THE ETERNAL TRIANGLE – A HISTORY OF A COUNTING PROBLEM

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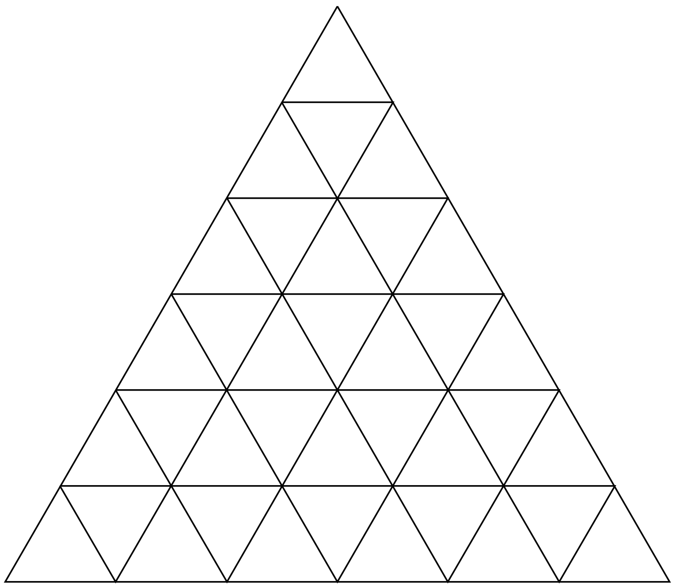
At least six times during the last 30 years the following problem was raised independently:

How many triangles can be counted in Figure 1, and what is the general formula for their number?



Figur 1

At first we think of counting all triangles, right–way–up, $\triangle$, upside–down, ∇, and of any size including the very frame of the pattern. But it is natural to ask for serarate countings of the right–way–up triangles of sides of length 1,2,...,n and of the upside–down triangles of sides of length 1,2,...,$[\frac{n}{2}]$, where n is the number of subintervals on the side of the big triangle and $[x]$ means the biggest integer less than or equal to x. Note that the side of the largest upside–down triangle cannot exceed $\frac{n}{2}$.

As an introduction to a course in combinatorics or discrete mathematics, this problem offers an excellent challenge to the students. None of the ways of counting is completely trivial, and several ideas lead to a solution, as we shall see. Actually, the answer has been published at least ten times with a variety of seven or eight proofs, so different from each other that they literally give a survey of combinatorics!

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The Formula.

The solution has been given in several forms of which the simplest [12] is

$$\left[\frac{n(n+2)(2n+1)}{8}\right] \tag{1}$$

where n is the number of subintervals of the side of the big triangle T_n, e.g. 6 in Figure 1. As a matter of fact, for n even the numerator is divisible by 8, but for n odd we get a remainder of 1. Using the formula we can check our counting of the triangles in Figure 1. The correct number is $\frac{6\cdot 8\cdot 13}{8} = 78$.

The Story.

The first appearence of the problem was in a note by J. Halsall in 1962 [7]. The formula was suggested without a rigorous proof. This paper went almost unnoticed except by N. J. A. Sloane [14], probably because of its misleading title. The problem made a second appearance in a note by J. E. Brider [1] in 1966. He suggests the formula based on the difference tables, but gives no proof. The third appearence was a note by Hamberg and Green [8] in 1967, which was appreciated by many readers. F. Gerrish [6] posed and solved the problem in 1970 in a rather complicated way, which gave rise to a couple of simplifications by Mastrantione [11], Martin [10] and by C. Wells [15] in 1971. A further improvement came from Moon and Pullman [12] in 1973. The problem was posed independently in 1974 by Edwards in Mathematics Magazine [4]. This time it immediately gave rise to three different solutions. In [13] Prielipp and Kuenzi gave the formula together with the references [6], [10], [11], and [12] while many other respondents gave the reference [8] and the formula. In [2] Carlitz and Scoville gave a new and elegant proof. Later, in 1976 Cormier and Eggleton [3] gave a different answer to [4]. Finally, in 1986, Garstang [5] gave a proof without any reference except to his wife. Eventually, in 1989 all the proofs gathered in this note [9]. In all the problem was posed at least six times: in 1962 in [7], in 1966 in [1], in 1967 in [8], in 1970 in [6], in 1974 in [4] and in 1986 in [5].

The Proofs.

Together the proofs cover most of the methods found in a textbook on combinatorics. They use the ideas of computing differences, branching and counting, the principle of correspondence, the solution of a difference equation using generating functions, induction or recursion, and finally, the adequate reference. I hope to present the proofs in enough detail for the reader to grasp them while reading. But even with this many solutions at hand the teacher presenting the problem in class may very well encounter a student with a completely new line of attack.

The Difference Method.

