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In Monthly 114,2 page 165 Donald Knuth, Stanford, asks for a proof ot the identity

$$S=\sum_{k=0}^{m}2^k\binom{2m-k}{m+n}=4^m-\sum_{j=1}^{n}\binom{2m+1}{m+j}$$

Partial summation of the left side summed backwards and omitting zeros yield

$$\sum_{k=n}^{m}2^{m-k}\binom{m+k}{m+n}=-\binom{2m+1}{m+n}+2^{m-n+1}+\sum_{k=n}^{m}2^{m-k}\binom{m+k}{m+n-1}=$$
$$=\sum_{k=n-1}^{m}2^{m-k}\binom{m+k}{m+n-1}-\binom{2m+1}{m+n}$$

As the left side for $n=m$ gives 1, we get

$$\sum_{k=n}^{m}2^{m-k}\binom{m+k}{m+n}=1+\sum_{j=n+1}^{m}\binom{2m+1}{m+j}=\sum_{j=n+1}^{m+1}\binom{2m+1}{m+j}=$$
$$=\sum_{j=1}^{m+1}\binom{2m+1}{m+j}-\sum_{j=1}^{n}\binom{2m+1}{m+j}=\frac{1}{2}2^{2m+1}-\sum_{j=1}^{n}\binom{2m+1}{m+j}$$

by the (half of) the binomial formula.