THE ART OF READY RECKONING

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Introduction. Recall Dustin Hoffman, who – as *Rainman* – throws numbers out of his sleeves. He looks at a box of matches dropt on the floor and immediately claims "246" (or whatever).
I feel rather like Alice in "Through the Looking Glass and What Alice Found There," Chapter IX, Queen Alice, (Lewis Carroll, 1871):
"Can you do Addition?" the White Queen asked.
"What's one and one and one and one and one and one and one and one and one and one?"
"I don't know," said Alice, "I lost count."
"She can't do Addition," the Red Queen interrupted.
I do not believe that autistic people have special gifts for dealing with numbers. But if they are interested in numbers, their capacity for monomania allows them to play with them day and night, so that numbers become their personal friends. In this respect they are like the famous Indian mathematician, S. Ramanujan (1887–1920). According to a well–known anecdote, G. H. Hardy's (1877–1947) once remarked to Ramanujan: "I think the number of my taxicab was 1729, it seemed to me a rather dull number." Ramanujan answered, "No, Hardy! It is a very interesting number. It is the smallest number expressible as the sum of two cubes in two different ways."

$$1729 = 10^3 + 9^3 = 12^3 + 1^3$$

Playing with numbers you denote that kind of coincidences.
Forty years ago, the main question was, whether you could reckon without using your head. Now we are so used to calculators that it becomes a challenge to manage without one.

The multiplication table. To reckon in your head you have to know by heart the tables of products (and sums) at least up to 9.
It is no big deal, altogether we must know 45 sums and 36 products (or, 31 prime resolutions).

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In this requirement we are lucky. In ancient Assyria or Babel a couple of thousend years BC., they used a positional system with base 60 (we still have the reminder of that system in our time units of minutes and seconds), so they needed to know the products up to 59. Remembering the rule of commutation this results in

$$\frac{58 \times 57}{2} + 58 = 30 \times 57 + 1 = 1711$$

products to remember.

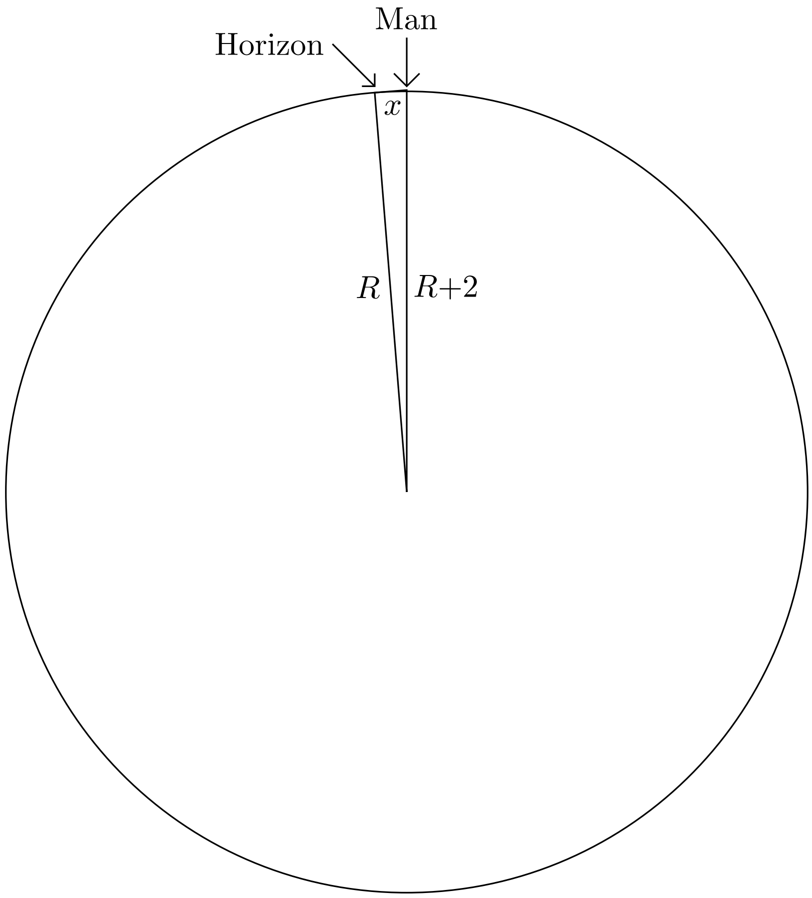
The multiplication table over 10. Multiplying a one digit number by a two digit number is hard to simplify. For example $7 \times 17 = 70 + 49 = 119$. Sometimes prime resolutions may give shortcuts, e.g., $9 \times 18 = 9 \times 9 \times 2 = 81 \times 2 = 162$. But for two numbers in the tens a rewriting may improve

$$(10 + x)(10 + y) = (10 + x + y)10 + xy$$

e.g.,

$$13 \times 19 = (13 + 9)10 + 3 \times 9 = 220 + 27 = 247$$
$$17 \times 19 = (17 + 9)10 + 7 \times 9 = 260 + 63 = 323$$

So, learning by heart is unnecessary.



