THE MONTHLY PROBLEM NO. 11212

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May 2, 2006

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In Monthly 113,3 page 268 David Beckwith, Sag Harbor, NY, asks for a proof ot the identity

$$S=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n-2k}{n-1}=0$$

For $n=0$ or $n>\lceil\frac{n}{2}\rceil$ we have $\binom{2n-2k}{n-1}=0$.
Let $\sim$ denote that a nonzero factor is omitted in the computation.
Define the descending factorial with stepsize as

$$[x,d]_n=x(x-d)\cdots(x-(n-1)d)$$

Then we procede for $n>0$

$$\begin{aligned}
S\sim&\sum_{k=0}^{\lceil\frac{n}{2}\rceil}(-1)^k\frac{[2n-2k,1]_{n-1}}{k!(n-k)!}=\\
&\sum_{k=0}^{\lceil\frac{n}{2}\rceil}(-1)^k\frac{[2n-2k,2]_{\lfloor\frac{n}{2}\rfloor}[2n-2k-1,2]_{\lceil\frac{n}{2}\rceil-1}}{k![n-k,1]_{\lfloor\frac{n}{2}\rfloor}\left(\lceil\frac{n}{2}\rceil-k\right)!}\sim\\
&\sum_{k=0}^{\lceil\frac{n}{2}\rceil}(-1)^k\frac{\left[n-k-\frac12,1\right]_{\lceil\frac{n}{2}\rceil-k}\left[\lfloor\frac{n}{2}\rfloor-\frac12,1\right]_{k-1}}{k!\left(\lceil\frac{n}{2}\rceil-k\right)!}\sim\\
&\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\frac{\left[-\lfloor\frac{n}{2}\rfloor-\frac12,1\right]_{\lceil\frac{n}{2}\rceil-k}}{\left(\lceil\frac{n}{2}\rceil-k\right)!}\cdot\frac{\left[\lfloor\frac{n}{2}\rfloor+\frac12,1\right]_k}{k!}=\\
&\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{-\lfloor\frac{n}{2}\rfloor-\frac12}{\lceil\frac{n}{2}\rceil-k}\binom{\lfloor\frac{n}{2}\rfloor+\frac12}{k}=\binom{0}{\lceil\frac{n}{2}\rceil}=0
\end{aligned}$$

by the Chu–Vandermonde convolution. During the computation we have omitted n, a power of 2, a power of -1 and $\lfloor\frac{n}{2}\rfloor+\frac12$.