COUPLED LINEAR DIFFERENTIAL EQUATIONS WITH REAL COEFFICIENTS

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Introduction. In a first course the case of two coupled linear differential equations tends to fall between two stools. The teacher's unrequited love for eigenvalues drives him into the complex domain, a maze in which he seldom finds the simple, real solutions of the original problem. And even if the complex numbers can be avoided he has difficulties returning through the coordinate transforms. It would seem that if the students had an adequate basis in algebra, everything would be easy. However, on the one hand, it is too much to include all that algebra. On the other hand, that particular subject is not something that can be used now and explained later.

Hence, it is tempting to look for a simple, direct solution, which works in the real domain and only requires straightforward ideas.

The Problem. We want to analyse an initial value problem: a couple of linear first-order differential equations with constant real coefficients in order to find the real solutions. The system is

$$\dot{x}_1 = ax_1 + bx_2 \tag{1}$$

$$\dot{x}_2 = cx_1 + dx_2 \tag{2}$$

where $a, b, c, d \in \mathbb{R}$, and x_1, x_2 are functions with initial values

$$x_1(0) = x_1^0 \tag{3}$$

$$x_2(0) = x_2^0 \tag{4}$$

with $x_1^0, x_2^0 \in \mathbb{R}$. We shall prefer to write it in matrix form. We define vectors

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \mathbf{x}^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}$$

and coefficient matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5}$$

Then we may replace (1), (2) by (6) and (3), (4) by (7):

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{6}$$

$$\mathbf{x}(0) = \mathbf{x}^0 \tag{7}$$

Motivation. In the traditional search for solutions we argue along the following lines: If $\mathbf{A}$ should happen to be a diagonal matrix, i.e., $b = c = 0$, then the system consists of two independent equations, namely

$$\dot{x}_1 = ax_1, \qquad \dot{x}_2 = dx_2,$$

with independent initial values

$$x_1(0) = x_1^0, \qquad x_2(0) = x_2^0.$$

If $\mathbf{A}$ is not of the wanted form, we look for a coordinate transformation

$$\mathbf{x} = \mathbf{S}\mathbf{y}$$

which changes the equation to

$$\dot{\mathbf{y}} = \mathbf{S}^{-1}A\mathbf{S}\mathbf{y}. \tag{8}$$

If the new matrix

$$\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$$

happens to be diagonal, then we are through. Unfortunately, we might need to extend the problem into the complex domain in order to obtain this diagonalization, and even so, as the matrix

$$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$$

shows, not all matrices can be diagonalized. In spite of the large amount of algebra employed we have hardly succeeded in finding the real solutions.

Alternative Analysis. The idea to be explained below is to argue slightly differently: If $\mathbf{A}^2$ should happen to be diagonal, then the system is easy to solve, even as an initial value problem. If $\mathbf{A}^2$ is not diagonal, then we are able to transform the problem, such that the new one has a coefficient matrix with diagonal square. As a matter of fact, neither of the above features needs complex numbers, and further, there are no exceptions to the procedure or even to the formulas for the solutions of the initial value problem.

The Solution with Trace Zero. If $\mathbf{A}$ is not already a diagonal matrix, then $\mathbf{A}^2$ is diagonal, if and only if the trace of $\mathbf{A}$ is zero. As we shall see, we can always transform the problem to the case where the trace of the new coefficient matrix is zero, even when $\mathbf{A}$ is diagonal. Hence we shall restrict our analysis to the case of trace zero.

Theorem 1. *If $\mathbf{A}$ has trace zero, then $-\mathbf{A}^2$ is the determinant of $\mathbf{A}$ times the unit matrix, i.e. $\mathbf{A}^2 = \Delta\mathbf{E}$ with $\Delta = a^2 + bc$.*

Theorem 2. *If* $\mathbf{A}^2$ *is diagonal, then either* $\mathbf{A}$ *is diagonal or the trace of* $\mathbf{A}$ *is zero.*