Without the pocket calculator. Think of yourself on the beach looking out over the ocean, in swim suits, and hence without a pocket for a calculator, wondering, how far away is the horizon? Can you calculate that in your head? Indeed!
The first problem is that you do not remember the radius of the earth!
But we recall from our history lessons that the meter was defined by setting the distance from the equator to the northe pole equal to 10000 km. So the circumference of the earth is by definition 40000 km! So, assuming that your eyes are approximately 2 m over the surface:

$$R^2 + x^2 = \left(R + \frac{2}{1000}\right)^2 = R^2 + \frac{4R}{1000} + \frac{4}{1000000}$$

Reckoning that the square $\frac{4}{1000000}$ is next to nothing, and observing that the the squares R^2 cancel, we get

$$x^2 = \frac{4R}{1000} = \frac{4 \times 6363}{1000} = 25.4$$

Hence we conclude that the distance is slightly more than 5 km.
How much more? Yes, e may get closer: if we add a little (ϵ) 5, we get the square

$$(5 + \epsilon)^2 = 25 + 10 \times \epsilon + \epsilon^2$$

so that if the last square is quite small, we may choose $\epsilon = 0.04$ and therefore the distance to the horizon in km is

$$5.04$$

How much did we calclutate? We divided 70 by 11 and multiplied 636 with 4.

Some useful formulas. A couple of algebraic formulas are extremely useful. They are the square of a sum of two terms and the product of a sum and a difference of the same two numbers:

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x + y)(x - y) = x^2 - y^2$$

e.g.,

$$7 \times 13 = (10 - 3)(10 + 3) = 100 - 9 = 91$$

and – slightly changed –

$$x^2 = y^2 + (x + y)(x - y)$$

e.g.,

$$59^2 = 60^2 + (59 + 60)(59 - 60) = 3600 - 119 = 3481$$

For $y = 1$ denote

$$(x + 1)^2 = x^2 + 2x + 1 = x^2 + x + (x + 1)$$

e.g.,

$$21^2 = 20^2 + 20 + 21 = 400 + 41 = 441$$

$$22^2 = 21^2 + 21 + 22 = 441 + 43 = 484$$

Squares. Numbers ending with the digit 5 er extremely easy to suare:

$$(x+5)^2 = 5^2 + (x-5)(x+5)$$

e.g.,

$$\begin{aligned} 25^2 &= 25 + 20 \times 30 = 625 \\ 85^2 &= 25 + 80 \times 90 = 7225 \\ 125^2 &= 25 + 120 \times 130 = 15625 \end{aligned}$$

Squares may also be easily calculated with a reference to 50 or 100. The following formulals may be used, if we know the squares of some smaller numbers:

$$\begin{aligned} (50+x)^2 &= 2500 + x \times 100 + x^2 = (25+x) \times 100 + x^2 \\ (100+x)^2 &= 10000 + 2x \times 100 + x^2 = (100+2x) \times 100 + x^2 \end{aligned}$$

If you remember or find easely

$$18^2 = 260 + 64 = 324$$

or perhaps

$$\begin{aligned} 23^2 &= 3^2 + (23-3)(23+3) = 9 + 20 \times 26 = 9 + 520 = 529 \\ 23^2 &= (25-2)^2 = 625 - 100 + 4 = 529 \end{aligned}$$

then the formulas above yield

$$\begin{aligned} 54^2 &= (25+4) \times 100 + 4^2 = 2916 \\ 26^2 &= (25-24) \times 100 + 24^2 = 100 + 576 = 676 \\ 37^2 &= (25-13) \times 100 + 13^2 = 1200 + 169 = 1369 \\ 92^2 &= (100-16) \times 100 + 8^2 = 8464 \\ 86^2 &= (100-28) \times 100 + 14^2 = 7200 + 196 = 7396 \\ 117^2 &= (100+34) \times 100 + 17^2 = 13400 + 289 = 13689 \end{aligned}$$