The difference method is found in [1], [7], [8], [10], and [13]. We count the triangles in T_n of sizes 1,2,3,... carefully. Then we find the differences Δ^1, their differences Δ^2 and so on hoping to reach a row of zeros. We obtain the following table:

n	1		2		3		4		5		6		7	
#	1		5		13		27		48		78		118	
Δ^1		4		8		14		21		30		40		52
Δ^2			4		6		7		9		10		12	
Δ^3				2		1		2		1		2		1

It didn't quite work. But if we consider the sequences of odd sizes alone and of even sizes alone, each of them will exhibit the desired behavior.

The triangles of odd size give:

m	0		1		2		3		4		5
$2m+1$	1		3		5		7		9		11
#	1		13		48		118		235		411
Δ^1		12		35		70		117		176	
Δ^2			23		35		47		59		
Δ^3				12		12		12			

And the triangles of even size give:

m	1		2		3		4		5		6
$2m$	2		4		6		8		10		12
#	5		27		78		170		315		525
Δ^1		22		51		92		145		210	
Δ^2			29		41		53		65		
Δ^3				12		12		12			

Each of these patterns allows us to guess a formula. Of course, we have no guarantee that the third differences Δ^3 will remain constant forever, so we have really *proved* nothing.

When the third differences are constant, then the second differences grow linearly, and the first differences grow quadratically, while eventually the very function might grow cubically. So, $f(2m+1)$ in particular must take the form

$$f(2m+1) = am^3 + bm^2 + cm + d \tag{2}$$

and the values 1,13,48,118 for $m = 0, 1, 2, 3$.

Thus we are led to solve the equations

$$\begin{aligned} d &= 1 \\ a+b+c+d &= 13 \\ 8a+4b+2c+d &= 48 \\ 27a+9b+3c+d &= 118. \end{aligned}$$

It is routine to see that the solution is $(a, b, c, d) = (2, \frac{11}{2}, \frac{9}{2}, 1)$. A reasonable suggestion for a formula for n odd, therefore, is:

$$f(2m+1) = \frac{1}{2}(4m^3 + 11m^2 + 9m + 2). \tag{3}$$

This is the form in [1] and [7].

Another way of finding the polynomial (2) or its analogue is by an interpolation formula. Using Lagrange's formula we get:

$$\begin{aligned}
f(2m) &= 5\frac{(m-2)(m-3)(m-4)}{(1-2)(1-3)(1-4)} \\
&+ 27\frac{(m-1)(m-3)(m-4)}{(2-1)(2-3)(2-4)} \\
&+ 78\frac{(m-1)(m-2)(m-4)}{(3-1)(3-2)(3-4)} \\
&+ 170\frac{(m-1)(m-2)(m-3)}{(4-1)(4-2)(4-3)} \\
&= \frac{1}{2}(4m^3+5m^2+m). \qquad (4)
\end{aligned}$$

This also is the form found in [1] and [7].

If we want to unify formulas (3) and (4), we might substitute $n = 2m+1$ in (3) and $n = 2m$ in (4). Then we get the forms [1] and [7]:

$$f(n) = \frac{2n^3+5n^2+2n-1}{8} \quad \text{for } n \text{ odd}, \qquad (5)$$

$$f(n) = \frac{2n^3+5n^2+2n}{8} \quad \text{for } n \text{ even}. \qquad (6)$$

Only the numerator of (6) has a first–degree factorization.

But if we define $\delta(n)$ by

$$\delta(n) = \begin{cases} 0 & \text{for } n \text{ even} \\ 1 & \text{for } n \text{ odd} \end{cases} \qquad (7)$$

then we can join (5) and (6):

$$f(n) = \frac{n(n+2)(2n+1)-\delta(n)}{8}, \qquad (8)$$

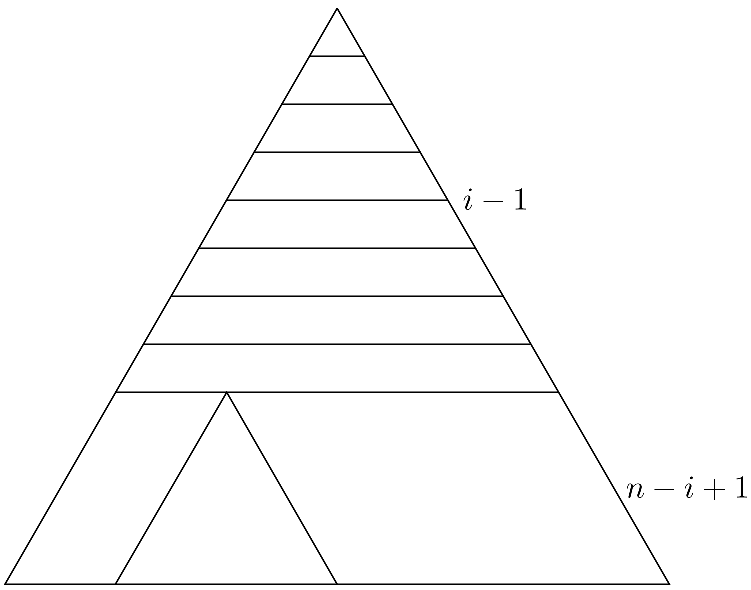
which of course is equivalent to (1).