Proofs. Elementary. $\square$

Under this assumption we shall analyse the solution of (6) and (7). Let $\mathbf{x}$ be a solution of (6)-(7). Then by Theorem 1:

$$\ddot{\mathbf{x}} = \Delta \mathbf{E} \mathbf{x}. \tag{9}$$

Let δ be the solution of the initial value problem

$$\ddot{\delta} = \Delta \delta; \tag{10}$$

$$\delta(0) = 0, \qquad \dot{\delta}(0) = 1. \tag{11}$$

Note that $\dot{\delta}$ solves (10), but not (11). Then $\dot{\delta}$ is not proportional to δ and hence the couple $(\delta, \dot{\delta})$ constitutes a basis for the solutions of (10). Because $\mathbf{x}$ solves (9), it must take the form

$$\mathbf{x} = \dot{\delta}\mathbf{v} + \delta\mathbf{w} \tag{12}$$

where $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^2$. As $\mathbf{x}$ satisfies (6), we have

$$\dot{\mathbf{x}} = \ddot{\delta}\mathbf{v} + \dot{\delta}\mathbf{w} = \mathbf{A}\mathbf{x} = \dot{\delta}\mathbf{A}\mathbf{v} + \delta\mathbf{A}\mathbf{w}.$$

Using (10) we get the equation

$$\dot{\delta}\mathbf{w} + \delta\Delta\mathbf{v} = \dot{\delta}\mathbf{A}\mathbf{v} + \delta\mathbf{A}\mathbf{w}.$$

At $t = 0$ we have, because of (11),

$$\mathbf{w} = \mathbf{A}\mathbf{v}. \tag{13}$$

Substitution of (13) in (12) yields

$$\mathbf{x} = (\dot{\delta}\mathbf{E} + \delta\mathbf{A})\mathbf{v}.$$

As $\mathbf{x}$ satisfies (7), we have

$$\mathbf{x}^0 = \mathbf{x}(0) = (1\mathbf{E} + 0\mathbf{A})\mathbf{v} = \mathbf{v}. \tag{14}$$

Hence the solution of (6) and (7) is of the form

$$\mathbf{x} = (\dot{\delta}\mathbf{E} + \delta\mathbf{A})\mathbf{x}^0. \tag{15}$$

This ends the analysis.

Now we can substitute (15) in (6) and (7) for verification. In the latter case we get (14), and in the former using (10)

$$\dot{\mathbf{x}} = (\ddot{\delta}\mathbf{E} + \dot{\delta}\mathbf{A})\mathbf{x}^0 = (\delta\Delta\mathbf{E} + \dot{\delta}\mathbf{A})\mathbf{x}^0$$

while using Theorem 1 yields

$$\mathbf{A}\mathbf{x} = \mathbf{A}(\dot{\delta}\mathbf{E} + \delta\mathbf{A})\mathbf{x}^0 = (\dot{\delta}\mathbf{A} + \delta\mathbf{A}^2)\mathbf{x}^0 = (\dot{\delta}\mathbf{A} + \delta\Delta\mathbf{E})\mathbf{x}^0.$$

The General Case. Without any assumptions about $\mathbf{A}$ we shall transform the equations (6) and (7) to the case of zero trace. This proves much easier than the transformation (8). Let $\Theta \in \mathbb{R}$ be a constant, and θ the solution of the initial value problem

$$\dot{\theta} = \Theta\theta; \qquad \theta(0) = 1 \tag{16}$$

(i.e. the exponential function $\theta(t) = e^{\Theta t}$).

We consider the coordinate transformation

$$\mathbf{x} = \theta\boldsymbol{\xi}.$$

Now (6) and (7) for $\mathbf{x}$ imply certain equations for $\boldsymbol{\xi}$. (7) is simple:

$$\mathbf{x}^0 = \mathbf{x}(0) = \theta(0)\boldsymbol{\xi}(0) = \boldsymbol{\xi}(0)$$

by (16). (6) is nicer:

$$\dot{\mathbf{x}} = \theta\dot{\boldsymbol{\xi}} + \dot{\theta}\boldsymbol{\xi} = \mathbf{A}\mathbf{x} = \theta\mathbf{A}\boldsymbol{\xi}.$$

Using (16) we get

$$\theta\dot{\boldsymbol{\xi}} + \Theta\theta\boldsymbol{\xi} = \theta\mathbf{A}\boldsymbol{\xi}.$$