More general formulas. Many of the formulas above are special cases of the following surprisingly useful "triviality";

$$(a+b)(a+c) = (a+b+c)a + bc$$

It's obvious that for $b = c$ or $b = -c$ we get some of the previous examples. But in general it is useful with a convenient choice of a, e.g., a multiple of

10 or 100.

$$\begin{aligned}
23 \times 29 &= 32 \times 20 + 3 \times 9 = 640 + 27 = 667 \\
23 \times 29 &= 22 \times 30 + (-7)(-1) = 660 + 7 = 667 \\
37 \times 34 &= 41 \times 30 + 7 \times 4 = 1230 + 28 = 1258 \\
37 \times 34 &= 31 \times 40 + (-3)(-6) = 1240 + 18 = 1258 \\
106 \times 97 &= (100 + 6 - 3) \times 100 + 6 \times (-3) = 10300 - 18 = 10282 \\
705 \times 694 &= (700 + 5)(700 - 6) = 699 \times 700 - 30 = 490000 - 700 - 30 = 489270 \\
12^3 &= 36 \times 48 = (40 - 4)(40 + 8) = 44 \times 40 - 4 \times 8 = 1760 - 32 = 1728
\end{aligned}$$

The cross product. The general case with no obvious shortcuts may be handled with the so called cross product. The numbers are imagined in two lines:

$$\begin{array}{ccc} x & & y \\ | & \times & | \\ z & & w \end{array}$$

Then you create the two products $xz \times 100$ and yw, to be imagined as a 4–digit number. Then you create the sum $xw + yz$ to be multiplied with 10 and added. E.g.,

$$\begin{aligned}
63 \times 46 &= 2418 + (6 \times 6 + 3 \times 4)10 = 2418 + (36 + 12)10 = 2418 + 480 = 2898 \\
36 \times 48 &= 1248 + (3 \times 8 + 6 \times 4)10 = 1248 + 480 = 1728
\end{aligned}$$

If $x = z$ or $y = w$, you may make the last part easier:

$$\begin{aligned}
34 \times 37 &= (900 + 28) + 3(4 + 7)10 = 928 + 330 = 1258 \\
63 \times 43 &= (2400 + 9) + 3(6 + 4)10 = 2409 + 300 = 2709
\end{aligned}$$

To juggle with many numbers. Once upon a time the later very famous mathematician, Carl Friedrich Gauß (1777-1855), annoyed the teacher in his first grade class being so brilliant. To get a rest the teacher asked him to add the numbers from 1 to 100. Gauß immediately anwered 5050.
He imagined them in two lines forward and backward:

$$\begin{array}{cccccc} 1 & 2 & 3 & ... & 99 & 100 \\ 100 & 99 & 98 & ... & 2 & 1 \end{array}$$

We note that there are 100 sums of 2 numbers, each given the result 101. So, twice the sum in question is 10100.
The general formula is

$$1 + 2 + \cdots n = \frac{n(n+1)}{2}$$

If we similarly added the odd numbers we would get

$$1+3=4, \quad 4+5=9, \quad 9+7=16, \quad 16+9=25$$

all squares. Is this a rule?
Following Gauß, we may write the sum of the first n odd numbers forward and backward:

$$\begin{array}{cccccc} 1 & 3 & 5 & ... & 2n-3 & 2n-1 \\ 2n-1 & 2n-3 & 2n-5 & ... & 3 & 1 \end{array}$$

and note that there are n sums of 2 numbers each adding to $2n$. Twice the sum is hence $2n^2$.
Formulas for the sums of powers are known for all exponents. The sum of the squares is

$$1+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

For example, we have

$$1+4+9+16+25+36 = \frac{6\times 7\times 13}{6} = 91$$

For the cubes we have a beauty:

$$1+8=9, \quad 9+27=36, \quad 36+64=100, \quad 100+125=225, \quad ...$$

The general formula is

$$1^3+2^3+3^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

For $n=6$ since $6^3 = 12\times 18 = 216$, we obtain $225+216 = 441 = 21^2$, or we use the formula $\left(\frac{6\times 7}{2}\right)^2 = 21^2$.
Note that

$$(1+2+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$$

The produkt of the 10 smallest primes. You can do that by now. We want

$$2\times 3\times 5\times 7\times 11\times 13\times 17\times 19\times 23\times 29$$