Branching and Counting.

The idea of branching into right-way-up triangles and upside-down triangles and counting them separately has been widely used. Let us denote these two numbers by $\triangle\ (n)$ and $\nabla(n)$ respectively. Then we have

$$f(n) = \triangle\ (n) + \nabla(n). \qquad (9)$$

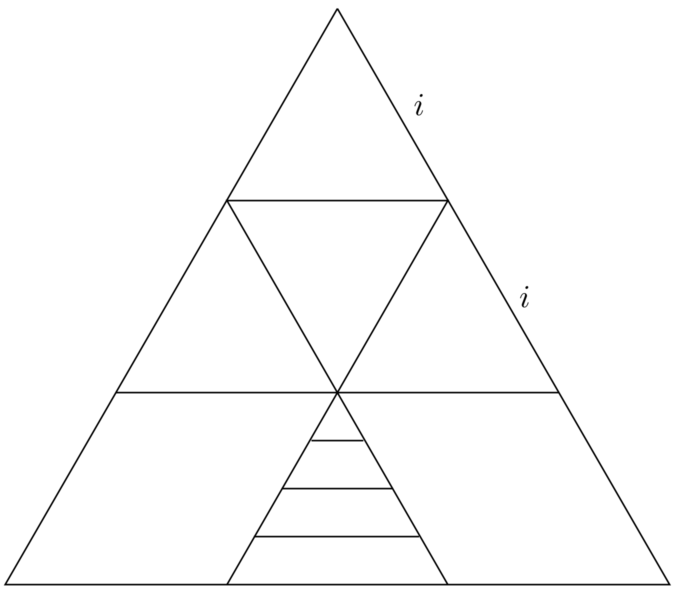
An even further branching is seen in [5], [6], [8] and [13]. While [6] and [8] branch so much that the final summation becomes complicated, [5] and [13] are satisfied with the natural branching by size.



Figur 2

The top of a $\triangle$-triangle of size $n-i+1$ must lie in the shaded region of Figure 2. Hence the number of $\triangle$-triangles of size $n-i+1$ equals the number of possible tops, i.e. $1+2+...+i=\frac{i(i+1)}{2}$. So,

$$\begin{aligned}\triangle\ (n) &= \sum_{i=1}^{n} \frac{i(i+1)}{2} = \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right) \\ &= \frac{n(n+1)(n+2)}{6} = \binom{n+2}{3}. \end{aligned} \tag{10}$$



Figur 3

Similarly the bottom vertex of a ∇-triangle of size i must lie in the shaded region of Figure 3. Hence the number of ∇-triangles of size i equals the number of possible bottoms, i.e. $1+2+...+(n+1-2i)=\frac{(n+1-2i)(n+2-2i)}{2}$. Hence the total number of ∇-triangles must be

$$\begin{aligned}\nabla(n) &= \sum_{i=1}^{\left[\frac{n}{2}\right]} \frac{(n+1-2i)(n+2-2i)}{2} \\ &= \frac{(n+1)(n+2)}{2}\left[\frac{n}{2}\right] - (n+1+n+2)\sum_{i=1}^{\left[\frac{n}{2}\right]} i + 2\sum_{i=1}^{\left[\frac{n}{2}\right]} i^2 \\ &= \frac{(n+1)(n+2)}{2}\left[\frac{n}{2}\right] - \frac{2n+3}{2}\left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right]+1\right) \\ &\quad + \frac{1}{3}\left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right]+1\right)\left(2\left[\frac{n}{2}\right]+1\right).\end{aligned}$$

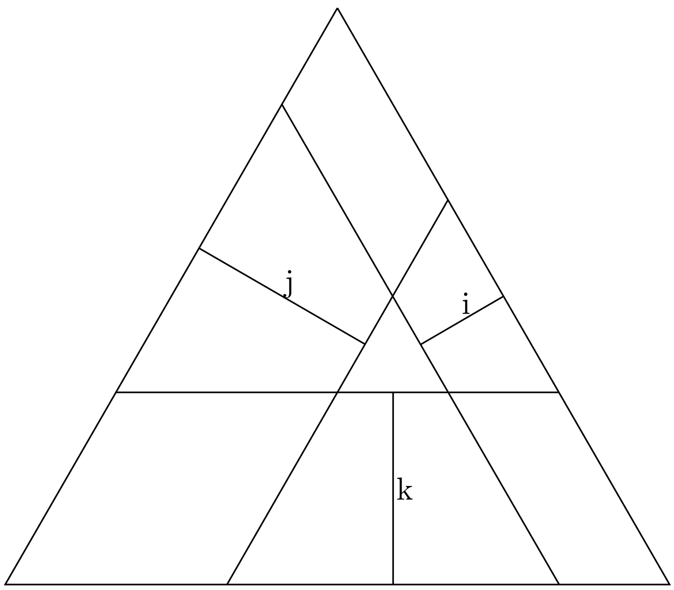
This expression can be simplified by the use of $\delta(n)$ defined by (7), using $\left[\frac{n}{2}\right]=\frac{n-\delta(n)}{2}$. Hence we can continue:

$$\begin{aligned}\nabla(n) &= \frac{n-\delta(n)}{2}\cdot \\ &\quad \left(\frac{n^2+3n+2}{2}+\frac{n+2-\delta(n)}{2}\left(\frac{n+1-\delta(n)}{3}-\frac{2n+3}{2}\right)\right) \\ &= \frac{2n^3+3n^2-2n-3\delta(n)}{24} \\ &= \frac{n(n+2)(2n-1)}{24}-\frac{\delta(n)}{8}, \qquad (11)\end{aligned}$$

using the simple fact $\delta(n)^2=\delta(n)$. The formula (8) follows by adding (10) and (11), according to (9).

Correspondence.

This principle is used in [3] in the following way. Each triangle is determined by a triple of integers i,j,k with $0\le i,j,k\le n$ where i,j,k are the "heights" of its three sides (number of rows from the sides of the large triangle, T_n). See Figure 4.



Figur 4

Each triple defines a triangle, except if it defines a point. The latter is the case if $i+j+k=n$. But the triangle will be inside T_n only if

$$i+j \le n, \tag{12}$$

$$i+k \le n, \tag{13}$$

$$j+k \le n; \tag{14}$$

hence the set of triangles corresponds to the set of triples

$$A = \{(i,j,k) \mid i+j \le n \,\text{and}\, j+k \le n \,\text{and}\, k+i \le n \,\text{and}\, i+j+k \ne n\}$$

A triple from A corresponds to a triangle that is right way up exactly if $i+j+k<n$. So in order to find $\triangle\,(n)$ we have to count

$$B = \{(i,j,k) \in A \mid i+j+k < n\}\,,$$

and to find $\nabla(n)$ we have to count

$$C = \{(i,j,k) \in A \mid i+j+k > n\}\,. \tag{15}$$

Before counting B we note that (12), (13) and (14) are superfluous - they follow from $i+j+k<n$. The trick now is to define the mapping on B

$$(i,j,k) \longrightarrow (i+1, i+j+2, i+j+k+3).$$

This mapping is a one-to-one correspondence from B onto the set

$$D = \{(a,b,c) \mid 1 \le a < b < c \le n+2\}\,.$$

But each triple of D corresponds to a subset of size 3 taken from the numbers $1, ..., n+2$, and thus D has the familiar binomial coefficient (10) as its cardinality

$$\triangle\,(n) = \binom{n+2}{3}.$$

To count C is not so easy. We define a mapping on C

$$(i, j, k) \longrightarrow (i+j+k-n, i+1, i+j+2). \tag{16}$$

This triple satisfies the following conditions

$$0 < i+j+k-n$$

by the defining inequality of C, (15),

$$i+j+k-n < i+1$$

by (14),

$$i+1 < i+j+2$$

because $0 \leq j$,

$$i+j+2 \leq n+2$$

by (12). Finally from (13) we get the additional condition

$$(i+j+k-n) + (i+1) < (i+j+2),$$

which actually is equivalent to (13). Hence the mapping (16) is a one-to-one correspondence from C onto the set

$$E = \{(a, b, c) \mid 1 \leq a < b < c \leq n+2 \text{ and } a+b < c\}$$

To count E we proceed as follows. For each choice of $a+b=d$, $d = 3, 4, ..., n+1$, we can choose a, $1 \leq a \leq \left[\frac{d-1}{2}\right]$, and c, $d < c \leq n+2$, while b must be $b = d-a$. Hence the number of triples in E is

