to \lim_I X$ for the unique morphism determined by the pair $(Y, (f_i\colon Y \to X(i))_{i \in \mathrm{ob}(I)})$; and we will write $\sum f_i\colon \operatorname{colim}_I X \to Y$ for the unique morphism determined by the pair $(Y, (f_i\colon X(i) \to Y)_{i \in \mathrm{ob}(I)})$.

Remark 1.7. The pair $(\lim_I X, (\mathrm{pr}_i\colon \lim_I X \to X(i))_{i \in \mathrm{ob}(I)})$ determines the unique morphism $(\mathrm{pr}_i)\colon \lim_I X \to \lim_I X$ such that for all $j \in \mathrm{ob}(I)$,

$$\mathrm{pr}_j = \mathrm{pr}_j \circ (\mathrm{pr}_i).$$

But clearly the identity map of $\lim_I X$ has this property. Hence, by the uniqueness, we conclude that $(\mathrm{pr}_i) = \mathrm{id}\colon \lim_I X \to \lim_I X$. More generally, if both the pairs $(\lim_I X, (\mathrm{pr}_i\colon \lim_I X \to X(i))_{i \in \mathrm{ob}(I)})$ and $(\lim'_I X, (\mathrm{pr}'_i\colon \lim'_I X \to X(i))_{i \in \mathrm{ob}(I)})$ are limits of the diagram $X\colon I \to \mathsf{C}$, then the unique morphisms

$$\lim_I X \underset{(\mathrm{pr}'_i)}{\overset{(\mathrm{pr}_i)}{\rightleftarrows}} \lim'_I X$$

are each other's inverses. Hence, the limit of a diagram is well-defined, up to unique isomorphism. Similar statements hold for colimits.

Example 1.8. Let $X\colon I \to \mathsf{C}$ be a diagram. An object 0 of I is *initial* if, for every object i of I, there exists a unique morphism $f_i\colon 0 \to i$. In this situation, the pair $(X(1), (X(f_i)\colon X(0) \to X(i))_{i \in \mathrm{ob}(I)})$ is a limit of X. Dually, an object 1 of I is *final* if, for every object i of I, there exists a unique morphism $f_i\colon i \to 1$. In this situation, the pair $(X(1), (X(f_i)\colon X(i) \to X(1))_{i \in \mathrm{ob}(I)})$ is a colimit of X.

If I is a small category, then the limit and colimit of a diagram $X\colon I \to \mathsf{Set}$ are given, up to unique isomorphism, as follows. The set $\lim_I X$ is the subset

$$\lim_I X \subset \textstyle\prod_{i \in \mathrm{ob}(I)} X(i)$$

of the product consisting of the tuples $(x_i)_{i \in \mathrm{ob}(I)}$ such that for every morphism $\alpha\colon i \to j$ in I, $X(\alpha)(x_i) = x_j$; and the map $\mathrm{pr}_i\colon \lim_I X \to X(i)$ takes the tuple $(x_j)_{j \in \mathrm{ob}(I)}$ to the ith component x_i. Dually, the set $\operatorname{colim}_I X$ is the quotient

$$\operatorname{colim}_I X = (\textstyle\coprod_{i \in \mathrm{ob}(I)} X(i))/\approx$$

of the disjoint union by the equivalence relation $\approx$ generated by the relation that identifies $x_i \in X(i)$ and $x_j \in X(j)$ if there exists a morphism $\alpha\colon i \to j$ in I such that $X(\alpha)(x_i) = x_j$; and the map $\mathrm{in}_i\colon X(i) \to \operatorname{colim}_I X$ takes $x_i \in X(i)$ to the class of (i, x_i). In general, the equivalence relation $\approx$ is extremely difficult to understand. However, it is much more well-behaved, if the category I is filtered.

DEFINITION 1.9. A category I is *filtered* if

(i) it is not the empty category;
(ii) for every pair of objects (i, j), there exists a pair of morphisms

$$\begin{array}{ccc} i & \xrightarrow{\alpha} & \\ & & k \\ j & \xrightarrow{\beta} & \end{array}$$

to a common target; and
(iii) for every pair of parallel morphisms

$$i \underset{\beta}{\overset{\alpha}{\rightrightarrows}} j,$$

there exists a morphism $\gamma \colon j \to k$ such that $\gamma \circ \alpha = \gamma \circ \beta$.

EXAMPLE 1.10. Let J be a topology on a category C. For every object X of C, we define a partial order $\leqslant$ on the set $J(X)$ by declaring that $S \leqslant S'$ if and only if $\mathrm{ob}(S) \subset \mathrm{ob}(S')$. We further define $J(X)_1$ to be the category with the set $J(X)$ as its set of objects, with the inclusion functor $j \colon S \to S'$ as the unique morphism from S to S' if $S \leqslant S'$, and with no morphisms from S to S' otherwise. We claim that the opposite category $J(X)_1^{\mathrm{op}}$ is filtered. Indeed, the requirement (i) that $J(X)_1^{\mathrm{op}} \neq \emptyset$ holds, since $\mathrm{id}_X \colon X \to X$ is a covering sieve; the requirement (ii) holds, since if both S and T are covering sieves, then the sieve $S \cap T$ with object set $\mathrm{ob}(S \cap T) = \mathrm{ob}(S) \cap \mathrm{ob}(T)$ also is a covering sieve; and the requirement (iii) is trivially satisfied, since parallel morphisms in $J(X)_1^{\mathrm{op}}$ necessarily are equal. To see that the intersection $S \cap T$ of two covering sieves S and T on X again is a covering sieve on X, it suffices, by axiom (T2), to show that for every object $f \colon Y \to X$ of S, the sieve $f^*(S \cap T)$ is a covering sieve on Y. But $f^*(S \cap T) = f^*(T)$, which is a covering sieve on Y by axiom (T1).

LEMMA 1.11. *Suppose that $X \colon I \to \mathsf{Set}$ is a diagram of sets indexed by a small filtered category I.*

(1) *Up to unique bijection, the set $\mathrm{colim}_I X$ is the quotient*

$$\mathrm{colim}_I X = (\textstyle\coprod_{i \in \mathrm{ob}(I)} X(i))/\sim$$

by the equivalence relation $\sim$ that identifies $x_i \in X(i)$ and $x_j \in X(j)$ if there exists morphisms $\alpha \colon i \to k$ and $\beta \colon j \to k$ in I such that

$$X(\alpha)(x_i) = X(\beta)(x_j) \in X(k).$$

(2) *Any two elements $x_i \in X(i)$ and $x_j \in X(j)$ are equivalent under $\sim$ to two elements of the same $X(k)$.*
(3) *Two elements $x_i, x_i' \in X(i)$ are equivalent under $\sim$ if and only if there exists a morphism $\alpha \colon i \to j$ in I such that*

$$X(\alpha)(x_i) = X(\alpha)(x_i') \in X(j).$$

PROOF. The statement (1) is clear once we show that $\sim$ is an equivalence relation. Only transitivity is at issue, and this follows by choosing morphisms

$$\begin{array}{ccccccccc} & & & & n & & & & \\ & & & \nearrow & & \nwarrow & & & \\ & & l & & & & m & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\ i & & & & j & & & & k. \end{array}$$

The statement (2) is immediate from I being filtered. Finally, to prove (3), suppose $x_i \sim x_i' \in X(i)$. By the definition of the relation $\sim$, there exists $\alpha, \beta \colon i \to j$ such that $X(\alpha)(x_i) = X(\beta)(x_i')$. Hence, choosing $\gamma \colon j \to k$ with $\gamma \circ \alpha = \gamma \circ \beta = \delta \colon i \to k$, we have $X(\delta)(x_i) = X(\delta)(x_i') \in X(k)$ as desired. $\square$

Let I and J be two small categories, and let $X \colon I \times J \to \mathsf{C}$ be a diagram indexed by the product category. By the universal property of limit and colimit we obtain a canonical morphism

$$\operatorname{colim}_I \lim_J X(i,j) \longrightarrow \lim_J \operatorname{colim}_I X(i,j).$$

In general, however, this morphism is not an isomorphism. To wit, if I and J are both the empty category, then the left-hand side is an initial object of C while the right-hand side is a terminal object of C. We say that a category J is *finite* if the sets $\operatorname{ob}(J)$ and $\operatorname{mor}(J)$ both are finite sets. The following result is extremely useful.

PROPOSITION 1.12. *Let I be a small filtered category, let J be a finite category, and let $X \colon I \times J \to \mathsf{Set}$ be a diagram of sets. In this situation, the canonical map*

$$\operatorname{colim}_I \lim_J X(i,j) \longrightarrow \lim_J \operatorname{colim}_I X(i,j).$$

is a bijection.

PROOF. We leave the case $J = \emptyset$ as an exercise. Non-empty finite limits can be obtained by iterated fiber products. For non-empty finite products are clearly obtained in this way and the equalizer of $f, g \colon X \to Y$ is the fiber product

$$\begin{array}{ccc} X \times_{(X \times Y)} X & \xrightarrow{\ \mathrm{pr}_1\ } & X \\ \Big\downarrow{\scriptstyle \mathrm{pr}_2} & & \Big\downarrow{\scriptstyle (\mathrm{id}_X, f)} \\ X & \xrightarrow{\ (\mathrm{id}_X, g)\ } & X \times Y. \end{array}$$

So it suffices to show that the canonical map

$$\operatorname{colim}_I (X(i) \times_{S(i)} Y(i)) \longrightarrow (\operatorname{colim}_I X(i)) \times_{(\operatorname{colim}_I S(i))} (\operatorname{colim}_I Y(i))$$

which to the class of (x_i, y_i) assigns (class of x_i, class of y_i) is a bijection. To construct the inverse map, we let $(\bar{x}, \bar{y})$ be an element of the right-hand side. If $x_i \in X(i)$ represents $\bar{x}$ and $y_j \in Y(j)$ represents $\bar{y}$, then

$$f_i(x_i) \in S_i \quad \sim \quad g_j(y_j) \in S_j.$$

We choose $\alpha \colon i \to k$ and $\beta \colon j \to k$; then $x_k = X(\alpha)(x_i) \in X_k$ represents $\bar{x}$ and $y_k = Y(\beta)(y_j) \in Y_k$ represents $\bar{y}$, and $f_k(x_k) \sim g_k(y_k) \in S(k)$. Finally, we choose a morphism $\gamma \colon k \to m$ such that $S(\gamma)(f_k(x_k)) = S(\gamma)(g_k(y_k)) \in S(m)$. The elements $x_m = X(\gamma)(x_k) \in X(m)$ and $y_m = Y(\gamma)(y_k) \in Y(m)$ again represent $\bar{x}$ and $\bar{y}$, respectively. But we now have $(x_m, y_m) \in X(m) \times_{S(m)} Y(m)$, and moreover, the class of (x_m, y_m) in the domain of the canonial map depends only on the given element $(\bar{x}, \bar{y})$ of the target. Hence, we have a well-defined map

$$(\operatorname{colim}_I X(i)) \times_{(\operatorname{colim}_I S(i))} (\operatorname{colim}_I Y(i)) \longrightarrow \operatorname{colim}_I (X(i) \times_{S(i)} Y(i))$$

that to the element $(\bar{x}, \bar{y})$ assigns the class of (x_m, y_m), and by construction, this map is inverse to the canonical map. □

The following result is a typical application of Proposition 1.12.

Corollary 1.13. *The underlying set of a colimit of a filtered diagram of rings is a colimit of the underlying filtered diagram of sets.*

Proof. A ring is a sextuple $(A, +, 0, -, *, 1)$ of a set A and five maps

$$A^2 \xrightarrow{+,*} A^1, \qquad A^0 \xrightarrow{0,1} A^1, \qquad A^1 \xrightarrow{-} A^1$$

between finite products of copies of the set A. In general, if we are given a diagram of sets $A\colon I \to \mathsf{Set}$ and a natural transformation $f_i\colon A(i)^m \to A(i)^n$ between the diagrams given by the termwise m-fold and n-fold products, then, by the universal property of colimits, the natural transformation induces a map

$$\operatorname{colim}_I(A^m) \xrightarrow{\operatorname{colim}_I(f)} \operatorname{colim}_I(A^n).$$

If I is filtered, then the canonical maps depicted vertically in the following diagram

$$\begin{array}{ccc} \operatorname{colim}_I(A^m) & \xrightarrow{\operatorname{colim}_I(f)} & \operatorname{colim}_I(A^n) \\ \downarrow & & \downarrow \\ (\operatorname{colim}_I A)^m & \xrightarrow{\text{“}f\text{”}} & (\operatorname{colim}_I A)^n \end{array}$$

are bijections, and therefore, there exists a unique map “f” making the diagram commute. In particular, if $(A, +, 0, -, * 1)$ is a diagram of rings indexed by a filtered category I, then the natural transformations $+$, 0, $-$, $*$, and 1 give rise to a ring structure on the set $\operatorname{colim}_I A$. Indeed, the ring axioms are identities between various natural transformations, and therefore, the corresponding identities hold for the induced maps. Moreover, the maps in the family

$$(\, A(i) \xrightarrow{\operatorname{in}_i} \operatorname{colim}_I A \,)_{i \in I}$$

are ring homomorphisms with respect to this ring structure. Therefore, this family is the colimit in the category of rings. □

Example 1.14. Let us point out that the conclusion of Corollary 1.13 generally does not hold if I is not filtered. For example, if $I = \emptyset$, then the colimit in the category of sets is the initial set $\emptyset$, while the colimit in the category of rings is the initial ring $\mathbb{Z}$.

We are now ready to give the definition of sheaves on a category C with respect to a topology J. If F is a presheaf on C, then for every object X of C and every sieve S on X, we have a functor

$$S^{\mathrm{op}} \xrightarrow{F_{X,S}} \mathsf{Set}$$

that to the object $f\colon Y \to X$ of S assigns the set $F_{X,S}(f) = F(Y)$ and that to the morphism $g\colon Z \to Y$ from $f \circ g\colon Z \to X$ to $f\colon Y \to X$ in S assigns the map $F_{X,S}(g) = F(g)\colon F_{X,S}(f) \to F_{X,S}(f \circ g)$.

DEFINITION 1.15. Let (C, J) be a site. A presheaf F on C is a *sheaf* with respect to J if for every object X of C and every sieve S in $J(X)$, the pair

$$(F(X), (\, F(X) \xrightarrow{F(f)} F_{X,S}(f)\,)_{f \in \mathrm{ob}(S)})$$

is a limit of the diagram $F_{X,S}\colon S^{\mathrm{op}} \to \mathsf{Set}$.

EXAMPLE 1.16. Let C be a category. Every presheaf F on C is a sheaf for the chaotic topology. Indeed, for every object X of C, only the maximal sieve C/X is a covering sieve, and the object $\mathrm{id}_X\colon X \to X$ is terminal in C/X, hence initial in $(\mathsf{C}/X)^{\mathrm{op}}$. A presheaf F on C is a sheaf for the trivial topology if and only for every object X of C, the set $F(X)$ is a singleton. Indeed, since the empty sieve is a covering sieve, the set $F(X)$ is a limit of the empty diagram.