Let's divide the problem into small parts:

$$\begin{aligned} &2\times 3\times 5 \\ &7\times 11\times 13 \\ &17\times 19 \\ &23\times 29 \end{aligned}$$

The first product is within the table, $2 \times 3 \times 5 = 30$. Let's use our methods to the next product. First we remember $7 \times 13 = 91$. Then we compute $91 \times 11 = 901 + 100 = 1001$.
For the next case we remember $17 \times 19 = 323$ and $23 \times 29 = 667$.
So, without much trouble we got

$$
\begin{aligned}
2 \times 3 \times 5 &= 30 \\
7 \times 11 \times 13 &= 1001 \\
17 \times 19 &= 323 \\
23 \times 29 &= 667
\end{aligned}
$$

Now all we need is to find

$$30 \times 1001 \times 323 \times 667$$

You may be able to remember these four numbers. Now the last one is approximately two–thirds of 1000. This is actually convenient:

$$30 \times 667 = 30 \times \left(\frac{2}{3} \times 1000 + \frac{1}{3}\right) = 20 \times 1000 + 10 = 20010$$

The product of this one with 1001 is easy enough to do in your head

$$1001 \times 20010 = 20030010$$

Now we multiply this by 323, which is particularly easy:

$$323 \times 20030010 = 6469693230$$

The trouble is to give it a name: we get six billions, four hundred and sixty–nine millions, six hundred and ninety–three thousands and two hundred and thirty.

Divisibility with a prime. 2, 3 and 5 are easy enough just looking at the last digit or taking the sum of the digits. The first problem is dividing by 7. The theorem says that if the number is written $a \times 10 + b$ (with b as the last digit), then it is divisible by 7, exactly when the number $a - 2 \times b$ is divisible by 7.
The trick works for all primes. We look for the smallest multiple of the prime which differs by only 1 from a multiple of 10.

$$
\begin{aligned}
3 \times 7 &= 21 \\
1 \times 11 &= 11 \\
3 \times 13 &= 39 \\
3 \times 17 &= 51 \\
1 \times 19 &= 19 \\
3 \times 23 &= 69 \\
1 \times 29 &= 29
\end{aligned}
$$

Now we add or subtract this multiple of the prime, a process that does not alter the possibility of divisibility by the prime.

$$\begin{aligned}
10a+b-21b &= 10a-20b = 10(a-2b)\\
10+b-11b &= 10a-10b = 10(a-b)\\
10a+b+39b &= 10a+40b = 10(a+4b)\\
10a+b-51b &= 10a-50b = 10(a-5b)\\
10a+b+19b &= 10a+20b = 10(a+2b)\\
10a+b+69b &= 10a+70b = 10(a+7b)\\
10a+b+29b &= 10a+30b = 10(a+3b)
\end{aligned}$$

For example,

$$\begin{array}{rcccc}
7: & (-2) & 1001 & 100-2=98 & 9-16=-7\\
11: & (-1) & 1001 & 100-1=99 & 9-9=0\\
13: & (+4) & 1001 & 100+4=104 & 10+16=26\\
17: & (-5) & 323 & 32-15=17 & \\
19: & (+2) & 323 & 32+6=38 & 3+16=19\\
23: & (+7) & 667 & 66+49=115 & 11+35=46\\
29: & (+3) & 667 & 66+21=87 & 8+21=29
\end{array}$$

For 11 this leads to the general rule that 11 divides a number if it divides the alternating sum of the digits, i.e. the sum of every second digit minus the sum of the rest.

The Josephus problem. It is told that after the Romans conquered Jerusalem in the year 70 A.D., the Jewish writer Josephus, with 40 other Jews, did hide in a cellar. They agreed to commit suicide to escape capture by the Romans. Josephus did not really agree, so he suggested the following procedure: They should count 1–2, 1–2, ... and each time one said “2” he was decapitated. So they lined up in a circle and the leader started to count. The problem for Josephus was to position himself so that he would be the last to count, so that he could change his mind.

Now you may learn a piece of mathematics. The question is, in which cases is the problem almost trivial? The answer is that for any power of 2, it is advantageous to be no. 1, since after one round you are again no. 1, and the number of those who remain is the half of a power of 2, which is again a power of 2.

But seeing this it is easy to solve the general problem: You want to be the one to count when the number of Jews is reduced to a power of 2. In our case of 41, the power is 32, so you must be the one to count after the suicide of 9 others. So you want to be no. 19.