$$\nabla(n) = \sum_{d=3}^{n+1} (n+2-d)\left[\frac{d-1}{2}\right].$$

One way to proceed is to consider the difference,

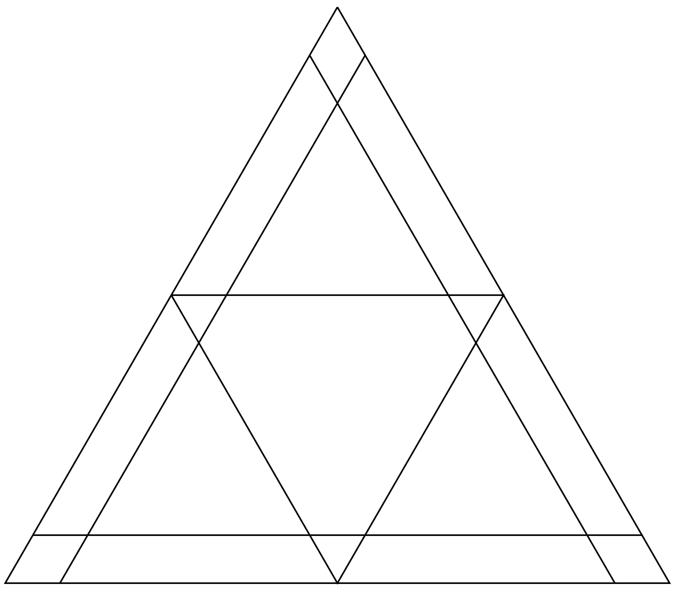
$$\begin{aligned}
\nabla(n) - \nabla(n-1) &= \sum_{d=3}^{n+1} (n+2-d)\left[\frac{d-1}{2}\right] - \sum_{d=3}^{n} (n+1-d)\left[\frac{d-1}{2}\right] \\
&= \left[\frac{n}{2}\right] + \sum_{d=3}^{n}\left[\frac{d-1}{2}\right] = \left[\frac{n}{2}\right] + \sum_{d=3}^{n} \frac{d-1-\delta(d-1)}{2} \\
&= \left[\frac{n}{2}\right] + \frac{1}{2}\left(\frac{n(n-1)}{2} - 1\right) - \frac{1}{2}\left[\frac{n-2}{2}\right] \\
&= \frac{n^2}{4} - \frac{\delta(n)}{4} = \left[\frac{n^2}{4}\right].
\end{aligned}$$

Then we obtain

$$\begin{aligned}\nabla(n) &= \sum_{\nu=1}^{n}\left[\frac{\nu^2}{4}\right] = \sum_{\nu=1}^{n}\frac{\nu^2}{4} - \sum_{\nu=1}^{n}\frac{\delta(\nu)}{4} \\ &= \frac{1}{4}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{1}{4}\left[\frac{n+1}{2}\right] \\ &= \frac{n(n+1)(2n+1)}{24} - \frac{1}{4}\cdot\frac{n+1-\delta(n+1)}{2} \\ &= \frac{n(n+2)(2n-1)}{24} - \frac{\delta(n)}{8},\end{aligned} \tag{17}$$

which is (11).

The Difference Equation.



Figur 5

The triangle T_n of size n contains exactly three triangles T_{n-1}^i $i = 1, 2, 3$ of size $n-1$. The intersection of two triangles, $T_{n-1}^i \cap T_{n-1}^j$ of size $n-1$ is a triangle, T_{n-2}^{ij} of size $n-2$, so the three pairs form three such triangles. Fortunately, the intersection of all three - or all six - is one triangle of size $n-3$, T_{n-3}. Their relations are illustrated in the following Diagram 1:

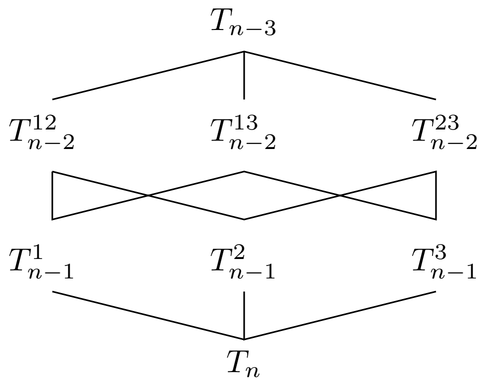


Diagram 1

First we will count the number of triangles inside T_n that are also inside at least one of the triangles T^i_{n-1}. Recall that $f(n-1)$ is the number of triangles in each of the three triangles of size $n-1$. However $3f(n-1)$ counts each triangle in the three triangles T^{ij}_{n-2} twice, and each triangle in T_{n-3} three times. Hence we must subtract the number of triangles $f(n-2)$ in T^{ij}_{n-2} three times. But now the number of triangles in T_{n-3}, $f(n-3)$, has been added three times and subtracted three times. Therefore we must add in the number of triangles in T_{n-3}, $f(n-3)$.

So, the number of triangles in T_n inherited from the smaller triangles T_{n-1} must be

$$3f(n-1) - 3f(n-2) + f(n-3)$$

To obtain the total number of triangles in T_n we must add one or two of its own: The very triangle T_n is always there, and for n even, the upside-down triangle of size $\frac{n}{2}$ (the dottet triangle in Figure 5) is not contained in any smaller ones. Hence the difference equation for $f(n)$ is

$$f(n) = 3f(n-1) - 3f(n-2) + f(n-3) + 2 - \delta(n). \tag{18}$$

This is the formula found in [11].

This equation can be solved in several ways. As we shall see in a later section, Moon and Pullman in [11] use the elegant method of an exponential generating function, but one could as well use an ordinary generating function as in the next section. Here, we will proceed in analogy to the solution of higher order differential equations.

