THE MONTHLY PROBLEM 11164

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Solution to the problem 11164 from José Luis Díaz-Barrero in Amer. Math. Monthly 112,6 p. 568.

Proof of the formula

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\sum_{1\le i\le j\le k}\frac{1}{ij}=\frac{1}{n^2}$$

Introducing harmonic numbers as

$$H_n=\sum_{k=1}^{n}\frac{1}{k}$$

I shall prefer to write the formula as

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\sum_{j=1}^{k}\frac{1}{j}H_j=\frac{1}{n^2}$$

Now, summation by parts yields this sum to equal

$$1-\sum_{k=1}^{n}(-1)^{k+1}\binom{n-1}{k}\frac{1}{k+1}H_{k+1}=1-\frac{1}{n}\sum_{k=2}^{n+1}(-1)^{k}\binom{n}{k}H_k$$

As we may write the terms as a difference of the function

$$g(n,k)=(-1)^{k-1}\left(\binom{n-1}{k-1}H_k-\frac{1}{n}\binom{n-1}{k}\right)$$

that is

$$(-1)^k\binom{n}{k}H_k=g(n,k+1)-g(n,k)$$

the sum becomes telescoping to yield only the last term for $k=2$:

$$1+\frac{1}{n}g(n,2)=1-\frac{1}{n}\left(\binom{n-1}{1}H_2-\frac{1}{n}\binom{n-1}{2}\right)=\frac{1}{n^2}$$