2. Sheafification

We first discuss the functoriality of limits and colimits. Given categories I and C, we write $[I, \mathsf{C}]$ for the category whose objects are the functors $F\colon I \to \mathsf{C}$ and whose morphisms are the natural transformations $\alpha\colon F \Rightarrow F'$ between such functors. The diagonal functor $\Delta_I\colon \mathsf{C} \to [I, \mathsf{C}]$ takes an object X of C to the constant functor $\Delta_I(X)\colon I \to \mathsf{C}$ with value X and takes a morphism $f\colon Y \to X$ to the constant natural transformation $\Delta_I(f)\colon \Delta_I(F) \Rightarrow \Delta_I(F')$ with value $f\colon X \to X'$. Choosing a limit of every diagram $F\colon I \to \mathsf{C}$, we obtain a functor

$$[I, \mathsf{C}] \xrightarrow{\lim_I} \mathsf{C}$$

defined as follows. Given a natural transformation $\alpha\colon F \Rightarrow F'$ and choices of limits $(\lim_I F, (\mathrm{pr}_i\colon \lim_I F \to F(i))_{i \in \mathrm{ob}(I)})$ and $(\lim_I F, (\mathrm{pr}_i\colon \lim_I F \to F(i))_{i \in \mathrm{ob}(I)})$, we define $\lim_I \alpha\colon \lim_I F \to \lim_I F'$ be the morphism in C determined by the pair

$$(\lim_I F, (\lim_I F \xrightarrow{\mathrm{pr}_i} F(i) \xrightarrow{\alpha_i} F'(i).)_{i \in \mathrm{ob}(I)})$$

The functor identities $\lim_I(\alpha \circ \beta) = \lim_I \alpha \circ \lim_I \beta$ and $\lim_I \mathrm{id}_F = \mathrm{id}_{\lim_I F}$ hold by the uniqueness of morphisms to limits, proving the claim for limits. Moreover, the pairs $(\lim_I F, (\mathrm{pr}_i\colon \lim_I F \to F(i))_{i \in \mathrm{ob}(I)})$ and $(X, (\mathrm{id}_X\colon X \to X)_{i \in \mathrm{ob}(I)})$ define natural transformations $\epsilon\colon \Delta_I \circ \lim_I \Rightarrow \mathrm{id}_{[I,\mathsf{C}]}$ and $\eta\colon \mathrm{id}_{\mathsf{C}} \Rightarrow \lim_I \circ \Delta_I$, respectively, making the following quadruple an adjunction from C to $[I, \mathsf{C}]$,

$$(\Delta_I, \lim_I, \epsilon, \eta).$$

Finally, we let $G\colon I \to I'$ be a functor and fix choices of limits $\lim_I$ and $\lim_{I'}$ as above. If $F'\colon I' \to \mathsf{C}$ is a functor, then we have a natural restriction morphism

$$\lim_{I'} F' \xrightarrow{\mathrm{res}_{G,F'}} \lim_I (F' \circ G)$$

defined by the pair $(\lim_{I'} F', (\mathrm{pr}'_{G(i)}\colon \lim_{I'} F' \to (F' \circ G')(i))_{i \in \mathrm{ob}(I)})$. Moreover, by the uniqueness of maps to limits, we have the following identities

$$\mathrm{red}_{\mathrm{id}_I, F} = \mathrm{id}_{\lim_I F}, \qquad \mathrm{res}_{H \circ G, F''} = \mathrm{res}_{G, F'' \circ H} \circ \mathrm{res}_{H, F''}$$

among these natural transformations.

Similarly, choosing a colimit of every diagram $F\colon I \to \mathsf{C}$, we get a functor

$$[I, \mathsf{C}] \xrightarrow{\mathrm{colim}_I} \mathsf{C}$$

which is part of an adjunction of functors from $[I, \mathsf{C}]$ to C,

$$(\mathrm{colim}_I, \Delta_I, \epsilon, \eta).$$

Moreover, if $G\colon I \to I'$ is a functor, then we have a natural morphism

$$\mathrm{colim}_I (F' \circ G) \xrightarrow{\mathrm{ind}_{G,F'}} \mathrm{colim}_{I'} F'$$

defined by the pair $(\mathrm{colim}_{I'} F', (\mathrm{in}_{G(i)}\colon (F' \circ G)(i) \to \mathrm{colim}_{I'} F')_{i \in \mathrm{ob}(I)})$, and

$$\mathrm{ind}_{\mathrm{id}_I, F} = \mathrm{id}_{\mathrm{colim}_I F}, \qquad \mathrm{ind}_{F'', H \circ G} = \mathrm{ind}_{F'' \circ H, G} \circ \mathrm{ind}_{F'', H}\,.$$

This concludes the preliminaries on the functoriality of limits and colimits.

Let C be a category, and let J be a topology on C. We write

$$\mathsf{C}^{\wedge} = [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$$

for the category of presheaves on C and $(\mathsf{C}, J)^{\sim}$ or simply $\mathsf{C}^{\sim}$ for the full subcategory of sheaves on C with respect to the topology J. Finally, we write

$$\mathsf{C}^{\sim} \xrightarrow{\ i\ } \mathsf{C}^{\wedge}$$

for the canonical inclusion functor.

THEOREM 2.1. *Let (C, J) be a site. The functor $i\colon \mathsf{C}^{\sim} \to \mathsf{C}^{\wedge}$ admits a left adjoint functor $a\colon \mathsf{C}^{\wedge} \to \mathsf{C}^{\sim}$. Moreover, the functor a preserves finite limits.*

PROOF. We first define an auxiliary functor

$$\mathsf{C}^{\wedge} \xrightarrow{\ L\ } \mathsf{C}^{\wedge}$$

as follows. Let F be a presheaf on C and let X be an object of C. There is a functor

$$J(X)_1^{\mathrm{op}} \xrightarrow{\ F_X\ } \mathsf{Set}$$

that to a covering sieve S on X assigns the limit set $F_X(S) = \lim_{S^{\mathrm{op}}} F_{X,S}$ and that to the canonical inclusion functor $j\colon S \to S'$ between covering sieves assigns the map $F_X(j)\colon F_X(S') \to F_X(S)$ given by the composition

$$\lim_{S'^{\mathrm{op}}} F_{X,S'} \xrightarrow{\mathrm{res}_{j,F_{X,S'}}} \lim_{S^{\mathrm{op}}}(F_{X,S'} \circ j^{\mathrm{op}}) \xrightarrow{\lim_{S^{\mathrm{op}}} \alpha_{S,S'}} \lim_{S^{\mathrm{op}}} F_{X,S},$$

where $\alpha_{S,S'}\colon F_{X,S'} \circ j^{\mathrm{op}} \Rightarrow F_{X,S}$ is the natural transformation whose value at the object $f\colon Y \to X$ of S is the identity map $\mathrm{id}_{F(Y)}\colon (F_{X,S'} \circ j^{\mathrm{op}})(f) \to F_{X,S}(f)$. Given a morphism $h\colon X' \to X$ in C, we have the functor $h^*\colon J(X)_1 \to J(X')_1$ given by pullback of sieves and define a natural transformation

$$F_X \xRightarrow{\varphi(h)} F_{X'} \circ h^{*\mathrm{op}}$$

as follows. There is a functor $h_S\colon h^*(S) \to S$ that to the object $k\colon Y' \to X'$ assigns the object $h \circ k\colon Y' \to X$ and that to the morphism $g\colon Z' \to Y'$ from $k \circ g\colon Z' \to X'$ to $k\colon Y' \to X'$ assigns the same morphism $g\colon Z' \to Y'$ from $h \circ k \circ g\colon Z' \to X$ to $h \circ k\colon Y' \to X$, and moreover, there is a natural transformation $\beta_S\colon F_{X,S} \circ h_S^{\mathrm{op}} \Rightarrow F_{X',h^*(S)}$ whose value at the object $k\colon Y' \to X'$ is the identity map $\mathrm{id}_{F(Y')}$. The value of $\varphi(h)$ at S is now defined to be the composite map

$$\lim_{S^{\mathrm{op}}} F_{X,S} \xrightarrow{\mathrm{res}_{h_S^{\mathrm{op}},F_{X,S}}} \lim_{h^*(S)^{\mathrm{op}}}(F_{X,S} \circ h_S^{\mathrm{op}}) \xrightarrow{\lim_{h^*(S)^{\mathrm{op}}} \beta_S} \lim_{h^*(S)^{\mathrm{op}}} F_{X',h^*(S)}.$$

We now define $L(F)$ to be the presheaf that to an object X assigns the colimit set

$$L(F)(X) = \operatorname{colim}_{J(X)_1^{\mathrm{op}}} F_X$$

and that to a morphism $h\colon X' \to X$ assigns the map

$$L(F)(X) \xrightarrow{L(F)(h)} L(F)(X')$$

given by the composition

$$\operatorname{colim}_{J(X)_1^{\mathrm{op}}} F_X \xrightarrow{\operatorname{colim}_{J(X)_1^{\mathrm{op}}} \varphi(h)} \operatorname{colim}_{J(X)_1^{\mathrm{op}}}(F_{X'} \circ h^{*\mathrm{op}}) \xrightarrow{\mathrm{ind}_{h^{*\mathrm{op}},F_{X'}}} \operatorname{colim}_{J(X')_1^{\mathrm{op}}} F_{X'}.$$

We leave it as an exercise in the definitions to verify that $L(F)(\mathrm{id}_X) = \mathrm{id}_{L(F)(X)}$ and $L(F)(g \circ h) = L(F)(h) \circ L(F)(h)$, showing that $L(F)$ is a presheaf; that a natural transformation $f\colon F \Rightarrow F'$ induces a natural transformation $L(f)\colon L(F) \Rightarrow L(F')$; and that this defines a functor $L\colon \mathsf{C}^\wedge \to \mathsf{C}^\wedge$ as promised.

In the colimit which defines $L(F)(X)$, the domain of the map

$$F_X(\mathsf{C}/X) \xrightarrow{\mathrm{in}_{\mathsf{C}/X}} \mathrm{colim}_{J(X)_1^{\mathrm{op}}} F_X$$

is uniquely bijective to $F(X)$. Indeed, the set in question is the limit set of the diagram $F_{X,\mathsf{C}/X}\colon (\mathsf{C}/X)^{\mathrm{op}} \to \mathsf{Set}$, the index category of which has $\mathrm{id}_X\colon X \to X$ as an initial object, and $F_{X,\mathsf{C}/X}(\mathrm{id}_X) = F(X)$. This defines a map

$$F(X) \xrightarrow{\ell_{F,X}} L(F)(X).$$

The family of maps $(\ell_{F,X}\colon F(X) \to L(F)(X))_{X \in \mathrm{ob}(\mathsf{C})}$ is a map of presheaves $\ell_F\colon F \to L(F)$; and the family of maps $(\ell_F\colon F \to L(F))_{F \in \mathrm{ob}(\mathsf{C}^\wedge)}$, in turn, is a natural transformation $\ell\colon \mathrm{id}_{\mathsf{C}^\wedge} \Rightarrow L$.

We now recall that, by definition, a presheaf F on C is a sheaf on C, if for every object X of C and every covering sieve $S \in J(X)$, the map

$$F(X) \xrightarrow{\iota_{F,S}} F_X(S)$$

defined by the inclusion $j\colon S \to \mathsf{C}/X$ is a bijection. If the map $\iota_{F,S}$ is injective for all $X \in \mathrm{ob}(\mathsf{C})$ and $S \in J(X)$, then F is said to be separated. We will show:

(i) If F is any presheaf, then $L(F)$ is a separated presheaf.
(ii) If F is a separated presheaf, then $L(F)$ is a sheaf.

We first prove (i). Let X be an object of C, let $x, y \in L(F)(X)$, and suppose that there exists $S \in J(X)$ such that $\iota_{L(F),S}(x) = \iota_{L(F),S}(y)$ in $L(F)_X(S)$. We must show that $x = y$. Since $L(F)(X)$ is the filtered colimit of the $F_X(T)$ with $T \in J(X)$, we can find $T \in J(X)$ and $u, v \in F_X(T)$ with $\mathrm{in}_T(u) = x$ and $\mathrm{in}_T(v) = y$. Moreover, for every $f\colon Y \to X$ in $\mathrm{ob}(S)$, the diagram

$$\begin{array}{ccc} F_X(T) & \xrightarrow{\varphi(f)_T} & F_Y(f^*(T)) \\ \downarrow{\scriptstyle \mathrm{in}_T} & & \downarrow{\scriptstyle \mathrm{in}_{f^*(T)}} \\ L(F)(X) & \xrightarrow{L(F)(f)} & L(F)(Y) \end{array}$$

commutes, and we have

$$L(F)(f)(x) = \mathrm{pr}_f(\iota_{L(F),S}(x)) = \mathrm{pr}_f(\iota_{L(F),S}(y)) = L(F)(f)(y)$$

in $L(F)(Y)$. Since $L(F)(Y)$ is filtered colimit, we can find $j\colon U_f \to f^*(T)$ in $J(Y)_1$ such that images of $\varphi_{f,T}(u)$ and $\varphi_{f,T}(v)$ by $F_Y(j)\colon F_Y(f^*(T)) \to F_Y(U_f)$ are equal. We now consider the sieve T' on X with

$$\mathrm{ob}(T') = \{f \circ g \mid f \in \mathrm{ob}(S), g \in \mathrm{ob}(U_f)\},$$

which is a covering sieve on X by Lemma 1.5 (2). Moreover, by construction, we have a morphism $i\colon T' \to T$ in $J(X)_1$ and the images of u and v by the induced map $F_X(i)\colon F_X(T) \to F_X(T')$ are equal. Hence, so are the images x and y of u and v by the composite map

$$F_X(T) \xrightarrow{F_X(i)} F_X(T') \xrightarrow{\mathrm{in}_{T'}} L(F)(X).$$

This completes the proof of (i).

We next prove (ii). Let F be separated presheaf on C, let X be an object of C, and let S be a covering sieve on X. We must show that the map

$$L(F)(X) \xrightarrow{\iota_{L(F),S}} L(F)_X(S)$$

is surjective. So we let $(x_f)_{f \in \mathrm{ob}(S)}$ be an element of the target and define an element y of the domain such that maps to it. For every $f \in \mathrm{ob}(S)$, we choose a family

$$(x_{f,g})_{g \in \mathrm{ob}(T_f)} \in F_Y(T_f)$$

with $T_f \in J(Y)$ that represents $x_f \in L(F)(Y)$. Since for every morphism $h\colon Z \to Y$ in C, the map $L(F)(h)\colon L(F)(Y) \to L(F)(Z)$ takes x_f to $x_{f \circ h}$, the chosen family

$$(x_{f\circ h,g})_{g \in \mathrm{ob}(T_{f\circ h})} \in F_Z(T_{f\circ h})$$

and the induced family

$$(x_{f,h\circ g'})_{g \in h^*(T_f)} \in F_Z(h^*(T_f))$$

both represent $x_{f\circ h}$. Hence, there is a sieve $U_{f,h} \subset T_{f\circ h} \cap h^*(T_f)$ in $J(Z)$ such that

$$x_{f\circ h,g''} = x_{f,h\circ g''},$$

for all $g'' \in \mathrm{ob}(U_{f,h})$. Moreover, since S and all T_f with $f \in \mathrm{ob}(S)$ are covering sieves, we conclude from Lemma 1.5 (2) that the sieve T on X with

$$\mathrm{ob}(T) = \{f \circ g \mid f \in S, g \in T_f\}$$

is in $J(X)$. We now define $y \in L(F)(X)$ to be the class represented by the family

$$(y_h)_{h \in \mathrm{ob}(T)} \in F_X(T)$$

with $y_{f\circ g} = x_{f,g}$. To see that this is well-defined, we must show that, given

$$\begin{array}{ccccc} & & Y & & \\ & \overset{g}{\nearrow} & & \overset{f}{\searrow} & \\ Z & & & & X \\ & \underset{g'}{\searrow} & & \underset{f'}{\nearrow} & \\ & & Y' & & \end{array}$$

with $f, f' \in \mathrm{ob}(S)$ such that the two composite morphisms are equal, we have

$$x_{f,g} = x_{f',g'} \in F(Z).$$

Now, for every $k \in \mathrm{ob}(U_{f,g} \cap U_{f',g'})$, we have

$$F(k)(x_{f,g}) = x_{f,g\circ k} = x_{f\circ g,k} = x_{f'\circ g',k} = x_{f',g'\circ k} = F(k)(x_{f',g'}),$$

and hence, the map

$$F(Z) \xrightarrow{\iota_{F,U_{f,g}\cap U_{f',g'}}} F_Z(U_{f,g} \cap U_{f',g'})$$

takes $x_{f,g}$ and $x_{f',g'}$ to the same element. In addition, since $U_{f,g} \cap U_{f',g'}$ is a covering sieve on Z and since F is separated, this map is injective, so we conclude that $x_{f,g} = x_{f',g'} \in F(Z)$ as desired. It remains only to prove that

$$L(F)(X) \xrightarrow{\iota_{L(F),S}} L(F)_X(S)$$

indeed takes y to $(x_f)_{f\in\mathrm{ob}(S)}$, or equivalently, that for every $f \in \mathrm{ob}(S)$, the two families $(x_{f,g})_{g\in\mathrm{ob}(T_f)} \in F_Y(T_f)$ and $(y_{f\circ h})_{h\in\mathrm{ob}(f^*(T))} \in F_Y(f^*(T))$ represent the same class in $L(F)(Y)$. But $T_f \subset f^*(T)$ and, for every $g \in \mathrm{ob}(T_f)$, $x_{f,g} = y_{f\circ g}$. This completes the proof of (ii).