Because $\theta \neq 0$, we can divide by it, hence

$$\dot{\boldsymbol{\xi}} = (\mathbf{A} - \Theta\mathbf{E})\boldsymbol{\xi}.$$

Now, if we choose Θ correctly, the new matrix will have trace zero. We define Θ as

$$\Theta = \frac{a+d}{2},$$

half of the trace of $\mathbf{A}$. Then the system gets the matrix

$$\mathbf{B} = (\mathbf{A} - \Theta\mathbf{E}) = \begin{pmatrix} \frac{a-d}{2} & b \\ c & \frac{d-a}{2} \end{pmatrix}$$

so $\boldsymbol{\xi}$ solves the initial value problem:

$$\dot{\boldsymbol{\xi}} = \mathbf{B}\boldsymbol{\xi}; \qquad \boldsymbol{\xi}(0) = \mathbf{x}^0 \tag{17}$$

of the type of trace zero. (17) is then solved by (15), where δ solves (10), (11) with

$$\Delta = \left(\frac{a-d}{2}\right)^2 + bc.$$

Conclusion. We can write down the solution of (6) and (7) explicitly. Let the half-trace Θ and the discriminant Δ of the matrix $\mathbf{A}$ be defined as

$$\Theta = \frac{a+d}{2}, \tag{18}$$

$$\Delta = \left(\frac{a-d}{2}\right)^2 + bc. \tag{19}$$

Let θ be the solution of the initial value problem

$$\dot{\theta} = \Theta\theta; \qquad \theta(0) = 1.$$

Let δ be the solution of the initial value problem

$$\ddot{\delta} = \Delta\delta; \quad \delta(0) = 0; \quad \dot{\delta}(0) = 1.$$

Then the solution of (6) and (7) is:

$$\mathbf{x} = \theta(\dot{\delta}\mathbf{E} + \delta(\mathbf{A} - \Theta\mathbf{E}))\mathbf{x}^0. \tag{20}$$

In coordinates this becomes

$$x_1 = \theta\left(x_1^0\dot{\delta} + \left(\frac{a-d}{2}x_1^0 + bx_2^0\right)\delta\right),$$
$$x_2 = \theta\left(x_2^0\dot{\delta} + \left(\frac{d-a}{2}x_2^0 + cx_1^0\right)\delta\right).$$

The functions θ and δ can be explicitly written down. They are

$$\theta(t) = e^{\Theta t} = e^{\frac{a+d}{2}t};$$
$$\delta(t) = \begin{cases} \frac{1}{\sqrt{\Delta}}\sinh(\sqrt{\Delta}t) & \text{for } \Delta > 0, \\ t & \text{for } \Delta = 0, \\ \frac{1}{\sqrt{-\Delta}}\sin(\sqrt{-\Delta}t) & \text{for } \Delta < 0. \end{cases}$$

Afterthought. From a higher point of view, the methods applied here are examples of more sophisticated analytic methods in algebraic disguise, to be compared with the standard sophisticated algebra. If Sophus Lie could have asked Jean B. J. Fourier to solve the equations, he would have done so as follows:

The system (1)-(2) should be transformed into one equation of second order, i.e.,

$$\ddot{x} - (a+d)\dot{x} + (ad - bc) = 0.$$

Fourier, of course, would have transformed the operator to a polynomial,

$$\xi^2 - (a+d)\xi + (ad - bc);$$

then he would have translated this by the distance Θ (from (18)) , say

$$\eta = \xi - \Theta, \tag{21}$$

and hence obtained

$$\eta^2 - \Theta^2 + (ad - bc) = \eta^2 - \Delta$$

with Δ defined by (19).

By the inverse Fourier transformation, η is transformed into y, satisfying

$$\ddot{y} = \Delta y$$

and related to x by the transform of (21), i.e.,

$$y = x \cdot e^{-\Theta t}.$$

Further, he would have formulated the results of his efforts in the form of (20). For then Sophus Lie could have extracted the matrix

$$\mathbf{C}(t) = \theta(t)(\dot{\delta}(t)\mathbf{E} + \delta(t)(\mathbf{A} - \Theta\mathbf{E})),$$

which is a handy representation of the *Lie group* of the flow of solutions of (1)-(4).

Hence $\mathbf{C}(t)$ must satisfy the relation

$$\mathbf{C}(t+s) = \mathbf{C}(t)\mathbf{C}(s).$$

We know this relation from the theory of Lie groups, but I shall leave the verification by the elementary trigonometric formulas for addition and their analogues as an exercise.

References

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