We first reformulate the equation as

$$f(n) - 3f(n-1) + 3f(n-2) - f(n-3) = 2 - \delta(n). \tag{19}$$

To solve this difference equation of higher degree we decompose it into a system of difference equations of degree 1. This is always possible. The system of difference equations is obtained by the definitions:

$$g(n) = f(n) - f(n-1) \text{ and } h(n) = g(n) - g(n-1).$$

In fact, then we find by direct calculation

$$h(n) - h(n-1) = f(n) - 3f(n-1) + 3f(n-2) - f(n-3) = 2 - \delta(n).$$

Therefore equation (19) can be replaced by the system

$$\begin{aligned} h(n) - h(n-1) &= 2 - \delta(n), \\ g(n) - g(n-1) &= h(n), \\ f(n) - f(n-1) &= g(n). \end{aligned}$$

The solutions are found successively:

$$\begin{aligned} h(n) &= \alpha + \sum_{\nu=1}^{n}(2 - \delta(\nu)) = \alpha + \frac{3n - \delta(n)}{2}, \\ g(n) &= \beta + \sum_{\nu=1}^{n}\left(\alpha + \frac{3\nu - \delta(\nu)}{2}\right) = \beta + \alpha \cdot n + \frac{3n^2 + 2n - \delta(n)}{4}, \\ f(n) &= \sum_{\nu=1}^{n}\left(\beta + \alpha \cdot \nu + \frac{3\nu^2 + 2\nu - \delta(\nu)}{4}\right) \\ &= \frac{3}{4} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{1}{2} \cdot \frac{n(n+1)}{2} - \frac{1}{4} \cdot \frac{n + \delta(n)}{2} + \\ &\quad + \beta \cdot n + \alpha \cdot \frac{n(n+1)}{2} \\ &= \frac{n(n+2)(2n+1) - \delta(n)}{8} + \beta \cdot n + \alpha \cdot \frac{n(n+1)}{2}. \end{aligned} \tag{20}$$

From $f(1) = 1$ and $f(2) = 5$ we obtain $\alpha = \beta = 0$. Hence (20) is easily recognized as (8).

The Generating Function.

We define a formal power series $F(x)$ with n-th coefficient equal to $f(n)$, where $f(n)$ is supposed to solve (19).

$$F(x) = \sum_{n=1}^{\infty} f(n)x^n. \tag{21}$$

By successive multiplication of (21) with x we obtain

$$\begin{aligned} xF(x) &= \sum_{n=2}^{\infty} f(n-1)x^n \\ x^2F(x) &= \sum_{n=3}^{\infty} f(n-2)x^n \\ x^3F(x) &= \sum_{n=4}^{\infty} f(n-3)x^n. \end{aligned}$$

The difference equation (19) gives

$$\begin{aligned} F(x) - 3xF(x) + 3x^2F(x) - x^3F(x) &= \sum_{n=1}^{\infty}(2 - \delta(n))x^n \\ &= 2\sum_{n=1}^{\infty} x^n - \sum_{n=1}^{\infty} \delta(n)x^n = 2x\sum_{n=0}^{\infty} x^n - x\sum_{n=0}^{\infty} x^{2n} \\ &= \frac{2x}{1-x} - \frac{x}{1-x^2} = \frac{x + 2x^2}{1-x^2}. \end{aligned} \tag{22}$$

We obtain from (22)

$$\begin{aligned} F(x) &= \frac{2x+x^2}{(1-x)^3(1-x^2)} \\ &= \frac{\frac{3}{2}}{(1-x)^4} - \frac{\frac{7}{4}}{(1-x)^3} + \frac{\frac{1}{8}}{(1-x)^2} + \frac{\frac{1}{16}}{1-x} + \frac{\frac{1}{16}}{1+x}. \end{aligned} \tag{23}$$

Each of these partial fractions has a well-known power series expansion, obtained by successive differentiation of the geometric series:

$$\begin{aligned} \frac{1}{1+x} &= \sum_{n=0}^{\infty} (-x)^n \\ \frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\ \frac{1}{(1-x)^2} &= \sum_{n=0}^{\infty} (n+1)x^n \\ \frac{1}{(1-x)^3} &= \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n \\ \frac{1}{(1-x)^4} &= \sum_{n=0}^{\infty} \frac{(n+1)(n+2)(n+3)}{6} x^n. \end{aligned} \tag{24}$$

Substituting the expressions in (24) into (23) and comparing coefficients with (21) gives

$$\begin{aligned} f(n) &= \frac{3}{2}\frac{(n+1)(n+2)(n+3)}{6} - \frac{7}{4}\frac{(n+1)(n+2)}{2} \\ &+ \frac{1}{8}(n+1) + \frac{1}{16} + \frac{1}{16}(-1)^n = \frac{2n^3+5n^2+2n-\delta(n)}{8}, \end{aligned}$$

easily recognized as (8).

The Exponential Generating Function.