Finally, we prove the theorem. Let F and F' be a presheaf and a sheaf on C, respectively, and let $f\colon F \to i(F')$ be a map of presheaves on C. In the diagram

$$\begin{array}{ccccc} F & \xrightarrow{\ell_F} & L(F) & \xrightarrow{\ell_{L(F)}} & L(L(F)) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle L(f)} & & \downarrow{\scriptstyle L(L(f))} \\ i(F') & \xrightarrow{\ell_{i(F')}} & L(i(F')) & \xrightarrow{\ell_{L(i(F'))}} & L(L(i(F'))), \end{array}$$

the lower horizontal maps are isomorphisms, since F' is a sheaf, and $L(L(F))$ is a sheaf by (i)–(ii) above. Hence, we conclude that the unique functor $a\colon \mathsf{C}^{\wedge} \to \mathsf{C}^{\sim}$ such that $i \circ a = L \circ L$ is left adjoint to i with $\eta_F = \ell_{L(F)} \circ \ell_F$ as the unit of the adjunction. The functor a preserves finite limits, since limits preserve all limits and filtered colimits preserve finite limits. □

Remark 2.2. A category C is defined to be a sextuple

$$\mathsf{C} = (\mathrm{ob}(\mathsf{C}), \mathrm{mor}(\mathsf{C}), s, t, \circ, \mathrm{id})$$

consisting of a set of objects $\mathrm{ob}(\mathsf{C})$; a set of morphisms $\mathrm{mor}(\mathsf{C})$; source and target maps $s, t\colon \mathrm{mor}(\mathsf{C}) \to \mathrm{ob}(\mathsf{C})$; a composition map $\circ\colon \mathrm{mor}(\mathsf{C}) \times_{\mathrm{ob}(\mathsf{C})} \mathrm{mor}(\mathsf{C}) \to \mathrm{mor}(\mathsf{C})$ from a choice a pullback

$$\begin{array}{ccc} \mathrm{mor}(\mathsf{C}) \times_{\mathrm{ob}(\mathsf{C})} \mathrm{mor}(\mathsf{C}) & \xrightarrow{\mathrm{pr}_1} & \mathrm{mor}(\mathsf{C}) \\ \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle s} \\ \mathrm{mor}(\mathsf{C}) & \xrightarrow{t} & \mathrm{ob}(\mathsf{C}) \end{array}$$

to $\mathrm{mor}(\mathsf{C})$; and an identity map $\mathrm{id}\colon \mathrm{ob}(\mathsf{C}) \to \mathrm{mor}(\mathsf{C})$; and this data is subject to the category axioms, which we leave it to the reader to formulate. However, with this definition, there is no category of sets Set, since there is no set of all sets to serve as the set $\mathrm{ob}(\mathsf{Set})$ of objects, nor is there a set $\mathrm{mor}(\mathsf{Set})$ of all maps between sets. There are several ways to deal with this problem. One way is to treat Set as a metamathematical object in the underlying language of ZFC set theory. Another way is to limit the meaning of "all" and let $\mathrm{ob}(\mathsf{Set})$ consist only of sets that are elements of a large fixed set U called the universe. We will implicitly take this second route and will assume U to be a Grothendieck universe such that its elements constitute a model of ZFC set theory. The existence of such a universe cannot be proved within ZFC set theory, so we will work in ZFCU theory, which is ZFC set theory together with the axiom of universe that for every set x, there exists a Grothendieck universe U such that $x \in U$.

3. The functoriality of categories of sheaves

In general, pretty much every functor between categories of sheaves is defined by means of two constructions: sheafification and Kan extensions. Both are examples of adjunctions, which we first discuss.

An *adjunction* from a category C to a category C' is a quadruple (F, G, ϵ, η) of two functors $F\colon \mathsf{C} \to \mathsf{C}'$ and $G\colon \mathsf{C}' \to \mathsf{C}$ and two natural transformations $\epsilon\colon F \circ G \Rightarrow \mathrm{id}_{\mathsf{C}'}$ and $\eta\colon \mathrm{id}_{\mathsf{C}} \Rightarrow G \circ F$ such that the composite natural transformations

$$F \xRightarrow{F \circ \eta} F \circ G \circ F \xRightarrow{\epsilon \circ F} F \qquad \text{and} \qquad G \xRightarrow{\eta \circ G} G \circ F \circ G \xRightarrow{G \circ \epsilon} G$$

are equal to the respective identity natural transformations. We call ϵ and η the *counit* and the *unit* of the adjunction, respectively, and refer to the identities above as the *triangle identities*. A functor $G\colon \mathsf{C}' \to \mathsf{C}$ is said to admit a *left adjoint*, if there exits an adjunction (F, G, ϵ, η) with G as its second component. Moreover, if also $(F', G, \epsilon', \eta')$ is such an adjunction, then the composite

$$F \xRightarrow{F \circ \eta'} F \circ G \circ F' \xRightarrow{\epsilon \circ F'} F'$$

is the unique natural transformation $\sigma\colon F \Rightarrow F'$ such that the diagrams

$$\begin{array}{ccc} F \circ G & \xRightarrow{\epsilon} & \mathrm{id}_{\mathsf{C}'} \\ \Downarrow{\scriptstyle \sigma \circ G} & & \Vert \\ F' \circ G & \xRightarrow{\epsilon'} & \mathrm{id}_{\mathsf{C}'} \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{id}_{\mathsf{C}} & \xRightarrow{\eta} & G \circ F \\ \Vert & & \Downarrow{\scriptstyle G \circ \sigma} \\ \mathrm{id}_{\mathsf{C}} & \xRightarrow{\eta'} & G \circ F' \end{array}$$

commutes and is an isomorphism; see [**3**, Theorem IV.7.2] for a proof. In this sense, a left adjoint functor of G, if it exists, is unique, up to unique isomorphism. This uniqueness result is extremely usual. The analogous uniqueness result holds for right adjoints.

Remark 3.1. We have already used the fact that if a functor $F\colon \mathsf{C} \to \mathsf{C}'$ has a right adjoint, then it preserves colimits in the sense that if the pair

$$(\operatorname{colim}_I X, (\operatorname{colim}_I X \xrightarrow{\mathrm{pr}_i} X(i))_{i \in \mathrm{ob}(I)})$$

is a colimit of a diagram $X\colon I \to \mathsf{C}$, then the pair

$$(F(\operatorname{colim}_I X), (F(\operatorname{colim}_I X) \xrightarrow{\mathrm{pr}_i} F(X(i)))_{i \in \mathrm{ob}(I)})$$

is a colimit of a diagram $F \circ X\colon I \to \mathsf{C}'$. Similarly, if a functor $G\colon \mathsf{C}' \to \mathsf{C}$ has a left adjoint, then it preserves limits.

Now let C be a category. A functor $u\colon I \to J$ gives rise to a functor

$$[J, \mathsf{C}] \xrightarrow{u^*} [I, \mathsf{C}]$$

defined on objects and morphisms, respectively, by $u(F) = F \circ u$ and $u(f) = f \circ u$. We call the functor u^* the *restriction along* u.

Definition 3.2. Let C be a category and let $u\colon I \to J$ be a functor. A left adjoint functor of u^* is called a *left Kan extension along* u and denoted $u_!$; and a right adjoint functor of u^* is called a *right Kan extension along* u and denoted u_*.

Let $u\colon I \to J$ be a functor and let j be an object of J. The slice category u/j is defined as follows. An object in u/j is a pair $(i, f\colon u(i) \to j)$ of an object i in I and a morphism $f\colon u(i) \to j$ in J, and a morphism in u/j from (i, f) to (i', f') is a morphism $g\colon i \to i'$ in I such that $f = f' \circ u(g)$. A morphism $h\colon j \to j'$ in J gives rise to a functor $u/h\colon u/j \to u/j'$ that takes the object (i, f) to the object $(i, h \circ f)$ and the morphism $g\colon (i, f) \to (i', f')$ to the morphism $g\colon (i, h \circ f) \to (i', h \circ f')$. Moreover, there is a forgetful functor $p\colon u/j \to I$ that takes the object (i, f) to the object i and takes the morphism $g\colon (i, f) \to (i', f')$ to the morphism $g\colon i \to i'$. We leave it to the reader to define the slice category j/u, the functor $h/u\colon j'/u \to j/u$ associated with a morphism $h\colon j \to j'$, and the forgetful functor $p\colon j/u \to I$.

Proposition 3.3. *Let C be a category and let $u\colon I \to J$ be a functor.*

(i) *The left Kan extension of $F\colon I \to \mathsf{C}$ along u is given by*

$$u_!(F)(j) = \operatorname{colim}_{u/j}(F \circ p),$$

provided that the indicate colimit exists, for all $j \in \operatorname{ob}(J)$.

(ii) *The right Kan extension of $F\colon I \to \mathsf{C}$ along any functor u is given by*

$$u_*(F)(j) = \lim_{j/u}(F \circ p),$$

provided that the indicated limit exists, for all $j \in \operatorname{ob}(J)$.

Proof. To prove (i), we construct an adjunction $(u_!, u^*, \epsilon, \eta)$ from $[J, \mathsf{C}]$ to $[I, \mathsf{C}]$ such that the diagram $u_!(F)$ is given on objects by the stated formula. To define $u_!(F)$ on morphisms, we note that if $h\colon j \to j'$ is a morphism in J, then the forgetful functors $p\colon u/j \to J$ and $p'\colon u/j' \to J$ satisfy $p = p' \circ u/h$. We then define $u_!(F)(h)\colon u_!(F)(j) \to u_!(F)(j')$ to be the induction morphism

$$\operatorname{colim}_{u/j}(F \circ p) = \operatorname{colim}_{u/j}(F \circ p' \circ u/h) \xrightarrow{\operatorname{ind}_{u/h, F\circ p'}} \operatorname{colim}_{u/j'}(F \circ p').$$

This defines the diagram $u_!(F)$. Next, if $\alpha\colon F \Rightarrow F'$ is a natural transformation, then we define $u_!(\alpha)\colon u_!(F) \Rightarrow u_!(F')$ to be the natural transformation, whose value at the object j in J is the morphism $u_!(\alpha)_j = \operatorname{colim}_{u/j}(\alpha \circ p)$. This defines the functor $u_!\colon [I, \mathsf{C}] \to [J, \mathsf{C}]$. Indeed, the functor axioms are satisfied by the uniqueness of morphisms from colimits.

We define $\epsilon\colon u_! \circ u^* \Rightarrow \mathrm{id}_{[J,\mathsf{C}]}$ to be the natural transformation such that

$$\operatorname{colim}_{u/j}(F \circ u \circ p) \xrightarrow{\epsilon_{F,j}} F(j)$$

is the morphism defined by the pair

$$(F(j), (\, F(u(i)) \xrightarrow{F(f)} F(j)\,)_{(i,f\colon u(i)\to j)\in \operatorname{ob}(u/j)});$$

and we define $\eta\colon \mathrm{id}_{[I,\mathsf{C}]} \Rightarrow u^* \circ u_!$ to the be natural transformation such that

$$F(i) \xrightarrow{\eta_{F,i}} \operatorname{colim}_{u/u(i)}(F \circ p)$$

is the morphism $\mathrm{in}_{(i, \mathrm{id}_{u(i)})}$, which is part of the choice of colimit. We leave it as an exercise in the definitions to verify that the triangle identities hold. This proves (i), and (ii) is proved analogously. □

Now let $u\colon \mathsf{C} \to \mathsf{C}'$ be a functor and let $u^{\mathrm{op}}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{C}'^{\mathrm{op}}$ be the induced functor between the opposite categories. The restriction along u^{op},

$$\mathsf{C}'^{\wedge} \xrightarrow{(u^{\mathrm{op}})^*} \mathsf{C}^{\wedge},$$

has both a left adjoint and a right adjoint given by the Kan extensions

$$\mathsf{C}^{\wedge} \xrightarrow{(u^{\mathrm{op}})_!} \mathsf{C}'^{\wedge} \qquad \text{and} \qquad \mathsf{C}^{\wedge} \xrightarrow{(u^{\mathrm{op}})_*} \mathsf{C}'^{\wedge}.$$

Suppose now that J and J' are topologies on C and C', respectively. We say that a functor $f\colon \mathsf{C}^{\wedge} \to \mathsf{C}'^{\wedge}$ *preserves sheaves*, if there exists a functor f', necessarily unique, making the diagram

$$\begin{array}{ccc} (\mathsf{C},J)^{\sim} & \xrightarrow{f'} & (\mathsf{C}',J')^{\sim} \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle i'} \\ \mathsf{C}^{\wedge} & \xrightarrow{f} & \mathsf{C}'^{\wedge} \end{array}$$

commute.

Definition 3.4. Let (C,J) and (C',J') be two sites, and let $u\colon \mathsf{C} \to \mathsf{C}'$ be a functor. The functor u is said to be *continuous* if the restriction along u^{op}, $(u^{\mathrm{op}})^*\colon \mathsf{C}'^{\wedge} \to \mathsf{C}^{\wedge}$, preserves sheaves. The functor u is said to be *cocontinuous* if the right Kan extension along u^{op}, $(u^{\mathrm{op}})_*\colon \mathsf{C}^{\wedge} \to \mathsf{C}'^{\wedge}$, preserves sheaves.

Example 3.5. Let $f\colon X' \to X$ be a continuous map between topological spaces, and let $f^{-1}\colon O(X) \to O(X')$ be the functor that to an open subset $U \subset X$ assigns its preimage $f^{-1}(U) \subset X'$. Let J be the topology on $O(X)$ generated by the pretopology K in which the set $K(U)$ of covering families of $U \subset X$ consist of the familes $(f_i\colon U_i \to U)_{i\in I}$ such that $\bigcup_{i\in I} U_i = U$, and let J' be the corresponding topology on $O(X')$. Then $f^{-1}\colon O(X) \to O(X')$ is a continuous functor.

Proposition 3.6. *If (C,J) and (C',J') are sites and if $u\colon \mathsf{C} \to \mathsf{C}'$ is a continuous functor, then the unique functor $u_s\colon (\mathsf{C}',J')^{\sim} \to (\mathsf{C},J)^{\sim}$ making the diagram*

$$\begin{array}{ccc} (\mathsf{C}',J')^{\sim} & \xrightarrow{u_s} & (\mathsf{C},J)^{\sim} \\ \big\downarrow{\scriptstyle i'} & & \big\downarrow{\scriptstyle i} \\ \mathsf{C}'^{\wedge} & \xrightarrow{(u^{\mathrm{op}})^*} & \mathsf{C}^{\wedge} \end{array}$$

commute has a left adjoint functor $u^s\colon (\mathsf{C},J)^{\sim} \to (\mathsf{C}',J')^{\sim}$.

Proof. Let $(a', i', \epsilon', \eta')$ be the sheafification-inclusion adjunction from $\mathsf{C}'^{\wedge}$ to (C',J') and let $((u^{\mathrm{op}})_!, (u^{\mathrm{op}})^*, \alpha, \beta)$ be the left Kan extension-restriction along u^{op} adjunction from $\mathsf{C}^{\wedge}$ to $\mathsf{C}'^{\wedge}$. We define an adjunction $(u^s, u_s, \epsilon, \eta)$ from $(\mathsf{C},J)^{\sim}$ to $(\mathsf{C}',J')^{\sim}$ as follows. The functor u^s is the composite functor $a' \circ (u^{\mathrm{op}})_! \circ i$ and the counit $\epsilon\colon u^s \circ u_s \Rightarrow \mathrm{id}_{(\mathsf{C}',J')^{\sim}}$ is the composite natural transformation

$$a' \circ (u^{\mathrm{op}})_! \circ i \circ u_s = a' \circ (u^{\mathrm{op}})_! \circ (u^{\mathrm{op}})^* \circ i' \xRightarrow{a' \circ \alpha \circ i'} a' \circ i' \xRightarrow{\epsilon'} \mathrm{id}_{(\mathsf{C}',J')^{\sim}}.$$

Finally, since the functor $i\colon (\mathsf{C},J)^{\sim} \to \mathsf{C}^{\wedge}$ is fully faithful, there is a unique natural transformation $\eta\colon \mathrm{id}_{(\mathsf{C},J)^{\sim}} \Rightarrow u_s \circ u^s$ such that the composite natural transformation

$$i \xRightarrow{\beta \circ i} (u^{\mathrm{op}})^* \circ (u^{\mathrm{op}})_! \circ i \xRightarrow{(u^{\mathrm{op}})^* \circ \eta' \circ (u^{\mathrm{op}})_! \circ i} (u^{\mathrm{op}})^* \circ i' \circ a' \circ (u^{\mathrm{op}})_! \circ i = i \circ u_s \circ u^s$$

is equal to $i \circ \eta$. We again leave it as an exercise to verify the triangle identities. □

REMARK 3.7. If u is a continuous functor, then the functor u^s need not preserve finite limits.

