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Prove that for any positive integer, n,

$$\sum_{k=0}^{n} \frac{\binom{n}{k}^2}{(2k+1)\binom{2n}{2k}} = \frac{2^{4n}(n!)^4}{(2n)!(2n+1)!}$$

Proof:

Introducing the notation of a descending factorial with specified stepsize, in casu 1 or 2:

$$[x,d]_n := \begin{cases} \prod_{j=0}^{n-1}(x-jd) & n \in \mathbb{N} \\ 1 & n=0 \\ \prod_{j=1}^{-n}\frac{1}{x+jd} & -n\in\mathbb{N},\ -x \notin \{d, 2d, \cdots, -nd\} \end{cases}$$

we proceed from the left side writing the binomial coefficients out as factorials

$$\sum_{k=0}^{n} \binom{n}{k} \frac{n!(2k)!(2n-2k)!}{k!(n-k)!(2n)!(2k+1)}$$

Splitting the factorials with a factor 2 in two taking every second factor to get factorials with stepsize 2 we get

$$\sum_{k=0}^{n} \binom{n}{k} \frac{n![2k,2]_k[2k-1,2]_k[2n-2k,2]_{n-k}[2n-2k-1,2]_{n-k}}{k!(n-k)![2n,2]_n[2n-1,2]_n(2k+1)}$$

Further we change the factorials to stepsize 1 by dividing with the appropriate powers of 2:

$$\sum_{k=0}^{n} \binom{n}{k} \frac{n!k!2^k[k-\frac{1}{2},1]_k 2^k (n-k)!2^{n-k}[n-k-\frac{1}{2},1]_{n-k}2^{n-k}}{k!(n-k)!n!2^n[n-\frac{1}{2},1]_n 2^n(2k+1)}$$

Cancelling common factors we get

$$\sum_{k=0}^{n} \binom{n}{k} \frac{[k-\frac{1}{2},1]_k[n-k-\frac{1}{2},1]_{n-k}}{[n-\frac{1}{2},1]_n(2k+1)} = \frac{1}{[n-\frac{1}{2},1]_n}\sum_{k=0}^{n} \binom{n}{k} \frac{[k-\frac{1}{2},1]_k[n-k-\frac{1}{2},1]_{n-k}}{2k+1}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

To get rid of the denominator we introduce the factorial

$$[n+\tfrac{1}{2},1]_n \cdot \tfrac{1}{2} = [n+\tfrac{1}{2},1]_{n+1} = [n+\tfrac{1}{2},1]_{n-k}(k+\tfrac{1}{2})[k-\tfrac{1}{2},1]_k$$

to get

$$\frac{1}{2k+1} = \frac{[n+\frac{1}{2},1]_{n-k}[k-\frac{1}{2},1]_k}{[n+\frac{1}{2},1]_n}$$

Then we may write

$$\frac{1}{[n-\frac{1}{2},1]_n[n+\frac{1}{2},1]_n}\sum_{k=0}^{n}\binom{n}{k}[k-\tfrac{1}{2},1]_k[n-k-\tfrac{1}{2},1]_{n-k}[n+\tfrac{1}{2},1]_{n-k}[k-\tfrac{1}{2},1]_k$$

Now we change the signs of all factors in the factorials containing the variable k:

$$\frac{1}{[n-\frac{1}{2},1]_n[n+\frac{1}{2},1]_n}\cdot$$
$$\sum_{k=0}^{n}\binom{n}{k}[-\tfrac{1}{2},1]_k(-1)^k[-\tfrac{1}{2},1]_{n-k}(-1)^{n-k}[n+\tfrac{1}{2},1]_{n-k}[-\tfrac{1}{2},1]_k(-1)^k$$

Organized nicely to

$$\frac{(-1)^n}{[n-\frac{1}{2},1]_n[n+\frac{1}{2},1]_n}\sum_{k=0}^{n}\binom{n}{k}[-\tfrac{1}{2},1]_k^2[-\tfrac{1}{2},1]_{n-k}[n+\tfrac{1}{2},1]_{n-k}(-1)^k$$

This may be recognized as the Pfaff-Saalschütz formula, (9.1), in my recent textbook, *Summa Summarum,* A K Peters 2007:

Theorem 9.1. *If the numbers satisfy $a_1+a_2+b_1+b_2 = n-1$ we have the Pfaff-Saalschütz formula*

$$\sum_{k=0}^{n}\binom{n}{k}[a_1,1]_k\,[a_2,1]_k\,[b_1,1]_{n-k}\,[b_2,1]_{n-k}\,(-1)^k = [b_1+a_1,1]_n\,[b_1+a_2,1]_n\,(-1)^n$$

So we obtain:

$$\frac{1}{[n-\frac{1}{2},1]_n[n+\frac{1}{2},1]_n}[-1,1]_n^2 = \frac{n!^2}{[n-\frac{1}{2},1]_n[n+\frac{1}{2},1]_n} =$$
$$=\frac{2^{2n}n!^2}{[2n-1,2]_n[2n+1,2]_n} = \frac{2^{4n}n!^4}{[2n-1,2]_n[2n,2]_n^2[2n+1,2]_n} = \frac{2^{4n}(n!)^4}{(2n)!(2n+1)!}$$