In [11] Moon and Pullman solve equation (18) with the sophisticated technique of exponential generating functions. We replace n with $n+3$ in (19) to get

$$f(n+3) - 3f(n+2) + 3f(n+1) - f(n) = 1 + \delta(n). \tag{25}$$

We define a formal power series $F(x)$ with n-th coefficient equal to $\frac{f(n)}{n!}$, where $f(n)$ is supposed to solve (25);

$$F(x) = \sum_{n=1}^{\infty} \frac{f(n)}{n!} x^n. \tag{26}$$

By successive differentiation of (26) we obtain

$$\begin{aligned} F'(x) &= \sum_{n=0}^{\infty} \frac{f(n+1)}{n!} x^n, \\ F''(x) &= \sum_{n=0}^{\infty} \frac{f(n+2)}{n!} x^n, \\ F'''(x) &= \sum_{n=0}^{\infty} \frac{f(n+3)}{n!} x^n. \end{aligned}$$

The difference equation (25) gives the differential equation

$$\begin{aligned} F'''(x) - 3F''(x) + 3F'(x) - F(x) &= \sum_{n=0}^{\infty} \frac{1+\delta(n)}{n!} x^n \\ &= e^x + \sinh(x) = \frac{3}{2}e^x - \frac{1}{2}e^{-x}. \end{aligned} \tag{27}$$

It is easy enough to solve (27) using standard techniques; we obtain the function $F(x)$ in the form

$$F(x) = \frac{1}{4}x^3 e^x + \alpha x^2 e^x + \beta x e^x + \gamma e^x + \frac{1}{16}e^{-x}$$

with α, β and γ to be chosen. When we substitute the series for e^x and e^{-x} we get

$$F(x) = \sum_{n=0}^{\infty} \left(\frac{1}{4}n(n-1)(n-2) + \alpha n(n-1) + \beta n + \gamma + \frac{(-1)^n}{16} \right) \frac{x^n}{n!}.$$

As we know the first few values of $f(n)$ we can easily deduce the values of the constants. We find that when

$$\begin{aligned} & n = 0, & \gamma + \frac{1}{16} &= f(0) = 0; \\ \text{when} \quad & n = 1, & \beta + \gamma - \frac{1}{16} &= f(1) = 1; \\ \text{and when } & n = 2, & 2\alpha + 2\beta + \gamma + \frac{1}{16} &= f(2) = 5. \end{aligned}$$

Hence

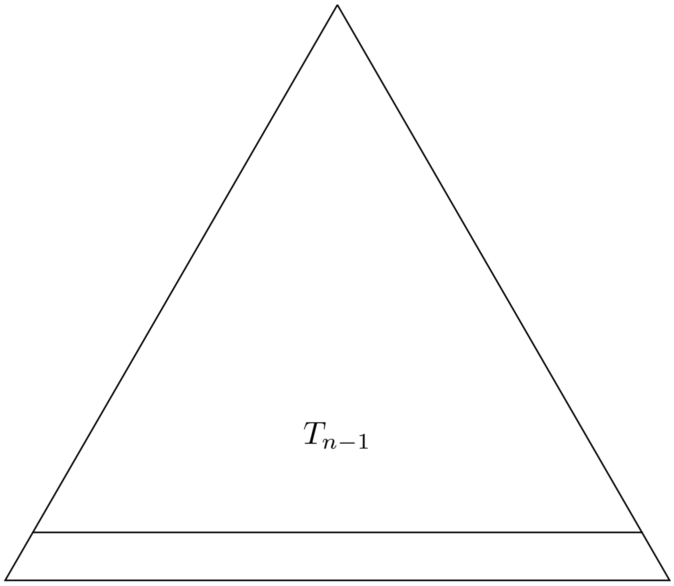
$$\gamma = -\frac{1}{16}, \quad \beta = \frac{9}{8}, \quad \alpha = \frac{11}{8};$$

so finally

$$\begin{aligned} f(n) &= \frac{2n(n-1)(n-2) + 11n(n-1) + 9n}{8} - \frac{1-(-1)^n}{16} \\ &= \frac{n(2n^2+5n+2)}{8} - \frac{\delta(n)}{8}, \end{aligned}$$

easily recognized as (8).

Induction or Recursion.



Figur 6

In this section we use a first order difference equation to find $f(n)$. Compared to (19), the gain in simplification on the left of the equality may be offset by complications on the right side:

$$f(n) - f(n-1) = F(n)$$

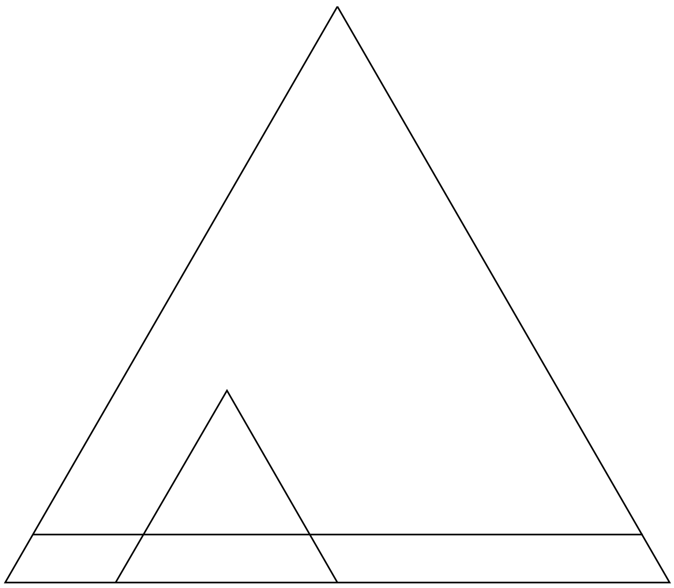
where $F(n)$ is some possibly complicated function. This approach is considered in [2].