PROPOSITION 3.8. *If* (C, J) *and* (C', J') *are sites and* $v\colon \mathsf{C} \to \mathsf{C}'$ *a cocontinuous functor, then the unique functor* $v_*\colon (\mathsf{C}, J)^{\tilde{}} \to (\mathsf{C}', J')^{\tilde{}}$ *making the diagram*

$$\begin{array}{ccc} (\mathsf{C}, J)^{\tilde{}} & \xrightarrow{v_*} & (\mathsf{C}', J')^{\tilde{}} \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle i'} \\ \mathsf{C}^{\wedge} & \xrightarrow{(v^{\mathrm{op}})_*} & \mathsf{C}'^{\wedge} \end{array}$$

commute has a left adjoint functor $v^*\colon (\mathsf{C}', J')^{\tilde{}} \to (\mathsf{C}, J)^{\tilde{}}$. *Moreover, the functor* v^* *preserves finite limits.*

PROOF. Let (a, i, α, β) be the sheafification-inclusion adjunction from $\mathsf{C}^{\wedge}$ to $(\mathsf{C}, J)^{\tilde{}}$ and let $((v^{\mathrm{op}})^*, (v^{\mathrm{op}})_*, \gamma, \delta)$ the the restriction-right Kan extension along v^{op} adjunction from $\mathsf{C}'^{\wedge}$ to $\mathsf{C}^{\wedge}$. We define an adjunction $(v^*, v_*, \epsilon, \eta)$ from $(\mathsf{C}', J')^{\tilde{}}$ to $(\mathsf{C}, J)^{\tilde{}}$ as follows. The functor v^* is the composite functor $a \circ (v^{\mathrm{op}})^* \circ i'$ and the counit $\epsilon\colon v^* \circ v_* \Rightarrow \mathrm{id}_{(\mathsf{C},J)^{\tilde{}}}$ is the composite natural transformation

$$a \circ (v^{\mathrm{op}})^* \circ i' \circ v_* = a \circ (v^{\mathrm{op}})^* \circ (v^{\mathrm{op}})_* \circ i \xRightarrow{a \circ \gamma \circ i} a \circ i \xRightarrow{\alpha} \mathrm{id}_{(\mathsf{C},J)^{\tilde{}}} .$$

Moreover, since $i'\colon (\mathsf{C}', J')^{\tilde{}} \to \mathsf{C}'^{\wedge}$ is fully faithful, there is a unique natural transformation $\eta\colon \mathrm{id}_{(\mathsf{C},J)^{\tilde{}}} \Rightarrow v_* \circ v^*$ for which the composite natural transformation

$$i' \xRightarrow{\delta \circ i'} (v^{\mathrm{op}})_* \circ (v^{\mathrm{op}})^* \circ i' \xRightarrow{(v^{\mathrm{op}})_* \circ \beta \circ (v^{\mathrm{op}})^* \circ i'} (v^{\mathrm{op}})_* \circ i \circ a \circ (u^{\mathrm{op}})^* \circ i' = i' \circ v_* \circ v^*$$

is equal to $i' \circ \eta$. Finally, the functor v^* preserves finite limits, because a does so and because $(v^{\mathrm{op}})^*$ and i' preserve all limits. □

4. Abelian categories and derived functors

We first introduce the notion of an abelian category, emphasizing that for a category to be abelian is a property of the category and does not require additional structure. A category C is a *abelian* if it has the following properties (i)–(v):

(i) The category C has finite limits and finite colimits.
(ii) The unique map $0 \to 1$ from a initial object to a final object is an isomorphism. Hence, every an initial object is automatically final and vice versa; such an object is said to be a *null object* of C.

It follows from (ii) that, for every pair (c_0, c_1) of objects in C, the set of morphisms $\mathsf{C}(c_0, c_1)$ has a distinguished element defined as the composition

$$c_0 \longrightarrow 1 \xleftarrow{\sim} 0 \longrightarrow c_1$$

of the unique morphisms to a final object 1, the inverse of the unique morphism from an initial object 0 to 1, and the unique morphism from 0 to c_1. The composite morphism does not depend on the choices of initial object and final object. We write $0\colon c_0 \to c_1$ for this morphism and call it the zero morphism from c_0 to c_1.

(iii) For every pair (c_1, c_2) of objects in C, the morphism

$$c_1 \sqcup c_2 \xrightarrow{(\mathrm{id}_{c_1}+0, 0+\mathrm{id}_{c_2})} c_1 \times c_2$$

from a coproduct of c_1 and c_2 to a product of c_1 and c_2 is an isomorphism.

We call the morphism in (iii) the canonical isomorphism. We also note that it is equal to $(\mathrm{id}_{c_1}, 0) + (0, \mathrm{id}_{c_2})$ and that it is a natural isomorphism of functors from $\mathsf{C} \times \mathsf{C}$ to C. We use the canonical isomorphism in (iii) to define, for every pair (c_0, c_1) of objects in C, a composition law "+" on the set of morphisms $\mathsf{C}(c_0, c_1)$. Given two morphism $f, g\colon c_0 \to c_1$, we define $f + g\colon c_0 \to c_1$ to be the composition

$$\begin{array}{ccccccc} c_0 & \xrightarrow{\Delta_{c_0}} & c_0 \times c_0 & \xleftarrow{\sim} & c_0 \sqcup c_0 & & \\ & & \downarrow{\scriptstyle f \times g} & & \downarrow{\scriptstyle f \sqcup g} & & \\ & & c_1 \times c_1 & \xleftarrow{\sim} & c_1 \sqcup c_1 & \xrightarrow{\nabla_{c_1}} & c_1, \end{array}$$

where the unmarked arrows are the canonical isomorphisms, and where the square commutes by the naturality of the canonical isomorphism. The composition law "+" is an abelian monoid structure with the zero morphism as identity element.

(iv) For every pair (c_0, c_1) of objects in C, the composition law "+" on $\mathsf{C}(c_0, c_1)$ is an abelian group structure.

If $f\colon c_0 \to c_1$ is a morphism in C, then an equalizer

$$\ker(f) \xrightarrow{i} c_0 \overset{f}{\underset{0}{\rightrightarrows}} c_1$$

is called a *kernel* of f, and a coequalizer

$$c_0 \overset{f}{\underset{0}{\rightrightarrows}} c_1 \xrightarrow{p} \operatorname{coker}(f)$$

is called a *cokernel* of f. Moreover, a coequalizer

$$\ker(f) \underset{0}{\overset{i}{\rightrightarrows}} c_0 \xrightarrow{q} \operatorname{coim}(f)$$

is called a *coimage* of f, and an equalizer

$$\operatorname{im}(f) \xrightarrow{j} c_1 \underset{0}{\overset{p}{\rightrightarrows}} \operatorname{coker}(f)$$

is called an *image* of f. There is a unique morphism $\bar{f}\colon \operatorname{coim}(f) \to \operatorname{im}(f)$ such that the diagram

$$\begin{array}{ccccccc} \ker(f) & \xrightarrow{i} & c_0 & \xrightarrow{f} & c_1 & \xrightarrow{p} & \operatorname{coker}(f) \\ & & \downarrow q & & \uparrow j & & \\ & & \operatorname{coim}(f) & \xrightarrow{\bar{f}} & \operatorname{im}(f) & & \end{array}$$

commutes.

(v) For every morphism $f\colon c_0 \to c_1$ in C, the induced morphism

$$\operatorname{coim}(f) \xrightarrow{\bar{f}} \operatorname{im}(f)$$

is an isomorphism.

This completes the definition of what it means for a category C to be abelian.

Let C be an abelian category, and let $h\colon \mathsf{C} \to \mathsf{C}^{\wedge}$ be the Yoneda embedding. For every object A of C, the functor $h(A)(-) = \mathsf{C}(-, A)$ from C^{op} to Set preserves all limits that exist in C. In particular, it preserves finite limits. An object I of C is defined to be *injective* if the functor $h(I)(-)$ preserves finite colimits. The category C is said to have *enough injectives* if for every object A in C, there exists a monomorphism $i\colon A \to I$ to an injective object. We recall that, in this case, every object A in C admits an injective resolution

$$A \xrightarrow{\eta} I^{\cdot},$$

which, by the fundamental lemma of homological algebra, is unique, up to chain homotopy equivalence. Let C and C' be abelian categories such that C has enough injectives, and let $F\colon \mathsf{C} \to \mathsf{C}'$ be a functor that preserves finite products. The nth derived functor of F is the functor

$$\mathsf{C} \xrightarrow{R^n F} \mathsf{C}',$$

well-defined, up to canonical natural isomorphism, which is given on objects by

$$(R^n F)(A) = H^n(F(I^{\cdot})),$$

where $\eta_A\colon A \to I^{\cdot}$ is a choice of injective resolution. In particular, the chosen chain map η_A induced a map, which, by abuse of notation, we write

$$F(A) \xrightarrow{\eta_A} (R^0 F)(A).$$

To define $R^n F$ on morphisms, let $f\colon A \to A'$ be a morphism in C. We choose injective resolutions $\eta\colon A \to I^{\cdot}$ and $\eta'\colon A' \to I'^{\cdot}$ and recall that, by the fundamental lemma of homological algebra, there exists a chain map $f^\cdot \colon I^\cdot \to I'^\cdot$, unique up to chain homotopy, such that $f^0 \circ \eta = \eta' \circ f$. In this situation, we define

$$(R^nF)(f) = H^n(F(f^\cdot)).$$

Using the fundamental lemma of homological algebra, we conclude that R^nF is indeed a functor.

PROPOSITION 4.1. *Let C be an abelian category with enough injectives, let C' be an abelian category, and let $F \colon \mathsf{C} \to \mathsf{C}'$ be a functor that preserves finite products.*

(i) *The maps $\eta_A \colon F(A) \to (R^0F)(A)$ constitute a natural transformation*

$$F \xRightarrow{\eta} R^0F,$$

which is a natural isomorphism, if F preserves finite limits.

(ii) *A short exact sequence in C,*

$$0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0,$$

gives rise to a long exact sequence in C',

$$\begin{array}{l} 0 \longrightarrow (R^0F)(A') \xrightarrow{(R^0F)(f)} (R^0F)(A) \xrightarrow{(R^0F)(g)} (R^0F)(A'') \\ \xrightarrow{\partial^0} (R^1F)(A') \xrightarrow{(R^1F)(f)} (R^1F)(A) \xrightarrow{(R^1F)(g)} (R^1F)(A'') \\ \xrightarrow{\partial^1} (R^2F)(A') \xrightarrow{(R^2F)(f)} (R^2F)(A) \xrightarrow{(R^2F)(g)} \cdots \end{array}$$

and this assignment is natural in the short exact sequence.

PROOF. The second statement in (i) follows immediately from the definition, and the second statement in (i) follows from the fundamental lemma of homological algebra. To prove the first statement in (ii), we choose a commutative diagram of cochain complexes in C,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{f} & A & \xrightarrow{g} & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \eta'} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta''} & & \\ 0 & \longrightarrow & I'^\cdot & \xrightarrow{f^\cdot} & I^\cdot & \xrightarrow{g^\cdot} & I''^\cdot & \longrightarrow & 0, \end{array}$$

such vertical morphisms are injective resolutions, and such that the sequences

$$0 \longrightarrow I'^n \xrightarrow{f^n} I^n \xrightarrow{g^n} I''^n \longrightarrow 0$$

all are split-exact, and choose sections s^n of g^n and retractions r^n of f^n in such a way that $s^n \circ g^n + f^n \circ r^n = \mathrm{id}_{I^n}$, for every non-negative integer n. This is possible, since C has enough injectives, but it is not possible, in general, to choose sections s^n and retractions r^n such that the families (s^n) and (r^n) form chain maps. Applying F term-wise, we obtain an sequence of cochain complexes in C',

$$0 \longrightarrow F(I'^\cdot) \xrightarrow{F(f^\cdot)} F(I^\cdot) \xrightarrow{F(g^\cdot)} F(I''^\cdot) \longrightarrow 0,$$

and a diagram chase shows that we obtain the long exact sequence in (ii) with

$$(R^nF)(A'') \xrightarrow{\partial^n} (R^{n+1}F)(A')$$

induced by the unique morphism $\tilde{\partial}^n$ that makes the diagram

$$\begin{array}{ccccccc}
\ker(d''^n) & & & \xrightarrow{\quad\quad\quad\quad\tilde{\partial}^n\quad\quad\quad\quad} & & & \ker(d'^n) \\
\big\downarrow{\scriptstyle i''^n} & & & & & & \big\downarrow{\scriptstyle i'^{n+1}} \\
F(I''^n) & \xrightarrow{F(s^n)} & F(I^n) & \xrightarrow{F(d^n)} & F(I^{n+1}) & \xrightarrow{F(r^{n+1})} & F(I'^{n+1})
\end{array}$$

commute. We leave it an exercise to prove the naturality statement in (ii). □

Remark 4.2. A word of warning is in order. While the long exact sequence in Proposition 4.1 is natural, it is by no means unique. For instance, keeping the terms in the sequence unchanged but replacing the morphisms $(R^nF)(f)$, $(R^nF)(g)$, and ∂^n by the morphisms $(-1)^n(R^nF)(f)$, $(-1)^n(R^nF)(g)$, and $(-1)^n\partial^n$, respectively, the resulting sequence is again long exact and natural. In homological algebra, the issue of signs is unavoidable. For example, the reader, who tries to define a natural isomorphism from $R^m(R^nF)$ to $R^{m+n}F$, will realize that sign choices are necessary. This author's preferred choices of signs is recorded in [**2**, Section 2].

Example 4.3. Let C be an abelian category with enough injectives. For every object A in C, the functor $\mathsf{C}(A,-)$ from C to the category Ab of abelian groups preserves limits. The derived functors are written

$$\mathrm{Ext}^n_{\mathsf{C}}(A,B) = R^n\mathsf{C}(A,-)(B)$$

and called the Ext-groups in C.

5. Grothendieck's small object argument

We will present the proof of the theorem in Grothendieck's Tohoku paper [**1**] concerning the existence of injective resolutions in abelian categories. It will be helpful first properly define some concepts that we have left undefined until now.

We recall that, in ZFC set theory, every term x is a set and $x \in y$ is the only relation among sets. A set U is a *universe* if it satisfies the following (U1)–(U5):

(U1) If $y \in U$ and $x \in y$, then $x \in U$.
(U2) If $x, y \in U$, then $\{x, y\} \in U$.
(U3) If $x \in U$, then the power set $P(x)$ is an element of U.
(U4) If $I \in U$ and $x\colon I \to U$ is a map, then $\bigcup_{i \in I} x(i) \in U$.
(U5) The first infinite ordinal ω is an element of U.

We remark that, in (U3), the power set $P(x)$ depends on the particular model of ZFC set theory. We also note that, in (U4), the map $x\colon I \to U$, which, by definition, is the subset $\{I\} \times \Gamma_x \times \{U\} \subset \{I\} \times I \times U \times \{U\}$, is not an element of U. If U is a universe, then its elements constitute a model of ZFC set theory. We will assume the *axiom of universe* which states that for every set x, there exists a universe U such that $x \in U$.

Let U be a universe. A set x is *U-small* if $x \in U$. We remark that a set bijective to a U-small set need not be U-small. For instance, the set $\{\emptyset\}$ is U-small, but the set $\{U\}$ is not. A category C is *locally U-small* if for every pair of objects (c, c'), the set of morphisms $\mathsf{C}(c, c')$ is U-small; and it is *U-small* if, in addition, the set of objects $\operatorname{ob}(\mathsf{C})$ is U-small. A category C is *U-complete* (resp. *U-cocomplete*) if every diagram in C indexed by a U-small category admits a limit (resp. a colimit).

