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Let $n!!$ denote the product of all positive integers not greater than n and congruent to n mod 2, and let $0!! = (-1)!! = 1$. Thus 7!!=105 and 8!!=384. For positive integer, n, find in closed form:

$$\sum_{i=0}^{n} \binom{n}{i} (2i-1)!!(2(n-i)-1)!!$$

Solution: $2^n n!$.

Proof:

We supply the producs of odd numbers with the missing even numbers to write the sum

$$\frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} \frac{(2i)!}{i!} \frac{(2(n-i)!}{(n-i)!}$$

Introducing the notation of a descending factorial:

$$[x]_n := x(x-1)\cdots(x-n+1)$$

(Of course, $[x]_0 = 0$) we may scip the common factors from the fractions and write

$$\frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} [2i]_i [2(n-i)]_{n-i}$$

Now we apply the well known identity

$$[2i]_i = (-4)^i [-\tfrac{1}{2}]_i$$

to rewrite the sum as

$$(-2)^n \sum_{i=0}^{n} \binom{n}{i} [-\tfrac{1}{2}]_i [-\tfrac{1}{2}]_{n-i}$$

This sum may be recognized as the well known Chu–Vandermonde sum, so it equals

$$(-2)^n [-\tfrac{1}{2} - \tfrac{1}{2}]_n = 2^n n!$$

In my recent textbook, *Summa Summarum,* A K Peters 2007, we find the Chu–Vandermonde formula as no. 8.2 and the transformation as formula 5.12.

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