THE MONTHLY PROBLEM 11459, OCTOBER 2009.

Mogens Esrom Larsen
November 18, 2009

Department for Mathematical Sciences
University of Copenhagen

Proposed by Pál Péter Dályay, Deák Ferenc High School, Szeged, Hungary.

Find all pairs (s, z) of complex numbers such that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\left(\prod_{j=1}^{k}(sj-z)\right)\left(\prod_{j=0}^{n-k-1}(sj+z)\right)$$

converges.

Solution: $|s| < 1$.

Proof: Chu–Vandermonde strikes again! Dividing each term in the products with $-s$ assuming $s \neq 0$ the terms of the infinite sum takethe form

$$(-s)^n\sum_{k=0}^{n}\binom{z/s-1}{k}\binom{-z/s}{n-k} = (-s)^n\binom{-1}{n} = s^n$$

If $s = 0$ the terms become for $n > 0$

$$\frac{z^n}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^k = (1-1)^n = 0$$