The category U-Set of all U-small sets and maps is locally U-small, U-complete, and U-cocomplete. As the following example, which I learned from Michael Shulman shows, this is typically the best situation that one can hope for.

EXAMPLE 5.1. We claim that if a category C is both U-small and U-complete, then C is a preorder. To prove this, we must show that if $f, g\colon c \to c'$ are parallel morphisms in C, then $f = g$. So assume that $f \neq g$. Since C is U-small, the total set of morphisms $\operatorname{mor}(\mathsf{C})$ is U-small, and since C is U-complete, the product $c'' = \prod_{h \in \operatorname{mor}(\mathsf{C})} c'$ exists. Now, since $f \neq g$, the cardinality of the set

$$\mathsf{C}(c, c'') = \prod_{h \in \operatorname{mor}(\mathsf{C})} \mathsf{C}(c, c')$$

is at least $2^{\operatorname{card}(\operatorname{mor}(\mathsf{C}))}$. But $\mathsf{C}(c, c'')$ is a subset of $\operatorname{mor}(\mathsf{C})$, so its cardinality is at most $\operatorname{card}(\operatorname{mor}(\mathsf{C}))$, which is a contradiction. Hence, we have $f = g$ as claimed.

An object G of a locally U-small category C is said to be a *generator* if the functor $\mathsf{C}(G, -)\colon \mathsf{C} \to U$-$\mathsf{Set}$ is faithful.

THEOREM 5.2 (Grothendieck). *Let U be a universe, let C be a locally U-small and U-cocomplete abelian category, and assume, in addition, that C admits a generator and that U-small filtered colimits and finite limits in C commute. In this situation, the category C has enough injectives.*

PROOF. Let A be an object of C. The (large) set of subobjects of A is the quotient of the (large) set of monomorphisms $i\colon A' \to A$ with target A by the equivalence relation that identifies $j\colon A' \to A$ and $j'\colon A'' \to A$ if there exists an isomorphism, necessarily unique, $f\colon A'' \to A'$ such that $j' = j \circ f$. We claim that, in the situation at hand, there exists a family $(j_t\colon A_t \to A)_{t\in T}$ of monomorphisms in C indexed by a U-small set T which contains a representative of every subobject of A. For, let G be a generator of C. One may show that two monomorphisms $j\colon A' \to A$ and $j'\colon A'' \to A$ are equivalent if and only if the images of the induced injections $\mathsf{C}(G,j)\colon \mathsf{C}(G,A') \to \mathsf{C}(G,A)$ and $\mathsf{C}(G,j')\colon \mathsf{C}(G,A'') \to \mathsf{C}(G,A)$ are equal. Hence, the subset T of the U-small set $\mathsf{C}(G,A)$ consisting of these images will do. Note that we use that the axiom of choice also holds for large sets to choose the family.

We write $(j_t\colon G_t \to G)_{t\in T}$ for the family chosen above in the case $A = G$. One may show that an object I of C is injective if and only if, for every commutative diagram in C of the form

$$\begin{array}{ccc} G_t & \xrightarrow{f_t} & I \\ \big\downarrow{\scriptstyle j_t} & & \big\downarrow \\ G & \longrightarrow & 0, \end{array}$$

there exists a morphism $f\colon G \to I$ such that $f_t = f \circ j_t$. (Here we have put in the obviously redundant 0 only to stress the analogue with Quillen's small object argument.)

We now let A be any object in C and proceed to construct a monomorphism

$$A \xrightarrow{i} I$$

to an injective object of C. We choose a limit ordinal $\beta \in U$ whose cardinality is strictly larger than the cardinality γ of the U-small indexing set T. Such an ordinal is known to be γ-filtered, which means that the supremum of any subset $A \subset \beta$ of cardinality at most γ is strictly less than β. We define, by transfinite induction, a $(\beta+1)$-indexed diagram of monomorphisms

$$A \xrightarrow{i_\alpha} I_\alpha,$$

beginning with $i_0 = \mathrm{id}_A\colon A \to I_0$. Here, we recall, the ordinal β, as every ordinal in U, is the hereditarily transitive set of all smaller ordinals, and we view β as a category with one arrow from α to α' if $\alpha \in \alpha'$. If $\alpha + 1 = \alpha \cup \{\alpha\}$ is a successor ordinal, then we let S_α be the set of all diagrams of the form

$$\begin{array}{ccc} G_{t(s)} & \xrightarrow{f_s} & I_\alpha \\ \big\downarrow{\scriptstyle j_{t(s)}} & & \big\downarrow \\ G & \longrightarrow & 0, \end{array}$$

and choose a pushout

$$\begin{array}{ccc} \coprod_{s\in S_\alpha} G_{t(s)} & \xrightarrow{\sum_{s\in S_\alpha} f_s} & I_\alpha \\ \big\downarrow{\scriptstyle \coprod_{s\in S_\alpha} j_{t(s)}} & & \big\downarrow{\scriptstyle I(\alpha\in\alpha+1)} \\ \coprod_{s\in S_\alpha} G & \longrightarrow & I_{\alpha+1}, \end{array}$$

where we have neglected writing the redundant morphisms to 0. And if α is a limit ordinal, then we define I_α to be the filtered colimit

$$I_\alpha = \operatorname*{colim}_{\alpha' \in \alpha} I_{\alpha'}$$

and define $I(\alpha' \in \alpha) = \mathrm{in}_{\alpha'} \colon I_{\alpha'} \to I_\alpha$. We claim that $i_\beta \colon A \to I_\beta$ is the desired monomorphism $i \colon A \to I$ to an injective object. Indeed, it is a monomorphism, as one sees by showing that its kernel is zero. This uses the assumption that filtered colimits commute with finite limits in C. Finally, to prove that I is injective, we must show that for every commutative diagram in C of the form

$$\begin{array}{ccc} G_t & \xrightarrow{f_t} & I \\ {\scriptstyle j_t}\downarrow & & \downarrow \\ G & \longrightarrow & 0, \end{array}$$

there exists a morphism $f \colon G \to I$ such that $f_t = f \circ j_t$. To this end, we choose for every $\alpha \in \beta$ a pullback diagram

$$\begin{array}{ccc} G_{t,\alpha} & \xrightarrow{f_{t,\alpha}} & I_\alpha \\ {\scriptstyle k_\alpha}\downarrow & & \downarrow{\scriptstyle I(\alpha\in\beta)} \\ G_t & \xrightarrow{f_t} & I. \end{array}$$

We show that there exists $\alpha \in \beta$ such that $k_\alpha \colon G_{t,\alpha} \to G_t$ is an isomorphism. First, since filtered colimits and finite limits in C commute, we find that the monomorphims k_α with $\alpha \in \beta$ exhibit G_t as the colimit of the $G_{t,\alpha}$. Second, the subset $A \subset \beta$ of all $\alpha \in \beta$ such that the structure morphism $G_{t,\alpha} \to G_{t,\alpha+1}$ in the limit system is not an isomorphism is bijective to a subset of T, and hence, has cardinality at most γ. Hence, the supremum α_0 of A is strictly less than β, which shows that for any $\alpha_0 \in \alpha \in \beta$, the morphism k_α is an isomorphism. Now, fix such an $\alpha \in \beta$. Since β is a limit ordinal, we also have $\alpha + 1 \in \beta$, and by the construction of $I_{\alpha+1}$, there exists a morphism $f_{\alpha+1} \colon G \to I_{\alpha+1}$ making the diagram

$$\begin{array}{ccc} G_t & \xrightarrow{f_{t,\alpha}\circ k_\alpha^{-1}} & I_\alpha \\ {\scriptstyle j_t}\downarrow & & \downarrow{\scriptstyle I(\alpha\in\alpha+1)} \\ G & \xrightarrow{f_{\alpha+1}} & I_{\alpha+1} \end{array}$$

commute. Hence, the composite morphism $f = \mathrm{in}_{\alpha+1} \circ f_{\alpha+1} \colon G \to I$ is the desired morphism satisfying $f_t = f \circ j_t$. This shows that I is injective. □

6. Cohomology

Let U be a universe, let C be a U-small category, and let U-Set be the locally U-small category of U-small sets. We claim that the category

$$\mathsf{C}^\wedge = [\mathsf{C}^{\mathrm{op}}, U\text{-}\mathsf{Set}]$$

of U-Set valued presheaves on C is isomorphic to a locally U-small category. Indeed, if $F, F' \colon \mathsf{C}^{\mathrm{op}} \to U$-$\mathsf{Set}$ are two functors, then the set of natural transformations from F to F' is defined to be a choice of equalizer

$$\mathsf{C}^\wedge(F,F') \xrightarrow{\ i\ } \prod U\text{-}\mathsf{Set}(F(c),F'(c)) \overset{a}{\underset{b}{\rightrightarrows}} \prod U\text{-}\mathsf{Set}(\mathsf{C}(c',c), U\text{-}\mathsf{Set}(F(c),F'(c')))$$

with the products indexed by the sets $\mathrm{ob}(\mathsf{C})$ and $\mathrm{ob}(\mathsf{C}) \times \mathrm{ob}(\mathsf{C})$ of objects and pairs of objects in C, respectively. Therefore, the two products and the equalizer of the maps a and b, which we leave it as an exercise to define, may be chosen to be U-small as claimed. Choosing $\mathsf{C}^\wedge$ locally U-small, the full subcategory $(\mathsf{C},J)^\sim$ of $\mathsf{C}^\wedge$ of sheaves for a topology J again is locally U-small.

Definition 6.1. Let U be a universe. A category X is a U-topos if it is locally U-small category and equivalent to a category of U-Set valued sheaves on a U-small site (C,J). A morphism of U-topoi $f \colon X \to X'$ is an adjunction $(f^*, f_*, \epsilon, \eta)$ from X' to X such that f^* preserves finite limits.

If $f \colon X \to X'$ is a morphism of topoi, then the functors f^* and f_* are called the inverse image functor and the direct image functor, respectively. We note that the direction of the morphism $f \colon X \to X'$ is the same as the direction of the direct image functor $f_* \colon X \to X'$, as the notation suggests. If (C,J) is a U-small site, then the adjunction (a, i, ϵ, η) from $\mathsf{C}^\wedge$ to $(\mathsf{C},J)^\sim$ with a the sheafification functor and i the canonical inclusion functor is a morphism of topoi $(\mathsf{C},J)^\sim \to \mathsf{C}^\wedge$.

Remark 6.2. Let X be a U-topos and suppose that X is equivalent to the category of U-Set valued sheaves on a U-small site (C,J). We can always find a morphism of topoi $f \colon (\mathsf{C},J)^\sim \to X$ such that $(f^*, f_*, \epsilon, \eta)$ is an adjoint equivalence of categories. Indeed, an equivalence of categories can always be replaced by an adjoint equivalence of categories. Moreover, if $(f^*, f_*, \epsilon, \eta)$ is an adjoint equivalence of categories, then so is $(f_*, f^*, \eta^{-1}, \epsilon^{-1})$, which shows that f^* and f_* both preserve all limits and colimits. In this situation, we say that the pair $((\mathsf{C},J), f)$ is a site of definition for the topos X.

The category U-Set is itself a U-topos, called the punctual topos. Morphisms of topoi from the punctual topos are points.

Definition 6.3. Let U be a universe.

(i) A point of a U-topos X is a morphism of topoi $x \colon U$-$\mathsf{Set} \to X$.
(ii) A family of points $(x_i)_{i \in I}$ of a U-topos X is conservative if it has the property that a morphism $f \colon F \to F'$ in X is an isomorphism if and only if the induced maps $x_i^*(f) \colon x_i^*(F) \to x_i^*(F')$ are bijections, for all $i \in I$.
(iii) A U-topos X has enough points it has a conservative family of points.

One can show that if a U-topos has a conservative family of points, then it has a conservative family of points indexed by a U-small set. We often write F_x for the set $x^*(F)$ and call it the stalk of F at x; similarly, we write $f_x\colon F_x \to F'_x$ for the map $x^*(f)\colon x^*(F) \to x^*(F')$ and call it the induced map of stalks at x.

Definition 6.4. Let U be a universe. A ringed U-topos is a pair $(X, \mathscr{O}_X)$ of a U-topos X and a unital and associative ring object $\mathscr{O}_X$ in X. A morphism of ringed U-topoi $u\colon (X, \mathscr{O}_X) \to (X', \mathscr{O}_{X'})$ is a pair (f, φ) of a morphism of topoi $f\colon X \to X'$ and a ring object homomorphism $\varphi\colon f^*(\mathscr{O}_{X'}) \to \mathscr{O}_X$.

Given a ringed topos $(X, \mathscr{O}_X)$, we define $\mathsf{Mod}_{\mathscr{O}_X}$ to be category whose objects are the right $\mathscr{O}_X$-module objects in X and whose morphisms are the $\mathscr{O}_X$-linear morphisms between them. Here, we require ring objects to be unital and associative, and we require module objects to be unital.

Example 6.5. Every topos X has an initial ring object $\mathbb{Z}_X$ called an integers object, which, as any initial object, is unique, up to unique isomorphism. Moreover, the forgetful functor from the category $\mathsf{Mod}_{\mathbb{Z}_X}$ of right $\mathbb{Z}_X$-module objects in X to the category $\mathrm{Ab}(X)$ of abelian group objects in X is an equivalence of categories.

Proposition 6.6. *Let U be a universe. For every ringed U-topos $(X, \mathscr{O}_X)$, the category $\mathsf{Mod}_{\mathscr{O}_X}$ is abelian and has enough injectives.*

Proof. We leave it as an exercise to show that $\mathsf{Mod}_{\mathscr{O}_X}$ is abelian and apply Grothendieck's theorem, Theorem 5.2, to show that it has enough injectives. We may assume that X is equal to the category $(\mathsf{C}, J)^{\sim}$ of U-Set valued sheaves on a U-small site (C, J). In this situation, the category $\mathsf{Mod}_{\mathscr{O}_X}$ is locally U-small (or at least isomorphic to a locally U-small category) and U-cocomplete, and moreover, U-small filtered colimits and finite limits in $\mathsf{Mod}_{\mathscr{O}_X}$ commute. Finally, given $c \in \mathrm{ob}(\mathsf{C})$, let $\bar{c}$ be the image of c in $\mathrm{ob}(X)$ by the composition of the Yoneda embedding $h_{\mathsf{C}}\colon \mathsf{C} \to \mathsf{C}^{\wedge}$ and the sheafification functor $a\colon \mathsf{C}^{\wedge} \to \mathsf{C}^{\sim}$. The slice category $X/\bar{c}$ is again a topos and there is a morphism of topoi $j_c\colon X/\bar{c} \to X$ for which $j_{c*}\colon X/\bar{c} \to X$ is the forgetful functor. We view the ring object $j_{c*}\, j_c^*(\mathscr{O}_X)$ as a right $\mathscr{O}_X$-module via the unit morphism $\eta_c\colon \mathscr{O}_X \to j_{c*}j_c^*(\mathscr{O}_X)$, which is a ring homomorphism. Now, since $i\colon X \to \mathsf{C}^{\wedge}$ is faithful, the right $\mathscr{O}_X$-module

$$\mathscr{G}_X = \bigoplus\nolimits_{c \in \mathrm{ob}(C)} j_{c*} j_c^*(\mathscr{O}_X)$$

is readily seen to be a generator of Mod_X. □

Let $(X, \mathscr{O}_X)$ be a ringed U-topos. If $\mathscr{M}$ and $\mathscr{N}$ are two right $\mathscr{O}_X$-module objects in X, then we write

$$\mathrm{Ext}^n_{\mathscr{O}_X}(\mathscr{N}, \mathscr{M}) = \mathrm{Ext}^n_{\mathsf{Mod}_{\mathscr{O}_X}}(\mathscr{N}, \mathscr{M})$$

for the U-small Ext-groups in $\mathsf{Mod}_{\mathscr{O}_X}$. In particular, we may take $\mathscr{N}$ to be $\mathscr{O}_X$ with the right $\mathscr{O}_X$-module structure given by right multiplication.

