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Proposed by Pál Péter Dályay, Deák Ferenc High School, Szeged, Hungary.

Find all pairs (s, z) of complex numbers such that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\left(\prod_{j=1}^{k}(sj-z)\right)\left(\prod_{j=0}^{n-k-1}(sj+z)\right)$$

converges.

Solution: $|s|<1$.

Proof: Chu–Vandermonde strikes again! Dividing each term in the products with $-s$ assuming $s \neq 0$ the terms of the infinite sum takethe form

$$(-s)^n\sum_{k=0}^{n}\binom{z/s-1}{k}\binom{-z/s}{n-k}=(-s)^n\binom{-1}{n}=s^n$$

If $s=0$ the terms become for $n>0$

$$\frac{z^n}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^k=(1-1)^n=0$$