This time we consider only one triangle T_{n-1} of size $n-1$ inside T_n (Figure 6), and then we try to compute the number of newcomers $F(n)$ that are in T_n but not in T_{n-1}.

It is easier to do this if we branch into right-way-up triangles and upside-down triangles, so we shall solve the two problems:

$$\begin{aligned} \triangle\,(n) - \triangle\,(n-1) &= G(n), \\ \triangledown(n) - \triangledown(n-1) &= H(n), \end{aligned}$$

where we look for G and H.



Figur 7

There is one right-way-up newcomer in T_n for each dot in T_{n-1} (see Figure 7), giving a total of

$$G(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \binom{n+1}{2}.$$

From this follows immediately

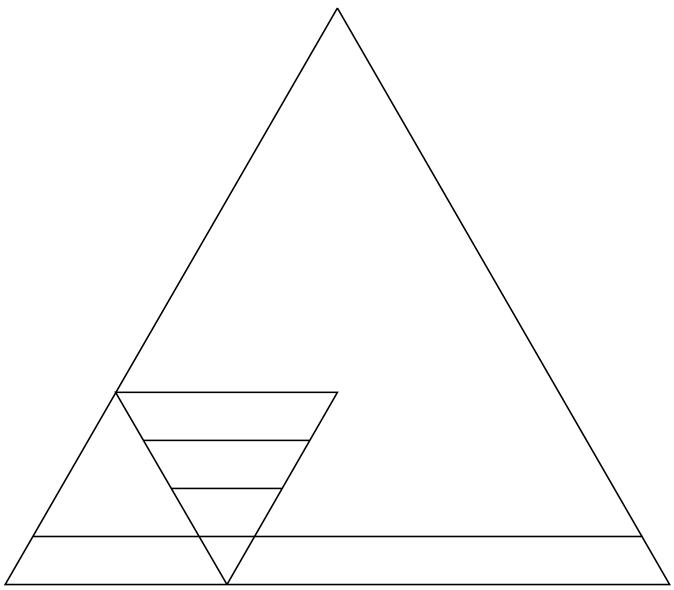
$$\triangle\,(n) = \sum_{\nu=1}^{n} G(\nu) = \sum_{\nu=1}^{n} \binom{\nu+1}{2} = \binom{n+2}{3}. \tag{28}$$

The sum (28) was verified directly in (10), but it can also be proved with induction using the formula

$$\binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i}$$

with $i = 3$. Actually, the induction step is:

$$\begin{aligned} \sum_{\nu=1}^{n} \binom{\nu+1}{2} &= \sum_{\nu=1}^{n-1} \binom{\nu+1}{2} + \binom{n+1}{2} \\ &= \binom{n+1}{3} + \binom{n+1}{2} = \binom{n+2}{3}. \end{aligned}$$



i

Figur 8

As we are used to by now, the function $H(n)$ is more tricky to find. All newcomers have their bottom vertex on the bottom side of the triangle T_n (Figure 8). For each dot i, $i = 1, ..., \left[\frac{n}{2}\right]$, there is room for i new triangles inside T_n. Hence we have by symmetry

$$H(n) = 2\sum_{i=1}^{\left[\frac{n}{2}\right]} i - (1-\delta(n))\frac{n}{2}$$

where we were careful not to count $i = \frac{n}{2}$ twice when n is even. We compute that

$$\begin{aligned} H(n) &= \left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right]+1\right) - (1-\delta(n))\,\frac{n}{2} \\ &= \frac{n-\delta(n)}{2}\left(\frac{n-\delta(n)}{2}+1\right) - \frac{n}{2} + \frac{n}{2}\delta(n) \\ &= \frac{n^2}{4} + \delta(n)\left(\frac{n}{2} - \frac{n}{4} - \frac{n}{4} + \frac{1}{4} - \frac{1}{2}\right) \\ &= \frac{n^2}{4} - \frac{\delta(n)}{4}. \end{aligned}$$

We proceed as in (17) and get

$$\nabla(n) = \frac{n(n+2)(2n-1)}{24} - \frac{\delta(n)}{8},$$

and so eventually, (11).

The Good Reference.

One way to reach a solution is to ask the public. When the question "How many triangles?" was asked in [4], approximately 60 persons replied with the formula, most of them referring to [8]. In [2] and [13] the problem was called "well-known," in [13] based on the references [6], [10], [11], and [12]. (We may now think of changing the attribute to "well-solved.")

Conclusion.

This is a story about the power of mathematics. Each of the well-established methods of counting proves its worth in the hunt for the tricky triangles by allowing the escape of not even one triangle.

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