Definition 6.7. Let U be a universe, let $(X, \mathscr{O}_X)$ be a ringed U-topos, and let $\mathscr{M}$ be a right $\mathscr{O}_X$-module. The U-small abelian group

$$H^n(X, \mathscr{M}) = \mathrm{Ext}^n_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{M})$$

is called the nth cohomology group of X with coefficients in $\mathscr{M}$.

By definition, the cohomology groups $H^n(X, \mathscr{M})$ measure the extend to which the free $\mathscr{O}_X$-module $\mathscr{O}_X$ fails to be projective. (The notation suggests that these groups only depend on X and the underlying abelian group object of $\mathscr{M}$, which is indeed the case.) In other words, the cohomology groups $H^n(X, \mathscr{M})$ measure the extend to which the axiom of choice fails to hold in the topos X.

Let $(X, \mathscr{O}_X)$ be a ringed U-topos and let $\mathscr{M}$ and $\mathscr{N}$ be a right $\mathscr{O}_X$-module and a left $\mathscr{O}_X$-module, respectively. In this situation, we define the tensor product $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$, following Bourbaki. It is a an abelian group object in X; is well-defined up to unique isomorphism; and has, if $\mathscr{O}_X$ is commutative, a canonical $\mathscr{O}_X$-module structure. In the latter case, the tensor product is the monoidal product in a closed symmetric monoidal structure $(\otimes_{\mathscr{O}_X}, \mathscr{O}_X, \alpha, \lambda, \gamma, [-,-], \epsilon, \eta)$ on $\mathsf{Mod}_{\mathscr{O}_X}$. As in any category with a symmetric monoidal structure, we say that an $\mathscr{O}_X$-module $\mathscr{L}$ is invertible with respect to the tensor product, if there exists an $\mathscr{O}_X$-module $\mathscr{L}'$ and an isomorphism between $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{L}'$ and the unit $\mathscr{O}_X$ for the tensor product. The full subcategory of $\mathsf{Mod}_{\mathscr{O}_X}$ whose objects are the invertible $\mathscr{O}_X$-modules is called the Picard category of $(X, \mathscr{O}_X)$; its maximal subgroupoid is called the Picard groupoid of $(X, \mathscr{O}_X)$; and its set of isomorphism classes of objects is called the Picard group of $(X, \mathscr{O}_X)$ and written $\mathrm{Pic}(X, \mathscr{O}_X)$.

We will next discuss cohomology of coherent modules. It will not be the main focus of this course, but it is useful to have some familiarity with it.

Definition 6.8. Let U be a universe and let $(X, \mathscr{O}_X)$ be a ringed U-topos.

(i) A right $\mathscr{O}_X$-module $\mathscr{M}$ is quasi-coherent if it admits a presentation

$$F_1 \longrightarrow F_0 \longrightarrow \mathscr{M} \longrightarrow 0$$

with F_0 and F_1 free right $\mathscr{O}_X$-modules generated by U-small sets.

(ii) A right $\mathscr{O}_X$-module $\mathscr{M}$ is finite if there exists an epimorphism

$$F \longrightarrow \mathscr{M} \longrightarrow 0$$

with F a free $\mathscr{O}_X$-module generated by a finite set.

(iii) A right $\mathscr{O}_X$-module $\mathscr{M}$ is coherent if it is finite and if for every open immersion of topoi $j\colon U \to X$ and every morphism $f\colon F \to j_U^*(\mathscr{M})$ from a finite free $j_U^*(\mathscr{O}_X)$-module, the kernel is a finite $j_U^*(\mathscr{O}_X)$-module.

Suppose that $(X, \mathscr{O}_X)$ is a scheme. We consider the category $O(X)$ of open subsets in the topological space X with the topology J generated by the pretopology consisting of the families $(U_i \to U)_{i \in I}$ such that $\bigcup_{i \in I} U_i = U$. We write X_{Zar} for the topos of sheaves on $(O(X), J)$ and call it the Zariski topos of X. The structure sheaf $\mathscr{O}_X$ is a commutative ring object in X_{Zar}, and hence, the pair $(X_{\mathrm{Zar}}, \mathscr{O}_X)$ is a commutative ringed topos. A morphism of schemes $f\colon (X, \mathscr{O}_X) \to (X', \mathscr{O}_{X'})$ induces a morphism of ringed topoi $f\colon (X_{\mathrm{Zar}}, \mathscr{O}_X) \to (X'_{\mathrm{Zar}}, \mathscr{O}_{X'})$.

The following flat base-change theorem is EGA III, Corollary 6.9.9. As usual, we abbreviate and write X for the scheme $(X, \mathscr{O}_X)$ and $f\colon X \to X'$ for the morphism of schemes $f\colon (X, \mathscr{O}_X) \to (X', \mathscr{O}_{X'})$.

Theorem 6.9. *Let S be a scheme, let $f\colon X \to S$ and $g\colon S' \to S$ be morphisms of schemes with f separated and quasi-compact, and let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

be a choice of pullback. Then, for every flat quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and every non-negative integer n, there is a canonical natural isomorphism

$$g^*(R^n f_*(\mathscr{F})) \otimes_{g^*(\mathscr{O}_S)} \mathscr{O}_{S'} \xrightarrow{\sim} R^n f'_*(g'^*(\mathscr{F}) \otimes_{g'^*(\mathscr{O}_X)} \mathscr{O}_{X'})$$

of $\mathscr{O}_{S'}$-modules.

We proceed to illustrate the usefulness of such base-change theorems by means of an example. Let k be a field, let $S = \operatorname{Spec}(k)$, let $f\colon X \to S$ be a morphism of schemes, and let $\mathscr{M}$ be an $\mathscr{O}_X$-module. The Zariski topos S_{Zar} is canonically isomorphic to the punctual topos, and moreover, under this isomorphism, we may identify the $\mathscr{O}_S$-module $R^n f_*(\mathscr{M})$ of S_{Zar} with the k-vector space $H^n(X, \mathscr{M})$.

Example 6.10. Let k be a field, let $S = \operatorname{Spec}(k)$, and let $f\colon X \to S$ be a smooth and proper morphism or relative dimension 1. (We say that X is a curve over S.) Given an invertible $\mathscr{O}_X$-module $\mathscr{L}$, its degree is the integer defined by

$$\deg(\mathscr{L}) = \chi(\mathscr{L}) - \chi(\mathscr{O}_X),$$

where, for $\mathscr{M}$ a coherent $\mathscr{O}_X$-module, its Euler characteristic is defined by

$$\chi(\mathscr{M}) = \dim_k H^0(X, \mathscr{M}) - \dim_k H^1(X, \mathscr{M}).$$

The k-vector spaces $H^n(X, \mathscr{M})$ are finite dimensional by Serre duality and vanish for $n > 1$. The degree defines a map of abelian groups

$$\operatorname{Pic}(X_{\mathrm{Zar}}, \mathscr{O}_X) \xrightarrow{\deg} \mathbb{Z}$$

from the Picard group of X.

We now let k' be any field extension of k, let $S' = \operatorname{Spec}(k')$, let $g\colon S' \to S$ be the morphism induced by the inclusion of k in k', and consider the pullback

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S. \end{array}$$

An invertible $\mathscr{O}_X$-module $\mathscr{L}$ is locally free, and hence, flat. Indeed, being flat is a local property, since the exactness of a sequence of $\mathscr{O}_X$-modules can be checked stalkwise, the Zariski topos X_{Zar} having enough points. Therefore, it follows from Theorem 6.9 that there is a canonical isomorphism of $\mathscr{O}_{S'}$-modules

$$g^*(R^n f_*(\mathscr{L})) \otimes_{g^*(\mathscr{O}_S)} \mathscr{O}_{S'} \xrightarrow{\sim} R^n f'_*(g'^*(\mathscr{L}) \otimes_{g'^*(\mathscr{O}_X)} \mathscr{O}_{X'}),$$

and taking global sections, we obtain a canonical isomorphism of k'-vector spaces

$$H^n(X, \mathscr{L}) \otimes_k k' \xrightarrow{\sim} H^n(X', g'^*(\mathscr{L}) \otimes_{g'^*(\mathscr{O}_X)} \mathscr{O}_{X'}).$$

It follows that for all non-negative integers n,

$$\dim_{k'}(H^n(X', g'^*(\mathscr{L}) \otimes_{g'^*(\mathscr{O}_X)} \mathscr{O}_{X'})) = \dim_k(H^n(X, \mathscr{L})).$$

In particular, we conclude that

$$\deg(g'^*(\mathscr{L}) \otimes_{g'^*(\mathscr{O}_X)} \mathscr{O}_{X'}) = \deg(\mathscr{L}),$$

showing that the degree of invertible $\mathscr{O}_X$-modules is preserved under base-change corresponding to an extension of the base-field.

7. The étale topology

We recall the notion of an étale morphism of schemes. Informally, a morphism of schemes is étale if it satisfies the hypothesis of the inverse function theorem, and the purpose of the étale topology is to make this theorem valid.

A morphism of schemes $j\colon (U, \mathscr{O}_U) \to (X, \mathscr{O}_X)$ is an *open immersion* if the continuous map $j\colon U \to X$ is a homeomorphism onto an open subset of X and if the ring homomorphism $j^{\#}\colon j^*(\mathscr{O}_X) \to \mathscr{O}_U$ is an isomorphism. A morphism of schemes $i\colon (Z, \mathscr{O}_Z) \to (X, \mathscr{O}_X)$ is a *closed immersion* if the map $i\colon Z \to X$ is a homeomorphism onto a closed subset of X and if the ring homomorphism $i^{\#}\colon \mathscr{O}_X \to i_*(\mathscr{O}_Z)$ is surjective and has kernel a quasi-coherent ideal of $\mathscr{O}_X$. The quasi-coherent ideal $\mathscr{I} \subset \mathscr{O}_X$ determines $i\colon (Z, \mathscr{O}_Z) \to (X, \mathscr{O}_X)$, up to isomorphism over $(X, \mathscr{O}_X)$, and is called the *ideal of definition* of $i\colon (Z, \mathscr{O}_Z) \to (X, \mathscr{O}_X)$. A closed immersion $i\colon (Z, \mathscr{O}_Z) \to (X, \mathscr{O}_X)$ is nilpotent is an *infinitesimal thickening* if its ideal of definition $\mathscr{I} \subset \mathscr{O}_X$ is nilpotent. In this case, the continuous map $i\colon Z \to X$ is a homeomorphism. We will now abbreviate and write X instead of $(X, \mathscr{O}_X)$. A morphism of schemes $f\colon Y \to X$ is defined to be *formally étale* if for every commutative diagram of schemes

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f} \\ \tilde{Z} & \xrightarrow{h} & X, \end{array}$$

in which i is an infinitesimal thickening, there exists a unique morphism $k\colon \tilde{Z} \to Y$ such that $g = k \circ i$ and $h = f \circ k$. A morphism of schemes $f\colon Y \to X$ is *locally presentable* if for every point $y \in Y$, there exists a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}(B) & \xrightarrow{j} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \operatorname{Spec}(A) & \xrightarrow{j'} & X \end{array}$$

such that j and j' are open immersions, such that y is contained in the image of j, and such that $f' = \operatorname{Spec}(g)$ with $g\colon A \to B$ a ring homomorphism that makes B a finitely presented A-algebra. A morphism of schemes $f\colon Y \to X$ is *étale* if it is formally étale and locally finitely presented.

One can show that $f\colon Y \to X$ is étale if and only if, for every $y \in Y$, one can find a diagram as above such that $g\colon A \to B$ induces an isomorphism

$$A[x_1, \dots, x_n]/(f_1, \dots, f_n) \xrightarrow{\tilde{g}} B$$

and such that the image of the Jacobian $(\partial f_i/\partial x_j)$ by $\tilde{g}$ is a unit in B. We note that étale morphisms need not be surjective. To wit, an open immersion is étale.

EXAMPLE 7.1. Let k be a field and let $(k_i)_{i \in I}$ be a family of finite separable field extension $g_i\colon k \to k_i$. In this situation, the morphism of schemes

$$\textstyle\coprod_{i \in I} \operatorname{Spec}(k_i) \xrightarrow{\sum_{in \in I} \operatorname{Spec}(g_i)} \operatorname{Spec}(k)$$

is étale, and conversely, every étale morphism $f\colon Y \to \mathrm{Spec}(k)$ is, up to isomorphism over $\mathrm{Spec}(k)$, of this form. The indexing set I can be arbitrarily large, since the requirement that f be locally finitely presented is local on Y. The domain of the étale map above is affine if and only if the canonical map

$$\coprod_{i\in I} \mathrm{Spec}(k_i) \longrightarrow \mathrm{Spec}(\prod_{i\in I} k_i)$$

is an isomorphism, which, in turn, happens if and only if I is finite. In general, the map of spaces underlying this map is a Stone-Čech compactification.

Finally, we recall that étale morphisms are étale and that if $f\colon Y \to X$ is an étale morphism, then a morphism $g\colon Z \to Y$ is étale if and only if the composite morphism $f \circ g\colon Z \to X$ is étale.

Remark 7.2. A morphism $f\colon X \to S$ is defined to be *separated* if the diagonal morphism $\Delta_{X/S}\colon X \to X \times_S X$ is a closed immersion. Any morphism between affine schemes is separated. For the standard example of a non-separated morphism, let k be a ring and let $\mathbb{A}^1_k = \mathrm{Spec}(k[x])$ be the affine line. The pushout

$$\begin{array}{ccc} \mathbb{A}^1_k \smallsetminus \{0\} & \xrightarrow{\ j\ } & \mathbb{A}^1_k \\ \big\downarrow{\scriptstyle j} & & \big\downarrow{\scriptstyle j_1} \\ \mathbb{A}^1_k & \xrightarrow{\ j_2\ } & X \end{array}$$

exists, since j is an open immersion. The scheme X is called the affine line with a double-point and the morphism $f = \mathrm{id}_{\mathbb{A}^1_k} + \mathrm{id}_{\mathbb{A}^1_k}\colon X \to \mathbb{A}^1_k$ is an example of a non-separated morphism. But this morphism is obviously étale, showing that étale morphisms need not be separated.

If U is a universe, then we say that a scheme $(X, \mathscr{O}_X)$ is a U-scheme if the topological space X is an element of U and if the sheaf $\mathscr{O}_X$ takes values in U-Set. We write U-Sch for the category of U-schemes and the morphisms between them. We recall from Problem Set 1 the topology J_K generated by a pretopology K.

Definition 7.3. If U is a universe, then the *étale topology* on U-Sch is the topology generated by the pretopology for which a family of morphisms

$$(\, X_i \xrightarrow{\ f_i\ } X\,)_{i\in I}$$

is a covering family if the set I is U-small; if f_i is an étale morphism, for every $i \in I$; and if the morphism $\sum_{i\in I} f_i\colon \coprod_{i\in I} X_i \to X$ is surjective.

We remark that the latter requirement is equivalent to the requirement that the requirement that the map of sets underlying the morphisms $f_i\colon X_i \to X$ cover the underlying set of X. We also recall that every base-change of a surjective morphism again is surjective.

Before we proceed, we need some general results about sites. Let U be a universe, let C be a locally U-small category, let let $\mathsf{C}^\wedge$ be the category of U-Set valued presheaves on C, and let $h\colon \mathsf{C} \to \mathsf{C}^\wedge$ be the Yoneda embedding. We say that a monomorphism $i\colon S \to T$ in $\mathsf{C}^\wedge$ is a *subfunctor* if for every object Y of C, the map $i_Y\colon S(Y) \to T(Y)$ is the canonical inclusion of a subset. We remark that, for every object T of $\mathsf{C}^\wedge$, the map that to a subfunctor $i\colon S \to T$ assigns the subobject containing it is a bijection. In addition, a sieve S on an object X of C determines and is determined by a subfunctor of $h(X)$ which, by abuse of notation, we write $i_S \colon S \to h(X)$ of simply $S \subset X$. Using this language, a topology on C is a functor J that to every object X of C assigns a set $J(X)$ of sieves on X such that the following axioms (T1)–(T3) hold:

(T1) If $f \colon Y \to X$ is a morphism of C and if $S \subset X$ is a sieve in $J(X)$, then the pullback sieve $S \times_X Y \subset Y$ is in $J(Y)$.
(T2) If X is an object of C, if $S \subset X$ is a sieve in $J(X)$, and if $T \subset X$ is a sieve such that for every morphism $f \colon Y \to X$ in C that factors through $S \subset X$, the pullback sieve $T \times_X Y \subset Y$ is in $J(Y)$, then $T \subset X$ is in $J(X)$.
(T3) If X is an object of C, then the maximal sieve $X \subset X$ is in $J(X)$.

Moreover, the presheaf F on C is a sheaf for the topology J precisely if for every object X of C and every sieve $S \subset X$ in $J(X)$, the map

$$\mathsf{C}^\wedge(X, F) \xrightarrow{i_S^*} \mathsf{C}^\wedge(S, F)$$

is a bijection. If both J and J' are topologies on C, and if $J(X) \supset J'(X)$, for every object X of C, then we say that J is *finer* than J'. Hence, in this situation, every sheaf for J is also a sheaf for J'. For example, the étale topology on U-Sch is finer than the *Zariski topology*, which we define to be the topology generated by the pretopology for which a family of morphisms $(f_i \colon X_i \to X)_{i \in I}$ is a covering family if I is a U-small set; if f_i is an open immersion, for every $i \in I$; and if the morphism $\sum_{i \in I} f_i \colon \coprod_{i \in I} X_i \to X$ is surjective.

Lemma 7.4. *Let U be a universe, let C be a locally U-small category, and let $(F_i)_{i \in I}$ be any family of U-Set valued presheaves on C. There exists a unique finest topology J on C such that, for every $i \in I$, the presheaf F_i is a sheaf for J.*

Proof. If X is an object of C, then we define $J(X)$ to be the set of all sieves $S \subset X$ such that, for every $i \in I$ and every morphism $f \colon Y \to X$ in C, the map

$$\mathsf{C}^\wedge(Y, F_i) \xrightarrow{(i_S \times \mathrm{id}_Y)^*} \mathsf{C}^\wedge(S \times_X Y, F_i)$$

is a bijection. It will suffice to show that J is a topology, since it then clearly is the finest topology with the stated property. Axioms (T1) and (T3) hold for trivial reasons, so it remains to show that also (T3) holds. So let X be an object of C, let $S \subset X$ be a sieve in $J(X)$, and let $T \subset X$ be a sieve such that for every morphism $f \colon Y \to X$ in C that factors through $S \subset X$, the pullback sieve $T \times_X Y \subset T$ is in $J(Y)$. We must show that $T \subset X$ is in $J(X)$. By the definition of $J(X)$, we must show that for every $i \in I$ and every morphism $f \colon Y \to X$ in C, the top horizontal map in the commutative diagram

$$\begin{array}{ccc} \mathsf{C}^\wedge(Y, F_i) & \xrightarrow{(i_T \times \mathrm{id}_Y)^*} & \mathsf{C}^\wedge(T \times_X Y, F_i) \\ \downarrow{\scriptstyle (i_S \times \mathrm{id}_Y)^*} & & \downarrow{\scriptstyle (i_S \times \mathrm{id}_T \times \mathrm{id}_Y)^*} \\ \mathsf{C}^\wedge(S \times_X Y, F_i) & \xrightarrow{(\mathrm{id}_S \times i_T \times \mathrm{id}_Y)^*} & \mathsf{C}^\wedge(S \times_X T \times_X Y, F_i) \end{array}$$

is a bijection. Now, the vertical maps are bijections, since $S \subset X$ is in $J(X)$, and the bottom horizontal map is a bijection, by the assumption on $T \subset X$. Hence, the top horizontal map is a bijection, as desired. □

In Lemma 7.4, the indexing set I is not required to be U-small. In particular, we can take I to be the set of objects in C.

DEFINITION 7.5. Let U be a universe, let C be a locally U-small category, and let $h\colon \mathsf{C} \to \mathsf{C}^\wedge$ be the Yoneda embedding. The *canonical topology* on C is the finest topology such that, for every $X \in \mathrm{ob}(\mathsf{C})$, the presheaf $h(X)$ is a sheaf.

We say that a topology J is *subcanonical* if it is coarser than the canonical topology, or equivalently, if every representable presheaf is a sheaf for J. We will show that, on the category U-Sch of U-schemes, the étale topology is subcanonical. To do so, we first introduce the *fidèlement plat et quasi-compact* topology.

DEFINITION 7.6. Let U be a universe. The *fpqc-topology* on U-Sch is the topology generated by the pretopology for which a family of morphisms

$$(X_i \xrightarrow{f_i} X)_{i \in I}$$

is a covering family if I is a U-small set; if f_i is flat, for every $i \in I$; if the morphism $\sum_{i \in I} f_i \colon \coprod_{i \in I} X_i \to X$ is surjective; and if for every affine open subscheme $V \subset X$, there exists a family of quasi-compact open subschemes $(V_i \subset X_i)_{i \in I_V}$ indexed by a finite subset $I_V \subset I$ such that $\sum_{i \in I_V} f_i|_{V_i} \colon \coprod_{i \in I_V} V_i \to V$ is surjective.

We remark that the first two requirement for a family $(f_i \colon X_i \to X)$ of morphisms in U-Sch to be a covering for the fpqc-topology are equivalent to the requirement that the morphism $\sum_{i \in I} f_i \colon \coprod_{i \in I} X_i \to X$ be flat and surjective, or equivalently, faithfully flat. Confusingly, however, the last requirement is *not* equivalent to this morphism being quasi-compact. Indeed, since I can be any U-small set, this morphism typically is not quasi-compact.

LEMMA 7.7. *The fpqc-topology on* U-Sch *is finer than the étale topology.*

PROOF. We must show that if $(f_i \colon X_i \to X)_{i \in I}$ is a covering family for the étale topology, then it is also a covering family for the fpqc-topology. We will use without proof that étale morphisms are flat and open. To verify the last requirement in Definition 7.6, we let $V \subset X$ be an affine open subscheme and write $f_i^{-1}(V) \subset X_i$ as the union of a family of open affine subschemes $(V_{i,j} \subset X_i)_{j \in J_i}$. Now, since the morphisms f_i are open, the family $(f_i(V_{i,j}) \subset X)_{(i,j) \in \coprod_{i \in I} J_i}$ is a Zariski covering of $V \subset X$, and since $V \subset X$ is affine and hence quasi-compact, there exists a finite subset $I_V' \subset \coprod_{i \in I} J_i$ such that the sub-family $(f_i(V_{i,j}) \subset X)_{(i,j) \in I_V'}$ is a Zariski covering. Therefore, if we let $I_V \subset I$ be the finite subset of all $i \in I$ for which there exists $j \in J_i$ with $(i,j) \in I_V'$; let $J_i' \subset J_i$ be the finite subset of all $j \in J_i$ such that $(i,j) \in I_V'$; and for $i \in I_V$, define $V_i = \bigcup_{j \in J_i'} V_{i,j}$, then $(V_i \subset X_i)_{i \in I_V}$ is a family of quasi-compact open subschemes indexed by a finite subset $I_V \subset I$ such that the morphism $\sum_{i \in I_V} f_i|_{V_i} \colon \coprod_{i \in I_V} V_i \to V$ is surjective. $\square$

REMARK 7.8. More generally, a flat morphism locally of finite presentation is open. Hence, the proof of Lemma 7.7 also shows that the fpqc-topology is finer than the fppf-topology, defined as the topology on U-Sch generated by the pretopology in which a family $(f_i \colon X_i \to X)_{i \in I}$ is a covering family if I is small; if f_i is flat and of finite presentation, for every $i \in I$; and if $\sum_{i \in I} f_i \colon \coprod_{i \in I} X_i \to X$ is surjective.

8. Faithfully flat descent

We will show that the fpqc-topology on U-Sch is subcanonical and begin by noting the following.

LEMMA 8.1. *Let U be a universe. A U-Set-valued on U-Sch is a sheaf for the fpqc-topology if and only if F is a sheaf for the Zariski topology and, for every faithfully flat morphism of affine U-schemes $f\colon W \to V$, the diagram*

$$F(V) \xrightarrow{F(f)} F(W) \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} F(W) \times_{F(V)} F(W)$$

is an equalizer.

PROOF. Exercise; or see Section 33.8 in the Stacks Project. □

LEMMA 8.2. *If U is a universe and if $f\colon A \to B$ is a faithfully flat morphism in the category U-Rng of commutative rings in U-Set, then the diagram*

$$A \xrightarrow{f} B \underset{\mathrm{in}_2}{\overset{\mathrm{in}_1}{\rightrightarrows}} B \otimes_A B$$

is an equalizer in U-Rng.

PROOF. Since $f\colon A \to B$ is faithfully flat, it suffices to show that the diagram

$$B \xrightarrow{e} B \otimes_A B \underset{d^1}{\overset{d^0}{\rightrightarrows}} B \otimes_A B \otimes_A B$$

in U-Rng obtained by applying the functor $B \otimes_A -$ to the diagram in the statement is an equalizer. Here $e(b_0) = b_0 \otimes 1$; $d^0(b_0 \otimes b_1) = b_0 \otimes 1 \otimes b_1$; and $d^1(b_1) = b_0 \otimes b_1 \otimes 1$. Now, the morphisms $s\colon B \otimes_A B \to B$ and $t\colon B \otimes_A B \otimes_A B \to B \otimes_A B$ defined by $s(b_0 \otimes b_1) = b_0 b_1$ and $t(b_0 \otimes b_1 \otimes b_2) = b_0 b_1 \otimes b_2$ satisfy that $d^0 e = d^1 e$; $se = \mathrm{id}_B$; $td^0 = \mathrm{id}_{B \otimes_A B}$; and $td^1 = es$. This shows that the diagram above is a split equalizer. In particular, it is an equalizer. □

THEOREM 8.3. *The fpqc-topology on U-Sch is subcanonical.*

PROOF. We must show that if the family of morphisms $(f_i\colon X_i \to X)_{i \in I}$ is a covering family for the fpqc-topology on U-Sch, then the diagram

$$\coprod_{(j,k) \in I \times I} X_j \times_X X_k \underset{b}{\overset{a}{\rightrightarrows}} \coprod_{i \in I} X_i \xrightarrow{f} X,$$

where $f \circ \mathrm{in}_i = f_i$, $a \circ \mathrm{in}_{(j,k)} = \mathrm{in}_j \circ \mathrm{pr}_1$, and $b \circ \mathrm{in}_{(j,k)} = \mathrm{in}_k \circ \mathrm{pr}_2$, is a coequalizer in U-Sch. By Lemma 8.1 it suffices the case of a Zariski covering and of a covering consisting of a single faithfully flat morphism $f\colon W \to V$ between affine U-schemes. The case of a Zariski covering follows rather immediately from the definition of a scheme; see EGA I.2.3.1 for details. It remains to prove that for $f\colon W \to V$ a faithfully flat morphism of affine schemes, the diagram

$$W \times_V W \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} W \xrightarrow{f} V$$

is a coequalizer in U-Sch. So we let $h\colon W \to X$ be a morphism of U-schemes such that $h \circ \mathrm{pr}_1 = h \circ \mathrm{pr}_2 \colon W \times_V W \to X$ and show that there exists a unique morphism $g\colon V \to W$ with $h = g \circ f$.

Suppose first that also $X = \mathrm{Spec}(A)$ is affine. We recall that the functor Spec from U-$\mathsf{Rng}^{\mathrm{op}}$ to U-Sch is right adjoint to the functor that to a scheme X assigns the ring $\Gamma(X, \mathscr{O}_X)$ of global sections of its structure sheaf and that the latter functor preserves finite limits. Therefore, it will suffice to show that the diagram

$$\Gamma(V, \mathscr{O}_V) \xrightarrow{f^\#} \Gamma(W, \mathscr{O}_W) \underset{\mathrm{in}_2}{\overset{\mathrm{in}_1}{\rightrightarrows}} \Gamma(W, \mathscr{O}_W) \times_{\Gamma(V,\mathscr{O}_V)} \Gamma(W, \mathscr{O}_W)$$

is an equalizer in U-Rng, which was proved in Lemma 8.2. Indeed, we obtain the desired morphism $g\colon V \to \mathrm{Spec}(A)$ as the adjunct of the unique ring homomorphism $g'\colon A \to \Gamma(V, \mathscr{O}_V)$ with $h' = f^\# \circ g'$.

We next let X be any U-scheme. We first show that a morphism $g\colon V \to X$ such that $h = g \circ f$ is necessarily unique. So let $g_1, g_2 \colon V \to X$ be two such morphisms. Since the underlying map of topological spaces associated with f is surjective, the morphisms g_1 and g_2 induce the same map of underlying topological spaces. Given $v \in V$, we first choose an affine open subset $X' \subset X$ containing $g_1(v) = g_2(v)$ and then choose $s \in \Gamma(V, \mathscr{O}_V)$ such that distinguished affine open $V_s \subset V$ is contained in both $g_1^{-1}(X')$ and $g_2^{-1}(X')$. Moreover, setting $t = f^\#(s) \in \Gamma(W, \mathscr{O}_W)$, the morphism $f|_{W_t} \colon W_t \to V_s$ again is faithfully flat. Therefore, by the case already proved, we conclude that the morphisms $g_1|_{V_s}$ and $g_2|_{V_s}$ are equal. This shows that there exists a Zariski covering $(f_i \colon V_i \to V)_{i \in I}$ such that the morphisms $g_1|_{V_i}$ and $g_2|_{V_i}$ are equal, for every $i \in I$, and since the diagram

$$\textstyle\coprod_{(j,k) \in I \times I} V_j \times_V V_k \underset{b}{\overset{a}{\rightrightarrows}} \coprod_{i \in I} V_i \xrightarrow{f} V$$

is a coequalizer in U-Sch, we conclude that $g_1 = g_2 \colon V \to X$.

It remains to construct a morphism $g\colon V \to X$ satisfying $h = g \circ f$. It follows from the construction of fiber-products in U-Sch that, at the level of underlying topological spaces, a continuous map with this property exists. Let us provisionally write $g\colon V \to X$ for this continuous map. Now, given $v \in V$, we first choose an affine open subset $X' \subset X$ with $f(v) \in X'$, and then choose global section $s \in \Gamma(V, \mathscr{O}_V)$ such that the distinguished affine open $V_s \subset V$ is contained in $g^{-1}(X')$. Setting $t = f^\#(s) \in \Gamma(W, \mathscr{O}_W)$, we have as before that $f|_{W_t} \colon W_t \to V_s$ is faithfully flat, and since $f|_{W_t} \circ \mathrm{pr}_1 = f|_{W_t} \circ \mathrm{pr}_2 \colon W_t \times_{V_s} W_t \to X'$, we conclude from the case already proved that there exists a morphism $g|_{V_s} \colon V_s \to X'$ with $h|_{W_t} = g|_{V_s} \circ f|_{W_t}$. This shows that there exists a Zariski covering $(f_i \colon V_i \to V)_{i \in I}$ together with morphisms $g_i \colon V_i \to X$ satisfying $g_i \circ f_i = h|_{W \times_V V_i}$. The uniqueness result proved earlier shows that for all $(j,k) \in I \times I$, we have $g_j \circ \mathrm{pr}_1 = g_k \circ \mathrm{pr}_2 \colon V_j \times_V V_k \to X$. Therefore, we obtain the desired map $g\colon V \to X$ with $h = g \circ f$ from the coequalizer diagram above. This completes the proof. □

Example 8.4. It follows from Theorem 8.3 that every representable presheaf on U-Sch is a sheaf for the fpqc-topology. In particular, every representable presheaf on U-Sch is a sheaf for the coarser étale topology and the even coarser Zariski topology; compare Lemma 7.7. For example, the presheaf F on U-Sch that takes a U-scheme X to the set $F(X)$ underlying the ring $\Gamma(X, \mathscr{O}_X)$ is a representable presheaf, and hence, is a sheaf for the fpqc-topology. A representation of F is given by the pair $(\mathbb{A}^1_{\mathbb{Z}}, a)$ of the affine line $\mathbb{A}^1_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[x])$ and the natural bijection

$$F(X) \xrightarrow{a_X} U\text{-}\mathsf{Sch}(X, \mathbb{A}^1_{\mathbb{Z}})$$

that to $s \in F(X)$ assigns the morphism $a_X(s)\colon X \to \mathbb{A}^1_{\mathbb{Z}}$ adjunct to the unique ring homomorphism $a'_X(s)\colon \mathbb{Z}[x] \to \Gamma(X, \mathscr{O}_X)$ with $a'_X(s)(x) = s$. (A different but perfectly valid choice would be to use $a'_X(s)(x) = -s$ to define a_X.) The natural additive group structure on $\Gamma(X, \mathscr{O}_X)$ defines an abelian group object structure $(+, 0, -)$ on the sheaf F. We write $\mathbb{G}_a$ for the abelian group object $(F, +, 0, -)$ in the category of U-Set valued fpqc-sheaves on U-Sch and call it the *additive group*. Moreover, under the representation $(\mathbb{A}^1_{\mathbb{Z}}, a)$ of F, the abelian group structure $(+, 0, -)$ determines and is determined by an abelian cogroup structure (Δ, ϵ, S) on the commutative ring $\mathbb{Z}[x]$. One finds that the abelian cogroup structure maps are ring homomorphisms given by $\Delta(x) = 1 \otimes x + x \otimes 1$; $\epsilon(x) = 0$; and $S(x) = -x$. We note that a commutative ring with an abelian cogroup structure is also said to be a commutative and cocommutative Hopf algebra over $\mathbb{Z}$.

9. The étale topos

Let U be universes and let U-Sch be the category of U-schemes. Let X be a U-scheme and let U-Sch/X be the slice category. A topology on U-Sch gives rise to an induced topology on U-Sch/X, characterized as being the finest topology among the topologies that render the forgetful functor

$$U\text{-}\mathsf{Sch}/X \xrightarrow{u} U\text{-}\mathsf{Sch}$$

continuous. By abuse of language, we refer to the topology on U-Sch/X induced by the étale topology on U-Sch as the étale topology on U-Sch/X, and similarly for the fpqc-topology, the fppf-topology, and the Zariski topology.

Definition 9.1. Let U be a universe and let X be a U-scheme. If U' is a universe such that $U \in U'$, then the U'-topos $(U\text{-}\mathsf{Sch}/X)^{\sim}_{\text{et}}$ of the U'-Set valued sheaves on U-Sch/X for étale topology is called a *big étale topos* on X.

We recall that the category $\mathsf{Mod}_{\mathbb{Z}_X}$ of abelian group objects in a big étale topos on X is abelian and has enough injectives; compare Proposition 6.6. Therefore, for every such abelian group object F, we may consider the cohomology groups

$$H^i_{\text{et}}(X,F) = \operatorname{Ext}^i_{\mathbb{Z}_X}(\mathbb{Z}_X, F).$$

It is not clear from the definition that these groups are independent, up to canonical isomorphism, of the choice of universe U' with $U \in U'$. We proceed to show that this is true, nevertheless.

In general, let U be a universe and let (C, J) and (C', J') be U-small sites. We write $i\colon (\mathsf{C},J)^{\sim} \to \mathsf{C}^{\wedge}$ and $i'\colon (\mathsf{C}',J')^{\sim} \to \mathsf{C}'^{\wedge}$ for the morphisms between the topoi of U-Set valued sheaves and presheaves given by the sheafification/inclusion adjunctions. A cocontinuous functor $u\colon \mathsf{C} \to \mathsf{C}'$ gives rise to a morphism of topoi

$$(\mathsf{C},J)^{\sim} \xrightarrow{g} (\mathsf{C}',J')^{\sim},$$

where g_* is the unique functor such that

$$\begin{array}{ccc} (\mathsf{C},J)^{\sim} & \xrightarrow{g_*} & (\mathsf{C}',J')^{\sim} \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i'_*} \\ \mathsf{C}^{\wedge} & \xrightarrow{(u^{\text{op}})_*} & \mathsf{C}'^{\wedge} \end{array}$$

commutes, and where $g^* = i^* \circ (u^{\text{op}})^* \circ i'_*$. If the functor u is also continuous, then the sheafification in the definition of g^* is unnecessary, and we may instead define g^* to be the unique functor such that

$$\begin{array}{ccc} (\mathsf{C},J)^{\sim} & \xleftarrow{g^*} & (\mathsf{C}',J')^{\sim} \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i'_*} \\ \mathsf{C}^{\wedge} & \xleftarrow{(u^{\text{op}})^*} & \mathsf{C}'^{\wedge} \end{array}$$

commutes. In this situation, the composite functor $g_! = i'^* \circ (u^{\text{op}})_! \circ i_*$ is a left adjoint of g^*. The adjunction $(g_!, g^*, \epsilon, \eta)$ need not be a morphism of topoi, since the functor $g_!$ may not preserve finite limits. However, if C has finite limits and if the functor $u\colon \mathsf{C} \to \mathsf{C}'$, in addition to being continuous and cocontinuous, is fully faithful and preserves finite limits, then $g_!$ preserves finite limits. Hence, in this situation, we have a morphism of topoi

$$(\mathsf{C}, J)^{\sim} \xleftarrow{\ f\ } (\mathsf{C}', J')^{\sim}$$

with $f^* = g_!$ and $f_* = g^*$. In addition, the composite $f \circ g$ is naturally isomorphic to the identity of $(\mathsf{C}, J)^{\sim}$, and the functor f^* is fully faithful. We refer to the Stacks Project, Lemma 7.20.8 for a proof.

We define $U\text{-}\mathsf{Et}/X$ to be the full subcategory of $U\text{-}\mathsf{Sch}/X$ whose object set consists of the pairs $(Y, f \colon Y \to X)$ such that f is an étale morphism. We give $U\text{-}\mathsf{Et}/X$ the topology induced by the étale topology on $U\text{-}\mathsf{Sch}/X$, which we will refer to as the étale topology on $U\text{-}\mathsf{Et}/X$. Since all morphism in $U\text{-}\mathsf{Et}/X$ are themselves étale, this topology may be defined directly as the topology generated by the pretopology in which a family of morphisms

$$\big((Y_i, Y_i \xrightarrow{f \circ g_i} X) \xrightarrow{g_i} (Y, Y \xrightarrow{\ f\ } X) \big)_{i \in I}$$

is a covering family if I is U-small and $\sum_{i \in I} g_i \colon \coprod_{i \in I} Y_i \to Y$ surjective. In this situation, the canonical inclusion functor

$$U\text{-}\mathsf{Et}/X \xrightarrow{\ u\ } U\text{-}\mathsf{Sch}/X$$

is fully faithful, continuous and cocontinuous, and preserves finite limits. (We note, in particular, that u takes the terminal object (X, id_X) in the domain category to the terminal object (X, id_X) in the target category.) Therefore, by the general theory discussed above, we obtain morphisms of topoi

$$(U\text{-}\mathsf{Sch}/X)^{\sim}_{\text{et}} \underset{g}{\overset{f}{\rightleftarrows}} (U\text{-}\mathsf{Et}/X)^{\sim}_{\text{et}}$$

between the respective categories of U'-Set valued sheaves for the étale topology such that the composite morphism $f \circ g$ is naturally isomorphic to the identity. Here U' is a universe with $U \in U'$.

PROPOSITION 9.2. *Let $U \in U'$ be universes, let X be a U-scheme, and let*

$$(U\text{-}\mathsf{Sch}/X)^{\sim}_{\text{et}} \underset{g}{\overset{f}{\rightleftarrows}} (U\text{-}\mathsf{Et}/X)^{\sim}_{\text{et}}$$

be the morphisms between the U'-topoi of U'-Set valued sheaves with respect to the étale topologies induced by the canonical inclusion $u \colon U\text{-}\mathsf{Et}/X \to U\text{-}\mathsf{Sch}/X$.

(i) *For every abelian group object F in $(U\text{-}\mathsf{Sch}/X)^{\sim}_{\text{et}}$, the canonical map*

$$H^s((U\text{-}\mathsf{Et}/X)^{\sim}_{\text{et}}, f_*(F)) \longrightarrow H^s((U\text{-}\mathsf{Sch}/X)^{\sim}_{\text{et}}, F)$$

is an isomorphism, for all non-negative integers s.

(ii) *For every abelian group object G in $(U\text{-}\mathsf{Et}/X)^{\sim}_{\text{et}}$, the canonical map*

$$H^s((U\text{-}\mathsf{Sch}/X)^{\sim}_{\text{et}}, f^*(G)) \longrightarrow H^s((U\text{-}\mathsf{Et}/X)^{\sim}_{\text{et}}, G)$$

is an isomorphism, for all non-negative integers s.

PROOF. We suppress the universe U in the notation. If e is a terminal object in $(\mathsf{Et}/X)^{\sim}_{\text{et}}$, then also $f^*(e)$ is a terminal object of $(\mathsf{Sch}/X)^{\sim}_{\text{et}}$, since f^* preserves finite limits. Now, to prove (i), we use the adjunction isomorphism

$$((\mathsf{Et}/X)^{\sim}_{\text{et}}(e,-)\circ f_*(-))(F) \xrightarrow{a} (\mathsf{Sch}/X)^{\sim}_{\text{et}}(f^*(e),-)(F)$$

and consider the Grothendieck spectral sequence of the left-hand composite functor. It takes the form

$$E_2^{s,t} = (R^s(\mathsf{Et}/X)^{\sim}_{\text{et}}(e,-)\circ R^t f_*(-))(F) \Rightarrow R^{s+t}(\mathsf{Sch}/X)^{\sim}_{\text{et}}(f^*(e),-)(F)$$

and it exists, since f_* has the left adjoint f^* which preserves finite limits, and hence, takes injectives to injectives. But $f_* = g^*$ also has the right adjoint functor g_*, and hence, preserves all limits and colimits. It follows that $E_2^{s,t}$ vanishes for $t > 0$, and therefore, the edge homomorphism

$$(R^s(\mathsf{Et}/X)^{\sim}_{\text{et}}(e,-)\circ f_*(-))(F) \longrightarrow R^s(\mathsf{Sch}/X)^{\sim}_{\text{et}}(f^*(e),-)(F)$$

is an isomorphism. This proves (i). To prove (ii), we use that f^* being fully faithful gives an isomorphism

$$(\mathsf{Et}/X)^{\sim}_{\text{et}}(e,-)(G) \xrightarrow{f^*} ((\mathsf{Sch}/X)^{\sim}_{\text{et}}(f^*(e),-)\circ f^*(-))(G)$$

and consider the Grothendieck spectral sequence of the composite functor on the right-hand side. It takes the form

$$E_2^{s,t} = (R^s(\mathsf{Sch}/X)^{\sim}_{\text{et}}(f^*(e),-)\circ R^t f^*(-))(G) \Rightarrow R^{s+t}(\mathsf{Et}/X)^{\sim}_{\text{et}}(e,-)(G),$$

provided that it exists. To show that it does, we must show that f^* takes injectives to $(\mathsf{Sch}/X)^{\sim}_{\text{et}}(f^*(e),-)$-acyclics. So let I be an injective object in the category of abelian group objects in $(\mathsf{Et}/X)^{\sim}_{\text{et}}$ and let $i\colon f^*(I) \to J^{\cdot}$ be an injective resolution in the category of abelian group objects in $(\mathsf{Sch}/X)^{\sim}_{\text{et}}$. Now, since f^* is fully faithfull, it gives the isomorphism

$$H^s(\mathsf{Sch}/X)^{\sim}_{\text{et}}(f^*(e),J^{\cdot}) \xleftarrow{f^*} H^s(\mathsf{Et}/X)^{\sim}_{\text{et}}(e,f_*(J^{\cdot})).$$

But f_* takes injectives to injectives and preserves both limits and colimits, and therefore $f_*(i)\colon f_*f^*(I) \to f_*(J^{\cdot})$ is an injective resolution in the category of abelian group objects in $(\mathsf{Et}/X)^{\sim}_{\text{et}}$. Moreover, since $f\circ g$ is naturally isomorphic to the identity and $f^* = g_*$, the unit $\eta\colon I \to f_*f^*(I)$ is an isomorphism. This shows that $f^*(I)$ is $(\mathsf{Sch}/X)^{\sim}_{\text{et}}$-acyclic as desired. Finally, since f^* preserves all colimits, we find that $E_2^{s,t}$ vanishes for $t > 0$. It follows again that the edge homomorphism

$$(R^s(\mathsf{Sch}/X)^{\sim}_{\text{et}}(f^*(e),-)\circ f^*(-))(G) \longrightarrow R^s(\mathsf{Et}/X)^{\sim}_{\text{et}}(e,-)(G)$$

is an isomorphims, which proves (ii). □

LEMMA 9.3. *Let U be a universe and let $u\colon \mathsf{C} \to \mathsf{C}'$ be a fully faithful functor from a U-small category to a locally U-small category. Let J' be a topology on C', let J be the induced topology on C, and let u_s be the unique functor that makes the following diagram of functors between the respective categories of U-Set valued* *presheaves and sheaves commute.*

$$\begin{array}{ccc} (\mathsf{C}', J')^{\sim} & \xrightarrow{u_s} & (\mathsf{C}, J)^{\sim} \\ \Big\downarrow{\scriptstyle i'} & & \Big\downarrow{\scriptstyle i} \\ \mathsf{C}'^{\wedge} & \xrightarrow{(u^{\mathrm{op}})^*} & \mathsf{C}^{\wedge} \end{array}$$

If every object X' of C' admits a covering family of the form

$$(\, u(X_i) \xrightarrow{f_i} X' \,)_{i \in I},$$

then the functor u_s is an equivalence of categories.

Proof. This "Lemme de comparaison" is proved in SGA 4, Théorème III.4.1. The requirement that C and C' be U-small and locally U-small, respectively, implies that the Kan extensions $(u^{\mathrm{op}})_!, (u^{\mathrm{op}})_* \colon \mathsf{C}^{\wedge} \to \mathsf{C}'^{\wedge}$ exists. The indexing set I is not required to be U-small? □

Proposition 9.4. *Let U be a universe and let X be a U-scheme. The category of U-Set valued sheaves on U-Et/X for the étale topology is a U-topos.*

Proof. The category $(U\text{-}\mathsf{Et}/X)^{\sim}$ is locally U-small, at least up to isomorphism of categories. We apply Lemma 9.3 to prove that it is equivalent to the category of U-Set valued sheaves on a U-small site (C, J).

We define the category C to be the full subcategory of U-Et/X whose objects are all pairs $(Y, f \colon Y \to X)$ for which the étale morphism f is the composition

$$\operatorname{Spec}(\Gamma(X', \mathscr{O}_X)[x_1, \dots, x_n]/(f_1, \dots, f_n)) \xrightarrow{g} \operatorname{Spec}(\Gamma(X', \mathscr{O}_X)) \xrightarrow{j} X$$

of a standard affine étale morphism and a standard open immersion onto an affine open subscheme X' of X. It is a U-small category, since the underlying topological space of X is U-small. We let $u \colon \mathsf{C} \to U\text{-}\mathsf{Et}/X$ be the canonical inclusion functor and let J be the topology on C induced from the étale topology on U-Et/X. By the definition of an étale morphism, every object $(Y, f \colon Y \to X)$ of U-Et/X admits a covering of the form

$$\big(u(Y_i, Y_i \xrightarrow{f_i} X) \xrightarrow{g_i} (Y, Y \xrightarrow{f} X)\big)_{i \in I}.$$

Therefore, the comparison lemma shows that the induced functor

$$(U\text{-}\mathsf{Et}/X)^{\sim}_{\mathrm{et}} \xrightarrow{u_s} (\mathsf{C}, J)^{\sim}$$

between the categories of U-Set valued sheaves is an equivalence of categories. □

Definition 9.5. Let U be a universe. The *small étale topos* of a U-scheme X is the U-topos X_{et} of U-Set valued sheaves on U-Et/X in the étale topology.

Corollary 9.6. *Let $U \in U'$ be universes, let X be a U-scheme, and let F be an abelian group object in the category of U-Set valued sheaves on U-Sch/X in the étale topology. In this situtation, the étale cohomology groups $H^i_{\mathrm{et}}(X, F)$, calculated in the larger category of abelian group objects in the category of U'-Set valued sheaves on U-Sch/X, are canonically isomorphic to U-small abelian groups and are independent, up to canonical isomorphism, of the choice of U'.*

Proof. ☐

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